CHARACTERIZATION OF n -DIMENSIONAL NORMAL AFFINE SL_n -VARIETIES

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Abstract. We show that any normal irreducible affine n -dimensional SL_n -variety X is determined by its automorphism group seen as an ind-group in the category of normal irreducible affine varieties. In other words, if Y is an irreducible affine normal algebraic variety such that $\mathrm{Aut}(Y) \simeq \mathrm{Aut}(X)$ as an ind-group, then $Y \simeq X$ as a variety. If we drop the condition of normality on Y , then this statement fails. In case $n \geq 3$, the result above holds true if we replace $\mathrm{Aut}(X)$ by $\mathcal{U}(X)$, where $\mathcal{U}(X)$ is the subgroup of $\mathrm{Aut}(X)$ generated by all one-dimensional unipotent subgroups. In dimension 2 we have some interesting exceptions.

1. Introduction and main results

Our base field is the field of complex numbers $\mathbb{C}$. For an affine variety X , the automorphism group $\mathrm{Aut}(X)$ has the structure of an ind-group. We will shortly recall the basic definitions and results in Section 2. The classical example is $\mathrm{Aut}(\mathbb{A}^n)$, $n > 1$, the group of automorphisms of the affine n -space $\mathbb{A}^n$. Recently, Hanspeter Kraft proved the following result which shows that the affine n -space is determined by its automorphism group (see [Kr15]).

Theorem 1.1. *Let Y be a connected affine variety. If $\mathrm{Aut}(Y) \simeq \mathrm{Aut}(\mathbb{A}^n)$ as ind-groups, then $Y \simeq \mathbb{A}^n$ as varieties.*

Note that this result was generalised in [CRX19] (see also [KRvS19] and in a similar spirit, see [LRU20, Thm. 1]) where the authors proved Theorem 1.1 under a weaker condition: namely, that groups $\mathrm{Aut}(Y)$ and $\mathrm{Aut}(\mathbb{A}^n)$ are isomorphic only as abstract groups. Moreover, recently Theorem 1.1 was generalized in [LRU18, Thm. 1.4] (see also [RvS21, Main Thm. 1]) where it was proved that an affine toric variety different from the algebraic torus is determined by its automorphism group seen as an ind-group in the category of normal affine irreducible varieties. If we drop the normality condition in [LRU18, Thm. 1.4], the situation changes. In this paper, we show that for “most” n -dimensional affine normal varieties X endowed with a nontrivial regular $\mathrm{SL}_n = \mathrm{SL}_n(\mathbb{C})$ -action, there are infinitely many

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affine varieties Y such that $\text{Aut}(Y) \simeq \text{Aut}(X)$ as an ind-group and we classify all such Y .

Let $d > 1$. Consider the action of $\mu_d = \{\xi \in \mathbb{C}^* \mid \xi^d = 1\}$ on $\mathbb{A}^n$ by scalar multiplication and denote by $A_{d,n}$ the quotient of $\mathbb{A}^n$ by μ_d . Note that $A_{d,n}$ is normal. Denote also by $\pi: \mathbb{A}^n \rightarrow A_{d,n}$ the quotient map. This means that $A_{d,n}$ is an affine variety with coordinate ring

$$\mathcal{O}(A_{d,n}) = \mathbb{C}[x_1, \dots, x_n]^{\mu_d} = \bigoplus_{k=0}^{\infty} \mathbb{C}[x_1, \dots, x_n]_{dk},$$

the algebra of invariants where $\mathbb{C}[x_1, \dots, x_n]_{dk}$ denotes the homogeneous polynomials of degree dk . Note that $A_{d,n}$ is indeed an orbit space, because μ_d is finite. For $d > 1$, $A_{d,n}$ has an isolated singularity in $\pi(0)$ and π induces an étale covering $\mathbb{A}^n \setminus \{0\} \rightarrow A_{d,n} \setminus \{p(0)\}$ with Galois group μ_d .

Remark 1.2. We will see in Lemma 6.1 and Proposition 5.4 that any affine normal variety endowed with a regular nontrivial SL_n -action is isomorphic to either SL_2/T , SL_2/N or to $A_{d,n}$ for some $d \in \mathbb{N}$, where $T \subset \text{SL}_2$ is the standard subtorus and $N \subset \text{SL}_2$ is the normalizer of T . This implies that Theorem 1.3 and Theorem 1.4 below indeed provide the characterization of n -dimensional normal affine SL_n -varieties.

Consider the affine variety $A_{d,n}^s$ with coordinate ring

$$\mathcal{O}(A_{d,n}^s) = \mathbb{C} \oplus \bigoplus_{k=s}^{\infty} \mathbb{C}[x_1, \dots, x_n]_{dk} \subset \mathcal{O}(A_{d,n}), \quad s \geq 1.$$

Then the induced morphism $\eta: A_{d,n} \rightarrow A_{d,n}^s$ is the normalization and has the property that the induced map $\eta': A_{d,n} \setminus \{\star\} \xrightarrow{\sim} A_{d,n}^s \setminus \{\star\}$ is an isomorphism, where $\star$ denotes the points corresponding to the homogeneous maximal ideals. In fact, η is SL_n -equivariant, and $A_{d,n} \setminus \{\star\}$ is an SL_n -orbit. We prove the following result.

Theorem 1.3. *Let X be an irreducible affine variety. Then $\text{Aut}(X)$ and $\text{Aut}(A_{d,n})$ are isomorphic as ind-groups if and only if $X \simeq A_{d,n}^s$ as a variety for some $s \in \mathbb{N}$.*

Theorem 1.3 and the following result shows that SL_2/T and SL_2/N are the only affine n -dimensional SL_n -varieties (except $\mathbb{A}^n$) that are determined by their automorphism groups in the category of affine irreducible varieties.

Theorem 1.4. *Let X be an irreducible variety such that $\text{Aut}(X) \simeq \text{Aut}(\text{SL}_2/T)$ respectively $\text{Aut}(X) \simeq \text{Aut}(\text{SL}_2/N)$ as ind-groups. Then $X \simeq \text{SL}_2/T$ respectively $X \simeq \text{SL}_2/N$ as varieties.*

For an affine variety X we denote by $\mathcal{U}(X) \subset \text{Aut}(X)$ the subgroup generated by the one-dimensional unipotent subgroups. We do not know whether $\mathcal{U}(X)$ has the structure of an ind-subgroup (i.e., whether $\mathcal{U}(X) \subset \text{Aut}(X)$ is closed). That is why we introduce the definition of an *algebraic isomorphism*. This is an isomorphism $\phi: \mathcal{U}(X) \xrightarrow{\sim} \mathcal{U}(Y)$ such that for any subgroup $U \subset \mathcal{U}(X)$, where U is a closed one-dimensional unipotent subgroup of $\text{Aut}(X)$, the image $\phi(U) \subset \text{Aut}(Y)$ is a closed one-dimensional unipotent subgroup and $\phi|_U: U \xrightarrow{\sim} \phi(U)$ is an isomorphism of algebraic groups.

Theorem 1.5. *Let X be $A_{d,n}$, SL_2/T or SL_2/N and Y be an irreducible affine variety. Assume that there is an algebraic isomorphism $\mathcal{U}(X) \xrightarrow{\sim} \mathcal{U}(Y)$. Then*

- (a) *if $X \simeq A_{2,2}$, then $Y \simeq \mathrm{SL}_2/T$ or $Y \simeq A_{2,2}^s$ for some $s \in \mathbb{N}$,*
- (b) *if $X \simeq \mathrm{SL}_2/T$, then $Y \simeq \mathrm{SL}_2/T$ or $Y \simeq A_{2,2}^s$ for some $s \in \mathbb{N}$,*
- (c) *if $X \simeq A_{4,2}$, then $Y \simeq \mathrm{SL}_2/N$ or $Y \simeq A_{4,2}^s$ for some $s \in \mathbb{N}$,*
- (d) *if $X \simeq \mathrm{SL}_2/N$, then $Y \simeq \mathrm{SL}_2/N$ or $Y \simeq A_{4,2}^s$ for some $s \in \mathbb{N}$,*
- (e) *if $X = A_{d,n}$, where $(d, n) \notin \{(2, 2), (2, 4)\}$, then $Y \simeq A_{d,n}^s$ for some $s \geq 1$.*

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2. Preliminaries

2.1. Ind-groups

The notion of an ind-group goes back to Shafarevich who called such objects *infinite dimensional groups* (see [Sh66]). We refer to [Kum02] and [FK18] for basic notions in this context.

Definition 2.1. By an *ind-variety* we mean a set V together with an ascending filtration $V_0 \subset V_1 \subset V_2 \subset \cdots \subset V$ such that the following holds:

- (1) $V = \bigcup_{k \in \mathbb{N}} V_k$,
- (2) each V_k has the structure of an affine algebraic variety,
- (3) for all $k \in \mathbb{N}$ the subset $V_k \subset V_{k+1}$ is closed in the Zariski-topology.

A morphism from an ind-variety $V = \bigcup_{k \in \mathbb{N}} V_k$ to an ind-variety $W = \bigcup_{m \in \mathbb{N}} W_m$ is a map $\phi: V \rightarrow W$ such that for any k there is an m such that $\phi(V_k) \subset W_m$ and such that the induced map $V_k \rightarrow W_m$ is a morphism of algebraic varieties. *Isomorphisms* of ind-varieties are defined in the obvious way.

Two filtrations $V = \bigcup_{k \in \mathbb{N}} V_k$ and $V = \bigcup_{k \in \mathbb{N}} V'_k$ are called *equivalent* if for every k there is an m such that $V_k \subset V'_m$ is a closed subvariety as well as $V'_k \subset V_m$.

An ind-variety V has a natural topology: a subset $S \subset V$ is open, (resp. closed), if $S_k = S \cap V_k \subset V_k$ is open, (resp. closed), for all k . A locally closed subset $S \subset V$ has the induced structure of an ind-variety. It is called an *ind-subvariety*. A subset $S \subset V$ that is a closed subset of some V_k is called an *algebraic subset*.

The product of two ind-varieties is defined in the usual way. This allows to give the following definition.

Definition 2.2. An ind-variety G is said to be an *ind-group* if the underlying set G is a group such that the map $G \times G \rightarrow G$, $(g, h) \mapsto gh^{-1}$, is a morphism.

An ind-group G is called *connected* if for every $g \in G$ there is an irreducible curve C and a morphism $C \rightarrow G$ whose image contains the neutral element e and g .

A closed subgroup H of G (i.e., H is a subgroup of G and is a closed subset) is again an ind-group under the closed ind-subvariety structure on G . A closed

subgroup H of an ind-group G is called an *algebraic subgroup* if and only if H is an algebraic subset of G .

Proposition 2.3 ([FK18, Thm. 0.3.1]). *Let X be an affine variety. Then $\mathrm{Aut}(X)$ has the structure of an ind-group such that for any algebraic group G , there is a correspondence between regular G -actions on X and ind-group homomorphisms $G \rightarrow \mathrm{Aut}(X)$.*

If G is an algebraic group acting regularly and faithfully on X , then by Proposition 2.3, we can consider G as an algebraic subgroup of $\mathrm{Aut}(X)$. We will often switch between these two points of view.

2.2. Locally nilpotent derivations and $\mathbb{G}_a$ -actions

Additive group actions on affine varieties can be described by a certain kind of derivations. We recall some of the basics here (see [Fre06] for details). Let $\lambda: \mathbb{G}_a \rightarrow \mathrm{Aut}(X)$ be a $\mathbb{G}_a$ -action on an affine variety X . Such an action induces a derivation on the level of regular functions $\mathcal{O}(X)$ by

$$\delta_\lambda: \mathcal{O}(X) \rightarrow \mathcal{O}(X), \quad f \mapsto \left[\frac{d}{ds} \lambda(s)^*(f) \right]_{s=0},$$

where $\mathbb{G}_a = \mathrm{Spec}(\mathbb{C}[s])$. We have that for every $f \in \mathcal{O}(X)$ there exists an $k \in \mathbb{N}$ with $\delta_\lambda^k(f) = 0$. Derivations that have such a property are called *locally nilpotent*. Moreover, every $\mathbb{G}_a$ -action on X arises from a certain locally nilpotent derivation δ and the $\mathbb{G}_a$ -action $\lambda_\delta: \mathbb{G}_a \times X \rightarrow X$ is obtained from δ via

$$(\alpha_\delta(s))^*: \mathcal{O}(X) \rightarrow \mathcal{O}(X)[s], \quad f \mapsto \exp(s\delta)(f) := \sum_{i=0}^{\infty} \frac{s^i \delta^i(f)}{i!}.$$

3. Automorphisms

Proposition 3.1. *Let $\pi: \mathbb{A}^n \rightarrow A_{d,n}$. Then every automorphism of $A_{d,n}$ lifts to an automorphism of $\mathbb{A}^n$ which commutes with each element of μ_d .*

Proof. The quotient map $\pi: \mathbb{A}^n \rightarrow A_{d,n}$ induces a natural embedding of $\mathcal{O}(A_{d,n})$ into $\mathcal{O}(\mathbb{A}^n) = \mathbb{C}[x_1, \dots, x_n]$. So, we assume that $\mathcal{O}(A_{d,n})$ is a subring of $\mathbb{C}[x_1, \dots, x_n]$ and equals $\bigoplus_{k=0}^{\infty} \mathbb{C}[x_1, \dots, x_n]_{dk}$, where $\mathbb{C}[x_1, \dots, x_n]_{dk}$ denotes the homogeneous polynomials of degree dk . Let $\phi \in \mathrm{Aut}(A_{d,n})$. First we claim that $p_i = \phi^*(x_i^d)$ and $p_j = \phi^*(x_j^d)$ are coprime in $\mathbb{C}[x_1, \dots, x_n]$, where $i \neq j$ and ϕ^* is the pull-back of ϕ . Let p be a common factor of p_i and p_j . Then $\tilde{p} = \prod_{g \in \mu_d} gp$ divides p_i^d and p_j^d . By construction it is clear that $\tilde{p} \in \mathcal{O}(A_{d,n})$. Hence, $(\phi^*)^{-1}(\tilde{p})$ is a common factor of $(\phi^*)^{-1}(p_i^d) = x_i^{d^2}$ and $(\phi^*)^{-1}(p_j^d) = x_j^{d^2}$. Therefore, $\tilde{p} \in \mathbb{C}$ and then, $p \in \mathbb{C}$.

We have

$$\phi^*(x_i^d) \phi^*((x_j^d)^{d-1}) = \phi^*(x_i^d x_j^{d(d-1)}) = \phi^*(x_i x_j^{d-1})^d,$$

which means that $p_i p_j^{d-1} = q^d$ for some $q \in \mathcal{O}(A_{d,n})$. Because p_i is coprime with p_j , it follows that $p_i = q_i^d$ for some $q_i \in \mathbb{C}[x_1, \dots, x_n]$.

Since for an automorphism $\phi: A_{d,n} \rightarrow A_{d,n}$ we have that $\phi^*(x_i^d) = q_i^d$ for some $q_i \in \mathbb{C}[x_1, \dots, x_n]$, we define the morphism

$$\tilde{\phi} = (q_1, \dots, q_n): \mathbb{A}^n \rightarrow \mathbb{A}^n$$

given by the map

$$(x_1, \dots, x_n) \mapsto (q_1, \dots, q_n).$$

If ϕ is the identity automorphism, then the restriction of $\tilde{\phi}^*$ to $\mathcal{O}(A_{d,n})$ is the identity and we have that $q_i^d = x_i^d$, which implies that $q_i = w_i x_i$ for some $w_i \in \mathbb{C}^*$, $w_i^d = 1$. In this case $\tilde{\phi}$ is an automorphism: i.e., the trivial automorphism of $A_{d,n}$ lifts to an automorphism of $\mathbb{A}^n$ which we denote by $\Delta(w_1, \dots, w_n)$.

Let now $\theta: A_{d,n} \rightarrow A_{d,n}$ be the inverse automorphism of $\phi \in \mathrm{Aut}(A_{d,n})$. Since $\phi \circ \theta$ is the trivial automorphism of $A_{d,n}$, it lifts to an automorphism $\tilde{\theta} \circ \Delta(w_1, \dots, w_n)^{-1}$ of $\mathbb{A}^n$ and so $\tilde{\phi}$ is an automorphism with the inverse $\tilde{\theta} \circ \Delta(w_1, \dots, w_n)^{-1}$. To finish the proof, we need to show that $\tilde{\phi}$ commutes with μ_d . Indeed, since $\tilde{\phi}^*$ preserves $\mathbb{C}[x_1, \dots, x_n]^{\mu_d}$, we have that $q_i^d(\xi x_1, \dots, \xi x_n) = q_i^d(x_1, \dots, x_n)$, where $\xi \in \mu_d$. Hence,

$$q_i(\xi x_1, \dots, \xi x_n) = \xi^l q_i(x_1, \dots, x_n)$$

for some $l = 1, \dots, d-1$. Since polynomial q_i has a linear summand it follows that $l = 1$. The proof follows. $\square$

Let X be an affine variety, H be a finite group that acts faithfully on X , and let $\pi: X \rightarrow X/H$ be the quotient morphism. Since H acts faithfully, H naturally embeds into $\mathrm{Aut}(X)$ and we identify H with its image in $\mathrm{Aut}(X)$. Denote by $\mathrm{Aut}^H(X) \subset \mathrm{Aut}(X)$ the subgroup of all automorphisms of X which normalize H : i.e., the subgroup of those automorphisms ϕ such that $\phi^{-1} \circ H \circ \phi = H$.

Lemma 3.2.

- (a) *There is a canonical homomorphism of groups $\phi: \mathrm{Aut}^H(X) \rightarrow \mathrm{Aut}(X/H)$,*
- (b) *if X is normal and contains only finitely many fixed points of H then every $\mathbb{C}^+$ -action on X/H lifts to a $\mathbb{C}^+$ -action on X .*

Proof. (a) Let $h \in H$, $f \in \mathcal{O}(X)^H$ and $\phi \in \mathrm{Aut}^H(X)$. Then $\phi^*: \mathcal{O}(X) \xrightarrow{\sim} \mathcal{O}(X)$ is an isomorphism and

$$h(\phi^*(f)) = \phi^*((\phi^*)^{-1} \circ h \circ \phi^*)(f) = (\phi^* \circ h')(f) = \phi^*(f)$$

for some $h' \in H$. Therefore $\phi^*(f) \in \mathcal{O}(X)^H$, which means that ϕ induces an automorphism of X/H .

(b) This follows from [MM09, Thm. 1.3]. $\square$

Let us recall that a closed algebraic subgroup U of $\mathrm{Aut}(X)$ is a 1-dimensional unipotent subgroup if $U \simeq \mathbb{C}^+$.

Proposition 3.3. *The homomorphism $\phi_d: \mathrm{Aut}^{\mu_d}(\mathbb{A}^n) \rightarrow \mathrm{Aut}(A_{d,n})$ is surjective with kernel μ_d . Moreover, every 1-dimensional unipotent subgroup of $\mathrm{Aut}(A_{d,n})$ is the image of some 1-dimensional unipotent subgroup of $\mathrm{Aut}^{\mu_d}(\mathbb{A}^n)$.*

Proof. The surjectivity of ϕ_d follows from Proposition 3.1. The last claim of the statement follows from Lemma 3.2 (b). What remains is to compute the kernel of ϕ_d .

It is clear that

$$\mathrm{Aut}^{\mu_d}(\mathbb{A}^n) = \left\{ f = (f_1, \dots, f_n) \in \mathrm{Aut}(\mathbb{A}^n) \mid f_i \in \bigoplus_{k=0}^{\infty} \mathbb{C}[x_1, \dots, x_n]_{kd+1}, i=1, \dots, n \right\}.$$

Now let $f = (f_1, \dots, f_n) \in \mathrm{Aut}^{\mu_d}(\mathbb{A}^n)$ be such that the map f' induced by f on $\mathbb{A}^n/\mu_d$ is the identity. This means that f' acts trivially on

$$\mathcal{O}(\mathbb{A}^n/\mu_d) = \mathbb{C} \oplus \bigoplus_{k \geq 1} \mathbb{C}[x_1, \dots, x_n]_{kd}.$$

Hence, $f'(x_i^d) = x_i^d$ for any i which implies that $f = (\xi_1 x_1, \dots, \xi_n x_n)$, where $\xi_i^d = 1$ for $i = 1, \dots, n$. In particular, $f'(x_i^{d-1} x_j) = x_i^{d-1} x_j$ which implies that $\xi_i^{d-1} \xi_j = 1$ for any i, j . Because $\xi_i^{d-1} \xi_i = 1$, we conclude that $\xi_i = \xi_j$. The claim follows. $\square$

4. Root subgroups

Let G be an ind-group, and let $T \subset G$ be a closed torus.

Definition 4.1. A closed subgroup $U \subset G$ isomorphic to $\mathbb{C}^+$ and normalized by T is called a *root subgroup* with respect to T . The *character* of T on $\mathrm{Lie} U \simeq \mathbb{C}$ (i.e., the algebraic action of T on $\mathrm{Lie} U$) is called the *weight character* of U .

Let X be an affine variety and consider a nontrivial algebraic action of $\mathbb{C}^+$ on X , given by $\lambda: \mathbb{C}^+ \rightarrow \mathrm{Aut}(X)$. If $f \in \mathcal{O}(X)$ is $\mathbb{C}^+$ -invariant, then the *modification* $f \cdot \lambda$ of λ is defined in as

$$(f \cdot \lambda)(s)x = \lambda(f(x)s)x$$

for $s \in \mathbb{C}$ and $x \in X$. It is easy to see that this is again a $\mathbb{C}^+$ -action. In fact, the corresponding locally nilpotent derivation to $f \cdot \lambda$ is $f\delta_\lambda$, where δ_λ is the locally nilpotent derivation that correspond to λ (see Section 2.2 for details). It is clear that if X is irreducible and $f \neq 0$, then $f \cdot \lambda$ and λ have the same invariants. If $U \subset \mathrm{Aut}(X)$ is a closed subgroup isomorphic to $\mathbb{C}^+$ and if $f \in \mathcal{O}(X)^U$ is a U -invariant, then in a similar way we define the modification $f \cdot U$ of U . Choose an isomorphism $\lambda: \mathbb{C}^+ \rightarrow U$ and set

$$f \cdot U = \{(f \cdot \lambda)(s) \mid s \in \mathbb{C}^+\}.$$

Note that $\mathrm{Lie}(f \cdot U) = f \mathrm{Lie} U \subset \mathrm{Vec}(X)$, where $\mathrm{Vec}(X)$ denotes the Lie algebra of (algebraic) vector fields on X , i.e., $\mathrm{Vec}(X) = \mathrm{Der}(\mathcal{O}(X))$, the Lie algebra of derivations of $\mathcal{O}(X)$.

If a torus T acts linearly and rationally on a vector space V , then we call V *multiplicity-free* if the weight spaces V_α are all of dimension less than or equal to 1.

Lemma 4.2 ([Kr15, Lem. 6.2]). *Let X be an irreducible affine variety and let $T \subset \mathrm{Aut}(X)$ be a torus. Assume that there exists a root subgroup $U \subset \mathrm{Aut}(X)$ with respect to T such that the T -module $\mathcal{O}(X)^U$ is multiplicity-free. Then $\dim T \leq \dim X \leq \dim T + 1$.*

The next result is going to be of use in the sequel and can be found in [Lie11, Thm. 1]. We denote by $\mathcal{U}(\mathbb{A}^n)$ the subgroup of $\mathrm{Aut}(\mathbb{A}^n)$ of the following form

$$\left\{ f = (f_1, \dots, f_n) \in \mathrm{Aut}(\mathbb{A}^n) \mid \mathrm{jac}(f) = \det \left[\frac{\partial f_i}{\partial x_j} \right]_{i,j} = 1 \right\}$$

and by T'_n a maximal subtorus of $\mathcal{U}(\mathbb{A}^n)$ of the form

$$\{(t_1 x_1, \dots, t_n x_n) \mid t_i \in \mathbb{C}^*, t_1 \cdots t_n = 1\}.$$

Lemma 4.3. *Let $U \subset \mathcal{U}(\mathbb{A}^n)$ be a one-dimensional unipotent subgroup. Then U is a root subgroup with respect to T'_n if and only if $U = U_\lambda = \{(x_1, \dots, x_i + c m_i, \dots, x_n) \mid c \in \mathbb{C}\}$, where $m_i = x_1^{\lambda_1} \cdots x_{i-1}^{\lambda_{i-1}} x_{i+1}^{\lambda_{i+1}} \cdots x_n^{\lambda_n}$. The character ξ_λ corresponding to the root subgroup U is the following:*

$$\xi_\lambda: T'_n \rightarrow \mathbb{C}^*, \quad t = (t_1, \dots, t_n) \mapsto t_i t_1^{-\lambda_1} \cdots \widehat{t_i} \cdots t_n^{-\lambda_n}.$$

Remark 4.4. The last lemma can also be expressed in the following way (see [KS13, Remark 2]): there is a bijective correspondence between the T'_n -stable one-dimensional unipotent subgroups $U \subset \mathrm{Aut}(\mathbb{A}^n)$ and the characters of T'_n of the form $\lambda = \sum_j \lambda_j \epsilon_j$ where one λ_i equals 1 and the others are ≤ 0 . We will denote this set of characters by $X_u(T'_n)$:

$$X_u(T'_n) = \left\{ \lambda = \sum \lambda_j \epsilon_j \mid \lambda_i = 1 \text{ and } \lambda_j \leq 0 \text{ for } j \neq i \right\}.$$

If $\lambda \in X_u(T'_n)$, then U_λ denotes the corresponding one-dimensional unipotent subgroup normalized by T'_n .

5. A special subgroup of $\mathrm{Aut}(X)$

For any affine variety X consider the normal subgroup $\mathcal{U}(X)$ of $\mathrm{Aut}(X)$ generated by closed one-dimensional unipotent subgroups. The group $\mathcal{U}(X)$ was introduced and studied in [AFK13], where the authors called it the group of special automorphisms of X . Following [Kr15], we introduce the notion of an algebraic homomorphism between these groups.

Definition 5.1. A homomorphism $\phi: \mathcal{U}(X) \rightarrow \mathcal{U}(Y)$ is algebraic if for any subgroup $U \subset \mathcal{U}(X)$ such that $U \subset \mathrm{Aut}(X)$ is closed, $U \simeq \mathbb{C}^+$, the image $\phi(U) \subset \mathrm{Aut}(Y)$ is closed, and $\phi|_U: U \rightarrow \phi(U)$ is a homomorphism of algebraic groups. We say that ϕ is an algebraic isomorphism if ϕ is an isomorphism of groups and $\phi|_U: U \xrightarrow{\sim} \phi(U)$ is an isomorphism of algebraic groups.

A subgroup $G \subset \mathcal{U}(X)$ is called algebraic if $G \subset \mathrm{Aut}(X)$ is a closed algebraic subgroup. The next lemma can be found in [Kr15, Lem. 4.2].

Lemma 5.2. *Let $\phi: \mathcal{U}(X) \rightarrow \mathcal{U}(Y)$ be an algebraic homomorphism. Then, for any algebraic subgroup $G \subset \mathcal{U}(X)$ generated by one-dimensional unipotent subgroups of $\text{Aut}(X)$, the image $\phi(G)$ is an algebraic subgroup of $\mathcal{U}(Y)$ and $\phi|_G: G \rightarrow \phi(G)$ is a homomorphism of algebraic groups.*

Let X be an affine variety and let $\eta: \tilde{X} \rightarrow X$ be a normalization map. It is well known that any automorphism of X lifts uniquely to an automorphism of $\tilde{X}$. Indeed, for a given automorphism $\phi: X \rightarrow X$, the composition $\phi \circ \eta: \tilde{X} \rightarrow X$ is a morphism, which by the universal property of normalization factors through a morphism $\tilde{\phi}: \tilde{X} \rightarrow \tilde{X}$ such that $\phi \circ \eta = \eta \circ \tilde{\phi}$. It remains to argue that $\tilde{\phi}$ is an automorphism. But for the same reason, the inverse ϕ^{-1} lifts to an automorphism $\psi: \tilde{X} \rightarrow \tilde{X}$. Since $\eta: \tilde{X} \rightarrow X$ is birational, the compositions $\psi \circ \tilde{\phi}$ and $\tilde{\phi} \circ \psi$ are equal to the identity on a dense open subset of an irreducible variety, hence everywhere. This shows that η induces a well-defined injective homomorphism $\tilde{\eta}: \text{Aut}(X) \hookrightarrow \text{Aut}(\tilde{X})$. Moreover, in [FK18, Prop. 12.1.1] it is proved that $\tilde{\eta}$ is a *closed immersion* of ind-groups: i.e., $\tilde{\eta}(\text{Aut}(X)) \subset \text{Aut}(\tilde{X})$ is a closed subgroup and $\tilde{\eta}$ induces the isomorphism of ind-groups $\text{Aut}(X) \xrightarrow{\sim} \tilde{\eta}(\text{Aut}(X))$. Hence, we have the following statement.

Lemma 5.3. *Let X be an irreducible affine variety, and let $\eta: \tilde{X} \rightarrow X$ be its normalization. Then every automorphism of X lifts uniquely to an automorphism of $\tilde{X}$ and the induced map $\tilde{\eta}: \text{Aut}(X) \hookrightarrow \text{Aut}(\tilde{X})$ is a closed immersion of ind-groups.*

Proposition 5.4. *Let $n \geq 3$ and let X be an n -dimensional irreducible affine variety endowed with a nontrivial SL_n -action. Then*

$$\mathcal{O}(X) = \sum_{i=1, \dots, l} \bigoplus_{k \geq 0} \mathbb{C}[x_1, \dots, x_n]_{kd_i}$$

for some $d_1, \dots, d_l \in \mathbb{N}$, where $(d_1, \dots, d_l) = d$ and the normalization of X is isomorphic to $A_{d,n}$. The same holds when $n = 2$ and the normalization of X is $A_{d,2}$ for some $d \in \mathbb{N}$.

Proof. First, let $n \geq 3$. If X is normal, then from [KRZ20, Thm. 1.6 and Prop. 4.4(2); see also Example 4.5] it follows that $X \simeq A_{d,n}$ for some $d \in \mathbb{N}$. It is well known that

$$\mathcal{O}(A_{d,n}) = \bigoplus_{k=0}^{\infty} \mathbb{C}[x_1, \dots, x_n]_{kd}$$

is a direct sum of irreducible pairwise nonisomorphic SL_n -modules $\mathbb{C}[x_1, \dots, x_n]_{kd}$.

Now, consider any n -dimensional irreducible affine variety X endowed with a nontrivial SL_n -action and a normalization morphism $\eta: A_{d,n} \rightarrow X$. Since any SL_n -action on $\mathcal{O}(X)$ lifts to an SL_n -action on $\mathcal{O}(A_{d,n})$, it follows that $\mathcal{O}(X)$ is a SL_n -submodule of $\mathcal{O}(A_{d,n})$ and therefore

$$\mathcal{O}(X) = \bigoplus_{k \in \Omega} \mathbb{C}[x_1, \dots, x_n]_{kd},$$

where Ω is a submonoid of $\mathbb{N}$ under addition. Since $\mathcal{O}(X)$ is finitely generated, $\Omega \subset \mathbb{N}$ is a finitely generated submonoid: i.e., there exist $d_1, \dots, d_l \in \mathbb{N}$ such that $\Omega = d_1\mathbb{N} + \dots + d_l\mathbb{N}$. The claim follows. $\square$

6. 2-dimensional case

6.1. Two dimensional normal affine SL_2 -surfaces

The next result can be found in [Pop73], §3 (see also [Giz71] and [Kr84], §4).

Lemma 6.1. *Let X be an affine normal irreducible variety of dimension two endowed with a nontrivial SL_2 -action. Then X is SL_2 -equivariantly isomorphic to one of the following varieties:*

- (a) $A_{d,2}$ for some $d \in \mathbb{N}$, where the SL_2 -action on $A_{d,2}$ is induced by the standard SL_2 -action on $\mathbb{A}^2$,
- (b) SL_2/T , where T is the standard subtorus of SL_2 and SL_2 acts on SL_2/T by left multiplication,
- (c) SL_2/N , where N is the normalizer of T and SL_2 acts on SL_2/N by left multiplication.

The SL_2 -action on SL_2/T and on SL_2/N from the Lemma above is transitive. The SL_2 -variety $A_{d,2}$ is the union of a fixed point and the orbit $(\mathbb{A}^2 \setminus \{0\})/\mu_d \simeq \mathrm{SL}_2/U_d$, where μ_d acts by scalar multiplication on $\mathbb{A}^2 \setminus \{0\}$ and

$$U_d = \left\{ \begin{bmatrix} \xi & t \\ 0 & \xi^{-1} \end{bmatrix} \mid t \in \mathbb{C}, \xi \in \mathbb{C}^*, \xi^d = 1 \right\}.$$

Moreover, any closed subgroup of SL_2 of codimension less than or equal to 2 is conjugate to either T , or N , or U_d for some $d \geq 1$, or

$$B = \left\{ \begin{bmatrix} a & t \\ 0 & a^{-1} \end{bmatrix} \mid t \in \mathbb{C}, a \in \mathbb{C}^* \right\}$$

(see, for example, [Pop73, p. 803]).

The next result can be found, for example, in [Kr84, III.2.5, Folgerung 3].

Proposition 6.2. *If a reductive group G acts on an affine variety X and if the stabilizer of a point $x \in X$ contains a maximal torus, then the orbit Gx is closed.*

Proposition 6.3. *Let X be a two-dimensional affine SL_2 -variety and let $O = \mathrm{SL}_2 x$ be the orbit of $x \in X$. Then we are in one of the following cases:*

- (a) x is a fixed point,
- (b) the orbit O is closed and SL_2 -isomorphic to SL_2/T or SL_2/N ,
- (c) $\overline{O} = O \cup \{x_0\}$, where $\overline{O}$ is the closure of the orbit O and x_0 is a fixed point. Moreover, either $\overline{O} \simeq \mathbb{A}^2$ or x_0 is an isolated singular point.

Proof. If the stabilizer of x contains a maximal torus, then we are in case (a) or (b) by Proposition 6.2 and Lemma 6.1. Otherwise, from the classification of closed subgroups of SL_2 , it follows that the stabilizer of x coincides with U_d for some $d \geq 1$ and $\overline{O}$ does not contain orbits of dimension one. Hence, $\overline{O} = O \cup \{x_0\}$. It is clear that if $\overline{O}$ is singular, then x_0 is an isolated singular point. If $\overline{O}$ is smooth, then from Lemma 6.1 it follows that $\overline{O}$ is isomorphic to $\mathbb{A}^2$. $\square$

Remark 6.4. Note that $\mathrm{SL}_2/T \simeq \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta$, where Δ is the diagonal, and $\mathrm{SL}_2/N \simeq \mathbb{P}^2 \setminus C$, where C is a smooth conic (see [Pop73, Lem. 2]).

6.2. The structure of $\text{Aut}(\text{SL}_2/T)$

The variety SL_2/T is isomorphic to the following so-called Danielewski surface: i.e., the smooth 2-dimensional affine quadric $V(xz - y^2 + y) \subset \mathbb{A}^3$ (see [DP09]) and the quotient map $\pi: \text{SL}_2 \rightarrow \text{SL}_2/T$ is given by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto (ab, ad, cd)$. It is not difficult to see that $X = V(xz + y^2 - 1) \simeq V(xz - y^2 + y) \subset \mathbb{A}^3$. From now on and until end of Section 6, we identify SL_2/T with $X = V(xz + y^2 - 1)$.

Consider the orthogonal group $\text{O}_3 = \text{O}(3, \mathbb{C})$ associated with the quadratic form $y^2 + xz$ generated by $\tau: \mathbb{A}^3 \rightarrow \mathbb{A}^3$ given by the map $(x, y, z) \mapsto (-x, -y, -z)$ and by the group $\text{SO}_3 = \text{SO}(3, \mathbb{C})$ that is composed of the matrices

$$\frac{1}{ad - bc} \begin{pmatrix} a^2 & 2ab & -b^2 \\ ac & ad + bc & -bd \\ -c^2 & -2cd & d^2 \end{pmatrix} \text{ with } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2.$$

Following [Lam05, Thm. 6] (see also [MM90]), $\text{Aut}(X)$ is the amalgamated product of the orthogonal group $\text{O}_3 = \text{SO}_3 \rtimes \langle \tau \rangle$ and $\text{J} \rtimes \langle \tau \rangle$ along their intersection C , where J is the subgroup of $\text{Aut}(X)$ of the automorphisms of the form

$$(x, y, z) \mapsto \left(\alpha x + 2\alpha y P(z) - \alpha z P^2(z), y - z P(z), \frac{1}{\alpha} z \right); \alpha \in \mathbb{C}^*, P \in \mathbb{C}[z].$$

Note that SO_3 is generated by $\mathbb{C}^+$ -actions. Define $\tilde{\text{J}}$ to be the subgroup of J generated by $\mathbb{C}^+$ -actions. The subgroup of $\text{Aut}(X)$ generated by SO_3 and $\tilde{\text{J}}$ coincides with the subgroup of $\text{Aut}(X)$ generated by SO_3 and J and is a subgroup of $\mathcal{U}(X)$. Moreover, because τ normalizes $\langle \text{J}, \text{SO}_3 \rangle$, we have that $\text{Aut}(X) = \langle \text{J}, \text{SO}_3 \rangle \rtimes \langle \tau \rangle$. This implies that $\text{Aut}(X)$ is not connected, and $\tau \notin \text{Aut}^\circ(X)$. Since the closure of $\mathcal{U}(X) \subset \text{Aut}(X)$ is connected as $\mathcal{U}(X)$ is generated by connected subgroups and since $\langle \text{J}, \text{SO}_3 \rangle \subset \mathcal{U}(X)$ we have that $\mathcal{U}(X) \subset \text{Aut}(X)$ coincides with $\text{Aut}^\circ(X)$ and hence is closed. Moreover, $\text{Aut}(X) = \mathcal{U}(X) \rtimes \langle \tau \rangle$.

The following proposition is going to be of use later (see [Neu48, Cor. 8.11]).

Proposition 6.5. *In the amalgamated product $G = A *_C B$ with the unified subgroup $C = A \cap B$, consider two subgroups $\tilde{A} \subset A$ and $\tilde{B} \subset B$, and let $\tilde{G} = \langle \tilde{A}, \tilde{B} \rangle$. Assume that $\tilde{A} \cap C = \tilde{C} = \tilde{B} \cap C$. Then $\tilde{G} = \tilde{A} *_C \tilde{B}$.*

Lemma 6.6. *The group $\mathcal{U}(X)$ is the amalgamated product of SO_3 and J along their intersection.*

Proof. We know that $\text{Aut}(X)$ is the amalgamated product of $\text{O}_3 = \text{SO}_3 \rtimes \langle \tau \rangle$ and $\text{J} \rtimes \langle \tau \rangle$ along their intersection C . Moreover, since $\text{SO}_3 \cap C = \text{J} \cap \text{SO}_3 = \text{J} \cap C$, we have by Proposition 6.5 that $\langle \text{J}, \text{SO}_3 \rangle \subset \text{Aut}(X)$ is the amalgamated product of J and SO_3 along their intersection. As $\mathcal{U}(X) = \langle \text{J}, \text{SO}_3 \rangle$ the claim follows. $\square$

Lemma 6.7. *The subgroup $\text{Aut}_{\text{SO}_3}(X) \subset \text{Aut}(X)$ of those automorphisms that commute with SO_3 is $\langle \tau \rangle$. Moreover, $X/\langle \tau \rangle \simeq \text{SL}_2/N$.*

Proof. Let $\varphi \in \text{Aut}_{\text{SO}_3}(X)$. Since $\text{Aut}(X)$ is the amalgamated product of O_3 and J along their intersection, we can write φ as the product $a_1 \circ b_1 \circ \dots \circ a_k \circ b_k$,

where $a_i \in \mathrm{O}_3$ and $b_i \in \mathrm{J}$. Since $a = \varphi \circ a \circ \varphi^{-1}$ for any $a \in \mathrm{SO}_3$, we have that $b_k \in \mathrm{O}_3 \cap \mathrm{J}$ and φ can be written as $\tilde{a}_1 \circ \tilde{b}_1 \circ \cdots \circ \tilde{a}_k$ for some $\tilde{a}_i \in \mathrm{O}_3$ and $\tilde{b}_i \in \mathrm{J}$. Assume first that $k > 1$. Hence, the expression

$$\tilde{a}_1 \circ \tilde{b}_1 \circ \cdots \circ \tilde{a}_k \circ a \circ (\tilde{a}_1 \circ \tilde{b}_1 \circ \cdots \circ \tilde{a}_k)^{-1} = a$$

implies that $\tilde{a}_k \circ a \circ \tilde{a}_k^{-1} \in \mathrm{O}_3 \cap \mathrm{J}$. Since this should hold for any $a \in \mathrm{SO}_3$, we get a contradiction. Therefore, $k = 1$ and hence $\varphi \in \mathrm{O}_3$. This implies that $\mathrm{Aut}_{\mathrm{SO}_3}(X) = \langle \tau \rangle$ as the centralizer of SO_3 in O_3 is $\langle \tau \rangle$.

To finish the proof we need to argue that $X/\langle \tau \rangle \simeq \mathrm{SL}_2/N$. We first note that $X/\langle \tau \rangle$ is normal and since SL_2 -action on X is transitive, it follows that the induced action of SL_2 on $X/\langle \tau \rangle$ is transitive too. Hence, from Lemma 6.1 it follows that $X/\langle \tau \rangle \simeq \mathrm{SL}_2/N$ as $X \not\simeq X/\langle \tau \rangle$. $\square$

Note that the subgroup $\mathcal{U}(X) = \mathrm{Aut}^\circ(X) \subset \mathrm{Aut}(X)$ is closed (see [Kr15, Lem. 6.3]), where $\mathrm{Aut}^\circ(X)$ is the neutral component of $\mathrm{Aut}(X)$. Hence, $\mathcal{U}(X)$ is an ind-group.

Proposition 6.8. *We have the following properties.*

- (a) *All closed subgroups $S \subset \mathrm{Aut}(X)$ isomorphic to PSL_2 are conjugate.*
- (b) *The root subgroups with respect to a maximal torus $\tilde{T}$ of any $S \simeq \mathrm{PSL}_2$ are multiplicity-free with weights $1, 2, 3, \dots$ up to an automorphism of $\tilde{T}$.*

Proof. (a) Since $\mathrm{Aut}(X)$ is the amalgamated product of O_3 and J over their intersection, we have that by [Sr80] any closed subgroup $S \subset \mathrm{Aut}(X)$ isomorphic to S is conjugate to one of the factors O_3 or J . Since all unipotent subgroups of J commute, S can not be embedded into J and hence S is conjugate to a subgroup of O_3 : i.e., to SO_3 . The claim follows.

Now we are going to prove (b). Let $U \subset \mathrm{Aut}(X)$ be a root subgroup with respect to $\tilde{T}$. This means that $\tilde{T} \ltimes U$ is an algebraic subgroup of $\mathrm{Aut}(X)$ and by [Sr80], $\tilde{T} \ltimes U$ is conjugate to a subgroup of either O_3 or J . If $\tilde{T} \ltimes U$ is conjugate to a subgroup of O_3 , then the weight of U with respect to $\tilde{T}$ is either 1 or -1 , i.e., up to an automorphism of $\tilde{T}$ we can assume that the weight is 1. If $\tilde{T} \ltimes U$ is conjugate to a subgroup of J , then without loss of generality we can assume that $\tilde{T} \ltimes U$ is an algebraic subgroup of $\mathrm{J}_{\leq k}$ generated by elements of the form

$$(x, y, z) \mapsto \left(\alpha x + 2\alpha y P(z) - \alpha z P^2(z), (y - z P(z)), \frac{1}{\alpha} z \right); \quad \alpha \in \mathbb{C}^*, P \in \mathbb{C}[z]_{\leq k}$$

for some natural k , where $\mathbb{C}[z]_{\leq k}$ denotes the polynomials of degree less or equal than k . Moreover, since all tori in $\mathrm{J}_{\leq k}$ are conjugate we can assume that

$$\tilde{T} = \{(tx, y, t^{-1}z) \mid t \in \mathbb{C}^*\}.$$

By the following computation,

$$\begin{aligned} (tx, y, t^{-1}z) \circ (x + 2yP(z) - zP^2(z), (y - zP(z)), z) \circ (t^{-1}x, y, tz) \\ = (x + 2ytP(tz) - zt^2P^2(tz), (y - ztP(tz)), z), \end{aligned}$$

it is easy to see that a root subgroup $U \subset J_{\leq k}$ should have the form

$$U_i = \{(x + 2cyP_i(z) - c^2zP_i^2(z), (y - czP_i(z)), z) \mid c \in \mathbb{C}, P_i(z) = z^i\}$$

for some natural $i \leq k$. Note that the root subgroup U_i with respect to $\tilde{T}$ has the weight $i + 1$. The claim follows. $\square$

6.3. The structure of $\text{Aut}(\text{SL}_2/N)$

By Lemma 6.7, there exists an automorphism $\tau \in \text{Aut}_{\text{SO}_3}(X)$ such that the quotient $Y = X/\langle\tau\rangle$ is isomorphic to SL_2/N . In particular, $\mathcal{O}(Y) = \mathcal{O}(X)^\tau$. An automorphism ϕ of X descends to an automorphism on Y if and only if ϕ sends $\langle\tau\rangle$ -orbits to $\langle\tau\rangle$ -orbits. In fact, such an automorphism induces the automorphism of $\mathcal{O}(X)$ that sends $\langle\tau\rangle$ -invariant functions of $\mathcal{O}(X)$ to $\langle\tau\rangle$ -invariant functions of $\mathcal{O}(X)$. This condition for ϕ is equivalent to the condition that ϕ normalizes $\langle\tau\rangle$. Moreover, since τ has order two, ϕ commutes with τ . Recall that by $\text{Aut}^{(\tau)}(X)$ we denote the subgroup of those elements of $\text{Aut}(X)$ that normalize $\langle\tau\rangle$, but in this particular case $\text{Aut}^{(\tau)}(X)$ is even the subgroup of those automorphisms of $\text{Aut}(X)$ that commute with τ .

As we have mentioned above, $\phi \in \text{Aut}(X)$ induces an automorphism of $Y \simeq \text{SL}_2/N$ if and only if $\phi \in \text{Aut}^{(\tau)}(X)$. On the other hand, since $X \simeq \text{SL}_2/T$ is simply connected and the quotient map $\pi: X \rightarrow X/\langle\tau\rangle = Y$ is an étale covering, every automorphism φ of Y can be lifted to a continuous analytical automorphism of X and hence by [Sr58, Prop. 20], φ can be lifted to an automorphism $\tilde{\varphi}$ of X : i.e., $\tilde{\varphi} \in \text{Aut}^{(\tau)}(X)$. Hence, we have the surjective homomorphism $\text{Aut}^{(\tau)}(X) \rightarrow \text{Aut}(Y)$ with the kernel $\langle\tau\rangle$ and so

$$\text{Aut}(Y) \simeq \text{Aut}^{(\tau)}(X)/\langle\tau\rangle. \quad (1)$$

Observe that $\text{SO}_3 \times \langle\tau\rangle$ is the subgroup of $\text{Aut}^{(\tau)}(X)$. Define the subgroup $J^{(\tau)} \subset \text{Aut}(X)$ of those automorphisms from J which normalize $\langle\tau\rangle$. It is not difficult to see that $J^{(\tau)}$ is comprised of the following automorphisms:

$$\left\{ (x, y, z) \mapsto \left(\alpha x + 2\alpha y P(z) - \alpha z P^2(z), y - z P(z), \frac{1}{\alpha} z \right) \mid \alpha \in \mathbb{C}^*, P \in \bigoplus_{l=0}^{\infty} \mathbb{C} z^{2l} \right\}.$$

We have the following statement.

Lemma 6.9. *The subgroup $\text{Aut}^{(\tau)}(X) \subset \text{Aut}(X)$ is the direct product of $\langle\tau\rangle$ and the amalgamated product of SO_3 and $J^{(\tau)}$ along their intersection.*

Proof. As we have mentioned above, $\text{Aut}^{(\tau)}(X)$ is the subgroup of those automorphisms of $\text{Aut}(X)$ that commute with τ . Assume $\phi \in \text{Aut}(X)$ commutes with τ . Since $\text{Aut}(X)$ is the amalgamated product of $\text{SO}_3 \times \langle\tau\rangle$ and $J \rtimes \langle\tau\rangle$, one can write ϕ as a product $a_1 \circ b_1 \circ \dots \circ a_k \circ b_k$, where $a_i \in \text{SO}_3 \times \langle\tau\rangle$ and $b_i \in J \rtimes \langle\tau\rangle$. Further, because ϕ commutes with τ , $\tau\phi\tau = \phi$ or equivalently,

$$\tau \circ a_1 \circ b_1 \circ \dots \circ a_k \circ b_k \circ \tau = a_1 \circ b_1 \circ \dots \circ a_k \circ b_k.$$

Since τ commutes with SO_3 , one can rewrite this equation as follows:

$$a_1 \circ (\tau \circ b_1 \circ \tau) \circ \cdots \circ a_k \circ (\tau \circ b_k \circ \tau) = a_1 \circ b_1 \circ \cdots \circ a_k \circ b_k.$$

From the amalgamated product structure of $\mathrm{Aut}(X)$, it follows that $\tau \circ b_i \circ \tau = c_i \circ b_i$ for some $c_i \in (\mathrm{SO}_3 \times \langle \tau \rangle) \cap (J \rtimes \langle \tau \rangle)$. But this can happen only if b_i commutes with τ : i.e., $b_i \in J^{(\tau)} \times \langle \tau \rangle$. Therefore, $\mathrm{Aut}^{(\tau)}(X)$ is generated by $\mathrm{SO}_3 \times \langle \tau \rangle$ and $J^{(\tau)} \times \langle \tau \rangle$. Moreover, $\mathrm{Aut}^{(\tau)}(X) = \langle \mathrm{SO}_3, J^{(\tau)} \rangle \times \langle \tau \rangle$ and by Proposition 6.5, $\langle \mathrm{SO}_3, J^{(\tau)} \rangle$ is the amalgamated product of SO_3 and $J^{(\tau)}$ over their intersection. $\square$

From Lemma 6.9 and (1) we have the following statement.

Lemma 6.10. *The automorphism group $\mathrm{Aut}(Y)$ is isomorphic to the amalgamated product of SO_3 and $J^{(\tau)}$. In particular, $\mathrm{Aut}(Y) = \mathcal{U}(Y)$.*

Remark 6.11. Lemma 6.10 can also be retrieved from [KPZ17, Rem. 3.9] and Remark 6.4 (see also [DG77, (2.4.3; 1)]).

Corollary 6.12. *We have the following properties.*

- (a) *All closed subgroups $S \subset \mathrm{Aut}(Y)$ isomorphic to PSL_2 are conjugate.*
- (b) *The root subgroups of $\mathrm{Aut}(Y)$ with respect to a maximal torus $\tilde{T}$ of any $S \simeq \mathrm{PSL}_2$ are multiplicity-free with weights $1, 3, 5, \dots$ up to an automorphism of $\tilde{T}$. In particular, $\mathcal{U}(\mathrm{SL}_2/N)$ and $\mathcal{U}(\mathrm{SL}_2/T)$ are not algebraically isomorphic.*

Proof. (a) By Lemma 6.10, $\mathrm{Aut}(Y)$ is the amalgamated product of SO_3 and $J^{(\tau)}$ and by [Sr80] any algebraic subgroup of the amalgamated product is conjugate to one of the factors. Since $J^{(\tau)}$ does not contain a copy of PSL_2 , it follows that S is conjugate to a subgroup of SO_3 : i.e., to SO_3 itself.

(b) Without loss of generality, we can assume that S equals SO_3 and $\tilde{T} \subset \mathrm{SO}_3$ is the subtorus of the form

$$\{(tx, y, t^{-1}z) \mid t \in \mathbb{C}^*\}.$$

Any root subgroup of $\mathrm{Aut}(Y)$ with respect to $\tilde{T}$ lifts to a root subgroup U of $\mathrm{Aut}^{(\tau)}(X)$ (see Lemma 3.2) with respect to the subtorus $p^{-1}(\tilde{T})^\circ \subset \mathrm{Aut}^{(\tau)}(X)$. As it follows from the proof of Proposition 6.8(b), U coincides with

$$U_{2i} = \{(x + 2yP_i(z) - zP_i^2(z), (y - zP_i(z)), z) \mid P_i(z) = z^{2i}\}$$

for some $i \in \mathbb{N} \cup \{0\}$. The weight of the root subgroup $U_{2i} \subset \mathrm{Aut}^{(\tau)}(X)$ with respect to $p^{-1}(\tilde{T})^\circ$ is $2i + 1$. Since the kernel of $p^{-1}(\tilde{T}) \rightarrow \tilde{T}$ is trivial, we have that the set of weights of root subgroups of $\mathrm{Aut}(Y)$ with respect to $\tilde{T}$ is $\{2i + 1 \mid i \in \mathbb{N}\}$. This proves the first part of the statement. The second part follows because if there is an algebraic isomorphism $\varphi: \mathcal{U}(X) \rightarrow \mathcal{U}(Y)$, then φ maps root subgroups of $\mathcal{U}(X)$ with respect to a subtorus $\tilde{T} \subset \mathcal{U}(X)$ to root subgroups of $\mathcal{U}(Y)$ with respect to $\varphi(\tilde{T})$ that have the same weights. But as follows from Proposition 6.8 and the first part of this proof, it is not the case. $\square$

Remark 6.13. Analogously as in the proof of Lemma 6.7, using amalgamated product structure of $\text{Aut}(Y)$ described in Lemma 6.10, we can show that the subgroup $\text{Aut}_{\text{SO}_3}(X) \subset \text{Aut}(X)$ of those automorphisms that commute with SO_3 is trivial.

6.4. On the automorphism group of $A_{d,2}$

Recall that by Proposition 3.3, there is a surjective homomorphism $\phi_d: \text{Aut}^{\mu_d}(\mathbb{A}^n) \rightarrow \text{Aut}(A_{d,n})$ of groups. Consider now the maximal subtorus

$$T_n = \{(t_1 x_1, \dots, t_n x_n) \mid t_i \in \mathbb{C}^*\} \subset \text{Aut}(\mathbb{A}^n)$$

and recall that by T'_n we denote the subtorus of the form

$$\{(t_1 x_1, \dots, t_n x_n) \mid t_i \in \mathbb{C}^*, t_1 \cdots t_n = 1\} \subset \mathcal{U}(\mathbb{A}^n)$$

that has dimension $n-1$. Then $T'_{d,n} = \phi_d(T'_n)$ is a maximal subtorus of $\mathcal{U}(A_{d,n}) \subset \text{Aut}(A_{d,n})$.

Lemma 6.14. *Let $U \subset \text{Aut}(A_{d,n})$ be a root subgroup with respect to $T'_{d,n}$ which has a character χ . Then U lifts to a root subgroup $\tilde{U} = (\phi_d^{-1}(U))^\circ \subset \text{Aut}^{\mu_d}(\mathbb{A}^n)$ with respect to $T'_n = (\phi_d^{-1}(T'_{d,n}))^\circ$ with character $\tilde{\chi} = \psi^*(\chi)$ such that the following diagram*

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mu_{(n,d)} & \longrightarrow & T'_n & \xrightarrow{\psi} & T'_{d,n} \longrightarrow 1 \\ & & & & \downarrow \tilde{\chi} & & \downarrow \chi \\ & & & & \mathbb{C}^* & \xrightarrow{=} & \mathbb{C}^* \end{array}$$

commutes, where $\psi = \phi_d|_{T'_n}$ and $\psi^(\chi)$ is the pull-back of χ .*

Proof. From Proposition 3.1, it follows that any root subgroup U of $\text{Aut}(A_{d,n})$ with respect to $T'_{d,n}$ lifts to a unipotent subgroup $\tilde{U} = (\phi_d^{-1}(U))^\circ$ of $\text{Aut}^{\mu_d}(\mathbb{A}^n)$. Moreover, $\tilde{U}$ is normalized by $(\phi_d^{-1}(T'_{d,n}))^\circ = T'_n$. Now, let $\tilde{u} \in \tilde{U}$ be a nontrivial element and $u = \phi_d(\tilde{u}) \in U$. We have group isomorphisms

$$\mathbb{C}^+ \xrightarrow{\sim} \tilde{U}, \quad s \mapsto \tilde{u}(s)$$

and

$$\mathbb{C}^+ \xrightarrow{\sim} U, \quad s \mapsto u(s).$$

Now the proof follows from the formula

$$\phi_d(\tilde{u}(\chi \circ \psi(t)s)) = u(\chi(t)s). \quad \square$$

Observe that the homomorphism $\phi_d: \text{Aut}^{\mu_d}(\mathbb{A}^n) \rightarrow \text{Aut}(A_{d,n})$ induces the homomorphism $\tilde{\phi}_d: \mathcal{U}^{\mu_d}(\mathbb{A}^n) \rightarrow \mathcal{U}(A_{d,n})$ which has the kernel $\mu_{(n,d)}$, where $\mathcal{U}^{\mu_d}(\mathbb{A}^n) \subset \text{Aut}^{\mu_d}(\mathbb{A}^n)$ is a subgroup generated by $\mathbb{C}^+$ -actions.

In [BH03], it is proved that any faithful action of an $(n-1)$ -dimensional torus on an n -dimensional toric T_Z -variety Z is conjugate to a subtorus of the big torus T_Z . This result is used in order to prove the following lemma.

Lemma 6.15. *Let T be an algebraic subtorus of $\mathcal{U}(A_{d,n})$ of dimension $(n-1)$. Then there exists an algebraic isomorphism $F: \mathcal{U}(A_{d,n}) \xrightarrow{\sim} \mathcal{U}(A_{d,n})$ such that $F(T) = T'_{d,n}$.*

Proof. Since $T \subset \mathcal{U}(A_{d,n}) \subset \mathrm{Aut}(A_{d,n})$ is an algebraic subtorus of dimension $n-1$, and since $A_{d,n}$ is toric, by [BH03, Thm. p. 2] there exists $\varphi \in \mathrm{Aut}(A_{d,n})$ such that $\varphi \circ T \circ \varphi^{-1} \subset T'_{d,n}$. Moreover, since $\mathcal{U}(A_{d,n})$ is a normal subgroup of $\mathrm{Aut}(A_{d,n})$, $\varphi \circ T \circ \varphi^{-1} \subset T'_{d,n}$ and hence since $\dim T'_{d,n} = n-1$, $\varphi \circ T \circ \varphi^{-1} = T'_{d,n}$. This proves that an algebraic isomorphism $F: \mathcal{U}(A_{d,n}) \rightarrow \mathcal{U}(A_{d,n})$, $\psi \mapsto \varphi \circ \psi \circ \varphi^{-1}$ maps T to $T'_{d,n}$. $\square$

Let Z be an irreducible affine variety of dimension $n \geq 2$ and $\psi: \mathcal{U}(Z) \xrightarrow{\sim} \mathcal{U}(A_{d,n})$ be an algebraic isomorphism. Let T be an $(n-1)$ -dimensional algebraic subtorus of $\mathcal{U}(Z)$. Then, after composing ψ with a suitable algebraic isomorphism $F: \mathcal{U}(A_{d,n}) \rightarrow \mathcal{U}(A_{d,n})$ (see Lemma 6.15), we can assume that $\psi(T) = T'_{d,n}$.

Lemma 6.16. *Root subgroups U and $\psi(U)$ have the same weight characters with respect to T and $\psi(T) = T'_{d,n}$, respectively: i.e., if $\chi: T'_{d,n} \rightarrow \mathbb{C}^*$ is the weight of $\psi(U)$, then the weight of U is $\chi \circ \psi$.*

Proof. Let U be a root subgroup of $\mathcal{U}(Z)$ with respect to T and $\mathrm{Lie} U = \mathbb{C}\nu$, where ν is a generator. Then $\psi(U)$ is the root subgroup of $\mathcal{U}(A_{d,n})$ with respect to $T'_{d,n}$. The algebraic isomorphism ψ induces an isomorphism $d\psi_e^u: \mathrm{Lie} U \rightarrow \mathrm{Lie} \psi(U)$. Note that the action of T on U induces the action of T on $\mathrm{Lie} U$. Then

$$\psi(t) \circ d\psi_e^u(\nu) \circ \psi(t^{-1}) = \chi(\psi(t)) d\psi_e^u(\nu) = d\psi_e^u(\chi(\psi(t))\nu) = d\psi_e^u(\chi \circ \psi(t)\nu),$$

where $t \in T$. On the other hand,

$$\psi(t) \circ d\psi_e^u(\nu) \circ \psi(t^{-1}) = d\psi_e^u(t \circ \nu \circ t^{-1}).$$

The claim follows. $\square$

Lemma 6.17. *Let d be even. Then the set of weights of root subgroups of $\mathrm{Aut}(A_{d,2})$ with respect to $T'_{d,2}$ is $\{(kd+2)/2 \mid k \in \mathbb{N} \cup \{0\}\}$ up to an automorphism of $T'_{d,2}$.*

Proof. By Proposition 3.3, any root subgroup of $\mathrm{Aut}(A_{d,2})$ with respect to $T'_{d,2}$ lifts to a root subgroup of $\mathrm{Aut}^{\mu_d}(\mathbb{A}^2)$ with respect to $\phi_d^{-1}(T'_{d,2})^\circ = T'_2$. By Lemma 4.3, any root subgroup of $\mathrm{Aut}(\mathbb{A}^2)$ with respect to T'_2 is equal either to

$$U_s = \{(x + cy^s, y) \mid c \in \mathbb{C}\}$$

or to

$$\tilde{U}_l = \{(x, y + cx^l) \mid c \in \mathbb{C}\}$$

for some $s, l \in \mathbb{N} \cup \{0\}$. Root subgroups U_s and $\tilde{U}_l$ belong to $\mathrm{Aut}^{\mu_d}(\mathbb{A}^2)$ if and only if $s, l \in d\mathbb{N} + 1$. The weight of the action of $T'_2 = \{(cx, c^{-1}y) \mid c \in \mathbb{C}^*\}$ on U_s by $t \circ u \circ t^{-1}$, $t \in T'_2$ and $u \in U_s$ equals $s+1$. Analogously, the weight of T'_2 -action on $\tilde{U}_l$ is $-l-1$. Therefore, the set of weights of root subgroups of $\mathrm{Aut}^{\mu_d}(\mathbb{A}^2)$ with respect to T'_2 is $\{kd+2 \mid k \in \mathbb{N} \cup \{0\}\}$ up to an automorphism of T'_2 . Moreover, since the kernel of the map $\phi_d: T'_2 \rightarrow T_{d,2}$ is μ_2 as d is even, the statement follows from Lemma 6.14. $\square$

By the Jung-Van der Kulk Theorem (see [Ju42] and [Kul53]), $\text{Aut}(\mathbb{A}^2) = \text{Aff}_2 *_C J$, where Aff_2 is the group of affine transformations of $\mathbb{A}^2$ and

$$J = \{(ax + c, by + f(x)) \mid a, b \in \mathbb{C}^*, c \in \mathbb{C}, f(y) \in \mathbb{C}[x]\}$$

and $C = \text{Aff}_2 \cap J$. Now, the subgroup $\text{Aut}^{\mu_d}(\mathbb{A}^2) \subset \text{Aut}(\mathbb{A}^2)$ contains the standard $\text{GL}_2 \subset \text{Aut}(\mathbb{A}^2)$ and

$$J_d = \left\{ (ax + c, by + f(x)) \mid a, b \in \mathbb{C}^*, c \in \mathbb{C}, f(y) \in \bigoplus_{l \geq 0} \mathbb{C}x^{ld+1} \right\}.$$

By [AZ13, Thm. 4.2], $\text{Aut}(A_{d,2}) \simeq \text{Aut}^{\mu_d}(\mathbb{A}^2)/\mu_d$ is the amalgamated product of GL_2/μ_d and J_d/μ_d along their intersection. Moreover, as we will see in Lemma 6.18 below, such an amalgamated product structure induces the amalgamated product structure of $\mathcal{U}(A_{d,2})$. Denote by $\tilde{J}_d$ the subgroup of J_d of the form

$$\left\{ (ax + c, a^{-1}y + f(x)) \mid a \in \mathbb{C}^*, c \in \mathbb{C}, f(y) \in \bigoplus_{l \geq 0} \mathbb{C}x^{ld+1} \right\}$$

and by $\tilde{T}_{d,2}$ the one-dimensional subtorus of $\text{Aut}(A_{d,2})$ induced by the $\mathbb{C}^*$ -action on $\mathbb{A}^2$ given by the maps $\{(x, y) \mapsto (cx, y) \mid c \in \mathbb{C}^*\}$. We have the following statement.

Lemma 6.18. *The group $\text{Aut}(A_{d,2})$ is the semidirect product $\mathcal{U}(A_{d,2}) \rtimes \tilde{T}_{d,2}$. Moreover, $\mathcal{U}(A_{d,2})$ is the amalgamated product of $\text{SL}_2/(\mu_d \cap \text{SL}_2)$ and $\tilde{J}_d/(\mu_d \cap \tilde{J}_d)$ along their intersection.*

Proof. As $\text{Aut}(A_{d,2}) \simeq \text{Aut}^{\mu_d}(\mathbb{A}^2)/\mu_d$, it is clear that $\text{Aut}(A_{d,2})$ is generated by $\mathcal{U}(A_{d,2})$ and $\tilde{T}_{d,2}$. Further, the subgroup $\mathcal{U}(A_{d,2}) \subset \text{Aut}(A_{d,2})$ is normal and the subgroups $\mathcal{U}(A_{d,2})$ and $\tilde{T}_{d,2}$ do not intersect. Indeed, if the subgroups $\tilde{T}_{d,2}$ and $\mathcal{U}(A_{d,2})$ have a nontrivial intersection, then as $\text{Aut}(A_{d,2}) \simeq \text{Aut}^{\mu_d}(\mathbb{A}^2)/\mu_d$, the subgroups $\{(x, y) \mapsto (cx, y) \mid c \in \mathbb{C}^*\}$ and $\mathcal{U}(\mathbb{A}^2)$ of $\text{Aut}(\mathbb{A}^2)$ also have a nontrivial intersection, which is not the case. Hence, we conclude that $\text{Aut}(A_{d,2}) = \mathcal{U}(A_{d,2}) \rtimes \tilde{T}_{d,2}$.

Recall that $\text{Aut}(A_{d,2})$ is the amalgamated product of GL_2/μ_d and J_d/μ_d along their intersection and $\text{GL}_2/\mu_d = \text{SL}_2/(\mu_d \cap \text{SL}_2) \rtimes \tilde{T}_{d,2}$ and $J_d/\mu_d = \tilde{J}_d/(\mu_d \cap \tilde{J}_d) \rtimes \tilde{T}_{d,2}$. Since $\tilde{T}_{d,2}$ is contained in the intersection $\text{GL}_2/\mu_d \cap J_d/\mu_d$, it follows that

$$\text{Aut}(A_{d,2}) = \tilde{T}_{d,2} \rtimes (\text{SL}_2/(\mu_d \cap \text{SL}_2) *_C \tilde{J}_d/(\mu_d \cap \tilde{J}_d)), \quad (2)$$

where C is the intersection of $A = \text{SL}_2/(\mu_d \cap \text{SL}_2)$ and $B = \tilde{J}_d/(\mu_d \cap \tilde{J}_d)$. As both A and B are generated by unipotent subgroups it follows that $A *_C B \subset \mathcal{U}(A_{d,2})$. The other inclusion follows from (2). The claim follows. $\square$

Remark 6.19. Define the homomorphism of abstract groups

$$\text{Aut}(A_{d,2}) = \tilde{T}_{d,2} \rtimes \mathcal{U}(A_{d,2}) \rightarrow \tilde{T}_{d,2}$$

by projection onto the first factor. Such a homomorphism is a morphism of ind-groups which implies that $\mathcal{U}(A_{d,2}) \subset \text{Aut}(A_{d,2})$ is a closed subgroup.

Remark 6.20. By Lemma 6.18, $\mathcal{U}(\mathbb{A}_{d,2})$ is the amalgamated product of the groups $\mathrm{SL}_2/(\mu_d \cap \mathrm{SL}_2)$ and $\tilde{\mathrm{J}}_d/(\tilde{\mathrm{J}}_d \cap \mu_d)$ along their intersection. Note that if d is even, then $\mathrm{SL}_2/(\mu_d \cap \mathrm{SL}_2)$ is isomorphic to PSL_2 . If d is odd, then $\mathrm{SL}_2/(\mu_d \cap \mathrm{SL}_2)$ is isomorphic to SL_2 .

The following result was pointed out to me by Hanspeter Kraft.

Proposition 6.21. *Let Z be an irreducible affine normal variety of dimension 2.*

- (a) *The groups $\mathcal{U}(\mathrm{SL}_2/T)$ and $\mathcal{U}(Z)$ are algebraically isomorphic if and only if $Z \simeq \mathrm{SL}_2/T$ or $Z \simeq A_{2,2}$.*
- (b) *The groups $\mathcal{U}(\mathrm{SL}_2/N)$ and $\mathcal{U}(Z)$ are algebraically isomorphic if and only if $Z \simeq \mathrm{SL}_2/N$ or $Z \simeq A_{4,2}$.*

Proof. Let X be isomorphic either to SL_2/T or to SL_2/N . Then $\mathcal{U}(X)$ contains a copy of PSL_2 (see Lemma 6.6 and Lemma 6.10, respectively). Hence, by Lemma 6.1 Z is isomorphic either to SL_2/T , to SL_2/N , or to $A_{d,2}$ for some $d \in \mathbb{N}$. We claim that Z can be isomorphic to $A_{d,2}$ only if d is even. Indeed, assume that $\mathrm{Aut}(A_{d,2})$ contains an algebraic subgroup S isomorphic to PSL_2 . Since $S \subset \mathcal{U}(A_{d,2})$ it follows from Lemma 6.18 that S is conjugate either to a subgroup of $\mathrm{SL}_2/(\mu_d \cap \mathrm{SL}_2)$ or $\tilde{\mathrm{J}}_d/(\mu_d \cap \tilde{\mathrm{J}}_d)$ (see [Sr80]). Moreover, since $\tilde{\mathrm{J}}_d/(\mu_d \cap \tilde{\mathrm{J}}_d)$ does not contain a copy of PSL_2 , S should be conjugate to a subgroup of $\mathrm{SL}_2/(\mu_d \cap \mathrm{SL}_2)$. Hence, by Remark 6.20 we conclude that d is even.

By Corollary 6.12 we have $\mathcal{U}(\mathrm{SL}_2/T) \not\simeq \mathcal{U}(\mathrm{SL}_2/N)$. Hence, to prove (a) we first need to show that an algebraic isomorphism $\phi: \mathcal{U}(A_{d,2}) \xrightarrow{\sim} \mathcal{U}(\mathrm{SL}_2/T)$ implies that $d = 2$. By Lemma 6.17, the set of weights of root subgroups of $\mathcal{U}(A_{d,2})$ with respect to $T'_{d,2}$ is $\{(kd + 2)/2 \mid k \in \mathbb{N} \cup \{0\}\}$ up to an automorphism of $T'_{d,2}$. Since $T_{d,2}$ is a subgroup of some $S \subset \mathcal{U}(A_{d,2})$ isomorphic to PSL_2 we have by Proposition 6.8 that the set of weights of root subgroups of $\mathcal{U}(X \simeq \mathrm{SL}_2/T)$ with respect to $\phi(T_{d,2})$ is $\{1, 2, 3, \dots\}$ up to an automorphism of $\phi(T_{d,2})$. By Lemma 6.16, the set of weights of root subgroups of $\mathcal{U}(A_{d,2})$ with respect to $T_{d,2}$ and of $\mathcal{U}(\mathrm{SL}_2/T)$ with respect to $\phi(T_{d,2})$ are equal. Therefore, d indeed equals 2. To finish the proof of (a) we need to show that $\mathcal{U}(A_{2,2})$ and $\mathcal{U}(X \simeq \mathrm{SL}_2/T)$ are algebraically isomorphic. To do so, we first note that by Lemma 6.18 and Lemma 6.6 the first factors SL_2/μ_2 and SO_3 from the amalgamated product structure of $\mathcal{U}(A_{2,2})$ and of $\mathcal{U}(X)$ respectively are isomorphic to PSL_2 . Moreover, $\tilde{\mathrm{J}}_2$ and J are algebraically isomorphic, as both $\tilde{\mathrm{J}}_2$ and J are direct limits of isomorphic algebraic groups. Finally, the intersections $\mathrm{SL}_2/\mu_2 \cap \tilde{\mathrm{J}}_2 \subset \mathrm{Aut}(A_{2,2})$ and $\mathrm{SO}_3 \cap \mathrm{J} \subset \mathrm{Aut}(X)$ are also isomorphic as algebraic groups as they are both isomorphic to a Borel subgroup of PSL_2 .

Define a homomorphism $\varphi: \mathcal{U}(A_{d,2}) \rightarrow \mathcal{U}(X)$ that sends isomorphically the first factor SL_2/μ_2 of the amalgamated product of $\mathcal{U}(A_{d,2})$ to the first factor SO_3 of the amalgamated product of $\mathcal{U}(X)$ in a way that $\varphi(\mathrm{SL}_2/\mu_2 \cap \tilde{\mathrm{J}}_2) = \mathrm{SO}_3 \cap \mathrm{J} \subset \mathrm{Aut}(X)$ and the second factor $\tilde{\mathrm{J}}_2$ of the amalgamated product of $\mathcal{U}(A_{d,2})$ to the second factor J of the amalgamated product of $\mathcal{U}(X)$. Such a map is well defined and is an isomorphism as follows from the amalgamated product structure of $\mathcal{U}(A_{2,2})$ and $\mathcal{U}(X \simeq \mathrm{SL}_2/T)$. The proof of (a) follows.

To prove (b) we first need to show that an algebraic isomorphism $\phi: \mathcal{U}(A_{d,2}) \xrightarrow{\sim} \mathcal{U}(\mathrm{SL}_2/N)$ implies that $d = 4$. As we have already mentioned above in this proof,

the set of weights of root subgroups of $\mathcal{U}(A_{d,2})$ with respect to $T'_{d,2}$ is $\{(kd+2)/2 \mid k \in \mathbb{N} \cup \{0\}\}$ up to an automorphism of $T'_{d,2}$ (see Lemma 6.17). Further, analogously as in the first part of the proof we have that the set of weights of root subgroups of $\mathcal{U}(Y \simeq \mathrm{SL}_2/N)$ with respect to $\phi(T'_{d,2})$ is $\{1, 3, 5, \dots\}$ up to an automorphism of $\phi(T'_{d,2})$ (see Corollary 6.12). By Lemma 6.16, the set of weights of root subgroups of $\mathcal{U}(A_{d,2})$ with respect to $T_{d,2}$ and of $\mathcal{U}(Y)$ with respect to $\phi(T_{d,2})$ coincide, which implies that $d = 4$. To finish the proof of (b) we need to show that groups $\mathcal{U}(A_{d,2})$ and $\mathcal{U}(Y \simeq \mathrm{SL}_2/N)$ are algebraically isomorphic. This follows analogously as in the previous paragraph in the case of groups $\mathcal{U}(A_{2,2})$ and $\mathcal{U}(X \simeq \mathrm{SL}_2/T)$. $\square$

7. Higher-dimensional case

Consider the action of SL_n on $A_{d,n}$ induced by the standard SL_n -action on $\mathbb{A}^n$. Denote by $S_{d,n} \subset \mathrm{Aut}(A_{d,n})$ the image of SL_n under the natural homomorphism $\mathrm{SL}_n \rightarrow \mathrm{Aut}(A_{d,n})$.

Lemma 7.1. *We have an isomorphism $S_{d,n} \simeq \mathrm{SL}_n / \mu_{(d,n)}$, where (d,n) denotes the greatest common divisor of d and n . Moreover, $S_{d,n} \subset \mathcal{U}(A_{d,n})$.*

Proof. By Proposition 3.3, there is a surjective homomorphism $\phi_d: \mathrm{Aut}^{\mu_d}(\mathbb{A}^n) \rightarrow \mathrm{Aut}(A_{d,n})$ of groups with $\ker \phi_d = \mu_d$. Hence, $\mathrm{Aut}(A_{d,n}) \simeq \mathrm{Aut}^{\mu_d}(\mathbb{A}^n) / \mu_d$, which shows that $S_{d,n} \simeq \mathrm{SL}_n / (\mu_d \cap \mathrm{SL}_n) \simeq \mathrm{SL}_n / \mu_{(d,n)}$. The second claim is clear since $S_{d,n}$ is generated by unipotent subgroups. $\square$

Lemma 7.2. *If there is an injective algebraic homomorphism*

$$\varphi: S_{d,n} = \mathrm{SL}_n / \mu_{(n,d)} \hookrightarrow \mathcal{U}(A_{l,n}),$$

then $(n,d) = (n,l)$. In particular, if $\mathcal{U}(A_{d,n})$ and $\mathcal{U}(A_{l,n})$ are algebraically isomorphic, then $(d,n) = (l,n)$.

Proof. Applying Lemma 6.15, we can assume $\varphi(T_{d,n}) = T_{l,n}$. Hence, intersection $\varphi(S_{d,n}) \cap S_{l,n}$ contains $T_{l,n}$. We claim that $\varphi(S_{d,n}) = S_{l,n}$. To show this, we first note that the subgroup $T_{l,n} \subset \varphi(S_{d,n})$ lifts to T'_n and by Proposition 3.3, each root subgroup of $\varphi(S_{d,n})$ with respect to $T_{l,n}$ lifts to a one-dimensional unipotent subgroup of $\mathrm{Aut}(\mathbb{A}^n)$. Moreover, the subgroup G of $\mathrm{Aut}(\mathbb{A}^n)$ generated by all one-dimensional unipotent subgroups U_i lifted from root subgroups of $\varphi(S_{d,n})$ with respect to $T_{l,n}$ is an algebraic subgroup of $\mathrm{Aut}(\mathbb{A}^n)$. Indeed, if G is not algebraic, then G can not be written as a finite product of U_i . In contrast, $\varphi(S_{d,n})$ can be written as a finite product of root subgroups of $\varphi(S_{d,n})$ with respect to $T_{l,n}$. Moreover, $\tilde{\phi}_d$ induces a homomorphism of groups $G \rightarrow \varphi(S_{d,n})$ with a kernel $\mu_{(l,n)}$.

Hence, a homomorphism of groups $G \rightarrow \varphi(S_{d,n})$ is a homomorphism of algebraic groups with the kernel $\mu_{(l,n)}$ and G is isomorphic to SL_n that contains T'_n as a maximal subtorus. It follows from [KRZ20, Thm. 1.1] that all subgroups of $\mathrm{Aut}(\mathbb{A}^n)$ isomorphic to SL_n are conjugate. Therefore, G is conjugate to the standard $\mathrm{SL}_n \subset \mathrm{Aut}(\mathbb{A}^n)$: i.e., there exists $\psi \in \mathrm{Aut}(\mathbb{A}^n)$ such that $\psi^{-1} \circ G \circ \psi = \mathrm{SL}_n$. Since $T'_n \subset G$ we have that $\psi^{-1} \circ T'_n \circ \psi$ is a subtorus in SL_n which implies that ψ is a linear map that moreover belongs to GL_n . Now it is easy to see that G coincides with SL_n . Therefore, $\varphi(S_{d,n}) = S_{l,n}$.

Therefore, $S_{d,n}$ is isomorphic to $S_{l,n}$ as an algebraic group. Hence, from Lemma 7.1 it follows that $(d, n) = (l, n)$. The second part of the statement follows from the first one directly since $\mathcal{U}(A_{d,n})$ contains a copy of $S_{d,n}$. $\square$

Proposition 7.3. *Let X be $A_{d,n}$, SL_2/T or SL_2/N and Y be an irreducible affine variety. Assume that there is an algebraic isomorphism $\mathcal{U}(X) \xrightarrow{\sim} \mathcal{U}(Y)$. Then $\dim Y \leq \dim X$. Moreover, if additionally Y is normal, then*

- (a) *if $X \simeq \mathrm{SL}_2/T$, then $Y \simeq A_{2,2}$ or $Y \simeq \mathrm{SL}_2/T$,*
- (b) *if $X \simeq A_{2,2}$, then $Y \simeq A_{2,2}$ or $Y \simeq \mathrm{SL}_2/T$,*
- (c) *if $X \simeq \mathrm{SL}_2/N$, then $Y \simeq A_{4,2}$ or $Y \simeq \mathrm{SL}_2/N$,*
- (d) *if $X \simeq A_{4,2}$, then $Y \simeq A_{4,2}$ or $Y \simeq \mathrm{SL}_2/N$,*
- (e) *if $X = A_{d,n}$, where $(d, n) \notin \{(2, 2), (2, 4)\}$, $Y \simeq X$.*

Proof. Fix an algebraic isomorphism $\psi: \mathcal{U}(X) \xrightarrow{\sim} \mathcal{U}(Y)$ and denote by T' the image of $T'_{d,n}$ if $X = A_{d,2}$ or the image of a maximal subtorus T of $\mathcal{U}(X)$ if $X = \mathrm{SL}_2/T$ or SL_2/N . By Lemma 6.14, Proposition 6.8, and Corollary 6.12, all root subgroups $U \subset \mathcal{U}(Y)$ with respect to T' have different weights. In particular, the root subgroups $\mathcal{O}(Y)^U \cdot U \subset \mathcal{U}(Y)$ have different weights, which implies that $\mathcal{O}(Y)^U$ is multiplicity-free because the map $\mathcal{O}(Y)^U \rightarrow \mathcal{O}(Y)^U \cdot U$ is injective. Hence, by Lemma 4.2 we have that

$$\dim Y \leq \dim T' + 1 = n,$$

which proves the first part of the proposition.

Now (a), (b), (c) and (d) follow from Proposition 6.21.

To prove (e), we note that $\mathcal{U}(A_{d,n})$ contains a copy of $\mathrm{SL}_n/\mu_{(n,d)}$, which implies that SL_n acts nontrivially on Y and thus by Proposition 5.4, Lemma 6.1 and Proposition 7.3(a)-(d), $Y \simeq A_{l,n}$ for some $l \in \mathbb{N}$. Hence, $\psi: \mathcal{U}(A_{d,n}) \xrightarrow{\sim} \mathcal{U}(A_{l,n})$. By Lemma 6.15, there exists an algebraic isomorphism $F: \mathcal{U}(A_{l,n}) \xrightarrow{\sim} \mathcal{U}(A_{l,n})$ such that $F(\psi(T'_{d,n})) = T'_{l,n}$. Therefore, we can assume that $\psi(T'_{d,n}) = T'_{l,n}$.

Consider the $\mathbb{C}^+$ -action

$$\{(x_1 + cx_2^{d+1}, x_2, \dots, x_n) \mid c \in \mathbb{C}\} \text{ on } \mathbb{A}^n.$$

It induces the $\mathbb{C}^+$ -action U on $A_{d,n}$ which is normalized by $T'_{d,n}$. Hence, $\psi(U) \subset \mathcal{U}(A_{l,n})$ is a root subgroup with respect to $\psi(T'_{d,n}) = T'_{l,n}$. By Proposition 3.3, $\psi(U)$ lifts to a $\mathbb{C}^+$ -action on $\mathbb{A}^n$ normalized by T'_n . Since $(n, d) = (n, l)$ by Lemma 7.2 and because U and $\psi(U)$ have the same weight characters with respect to $T'_{d,n}$ and $T'_{l,n}$ respectively (see Lemma 6.16), Lemma 6.14 implies that $\psi(U)$ lifts to a root subgroup

$$\{(x_1 + cx_2^{d+1}, x_2, \dots, x_n) \mid c \in \mathbb{C}\}$$

of $\mathrm{Aut}(\mathbb{A}^n)$ with respect to T'_l . Therefore, $l \leq d$. Analogously, $d \leq l$, i.e., $d = l$. The proof follows. $\square$

Proof of Theorem 1.5. Let $\psi: \mathcal{U}(X) \xrightarrow{\sim} \mathcal{U}(Y)$ be an algebraic isomorphism. Proposition 7.3 implies that $\dim Y \leq \dim X$. Since SL_n acts regularly and nontrivially on X , SL_n also acts nontrivially and regularly on Y .

First, let X be isomorphic to $A_{d,n}$. Then by Lemma 6.1 and by Proposition 5.4, the normalization of Y , which we denote by $\tilde{Y}$, is isomorphic to SL_2/T , SL_2/N or $A_{l,n}$ for some $l \geq 1$. First, assume that $\tilde{Y} \simeq A_{l,n}$. Hence Proposition 5.4 implies that

$$\mathcal{O}(Y) = \sum_{i=1, \dots, l} \bigoplus_{k \geq 0} \mathbb{C}[x_1, \dots, x_n]_{kd_i}$$

for some $d_1, \dots, d_l \in \mathbb{N}$, where $(d_1, \dots, d_l) = l$

Let $\eta: A_{l,n} \rightarrow Y$ be the normalization morphism, which by Lemma 5.3 induces the algebraic homomorphism $\tilde{\eta}: \mathcal{U}(Y) \hookrightarrow \mathcal{U}(A_{l,n})$. Note that $\mathrm{SL}_n/\mu_{(n,d)}$ acts faithfully on X . Then $\mathrm{SL}_n/\mu_{(n,d)}$ also acts faithfully on Y and therefore on $A_{l,n}$. Hence, by Lemma 7.2 we have that $(n, d) = (n, l)$.

Consider the $\mathbb{C}^+$ -action

$$\{(x_1 + cx_2^{d+1}, x_2, \dots, x_n) \mid c \in \mathbb{C}\} \text{ on } \mathbb{A}^n.$$

It induces the $\mathbb{C}^+$ -action U on $A_{d,n}$ which is normalized by $T'_{d,n}$. Hence, $\psi(U) \subset \mathcal{U}(Y)$ is a root subgroup with respect to $\psi(T'_{d,n})$. By Lemma 6.15, there is an algebraic isomorphism $\mathcal{U}(A_{d,n}) \xrightarrow{\sim} \mathcal{U}(A_{d,n})$ that maps $T'_{d,n}$ to $\psi^{-1}(\tilde{\eta}^{-1}(T'_{l,n}))$ and so there is an isomorphism $\mathcal{U}(Y) \xrightarrow{\sim} \mathcal{U}(Y)$ that maps $\psi(T'_{d,n})$ to $\tilde{\eta}^{-1}(T'_{l,n})$. Hence, we can assume that $\psi(U)$ is a root subgroup with respect to $\tilde{\eta}^{-1}(T'_{l,n})$. By Lemma 5.3, $\psi(U)$ lifts to a $\mathbb{C}^+$ -action on $A_{l,n}$, which is normalized by $T'_{l,n}$ and then by Lemma 3.2(c), $\psi(U)$ lifts to a $\mathbb{C}^+$ -action on $\mathbb{A}^n$ normalized by T'_n . Since $(n, d) = (n, l)$ and because U and $\psi(U)$ have the same weight characters with respect to $\tilde{\eta}^{-1}(T'_{l,n})$ and $T'_{d,n}$ respectively (i.e., if $\chi: T'_{d,n} \rightarrow \mathbb{C}^*$ is the weight of $\psi(U)$, then the weight of U is $\chi \circ \psi$), Lemma 6.14 implies that $\psi(U)$ lifts to a root subgroup

$$\{(x_1 + cx_2^{d+1}, x_2, \dots, x_n) \mid c \in \mathbb{C}\}$$

of $\mathrm{Aut}(\mathbb{A}^n)$ with respect to T'_n . Hence

$$\{(x_1 + cx_2^{d+1}, x_2, \dots, x_n) \mid c \in \mathbb{C}\} \text{ induces an action on } \mathcal{O}(\mathbb{A}^n)^{\mu_l},$$

which implies that $l \mid d$. Moreover, $\{(x_1 + cx_2^{d+1}, x_2, \dots, x_n) \mid c \in \mathbb{C}\}$ induces an action on $\mathcal{O}(Y)$. This implies that

$$d + l_i \in \mathbb{N}l_1 + \dots + \mathbb{N}l_s$$

for any i .

The $\mathbb{C}^+$ action on $\mathbb{A}^n$ of the form $\{(x_1 + cx_2^{l_i+1}, x_2, \dots, x_n) \mid c \in \mathbb{C}\}$ induces an action on $\mathcal{O}(Y)$. Since $(d, n) = (l, n)$ it follows that

$$\{(x_1 + cx_2^{l_i+1}, x_2, \dots, x_n) \mid c \in \mathbb{C}\} \text{ induces an action on } \mathcal{O}(\mathbb{A}^n)^{\mu_d}.$$

Hence, $d \mid l_i$ for any i and then $d \mid (l_1, \dots, l_s) = l$. Therefore, $d = l$. Now, because $d \mid l_i$ for any i , $d + l_i \in \mathbb{N}l_1 + \dots + \mathbb{N}l_s$ implies that

$$\mathbb{N}l_1 + \dots + \mathbb{N}l_s = \mathbb{N}_{\geq \min_i \{l_i \mid i=1, \dots, l\}} d,$$

where $\mathbb{N}_{\geq k} = \{m \in \mathbb{N} \mid m \geq k\}$.

Now assume that $\tilde{Y}$ is isomorphic to SL_2/T or to SL_2/N , then by Proposition 6.3, $Y = \tilde{Y}$. Then (e) follows from Proposition 6.21.

Let now $X \simeq \mathrm{SL}_2/T$. Then by Lemma 6.1, $\tilde{Y}$ can only be isomorphic to SL_2/T , SL_2/N or $A_{2,2}$. By Proposition 6.21, $\tilde{Y}$ is isomorphic to SL_2/T or to $A_{2,2}$. If $\tilde{Y} \simeq \mathrm{SL}_2/T$, from Proposition 6.3, it follows that $Y = \tilde{Y}$. Hence, (b) follows from the first part of the proof. Statement (d) follows analogously. $\square$

Proof of Theorem 1.3. The isomorphism $\mathrm{Aut}(X) \xrightarrow{\sim} \mathrm{Aut}(A_{d,n})$ induces an algebraic isomorphism $\mathcal{U}(X) \xrightarrow{\sim} \mathcal{U}(A_{d,n})$. Note that X admits a torus action of dimension n . From Theorem 1.5, it follows that X can only be isomorphic to $A_{d,n}^s$. On the other hand, since normalization of $A_{d,n}^s$ is equal to $A_{d,n}$, it follows from Lemma 5.3 that there is a closed embedding $\mathrm{Aut}(A_{d,n}^s) \hookrightarrow \mathrm{Aut}(A_{d,n})$ of ind-groups. Now the proof follows from [RvS21, Prop. 9.1(3)]. $\square$

Proof of Theorem 1.4. Let Z be isomorphic either to SL_2/T or to SL_2/N . Then an isomorphism $\mathrm{Aut}(X) \xrightarrow{\sim} \mathrm{Aut}(Z)$ induces an algebraic isomorphism $\mathcal{U}(X) \xrightarrow{\sim} \mathcal{U}(Z)$. By Theorem 1.5, X is isomorphic either to Z or to $A_{2k,2}^s$ for some $s \in \mathbb{N}$ and $k \in \{1, 2\}$. To finish the proof we need to show that $\mathrm{Aut}(Z)$ can not be isomorphic to $\mathrm{Aut}(A_{2k,2}^s)$. But this is clear as all $A_{2k,2}^s$ admit an action of a two-dimensional torus and varieties SL_2/T and SL_2/N do not admit such an action. $\square$

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