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ON THE SATAKE ISOMORPHISM

G. LUSZTIG*

Department of Mathematics
MIT
Cambridge, MA 02139, USA

Institute for Advanced Study
Princeton, NJ 08450, USA

gyuri@mit.edu

Dedicated to the memory of my dear friend, Jim Humphreys

Abstract. In a 1983 paper, the author has established a (decategorified) Satake equivalence for affine Hecke algebras. In this paper, we give new proofs for some results of that paper, one based on the theory of J-rings and one based on the known character formula for rational representations of a reductive group in positive, large characteristic. We also give an extension of that formula to disconnected groups.

Introduction

0.1. Let H_q be the affine Hecke algebra over $\mathbf{C}$ (with equal parameters q, a prime power) associated to an affine Weyl group W (defined in terms of the dual G^* of an adjoint group G). Let $H_{0,q}$ be the Hecke algebra over $\mathbf{C}$ (with equal parameters q) associated to the corresponding finite Weyl group $W_0 \subset W$. Let H_q^{sph} be the vector subspace of H_q consisting of elements which are eigenvectors for the left and right multiplication by $H_{0,q}$, with eigenvalue defined by the one dimensional representation of $H_{0,q}$ corresponding to the unit representation of W_0. Then H_q^{sph} is an algebra for the product $f * f' = \left(\sum_{w\in W_0} q^{|w|}\right)^{-1} f f'$ where ff' is the product in H_q and $||$ is the standard length function on W_0. Let Q be the group of translations in W. The classical Satake isomorphism states that the algebra H_q^{sph} is isomorphic to the algebra of W_0-invariants in the group algebra $\mathbf{C}[Q]$. In [L83] we gave a refinement of this isomorphism in which the basis of $\mathbf{C}[Q]$ formed by the irreducible representations of a semisimple group with Weyl group W_0 and for which Q is the lattice of roots corresponds to a basis β of H_q^{sph} formed by certain elements of the basis [KL79] of H_q, suitably normalized. This shows in particular that

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Corresponding Author: G. Lusztig, e-mail: gyuri@mit.edu

(a) the structure constants of the algebra H_q^{sph} with respect to β are integers independent of q.

This is the starting point of "geometric Satake equivalence" (which we do not discuss in this paper).

0.2. In this paper, we show (see 1.5) that the structure constants in 0.1(a) can be interpreted as structure constants for a certain subring J_* of the J-ring attached to W with respect to the standard basis of J_*. (We actually prove a more general statement involving a weight function on W.) This gives a new (and simpler) proof of 0.1(a). We also give another approach to 0.1(a) based on the character formula for simple rational modules of a semisimple group in characteristic $p \gg 0$. At the time when [L83] was written, this character formula was only conjectured and providing evidence for the conjecture was one of the motivations which led the author to [L83]. We also state an extension of that character formula to certain disconnected groups.

0.3. The results in this paper hold with similar proofs also for extended affine Weyl groups; to simplify notation we do not treat this slightly more general case.

Contents

1. Weighted affine Weyl groups and the ring J_*

1.1. Let W be an irreducible affine Weyl group with a given set S of simple reflections assumed to have at least two elements. Let Q be the set of all translations in W: that is, the set of all $t \in W$ such that the W-conjugacy class of t is finite. It is known that Q is a normal free abelian subgroup of finite index of W. We write the group operation in Q as $+$. We fix $s_0 \in S$ such that W is generated by Q and by the finite subgroup W_0 generated by $S_0 = S - \{s_0\}$. (Such s_0 is said to be "special".) Let $w \mapsto |w|$ be the length function of W. Let $Q^+ = \{x \in Q; |sx| = |x| + 1 \text{ for any } s \in S_0\}$. We have $W = \bigsqcup_{x \in Q^+} W_0 x W_0$. For any $x \in Q^+$ we denote by M_x the unique element in $W_0 x W_0$ such that $|M_x|$ is maximal, or equivalently, such that $|sM_x| = |M_x| - 1 = |M_x s|$ for all $s \in S_0$. In particular, M_0 is the longest element in W_0. Let $L : W \to \mathbf{N}$ be a weight function: that is, a function such that $L(ww') = L(w) + L(w')$ whenever w, w' in W satisfy $|ww'| = |w| + |w'|$. We assume that $L(s) > 0$ for any $s \in S$. Let v be an indeterminate. Let H be the $\mathbf{Q}(v)$-vector space with basis $\{T_w; w \in W\}$. We can regard H as an associative algebra in which $T_w T_{w'} = T_{ww'}$ if w, w' in W satisfy $|ww'| = |w| + |w'|$ and $(T_s + v^{-L(s)})(T_s - v^{L(s)}) = 0$ for $s \in S$. Let $\{c_w; w \in W\}$ be the basis of H defined in [L83a] and [L03, 5.2]. (See [KL79] for the case $L = ||$.) We have $c_w = \sum_{y \in W} v^{-L(w)+L(y)} P_{y,w;L} T_y$ where $P_{y,w;L} \in \mathbf{Z}[v^2]$ is zero for all but finitely many y (see [L03, 5.4]). Let $\mathcal{A} = \mathbf{Z}[v, v^{-1}]$ and let $H_{\mathcal{A}}$ be the $\mathcal{A}$-submodule of H spanned by $\{T_w; w \in W\}$ or equivalently by $\{c_w; w \in W\}$. This is a subring of H.

We set $\pi_L = \sum_{e \in W_0} v^{2L(e)}$. For $x \in Q^+$, we set $\mathbf{c}_x = (v^{L(M_0)}/\pi_L) c_{M_x} \in H$.

1.2. From [L03, 8.6] we see that for $w \in W, x \in Q^+, y \in Q^+$ we have

$$c_w c_{M_y} \in \sum_{u \in W; |us|=|u|-1 \ \forall s \in S_0} \mathcal{A} c_u,$$
$$c_{M_x} c_w \in \sum_{u \in W; |su|=|u|-1 \ \forall s \in S_0} \mathcal{A} c_u.$$

It follows that

$$c_{M_x} c_{M_y} \in \sum_{u \in W; |su|=|u|-1=|us| \ \forall s \in S_0} \mathcal{A} c_u$$

so that

(a) $$c_{M_x} c_{M_y} = \sum_{z \in Q^+} \tilde{r}_{x,y,z;L} c_{M_z}$$

where $\tilde{r}_{x,y,z;L} \in \mathcal{A}$ is zero for all but finitely many z, hence

(b) $$\mathbf{c}_x \mathbf{c}_y = \sum_{z \in Q^+} r_{x,y,z;L} \mathbf{c}_z$$

where $r_{x,y,z;L} = v^{L(M_0)} {\pi_L}^{-1} \tilde{r}_{x,y,z;L}$.

1.3. Let $w \in W$. We write $c_w = \sum_{u \in W} l_u T_u$ with $l_u \in \mathcal{A}$. From [L03, 6.6a] we see by induction on $|w|$ that:

(a) If $s \in S$, $|sw| = |w| - 1$, then $c_w \in (T_s + v^{-L(s)}) H_{\mathcal{A}}$; in other words, $l_{su} = v^{L(s)} l_u$ for any $u \in W$ such that $|su| = |u| + 1$.

Assume now that $|w| = |M_0 w| + |M_0|$. We show

(b) For any $u \in W$ such that $|M_0 u| = |u| + |M_0|$ and any $e \in W_0$ we have $l_{eu} = v^{L(e)} l_u$.

We argue by induction on $|e|$. If $|e| = 0$, there is nothing to prove. Assume now that $|e| > 0$. We have $e = se'$ for some $s \in S_0, e' \in W_0$ with $|e| = |e'| + 1$. By the induction hypothesis we have $l_{e'u} = v^{L(e')} l_u$. We have $|se'u| = |e'u| + 1$ (both sides are equal to $|se'| + |u|$). Using (a) we have $l_{se'u} = v^{L(s)} l_{e'u}$, hence

$$l_{eu} = l_{se'u} = v^{L(s)} v^{L(e')} l_u = v^{L(s)+L(e')} l_u = v^{L(se')} l_u = v^{L(e)} l_u.$$

This proves (b).

In the setup of (b) we have

$$\begin{aligned} c_w &= \sum_{u \in W; |M_0 u| = |M_0| + |u|} l_u \sum_{e \in W_0} v^{L(e)} T_e T_u \\ &= \sum_{e \in W_0} v^{L(e)} T_e \sum_{u \in W; |M_0 u| = |M_0| + |u|} l_u T_u \\ &= v^{L(M_0)} c_{M_0} \sum_{u \in W; |M_0 u| = |M_0| + |u|} l_u T_u. \end{aligned}$$

It follows that

(c) If $|w| = |M_0 w| + |M_0|$, then $c_w \in c_{M_0} H_{\mathcal{A}}$.

Similarly, we have

(d) If $w' \in W$ satisfies $|w'| = |w' M_0| + |M_0|$ then $c_{w'} \in H_{\mathcal{A}} c_{M_0}$.

Taking $w' = M_x$, $w = M_y$ with x, y in Q^+, we see from (c), (d) that

$$c_{M_x} c_{M_y} \in H_{\mathcal{A}} c_{M_0} c_{M_0} H_{\mathcal{A}} \subset \pi_L H_{\mathcal{A}}$$

(we use that $c_{M_0} c_{M_0} \in \pi_L H_{\mathcal{A}}$). Combining this with 1.2(a), we see that $\tilde{r}_{x,y,z;L}$ in 1.2 is in $\pi_L \mathcal{A}$.

If $x \in Q^+$, then from (c), (d) we see that

$$c_{M_0} c_{M_x} = c_{M_x} c_{M_0} = v^{-L(M_0)} \pi_L c_{M_x}.$$

(We use that $c_{M_0} c_{M_0} = v^{-L(M_0)} \pi_L c_{M_0}$.)

1.4. From [L03, 13.4], for any w, w' in W we have

$$T_w T_{w'} \in v^{L(M_0)} \sum_{w'' \in W} \mathbf{Z}[v^{-1}] T_{w''}.$$

(In the case where $L = ||$ this is proved in [L85, §7]; the proof for general L is entirely similar.) As in [L03, 13.5], we deduce that for any w, w' in W we have

(a) $$c_w c_{w'} = \sum_{w'' \in W} h_{w,w',w''} c_{w''}$$

(finite sum) where $h_{w,w',w''} = N_{w,w',w'';L} v^{L(M_0)} \bmod v^{L(M_0)-1}$ with $N_{w,w',w'';L} \in \mathbf{Z}$.

Let J be the free abelian group with basis $\{\tau_w; w \in W\}$. We define a bilinear multiplication $J \times J \to J$ by

$$\tau_w \tau_{w'} = \sum_{w'' \in W} N_{w,w',w'';L} \tau_{w''},$$

(this is a finite sum). It is known [L03, 18.3] that this multiplication is associative if the conditions in [L03, 18.1] are satisfied.

Let J_* be the subgroup of J with basis $\{\tau_{M_x}; x \in Q^+\}$. From 1.2(b) we see that J_* is closed under the multiplication in J; thus for x, y in Q^+ we have

$$\tau_{M_x} \tau_{M_y} = \sum_{z \in Q^+} N_{M_x, M_y, M_z; L} \tau_{M_z}$$

(this is a finite sum).

1.5. Theorem.

(a) *For x, y in Q^+ we have*

$$\mathbf{c}_x \mathbf{c}_y = \sum_{z \in Q^+} N_{M_x, M_y, M_z; L} \mathbf{c}_z.$$

(b) *The subgroup R of H with $\mathbf{Z}$-basis $\{\mathbf{c}_x; x \in Q^+\}$ is closed under multiplication in H.*

(c) *The isomorphism of abelian groups $R \xrightarrow{\sim} J_*$ given by $\mathbf{c}_x \mapsto \tau_{M_x}$ is compatible with the multiplication. In particular, the multiplication in J_* is associative.*

For x, y, z in Q^+ we have

$$r_{x,y,z;L}\pi_L v^{-L(M_0)} = \tilde{r}_{x,y,z;L} = h_{M_x,M_y,M_z} = v^{L(M_0)}X$$
$$= \big(\textstyle\sum_{e\in W_0} v^{2L(e)}\big)Y = \big(\textstyle\sum_{e\in W_0} v^{-2L(e)}\big)Y'$$

where $X \in \mathbf{Z}[v^{-1}]$, $Y \in \mathcal{A}$, $Y' = v^{2L(M_0)}Y \in \mathcal{A}$. It follows that

$$\big(\textstyle\sum_{e\in W_0} v^{-2L(e)}\big)^{-1}X \in \mathbf{Z}[v, v^{-1}].$$

Since $X \in \mathbf{Z}[v^{-1}]$ and $\sum_{e\in W_0} v^{-2L(e)} \in 1 + v^{-1}\mathbf{Z}[v^{-1}]$, we have

$$\big(\textstyle\sum_{e\in W_0} v^{-2L(e)}\big)^{-1}X \in \mathbf{Z}[[v^{-1}]];$$

but this is also in $\mathbf{Z}[v, v^{-1}]$ hence it must be in $\mathbf{Z}[v^{-1}]$. Thus $(\sum_{e\in W_0} v^{-2L(e)})^{-1}X \in \mathbf{Z}[v^{-1}]$: that is,

(d) $$r_{x,y,z;L} \in \mathbf{Z}[v^{-1}].$$

From the definition of c_w, we have

$$\bar{h}_{w,w',w''} = h_{w,w',w''}$$

for any w, w', w'' in W, where $\bar{\ }: \mathcal{A} \to \mathcal{A}$ is the ring involution which takes v^n to v^{-n} for any n. Using this and the fact that $\pi_L v^{-L(M_0)}$ is fixed by $\bar{\ }$, we see that the left-hand side of (d) is fixed by $\bar{\ }$ and hence is necessarily in $\mathbf{Z}$.

Taking the coefficient of $v^{L(M_0)}$ in the two sides of the equality

$$r_{x,y,z;L}\pi_L v^{-L(M_0)} = h_{M_x,M_y,M_z}$$

in which $r_{x,y,z;L} \in \mathbf{Z}$, we see that $r_{x,y,z;L} = N_{M_x,M_y,M_z}$. This completes the proof of (a). Now (b), (c) are immediate consequences of (a). □

1.6. The ring R has unit element $\mathbf{c}_0$ and is known to be commutative; it follows that the ring J_* has unit element τ_{M_0} and is commutative. In the case where $L = ||$, 1.5(b) recovers a result in [L83]. For general L, 1.5(b) recovers a result in [K05]. But the present proof is simpler than that in these references.

1.7. In this subsection, we assume that $L = ||$. In this case, the ring J in [L03, 18.3] is associative. In [L97] we have categorified J to a monoidal tensor category with simple objects indexed by W. In particular J_* is categorified to a monoidal tensor category $\underline{J}_*$. It is known that (as a consequence of 1.5(b)) R can be also categorified to a monoidal category $\mathbf{S}$ known as the "Satake category". The ring isomorphism $R \xrightarrow{\sim} J_*$ in 1.5(c) gives rise to an equivalence of monoidal categories $\mathbf{S} \xrightarrow{\sim} \underline{J}_*$.

2. Use of modular representations

2.1. In this section, we assume that $L = || : W \to \mathbf{N}$. Let $\mathbf{k}$ be an algebraically closed field of characteristic $p \geq 0$. Let G be an adjoint semisimple group over $\mathbf{k}$ with a fixed pinning (involving a maximal torus T). We assume that the Weyl group of G is W_0, the lattice of roots of G with respect to T is Q, and that $W = W_0 Q$ is the affine Weyl group associated in the usual way to the dual group G^*. Then Q^+ is the set of dominant weights of G. For $x \in Q^+$, let V_x be a Weyl module of G over $\mathbf{k}$ with highest weight x; let L_x be a simple rational G-module with highest weight x.

Let $\rho \in Q_{\mathbf{R}} = \mathbf{R} \otimes Q$ be half the sum of all positive roots of G.

Let $\mathcal{H}$ be the set of hyperplanes $H_{\check{\alpha},m} = \{x \in Q_{\mathbf{R}}; \check{\alpha}(x+\rho) = mp\}$ for various coroots $\check{\alpha} : Q_{\mathbf{R}} \to \mathbf{R}$ and various $m \in \mathbf{Z}$. (When $p = 0$, $\mathcal{H}$ consists of the hyperplanes $H_{\check{\alpha},0}$.)

2.2. We now assume that p is a prime number, $p \gg 0$. Following Verma [Ve] we identify W with the subgroup W_p of the group of affine transformations of $Q_{\mathbf{R}}$ generated by the reflections in the hyperplanes in $\mathcal{H}$ which preserve the set $\mathcal{H}$.

Let $x \in Q^+$ be such that $x \notin \bigcup_{\check{\alpha},m} H_{\check{\alpha},m}$ and $\check{\alpha}_0(x) \leq p(p-h+2)$ where $\check{\alpha}_0$ is the highest coroot and h is the Coxeter number. It is known [AJS94], [KL94], [KT95] that, as virtual T-modules, we have

(a) $$L_x = \textstyle\sum_{y \in Z_x} (-1)^{|w_y w_x|} \dim(\mathcal{V}_{w_y,w_x}) V_y,$$

where Z_x is the set of all $y \in Q^+$ in the same W_p-orbit as x; w_x, w_y are certain well-defined explicit elements of W_p; $\mathcal{V}_{w_y,w_x}$ is a $\mathbf{C}$-vector space of dimension $P_{w_y,w_x;||}(1)$ defined in terms of the stalks of the intersection cohomology complex of an affine Schubert variety associated to G^*.

As shown in [L17, comments to [53]], from (a) with x of the form $x = px', x' \in Q^+$, one can deduce that for $y' \in Q^+$ we have

(b) $$P_{M_{y'},M_{x'};||}(1) = \dim(V_{x'}^{y'})$$

where $V_{x'}^{y'}$ is the y'-weight space of $V_{x'}$. (Note that in our case we have automatically $x \notin \bigcup_{\check{\alpha},m} H_{\check{\alpha},m}$.) This provides a new proof of one of the main results in [L83].

2.3. In this subsection, we assume that $p = 0$. Let $\mathbf{A}$ be the subring of $\mathbf{Q}(\mathbf{v})$ consisting of elements which have no pole for $v = 1$. Let $H_{\mathbf{A}}$ be the $\mathbf{A}$-submodule of H spanned by $\{T_w; w \in W\}$ or equivalently by $\{c_w; w \in W\}$. This is a subring of H. We define a group homomorphism ξ from $H_{\mathbf{A}}$ to the group ring $\mathbf{Q}[W]$ by $\sum_w f_w T_w \mapsto \sum_w f_w(1) w$; here $f_w \in \mathbf{A}$. This is a ring homomorphism. Recall that for $x \in Q^+$, $P_{w,M_x;||}(1)$ depends only on the (W_0, W_0) double coset of $w \in W$. Hence

$$\begin{aligned}\xi(\mathbf{c}_x) &= \sharp(W_0)^{-1} \textstyle\sum_{w \in W} P_{w,M_x;||}(1) w \\ &= \sharp(W_0)^{-1} \textstyle\sum_{x' \in Q^+} P_{M_{x'},M_x;||}(1) \sum_{w \in W_0 x' W_0} w \\ &= \sharp(W_0)^{-1} \textstyle\sum_{x' \in Q^+} \dim(V_x^{x'}) \sum_{w \in W_0 x' W_0} w\end{aligned}$$

(we have used 2.2(b)). We have also

$$\text{(a)}\qquad \begin{aligned}\xi(\mathbf{c}_x) &= \sharp(W_0)^{-1}\textstyle\sum_{e\in Q}\dim(V_x^e)\sum_{a\in W_0} ae\\ &= \sharp(W_0)^{-1}\textstyle\sum_{e'\in Q}\dim(V_x^{e'})\sum_{a\in W_0} e'a.\end{aligned}$$

Indeed,

$$\begin{aligned}\sum_{e\in Q}\dim(V_x^e)\sum_{a\in W_0} ae &= \sum_{x'\in Q^+,(a,b)\in W_0\times W_0}\frac{\dim(V_x^{x'})}{\sharp(b'\in W_0; b'x'=x'b')}abx'b^{-1}\\ &= \sum_{x'\in Q^+,w\in W_0x'W_0}\frac{\dim(V_x^{x'})\sharp((b,c)\in W_0\times W_0, w=cx'b^{-1})}{\sharp(b'\in W_0; b'x'=x'b')}w\\ &= \sum_{x'\in Q^+,w\in W_0x'W_0}\frac{\dim(V_x^{x'})\sharp((b,c)\in W_0\times W_0, cx'b^{-1}=x')}{\sharp(b'\in W_0; b'x'=x'b')}w\\ &= \sum_{x'\in Q^+,w\in W_0x'W_0}\dim(V_x^{x'})w=\sharp(W_0)\xi(\mathbf{c}_x)\end{aligned}$$

and the first equality in (a) is established. The second equality in (a) follows the first by the substitution $e'=aea^{-1}$.

Now let $x\in Q^+$, $y\in Q^+$. For $e''\in Q$, let $(V_x\otimes V_y)^{e''}$ be the e''-weight space of $V_x\otimes V_y$. We have

$$\text{(b)}\qquad \begin{aligned}\xi(\mathbf{c}_x\mathbf{c}_y) &= \xi(\mathbf{c}_x)\xi(\mathbf{c}_y)\\ &= \sharp(W_0)^{-2}\textstyle\sum_{e\in Q}\dim(V_x^e)\sum_{a\in W_0} ae\sum_{e'\in Q}\dim(V_y^{e'})\sum_{b\in W_0} e'b\\ &= \sharp(W_0)^{-2}\textstyle\sum_{(e,e')\in Q\times Q}\dim(V_x^e)\dim(V_y^{e'})\sum_{(a,b)\in W_0\times W_0} aee'b\\ &= \sharp(W_0)^{-2}\textstyle\sum_{e''\in Q}\dim(V_x\otimes V_y)^{e''}\sum_{(a,b)\in W_0\times W_0} ae''b\\ &= \sharp(W_0)^{-2}\textstyle\sum_{e''\in Q,z\in Q^+}(V_z:V_x\otimes V_y)\dim(V_z^{e''})\sum_{(a,b)\in W_0\times W_0} ae''b\\ &= \sharp(W_0)^{-1}\textstyle\sum_{e''\in Q,z\in Q^+}(V_z:V_x\otimes V_y)\dim(V_z^{e''})\sum_{a\in W_0} ae''.\end{aligned}$$

Here $(V_z:V_x\otimes V_y)$ is the multiplicity of V_z in $V_x\otimes V_y$. On the other hand, we have

$$\begin{aligned}\xi(\mathbf{c}_x\mathbf{c}_y) &= \textstyle\sum_{z\in Q^+} r_{x,y,z;||}\xi(\mathbf{c}_z)\\ &= \sharp(W_0)^{-1}\textstyle\sum_{z\in Q^+,e''\in Q} r_{x,y,z;||}\dim(V_z^{e''})\sum_{a\in W_0} ae''.\end{aligned}$$

Comparing with (b), we deduce

$$\textstyle\sum_{z\in Q^+}(V_z:V_x\otimes V_y)\dim(V_z^{e''})=\sum_{z\in Q^+} r_{x,y,z;||}\dim(V_z^{e''})$$

for any $e''\in Q$. Hence

$$\textstyle\sum_{z\in Q^+}(V_z:V_x\otimes V_y)V_z=\sum_{z\in Q^+} r_{x,y,z;||}V_z$$

in the Grothendieck group of representations of G. Since $(V_z)_{z\in Q^+}$ is a basis of this Grothendieck group, we see that

$$\text{(c)}\qquad (V_z:V_x\otimes V_y)=r_{x,y,z;||}$$

for any x,y,z in Q^+. Thus, we recover one of the main results in [L83].

3. Folding

3.1. In this section, we assume that W, S, s_0, W_0, Q, Q^+ in 1.1 are such that W is irreducible of simply laced type. We assume given an automorphism σ of (W, S) of order $\delta \in \{2, 3\}$ preserving s_0.

Let $'W = \{w \in W; \sigma(w) = w\}$. For each σ-orbit $\mathcal{O}$ in S let $s_{\mathcal{O}}$ be the longest element in the subgroup of W generated by the elements in $\mathcal{O}$. Let $'S$ be the subset of $'W$ consisting of the elements $s_{\mathcal{O}}$ for various $\mathcal{O}$ as above. Note that $('W, 'S)$ is an affine Weyl group. Let $L : 'W \to \mathbf{N}$ be the restriction to $'W$ of the usual length function of W; this is a weight function on $'W$. (These statements can be deduced from [L14, Appendix A8, A9]).

We preserve the setup of 2.1. We assume that G is simple of simply laced type. We fix an automorphism of G preserving the pinning of G which induces the automorphism σ of W considered above. This automorphism of G is denoted again by σ. If $x \in Q^+$ and $\sigma(x) = x$ then $\sigma : G \to G$ induces linear isomorphisms $V_x \to V_x$, $L_x \to L_x$ denoted again by σ (they act as identity on a highest weight vector). We have

$$V_x = \bigoplus_{\theta \in \mathbf{k}^*_\delta} V_{x,\theta}, L_x = \bigoplus_{\theta \in \mathbf{k}^*_\delta} L_{x,\theta}$$

where $\mathbf{k}^*_\delta = \{\theta \in \mathbf{k}^*; \theta^\delta = 1\}$ and $V_{x,\theta}, L_{x,\theta}$ are the θ-eigenspaces of σ.

3.2. We now assume that $p \gg 0$ and that x in 2.2(a) satisfies in addition $\sigma(x) = x$. The proof of 2.2(a) is sufficiently functorial to imply that we have also

(a) $$\sum_{\theta \in \mathbf{k}^*_\theta} \tilde{\theta} L_{x,\theta} = \sum_{y \in Z_x, \sigma(y) = y} (-1)^{L(w_y w_x)} \mathrm{tr}(\sigma, \mathcal{V}_{w_y, w_x}) \sum_{\theta \in \mathbf{k}^*_\theta} \tilde{\theta} V_{y,\theta}$$

(equality in the representation ring of $T/\{\sigma(t)t^{-1}; t \in T\}$ tensored with $\mathbf{C}$; here $\theta \mapsto \tilde{\theta}$ is an imbedding of $\mathbf{k}^*_\delta$ into $\mathbf{C}^*$). Note that $\sigma(w_x) = w_x$ and that when $y \in Z_x$, $\sigma(y) = y$, we have $\sigma(w_y) = w_y$, so that σ acts naturally on $\mathcal{V}_{w_y, w_x}$. We now substitute

(b) $$\mathrm{tr}(\sigma, \mathcal{V}_{w_y, w_x}) = P_{w_y, w_x; L}(1)$$

where $P_{w_y, w_x; L}$ is defined in terms of $'W$ and $L : 'W \to \mathbf{N}$ as in 3.1. (See 4.5, 4.6.) We obtain

(c) $$\sum_{\theta \in \mathbf{k}^*_\delta} \tilde{\theta} L_{x,\theta} = \sum_{y \in Z_x, \sigma(y) = y} (-1)^{L(w_y w_x)} P_{w_y, w_x; L}(1) \sum_{\theta \in \mathbf{k}^*_\delta} \tilde{\theta} V_{y,\theta}.$$

This is an extension of the character formula 2.2(a) to certain disconnected groups. Note that the coefficients $P_{w_y, w_x; L}(1)$ are computable by an algorithm in [L03, §6] (which is somewhat more involved than that for the unweighted case in [KL79]).

3.3. Note that σ acts naturally on G^*. Let $'G$ be the simply connected group over $\mathbf{k}$ isogenous to the dual group of the identity component of the σ-fixed point set on G^*. By a theorem of Jantzen [Ja73], the expression $\sum_{\theta \in \mathbf{k}^*_\delta} \tilde{\theta} V_{y,\theta}$ in (c) can be expressed in terms of the character of a Weyl module of $'G$. Using this one can deduce as in §2 the analogues of 2.3(b), 2.3(c) with $(W, S, ||)$ replaced by $('W, 'S, L)$. (This recovers in our case a result in [K05].)

3.4. Assume that (W,S) is of (affine) type A_2 with σ of order 2. In this case, $('W,'S)$ is of (affine) type A_1 and the values of $L|_{'S}$ are 1 and 3. In this case, the ring J_* associated to $('W,'S,L)$ in 1.5 is isomorphic together with its basis to the representation ring of $\mathrm{SL}_2(\mathbf{C})$ with its standard basis; see [L03, 18.5]. This shows that the group $'G$ in 3.3 cannot be replaced by the corresponding adjoint group (even though G was adjoint).

3.5. In the setup of 3.1, 3.2 with $\mathbf{k}=\mathbf{C}$, we identify W_0 with the group $\mathcal{W}_0$ of affine transformations of $Q_{\mathbf{R}}$ generated by the reflections in the (finitely many) hyperplanes in $\mathcal{H}$ and which preserve $\mathcal{H}$. Let $\mathfrak{g}$ be the Lie algebra of G. Let $x\in Q$ be such that $x\notin\bigcup_{\check\alpha}H_{\check\alpha,0}$. Let $z\in Q$. Then the Verma $\mathfrak{g}$-module $\mathbf{V}_x$, its irreducible quotient $\mathbf{L}_x$ and their z-weight spaces $\mathbf{V}_x^z$, $\mathbf{L}_x^z$ are defined. It is known that the following equality (conjectured in [KL79]) holds:

(a) $$\dim\mathbf{L}_x^z=\sum_{y\in\mathcal{Z}_x}(-1)^{|\omega_y\omega_x|}P_{\omega_y,\omega_x;||}(1)\dim\mathbf{V}_y^z$$

where $\mathcal{Z}_x$ is the set of all $y\in Q$ in the same $\mathcal{W}_0$-orbit as x; ω_x,ω_y are certain well-defined explicit elements of $\mathcal{W}_0$.

Now assume that x,z are fixed by σ. Then $\sigma:G\to G$ induces automorphisms of $\mathbf{L}_x^z$ and of $\mathbf{V}_x^z$ denoted again by σ. We have

(b) $$\mathrm{tr}(\sigma,\mathbf{L}_x^z)=\sum_{y\in\mathcal{Z}_x,\sigma(y)=y}(-1)^{L(\omega_y\omega_x)}P_{\omega_y,\omega_x;L}(1)\mathrm{tr}(\sigma,\mathbf{V}_y^z).$$

This follows from the proof of (a) in the same way as 3.2(c) follows from the proof of 2.2(a) (using 4.5).

4. A geometric interpretation of $P_{y,w;L}$

4.1. Let W_0 be a (finite) Weyl group with a set S_0 of simple reflections and let $\sigma:W_0\to W_0$ be an automorphism preserving S_0. For each σ-orbit $\mathcal{O}$ in S_0 we denote by $\sigma_{\mathcal{O}}$ the longest element in the subgroup of W_0 generated by the reflections in $\mathcal{O}$. Let $'W_0=\{w\in W_0;\sigma(w)=w\}$ and let $'S_0$ be the subset of $'W_0$ consisting of the elements $s_{\mathcal{O}}$ for various $\mathcal{O}$ as above. Then $'W_0$ is a Weyl group with set of simple reflections $'S_0$. Let $L:{}'W_0\to\mathbf{N}$ be the restriction to $'W_0$ of the standard length function of W_0; it is known that L is a weight function on $'W_0$ so that the Hecke algebra over $\mathcal{A}$ with its bases $\{T_w;w\in{}'W_0\}$, $\{c_w;w\in{}'W_0\}$ can be defined as in 1.1 (in terms of $'W_0,'S_0,L$ instead of W,S,L). (These statements can be deduced from [L14, Appendix A8, A9].) This Hecke algebra specialized at $v=\sqrt{q}$ with q a prime power is a $\mathbf{C}$-algebra denoted by $H_{0,q;L}$.

For $w\in{}'W_0$, we write

$$c_w=\sum_{y\in'W_0}v^{-L(w)+L(y)}P_{y,w;L}T_y$$

where $P_{y,w;L}\in\mathbf{Z}[v^2]$.

For $w\in{}'W_0$, we have

(a) $$T_wT_{w_0}=\sum_{y\in'W_0}v^{L(y)-L(w)}R_{y,w;L}T_{yw_0}$$

where $R_{y,w;L} \in \mathbf{Z}[v^2]$ is 0 unless $y \le w$ and w_0 is the longest element of W_0 (or $'W_0$). Note the following inductive formulas for $R_{y,w;L}$; see [L03, 4.4]. (Here $s \in S$.)

(b)
$$R_{y,w;L} = R_{sy,sw;L} \text{ if } |sy| < |y|, |sw| < |w|;$$
$$R_{y,w;L} = v^{2L(s)} R_{sy,w;L} + (v^{2L(s)} - 1) R_{sy,sw} \quad \text{if } |sy| > |y|, |sw| < |w|.$$

We have $P_{y,w;L} = 0$ unless $y \le w$ and $P_{w,w;L} = 1$. For y, w in $'W_0$ we have

(c)
$$v^{2L(w)-2L(y)} \bar{P}_{y,w;L} = \sum_{z \in 'W_0} R_{y,z;L} P_{z,w;L}.$$

See [L03, 5.3].

4.2. Let $\mathbf{k}$ be an algebraic closure of the finite prime field $\mathbf{F}_p$. Let G be a simply connected semisimple group over $\mathbf{k}$ with Weyl group (W_0, S_0) and with a fixed pinning involving a maximal torus T and a Borel subgroup B containing T. We fix an $\mathbf{F}_p$-rational structure on G (with Frobenius map $F : G \to G$) compatible with the pinning such that T is split over $\mathbf{F}_p$, hence B is defined over $\mathbf{F}_p$. We consider an automorphism of G preserving the pinning and compatible with the $\mathbf{F}_p$-structure; it induces an automorphism of W_0, which we assume to be σ. This automorphism of G is denoted again by σ; we have $\sigma F = F\sigma$. Hence, if $t \ge 1$, then $F_t := F^t \sigma = \sigma F^t$ is the Frobenius map for a rational structure over the subfield $\mathbf{F}_{p^t}$ with p^t elements of $\mathbf{k}$. Let $\mathcal{B}$ be the variety of Borel subgroups of G. Note that F_t acts naturally on $\mathcal{B}$ and defines a Frobenius map on $\mathcal{B}$. We say that B_1, B_2 in $\mathcal{B}$ are opposed if $B_1 \cap B_2$ is a maximal torus. We define $B^* \in \mathcal{B}$ by the conditions that $B \cap B^* = T$. For B_1, B_2 in $\mathcal{B}$, let $pos(B_1, B_2) \in W_0$ be the relative position of B_1, B_2. For $w \in W_0$, we set $\mathcal{B}_w = \{B' \in \mathcal{B}; pos(B, B') = w\}$. For $y \in W_0$, we define ${}^yB \in \mathcal{B}$ by the conditions $T \subset {}^yB, {}^yB \in \mathcal{B}_y$; we define ${}^yB^* \in \mathcal{B}$ by the conditions $T \subset {}^yB^*, pos(B^*, {}^yB^*) = y$. Let $\bar{\mathcal{B}}_w$ be the closure of $\mathcal{B}_w$ in $\mathcal{B}$. For $y \in W_0$, we set $A^y = \{B' \in \mathcal{B}; B', {}^yB^* \text{ opposed}\}$.

For any algebraic variety X of pure dimension, let $IC(X)$ be the intersection cohomology complex of X with coefficients in $\bar{\mathbf{Q}}_l$ (with l a prime $\ne p$). Let $\mathbf{H}^i(X)$ (resp. $\mathbf{H}^i_c(X)$) be the i-th cohomology (resp. i-th cohomology with compact support) of X with coefficients in $IC(X)$. For $x \in X$, let $\mathcal{H}^i_x(X)$ be the stalk at x of the i-th cohomology sheaf of $IC(X)$.

The following result gives a geometric interpretation of $P_{y,w;L}$ (stated without proof in [L83a, (8.1)]) extending the already known case where $\sigma = 1$ considered in [KL80]; see also [L03, §16].

4.3. Theorem. *Let $y \in 'W_0, w \in 'W_0$ be such that $y \le w$. We have*

$$P_{y,w;L} = \sum_{i \text{ } even} \mathrm{tr}(\sigma, \mathcal{H}^i_{{}^yB}(\bar{\mathcal{B}}_w)) v^{2i}.$$

(Note that σ acts naturally on $\mathcal{H}^i_{{}^yB}(\bar{\mathcal{B}}_w)$.) The proof will use the following result (analogous to [KL79, A4(a)]).

4.4. Lemma. *Let $y \in 'W_0, z \in 'W_0$ be such that $y \le z$. We have $\sharp((\mathcal{B}_z \cap A^y)^{F_t}) = R_{y,z;L}(p^t) p^{tL(y)}$.*

Let $\mathcal{F}$ be the vector space of functions $\mathcal{B}^{F_t} \to \mathbf{C}$. Then $\mathcal{F}$ is an $H_{0,p^t;L}$-module, in which for $w \in 'W_0$ and $f \in \mathcal{F}$, we have $T_w f = f'$ where for $B' \in \mathcal{B}^{F_t}$ we have

$f'(B') = p^{-tL(w)/2} \sum_{B'' \in \mathcal{B}^{F_t}; \mathrm{pos}(B', B'') = w} f(B'')$. Applying the equality 4.1(a) to $f \in \mathcal{F}$ and evaluating at B we see that for $z \in {}'W_0$ we have

$$\text{(a)} \quad \begin{aligned} &\textstyle\sum_{y' \in {}'W_0} p^{t(L(y')-L(z))/2} R_{y',z;L}(p^t) p^{t(L(y')-L(w_0))/2} \sum_{C \in \mathcal{B}^{F_t}; \mathrm{pos}(B,C) = y'w_0} f(C) \\ &\quad = p^{-tL(z)/2} \textstyle\sum_{B'' \in \mathcal{B}^{F_t}; \mathrm{pos}(B,B'') = z} p^{-tL(w_0)/2} \sum_{C \in \mathcal{B}^{F_t}; \mathrm{pos}(B'',C) = w_0} f(C). \end{aligned}$$

We now take f to be the function equal to 1 at $C_0 = {}^{yw_0}B$ and equal to 0 on $\mathcal{B}^{F_t} - \{C_0\}$. We obtain

$$\text{(b)} \quad \begin{aligned} &\sharp(B'' \in \mathcal{B}^{F_t}; \mathrm{pos}(B, B'') = z, \mathrm{pos}(B'', C_0) = w_0) p^{-tL(z)/2} \\ &\qquad = p^{t(L(y)-L(z)+L(y))/2} R_{y,z;L}(p^t), \end{aligned}$$

that is

$$\sharp(\mathcal{B}_z \cap A^y)^{F_t} = R_{y,z;L}(p^t) p^{tL(y)}.$$

The lemma is proved. □

4.5. We now prove the theorem. When $y = w$ the result is obvious. We can assume that $y < w$ and that

(a) the result is true when y, w is replaced by z, w with $z \in {}'W_0$ such that $y < z \leq w$.

Here the partial order refers to W_0'; it is the restriction of the partial order on W_0. Applying the Grothendieck-Lefschetz fixed point formula for F_t on the F_t-stable open subvariety $\bar{\mathcal{B}}_w \cap A^y$ of $\bar{\mathcal{B}}_w$ we obtain

$$\begin{aligned} &\mathrm{tr}\big(F_t, \textstyle\sum_i (-1)^i \mathbf{H}^i_c(\bar{\mathcal{B}}_w \cap A^y)\big) \\ &\qquad = \textstyle\sum_{z \in W_0; y \leq z \leq w} \sum_{B' \in (\mathcal{B}_z \cap A^y)^{F_t}} \mathrm{tr}(F_t, \sum_i (-1)^i \mathcal{H}^i_{B'}(\bar{\mathcal{B}}_w)). \end{aligned}$$

Now the fixed point set $(\mathcal{B}_z \cap A^y)^{F_t}$ is empty unless $\sigma(z) = z$. For such z we apply Lemma 4.4 and we obtain

$$\begin{aligned} &\mathrm{tr}\big(F_t, \textstyle\sum_i (-1)^i \mathbf{H}^i_c(\bar{\mathcal{B}}_w \cap A^y)\big) \\ &\qquad = \textstyle\sum_{z \in {}'W_0; y \leq z \leq w} R_{y,z;L}(p^t) p^{tL(y)} \mathrm{tr}(F_t, \sum_i (-1)^i \mathcal{H}^i_{{}^zB}(\bar{\mathcal{B}}_w)). \end{aligned}$$

By Poincaré duality on $\bar{\mathcal{B}}_w \cap A^y$ we have

$$\mathrm{tr}\big(F_t, \textstyle\sum_i (-1)^i \mathbf{H}^i_c(\bar{\mathcal{B}}_w \cap A^y)\big) = p^{tL(w)} \mathrm{tr}\big({F_t}^{-1}, \sum_i (-1)^i \mathbf{H}^i(\bar{\mathcal{B}}_w \cap A^y)\big).$$

Using [KL80, 4.5(a), 1.5] we have

$$\mathrm{tr}\big({F_t}^{-1}, \textstyle\sum_i (-1)^i \mathbf{H}^i(\bar{\mathcal{B}}_w \cap A^y)\big) = \mathrm{tr}\big({F_t}^{-1}, \sum_i (-1)^i \mathcal{H}^i_{{}^yB}(\bar{\mathcal{B}}_w)\big),$$

so that

$$\begin{aligned} &p^{tL(w)} \mathrm{tr}\big({F_t}^{-1}, \textstyle\sum_i (-1)^i \mathcal{H}^i_{{}^yB}(\bar{\mathcal{B}}_w)\big) \\ &\qquad = \textstyle\sum_{z \in {}'W_0; y \leq z \leq w} R_{y,z;L}(p^t) p^{tL(y)} \mathrm{tr}\big(F_t, \sum_i (-1)^i \mathcal{H}^i_{{}^zB}(\bar{\mathcal{B}}_w)\big). \end{aligned}$$

By [KL80, 4.2] we have $\mathcal{H}^i_{{}^yB}(\bar{\mathcal{B}}_w) = 0$ if i is odd, while if i is even the eigenvalues of F^t on $\mathcal{H}^i_{{}^yB}(\bar{\mathcal{B}}_w)$ are equal to $p^{it/2}$. It follows that

$$p^{tL(w)} \sum_{i \text{ even}} p^{-it/2} \text{tr}\big(\sigma^{-1}, \mathcal{H}^i_{{}^yB}(\bar{\mathcal{B}}_w)\big) = \sum_{z \in {}'W_0; y \le z \le w} R_{y,z;L}(p^t) p^{tL(y)} \sum_{i \text{ even}} p^{it/2} \text{tr}\big(\sigma, \mathcal{H}^i_{{}^zB}(\bar{\mathcal{B}}_w)\big).$$

Since this holds for $t = 1, 2$, we can replace p^t by v^2 where v is an indeterminate and we get an equality in $\bar{\mathbf{Q}}_l[v, v^{-1}]$:

$$v^{2L(w)} \sum_{i \text{ even}} v^{-i} \text{tr}(\sigma^{-1}, \mathcal{H}^i_{{}^yB}(\bar{\mathcal{B}}_w)) = \sum_{z \in {}'W_0; y \le z \le w} R_{y,z;L} v^{2L(y)} \sum_{i \text{ even}} v^i \text{tr}(\sigma, \mathcal{H}^i_{{}^zB}(\bar{\mathcal{B}}_w)).$$

Using the induction hypothesis (a) we obtain

$$v^{2L(w)} \sum_{i \text{ even}} v^{-i} \text{tr}(\sigma^{-1}, \mathcal{H}^i_{{}^yB}(\bar{\mathcal{B}}_w)) - v^{2L(y)} \sum_{i \text{ even}} v^i \text{tr}(\sigma, \mathcal{H}^i_{{}^yB}(\bar{\mathcal{B}}_w)) = \sum_{z \in {}'W_0; y < z \le w} R_{y,z;L} v^{2L(y)} P_{z,w;L}.$$

Using 4.1(c), the right-hand side of this equality is

$$v^{2L(w)} \bar{P}_{y,w;L} - v^{2L(y)} P_{y,w;L}.$$

Thus we have

$$\text{(b)} \quad v^{L(w)-L(y)} \Big(\sum_{i \text{ even}} v^{-i} \text{tr}\big(\sigma^{-1}, \mathcal{H}^i_{{}^yB}(\bar{\mathcal{B}}_w)\big) - \bar{P}_{y,w;L} \Big) = v^{L(y)-L(w)} \Big(\sum_{i \text{ even}} v^i \text{tr}\big(\sigma, \mathcal{H}^i_{{}^yB}(\bar{\mathcal{B}}_w)\big) - P_{y,w;L} \Big).$$

By the known properties of $\mathcal{H}^i$, in both sides of (b) we can assume that $i < \dim \mathcal{B}_w - \dim \mathcal{B}_y = L(w) - L(y)$. Moreover, we have $v^{L(w)-L(y)} \bar{P}_{y,w;L} \in v\mathbf{Z}[v]$ and $v^{L(y)-L(w)} P_{y,w;L} \in v^{-1}\mathbf{Z}[v^{-1}]$. Thus the left-hand side of (b) is in $v\bar{\mathbf{Q}}_l[v]$ while the right-hand side of (b) is in $v^{-1}\bar{\mathbf{Q}}_l[v^{-1}]$. We see that both sides of (b) are zero. The theorem is proved. □

4.6. The proof in 4.5 is written in such a way that it remains valid in the affine case so that it gives an analogous geometric interpretation for $P_{y,w;L}$ with y, w in ${}'W$ where ${}'W, L$ are as in 3.1. In this case, B is an Iwahori subgroup and B^* is an anti-Iwahori subgroup (opposed to B) as in [KL80, §5]. The definition of A^y still makes sense; it is the set of Iwahori subgroups opposed to a certain fixed anti-Iwahori subgroup. Now $R_{y,w;L}$ as defined by 4.1(a) does not make sense in the affine case; instead, one can use the inductive definition in 4.1(b). With this definition, the analogue of 4.1(c) remains valid; Lemma 4.4 remains valid but it is now proved by an (easy) induction on $|z|$.

4.7. Erratum to [L83].

On p. 212, line −3, the definition of K^1 should be

$$K^1 = \{x \in (1/|W|)\mathbf{Z}[\tilde{W}_a]; wx = x, xw = x \quad \forall w \in W\}.$$

On p. 212, line −1, the definition of J^1 should be

$$J^1 = \{x \in \mathbf{Z}[\tilde{W}_a]; wx = (-1)^{l(w)}x, xw = x \quad \forall w \in W\}.$$

On p. 212, line −9, the definition of K should be

$$K = \{x \in (1/\mathcal{P})H; T_w x = q^{l(w)}x, xT_w = q^{l(w)}x \quad \forall w \in W\}.$$

On p. 212, line −7, the definition of J should be

$$J = \{x \in H; T_w x = (-1)^{l(w)}x, xT_w = q^{l(w)}x \quad \forall w \in W\}.$$

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