CUBIC DIRAC OPERATORS AND THE STRANGE FREUDENTHAL–DE VRIES FORMULA FOR COLOUR LIE ALGEBRAS

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Abstract. The aim of this paper is to define cubic Dirac operators for colour Lie algebras. We give a necessary and sufficient condition to construct a colour Lie algebra from an ϵ -orthogonal representation of an ϵ -quadratic colour Lie algebra. This is used to prove a strange Freudenthal–de Vries formula for basic colour Lie algebras as well as a Parthasarathy formula for cubic Dirac operators of colour Lie algebras. We calculate the cohomology induced by this Dirac operator, analogously to the algebraic Vogan conjecture proved by Huang and Pandžić.

1. Introduction

The use of Dirac operators in representation theory started by Parthasarathy in [20] has been developed and generalised in many directions. An important advancement of this approach is the construction of discrete series representations of semisimple Lie groups as kernels of Dirac operators by Atiyah and Schmid ([1]). In [13], Kostant studies Dirac operators in a more general setting and shows that the introduction of a cubic term in the usual Dirac operator is necessary for non-symmetric pairs. Huang and Pandžić calculated the cohomology associated to Dirac operators as conjectured by Vogan ([9]) and then Dirac cohomologies have been studied in various situations (e.g., [15], [11], [16]). The theory of Dirac operators has been extended to other algebraic structures: for example this has been studied for Lie superalgebras in [21], [10], [12], [28] and for Hecke algebras and rational Cherednik algebras in [2], [5], [6].

Colour Lie algebras generalise both Lie algebras and Lie superalgebras and have been originally introduced by Ree in [22]. The terminology “colour” comes from their use in mathematical physics by Rittenberg and Wyler in [23], [24] and their study has been intensified by Scheunert in [25], [26], [27].

In this article we define and study cubic Dirac operators for colour Lie algebras.

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We first recall general properties of the multilinear algebra of vector spaces graded by an abelian group Γ with respect to a commutation factor ϵ of Γ (Section 2). In particular, we give properties of the generalised Clifford theory developed in [19] and [4] and we study some features of roots of “basic” colour Lie algebras similar to properties of semisimple complex Lie algebras.

In Section 3 we give a necessary and sufficient condition to extend the action of an ϵ -quadratic colour Lie algebra $\mathfrak{g}$ on an ϵ -orthogonal representation V to define a colour Lie algebra structure on $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus V$ (see Theorem 3.1). If this condition is satisfied we say that the representation V is of colour Lie type. This is similar to the conditions of [18] and generalises the results of Kostant ([13], [14]) for orthogonal and symplectic complex representations of quadratic Lie algebras and the results of Chen and Kang ([3]) for orthosymplectic complex representations of quadratic Lie superalgebras. If $\mathfrak{g}$ is a basic colour Lie algebra, then this condition for the adjoint representation of $\mathfrak{g}$ compared to the action of the Casimir element of $\mathfrak{g}$ on an oscillator representation of the ϵ -Clifford algebra of V ([19]) leads to the strange Freudenthal–de Vries formula for colour Lie algebras:

Theorem 1. *Let k be a field of characteristic 0. Let $\mathfrak{g}$ be a finite-dimensional basic colour Lie algebra in the sense of Subsection 2.4 and such that its ϵ -Killing form $B_{\mathfrak{g}}$ is non-degenerate. We have*

$$\dim_{\epsilon}(\mathfrak{g}) = 24B_{\mathfrak{g}}^*(\rho, \rho),$$

where ρ is the Weyl vector of $\mathfrak{g}$ with respect to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a choice of a full set of positive roots for the action of $\text{ad}(\mathfrak{h})$.

Note that the proof of Freudenthal and de Vries of this formula for Lie algebras uses the Weyl character formula ([8]), the proof of Kac, Möseneder, Frajria and Papi of this formula for Lie superalgebras uses vertex algebras ([12]), but our proof uses Clifford theory and is more in the spirit of Fegan and Steer ([7]).

Afterwards, in Section 4, we define a cubic Dirac operator for the ϵ -orthogonal representation of colour Lie type V of the ϵ -quadratic colour Lie algebra $\mathfrak{g}$. The operator D is defined to be the identity $\text{Id} \in \text{End}(V)$ under the canonical isomorphism between $\text{End}(V)$ and $V \otimes V$ and the cubic Dirac operator $\tilde{D}$ is defined by adding to D the cubic term ψ in the ϵ -Clifford algebra of V which is defined by the projection on V of the bracket of $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus V$ restricted to $V \times V$ similarly to [13]. We calculate the square of this cubic Dirac operator analogously to Parthasarathy’s formula ([20]) for the square of the Dirac operator of a Cartan decomposition of a reductive Lie algebra:

Theorem 2. *Let k be a field of characteristic not 2 or 3. Let $\pi : \mathfrak{g} \rightarrow \mathfrak{so}_{\epsilon}(V, (\ , \))$ be a finite-dimensional ϵ -orthogonal representation of colour Lie type of a finite-dimensional ϵ -quadratic colour Lie algebra $(\mathfrak{g}, B_{\mathfrak{g}})$. Let $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus V$ be the associated colour Lie algebra and let $\tilde{D}$ be the cubic Dirac operator. In $U_{\epsilon}(\tilde{\mathfrak{g}}) \otimes C_{\epsilon}(V, (\ , \))$, we have*

$$\tilde{D}^2 = \Omega(\tilde{\mathfrak{g}}) \otimes 1 - \Delta(\Omega(\mathfrak{g})) + \frac{1}{24}(\text{Tr}_{\epsilon}(\text{ad}_{\tilde{\mathfrak{g}}}(\Omega(\tilde{\mathfrak{g}}))) - \text{Tr}_{\epsilon}(\text{ad}_{\mathfrak{g}}(\Omega(\mathfrak{g}))))1 \otimes 1,$$

where $\Omega(\tilde{\mathfrak{g}})$ (resp. $\Omega(\mathfrak{g})$) is the Casimir element of $\tilde{\mathfrak{g}}$ (resp. $\mathfrak{g}$).

In this theorem, Δ is the diagonal embedding of $\mathfrak{g}$ in $U_\epsilon(\tilde{\mathfrak{g}}) \otimes C_\epsilon(V, (\cdot, \cdot))$ coming from the fact that the ϵ -orthogonal colour Lie algebra $\mathfrak{so}_\epsilon(V, (\cdot, \cdot))$ is canonically isomorphic to the second ϵ -exterior power $\Lambda_\epsilon^2(V)$ and that we can quantise $\Lambda_\epsilon^2(V)$ to the ϵ -Clifford algebra of V (Section 2).

Finally, this cubic Dirac operator induces a differential map and we calculate the cohomology of this differential complex in Section 5, similarly to the proof of Huang and Pandžić of the Vogan conjecture ([9]):

Theorem 3. *Let k be a field of characteristic 0. Let $\pi : \mathfrak{g} \rightarrow \mathfrak{so}_\epsilon(V, (\cdot, \cdot))$ be a finite-dimensional ϵ -orthogonal representation of colour Lie type of a finite-dimensional ϵ -quadratic colour Lie algebra $(\mathfrak{g}, B_\mathfrak{g})$. Let $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus V$ be the associated colour Lie algebra, let $\tilde{D}$ be the cubic Dirac operator and let d be its induced differential on $(U_\epsilon(\tilde{\mathfrak{g}}) \otimes C_\epsilon(V, (\cdot, \cdot)))^\mathfrak{g}$. Then we have*

$$\text{Ker}(d) = \mathcal{Z}_\epsilon(\Delta(\mathfrak{g})) \oplus \text{Im}(d).$$

To prove this theorem, we have to define a “standard resolution” of the field k similar to the Koszul and de Rham differentials and to prove the analogues of the Koszul cohomology and the Poincaré lemma in the context of ϵ -symmetric modules (Lemma 5.4).

Notation. Let k be a field of characteristic not 2 or 3, let Γ be an abelian group and let ϵ be a commutation factor of Γ . The reader who is only interested in supersymmetry and Lie superalgebras can assume $\Gamma = \mathbb{Z}_2$ and $\epsilon(a, b) = (-1)^{ab}$ for all $a, b \in \mathbb{Z}_2$.

2. Multilinear algebra over vector spaces graded by an abelian group

2.1. Basic definitions and properties about graded vector spaces

Let $V = \bigoplus_{\gamma \in \Gamma} V_\gamma$ be a Γ -graded vector space. For $v \in V_\gamma$ we set $|v| := \gamma$. For convenience, whenever the degree of an element is used in a formula, it is assumed that this element is homogeneous and that we extend by linearity the formula for non-homogeneous elements. For v, w in V we denote $\epsilon(|v|, |w|)$ by $\epsilon(v, w)$ and $\epsilon(|v|, |v|)$ by $\epsilon(v)$. We always assume that a basis of a Γ -graded vector space is composed of homogeneous elements and that the base field k has the trivial Γ -gradation.

If V and W are Γ -graded vector spaces, then $V \oplus W$ and $V \otimes W$ are Γ -graded in an obvious way. Define the Γ -graded vector space $\text{Homgr}(V, W)$ by

$$\text{Homgr}(V, W)_\gamma := \{f \in \text{Hom}(V, W) \mid f(V_a) \subseteq W_{a+\gamma} \quad \forall a \in \Gamma\} \quad \forall \gamma \in \Gamma.$$

If V and W are finite-dimensional, then $\text{Homgr}(V, W) = \text{Hom}(V, W)$ and so $V^* = \text{Hom}(V, k)$ is Γ -graded. If $\{e_i\}$ is a basis of V and $\{e_i^*\}$ is the basis of V^* dual to $\{e_i\}$ in the sense that $e_i^*(e_j) = \delta_{ij}$, then $|e_i^*| = -|e_i|$. Define also $\mathcal{E} \in \text{End}(V)$ by $\mathcal{E}(v) := \epsilon(v)v$, the ϵ -trace of $f \in \text{End}(V)$ by $\text{Tr}_\epsilon(f) := \text{Tr}(\mathcal{E} \circ f)$ and the ϵ -dimension of V by $\dim_\epsilon(V) := \text{Tr}_\epsilon(\text{Id}_V)$.

Remark 2.1. The isomorphism of Γ -graded vector spaces between $V \otimes W$ and $W \otimes V$ is given by $v \otimes w \mapsto \epsilon(v, w)w \otimes v$ and hence the isomorphism of Γ -graded vector spaces between $\text{End}(V)$ and $V \otimes V^*$ by $f \mapsto \sum_i f(e_i) \otimes e_i^*$.

It is known that if V is a Γ -graded vector space, there is a unique right group action $\pi : S_n \rightarrow \text{GL}(V^{\otimes n})$ of the permutation group S_n on $V^{\otimes n}$ such that the action of a transposition $\tau_{i,i+1} \in S_n$ is given by

$$\pi(\tau_{i,i+1})(v_1 \otimes \cdots \otimes v_n) = -\epsilon(v_i, v_{i+1})v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_n$$

for all $v_1, \dots, v_n \in V$. For arbitrary elements $\sigma, \sigma' \in S_n$, this action is given by

$$\pi(\sigma)(v_1 \otimes \cdots \otimes v_n) = p(\sigma; v_1, \dots, v_n)v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

where

$$p(\sigma; v_1, \dots, v_n) = \text{sgn}(\sigma) \prod_{\substack{1 \leq i < j \leq n \\ \sigma^{-1}(i) > \sigma^{-1}(j)}} \epsilon(v_i, v_j)$$

and satisfies to

$$p(\sigma\sigma'; v_1, \dots, v_n) = p(\sigma'; v_{\sigma(1)}, \dots, v_{\sigma(n)})p(\sigma; v_1, \dots, v_n), \quad p(\text{Id}; v_1, \dots, v_n) = 1.$$

Recall that for multilinear maps on Γ -graded vector spaces there is a notion of symmetry and antisymmetry with respect to the commutation factor ϵ .

Definition 2.2. Let V and W be Γ -graded vector spaces and let $B : V \times V \rightarrow W$ be a bilinear map.

- a) We say that B is ϵ -symmetric if $B(v, w) = \epsilon(v, w)B(w, v)$ for all $v, w \in V$.
- b) We say that B is ϵ -antisymmetric if $B(v, w) = -\epsilon(v, w)B(w, v)$ for all $v, w \in V$.
- c) If V is finite-dimensional, $\{e_i\}$ is a basis of V and $(\ , \)$ is a non-degenerate ϵ -symmetric bilinear form of degree 0 on V , then the unique basis $\{e^i\}$ of V such that $(e_i, e^j) = \delta_{ij}$ is called the dual basis of $\{e_i\}$.

Unless otherwise stated, we always assume the bilinear forms to be of degree 0. The basic features of a basis and its dual basis are given in the following remark.

Remark 2.3. Let V be a finite-dimensional Γ -graded vector space together with a non-degenerate ϵ -symmetric bilinear form $(\ , \)$, let $\{e_i\}$ be a basis of V and let $\{e^i\}$ be its dual basis.

- a) We have $|e^i| = -|e_i|$.
- b) As Γ -graded vector spaces, V is canonically isomorphic to V^* by $v \mapsto (v, \)$ and we have $e_i^* = \epsilon(e_i)(e^i, \)$.
- c) Using b) and Remark 2.1 the Γ -graded vector spaces $\text{End}(V)$ and $V \otimes V$ are canonically isomorphic and the identity map $\text{Id}_V \in \text{End}(V)$ corresponds to

$$\sum_i e^i \otimes e_i = \sum_i \epsilon(e_i)e_i \otimes e^i \in V \otimes V.$$

In particular, $\sum_i e^i \otimes e_i$ is independent of the choice of the basis $\{e_i\}$.

d) We have

$$v = \sum_i (v, e^i) e_i = \sum_i \epsilon(e_i)(v, e_i) e^i = \sum_i (e_i, v) e^i \quad \forall v \in V. \quad (1)$$

e) Let W be a Γ -graded vector space and let $f : V \otimes V \rightarrow W$ be an ϵ -antisymmetric bilinear map. Using c) we have

$$f\left(\sum_i e^i \otimes e_i\right) = 0.$$

2.2. ϵ -exterior algebra and ϵ -Clifford algebra

Let $(V, (\ , \))$ be a finite-dimensional Γ -graded vector space together with a non-degenerate ϵ -symmetric bilinear form. In this subsection we recall basic properties of the ϵ -exterior algebra V and of the ϵ -Clifford algebra of $(V, (\ , \))$ (see [4]).

Definition 2.4.

a) The ϵ -Clifford algebra $C_\epsilon(V, (\ , \))$ of V is defined by

$$C_\epsilon(V, (\ , \)) := T(V)/I(V),$$

where $T(V)$ is the tensor algebra over V and $I(V)$ is the two-sided ideal of $T(V)$ generated by elements of the form

$$x \otimes y + \epsilon(x, y) y \otimes x - 2(x, y) 1 \quad \forall x, y \in V.$$

b) The ϵ -exterior algebra $\Lambda_\epsilon(V)$ of V is defined to be $C_\epsilon(V, (\ , \))$ where $(\ , \)$ is totally degenerate.

The tensor algebra $T(V)$ is a $\mathbb{Z} \times \Gamma$ -graded algebra and then the ϵ -Clifford algebra $C_\epsilon(V, (\ , \))$ is a $\mathbb{Z}_2 \times \Gamma$ -graded algebra and the ϵ -exterior algebra $\Lambda_\epsilon(V)$ is a $\mathbb{Z} \times \Gamma$ -graded algebra. For the universal property of the ϵ -Clifford algebra, see [4]. The ϵ -Clifford algebra $C_\epsilon(V, (\ , \))$ is filtered and its associated graded algebra is isomorphic to $\Lambda_\epsilon(V)$. Define $e : V \rightarrow \text{End}(\Lambda_\epsilon(V))$ by $e(v)(w) := v \wedge w$ for all $v \in V$, for all $w \in \Lambda_\epsilon(V)$ and define $i : V \rightarrow \text{End}(T(V))$ by

$$i(v)(w_1 \otimes \cdots \otimes w_n) := \sum_i (-1)^{i-1} \epsilon(v, w_1 + \cdots + w_{i-1})(v, w_i) w_1 \otimes \cdots \otimes \hat{w}_i \otimes \cdots \otimes w_n$$

for all $v \in V$ and for all $w_1 \otimes \cdots \otimes w_n \in T(V)$. This descends to define $i : V \rightarrow \text{End}(\Lambda_\epsilon(V))$ and by the universal property of the ϵ -Clifford algebra, the map $\gamma : V \rightarrow \text{End}(\Lambda_\epsilon(V))$ defined by $\gamma(v) = e(v) + i(v)$ for all $v \in V$ extends to define $\gamma : C_\epsilon(V, (\ , \)) \rightarrow \text{End}(\Lambda_\epsilon(V))$. Using the maps e and i we can define oscillator representations for a particular class of ϵ -Clifford algebras.

Definition 2.5. Suppose that one of the following holds:

- a) there are two maximal isotropic Γ -graded subspaces W and W' such that $V = W \oplus W'$;
- b) there are two maximal isotropic Γ -graded subspaces W and W' , a one-dimensional subspace U of degree 0 orthogonal to W and W' , $u \in U$ such that $(u, u) = 1$ and $V = W \oplus W' \oplus U$.

Define the oscillator representation $m : V \rightarrow \text{End}(\Lambda_\epsilon(W))$ by

$$m(w + w') := e(w) + 2i(w') \quad \forall w \in W, \forall w' \in W',$$

and, if b) holds, by

$$m(u) := \text{Id}|_{\Lambda_\epsilon(W)_0} - \text{Id}|_{\Lambda_\epsilon(W)_1},$$

where $\Lambda_\epsilon(W)_0 := \bigoplus_{i \in \mathbb{N}} \Lambda_\epsilon^{2i}(W)$ and $\Lambda_\epsilon(W)_1 := \bigoplus_{i \in \mathbb{N}} \Lambda_\epsilon^{2i+1}(W)$.

Using the universal property of the ϵ -Clifford algebra, this extends to define $m : C_\epsilon(V, (,)) \rightarrow \text{End}(\Lambda_\epsilon(W))$. There is a quantisation map from the ϵ -exterior algebra to the ϵ -Clifford algebra:

Definition 2.6. Let $n \in \mathbb{N}$ and suppose that $\text{char}(k) = 0$ or $n < \text{char}(k)$. Define the quantisation map $Q_n : \Lambda_\epsilon^n(V) \rightarrow C_\epsilon(V, (,))$ by

$$Q_n(v_1 \wedge \cdots \wedge v_n) := \frac{1}{n!} \sum_{\sigma \in S_n} p(\sigma; v_1, \dots, v_n) v_{\sigma(1)} \cdots v_{\sigma(n)}.$$

Since the graded map $\text{gr}(Q_n) \in \text{End}(\Lambda_\epsilon^n(V))$ is the identity, the map Q_n is injective. Furthermore, if $\text{char}(k) = 0$, then the map $\sum_n Q_n : \Lambda_\epsilon(V) \rightarrow C_\epsilon(V, (,))$ is an isomorphism of Γ -graded vector spaces and its inverse map is $\gamma' : C_\epsilon(V, (,)) \rightarrow \Lambda_\epsilon(V)$ given by $\gamma'(v) = \gamma(v)(1)$ for all $v \in C_\epsilon(V, (,))$ (see [4]).

We now extend $(,)$ to $\Lambda_\epsilon(V)$. Let $u_1, \dots, u_n, v_1, \dots, v_n \in V$ and consider the bilinear forms on $T^n(V)$ defined by

$$B(u_1 \otimes \cdots \otimes u_n, v_1 \otimes \cdots \otimes v_n) := \prod_{i=0}^{n-1} (u_{n-i}, v_{1+i})$$

and

$$(u_1 \otimes \cdots \otimes u_n, v_1 \otimes \cdots \otimes v_n)_\Lambda := \sum_{\sigma \in S_n} B(\pi(\sigma)(u_1 \otimes \cdots \otimes u_n), v_1 \otimes \cdots \otimes v_n).$$

One can check that

$$(u_1 \otimes \cdots \otimes u_n, v_1 \otimes \cdots \otimes v_n)_\Lambda = \sum_{\sigma \in S_n} B(u_1 \otimes \cdots \otimes u_n, \pi(\sigma)(v_1 \otimes \cdots \otimes v_n))$$

and so this descends to define a bilinear form on $\Lambda_\epsilon^n(V)$.

Proposition 2.7. *Let $n \in \mathbb{N}$ and suppose that $\text{char}(k) = 0$ or $n < \text{char}(k)$.*

- a) *The bilinear form $(\ , \)_\Lambda$ of $\Lambda_\epsilon^n(V)$ defined above is ϵ -symmetric and non-degenerate.*
- b) *The linear map $\Lambda_\epsilon^n(V) \rightarrow \Lambda_\epsilon^n(V)^*$ given by $f \mapsto (f, \)_\Lambda$ is an isomorphism of Γ -graded vector spaces and for $f \in \Lambda_\epsilon^n(V)^*$, the element $\tilde{f} \in \Lambda_\epsilon^n(V)$ corresponding to f under this isomorphism satisfies to*

$$\tilde{f} = \frac{1}{n!} \sum_{i_1, \dots, i_n} f(e^{i_1} \wedge \dots \wedge e^{i_n}) e_{i_n} \wedge \dots \wedge e_{i_1} \in \Lambda_\epsilon^n(V)$$

and

$$Q_n(\tilde{f}) = \frac{1}{n!} \sum_{i_1, \dots, i_n} f(e^{i_1} \wedge \dots \wedge e^{i_n}) e_{i_n} \cdots e_{i_1} \in C_\epsilon(V, (\ , \))$$

where $\{e_i\}$ is a basis of V and $\{e^i\}$ its dual basis.

Proof. A straightforward calculation shows that $(\ , \)_\Lambda$ is ϵ -symmetric.

Let $(i'_1, \dots, i'_n) \in \llbracket 1, \dim(V) \rrbracket^n$. It is non-degenerate since

$$(e_{i'_n} \wedge \dots \wedge e_{i'_1}, e^{i'_1} \wedge \dots \wedge e^{i'_n})_\Lambda = |\{\sigma \in S_n \mid \sigma \cdot (i'_1, \dots, i'_n) = (i'_1, \dots, i'_n)\}|.$$

We want to show that

$$\left(\frac{1}{n!} \sum_{i_1, \dots, i_n} f(e^{i_1} \wedge \dots \wedge e^{i_n}) e_{i_n} \wedge \dots \wedge e_{i_1}, e^{i'_1} \wedge \dots \wedge e^{i'_n} \right)_\Lambda = f(e^{i'_1} \wedge \dots \wedge e^{i'_n}). \quad (2)$$

Let $(i_1, \dots, i_n)$ such that for all $\sigma \in S_n$ we have $\sigma \cdot (i_1, \dots, i_n) \neq (i'_1, \dots, i'_n)$. Hence

$$(e_{i_n} \wedge \dots \wedge e_{i_1}, e^{i'_1} \wedge \dots \wedge e^{i'_n})_\Lambda = 0$$

and so

$$\begin{aligned} & \left(\frac{1}{n!} \sum_{i_1, \dots, i_n} f(e^{i_1} \wedge \dots \wedge e^{i_n}) e_{i_n} \wedge \dots \wedge e_{i_1}, e^{i'_1} \wedge \dots \wedge e^{i'_n} \right)_\Lambda \\ &= \frac{1}{n!} \sum_{(i_1, \dots, i_n) \in S} f(e^{i_1} \wedge \dots \wedge e^{i_n}) (e_{i_n} \wedge \dots \wedge e_{i_1}, e^{i'_1} \wedge \dots \wedge e^{i'_n})_\Lambda \end{aligned}$$

where S is the set of all $(i_1, \dots, i_n)$ such that there exists $\sigma \in S_n$ such that $\sigma \cdot (i_1, \dots, i_n) = (i'_1, \dots, i'_n)$. For each $(i_1, \dots, i_n) \in S$ we have

$$f(e^{i_1} \wedge \dots \wedge e^{i_n}) e_{i_n} \wedge \dots \wedge e_{i_1} = f(e^{i'_1} \wedge \dots \wedge e^{i'_n}) e_{i'_n} \wedge \dots \wedge e_{i'_1}$$

and then

$$\begin{aligned} & \left(\frac{1}{n!} \sum_{i_1, \dots, i_n} f(e^{i_1} \wedge \dots \wedge e^{i_n}) e_{i_n} \wedge \dots \wedge e_{i_1}, e^{i'_1} \wedge \dots \wedge e^{i'_n} \right)_\Lambda \\ &= f(e^{i'_1} \wedge \dots \wedge e^{i'_n}) \frac{|S|}{n!} (e_{i'_n} \wedge \dots \wedge e_{i'_1}, e^{i'_1} \wedge \dots \wedge e^{i'_n})_\Lambda. \end{aligned}$$

Finally,

$$(e_{i'_n} \wedge \cdots \wedge e_{i'_1}, e^{i'_1} \wedge \cdots \wedge e^{i'_n})_\Lambda = |\{\sigma \in S_n \mid \sigma \cdot (i'_1, \dots, i'_n) = (i'_1, \dots, i'_n)\}|$$

and since

$$|S| |\{\sigma \in S_n \mid \sigma \cdot (i'_1, \dots, i'_n) = (i'_1, \dots, i'_n)\}| = n!$$

we obtain Equation (2) and the formula

$$\tilde{f} = \frac{1}{n!} \sum_{i_1, \dots, i_n} f(e^{i_1} \wedge \cdots \wedge e^{i_n}) e_{i_n} \wedge \cdots \wedge e_{i_1}.$$

Applying the quantisation map Q_n to $\tilde{f}$, we have

$$Q_n(\tilde{f}) = \frac{1}{(n!)^2} \sum_{\substack{i_1, \dots, i_n \\ \sigma \in S_n}} p(\sigma; e_{i_n}, \dots, e_{i_1}) f(e^{i_1} \wedge \cdots \wedge e^{i_n}) e_{\sigma(i_n)} \cdots e_{\sigma(i_1)}.$$

Since

$$p(\sigma; e_{i_n}, \dots, e_{i_1}) e^{i_1} \wedge \cdots \wedge e^{i_n} = e^{\sigma(i_1)} \wedge \cdots \wedge e^{\sigma(i_n)},$$

then, by reindexing, we obtain

$$\begin{aligned} \sum_{i_1, \dots, i_n} p(\sigma; e_{i_n}, \dots, e_{i_1}) f(e^{i_1} \wedge \cdots \wedge e^{i_n}) e_{\sigma(i_n)} \cdots e_{\sigma(i_1)} \\ = \sum_{i_1, \dots, i_n} f(e^{\sigma(i_1)} \wedge \cdots \wedge e^{\sigma(i_n)}) e_{\sigma(i_n)} \cdots e_{\sigma(i_1)} \\ = \sum_{i_1, \dots, i_n} f(e^{i_1} \wedge \cdots \wedge e^{i_n}) e_{i_n} \cdots e_{i_1} \end{aligned}$$

and so

$$Q_n(\tilde{f}) = \frac{1}{n!} \sum_{i_1, \dots, i_n} f(e^{i_1} \wedge \cdots \wedge e^{i_n}) e_{i_n} \cdots e_{i_1}. \quad \square$$

2.3. Colour Lie algebras

In this subsection, following [22], we define colour Lie algebras, representations of colour Lie algebras ([23], [24], [25]) and give some properties of the moment map of an ϵ -orthogonal representation of an ϵ -quadratic colour Lie algebra.

Definition 2.8. A colour Lie algebra is a Γ -graded vector space $\mathfrak{g}$ together with a bilinear map $\{ , \} : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ such that

- a) $\{\mathfrak{g}_\alpha, \mathfrak{g}_\beta\} \subseteq \mathfrak{g}_{\alpha+\beta}$ for all $\alpha, \beta \in \Gamma$,
- b) $\{x, y\} = -\epsilon(x, y)\{y, x\}$ for all $x, y \in \mathfrak{g}$ (ϵ -antisymmetry),
- c) $\epsilon(z, x)\{x, \{y, z\}\} + \epsilon(x, y)\{y, \{z, x\}\} + \epsilon(y, z)\{z, \{x, y\}\} = 0$ for all $x, y, z \in \mathfrak{g}$ (ϵ -Jacobi identity).

Lie (super)algebras are colour Lie algebras and here are some examples:

Example 2.9. Let V be a Γ -graded vector space.

a) The associative Γ -graded algebra $\text{Homgr}(V, V)$ is a colour Lie algebra for the bracket $\{a, b\} := ab - \epsilon(a, b)ba$ for a, b in $\text{Homgr}(V, V)$ and is denoted $\mathfrak{gl}_\epsilon(V)$.

b) Let $(\ , \)$ be an ϵ -symmetric bilinear form on V . We set $\mathfrak{so}_\epsilon(V, (\ , \)) := \bigoplus_{\gamma \in \Gamma} \mathfrak{so}_\epsilon(V, (\ , \))_\gamma$ where

$$\mathfrak{so}_\epsilon(V, (\ , \))_\gamma := \{f \in \text{Homgr}(V, V)_\gamma \mid (f(v), w) + \epsilon(f, v)(v, f(w)) = 0 \quad \forall v, w \in V\}.$$

One can show that $\mathfrak{so}_\epsilon(V, (\ , \))$ is stable under the bracket of $\mathfrak{gl}_\epsilon(V)$ and hence is a colour Lie algebra.

Let $\mathfrak{g}$ and $\mathfrak{g}'$ be colour Lie algebras. A degree 0 linear map $f \in \text{Homgr}(\mathfrak{g}, \mathfrak{g}')$ is a morphism of colour Lie algebras if $f(\{x, y\}) = \{f(x), f(y)\}$ for all $x, y \in \mathfrak{g}$ and we say that $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic if f is a linear isomorphism. A representation V of $\mathfrak{g}$ is a Γ -graded vector space V together with a morphism of colour Lie algebras $\pi : \mathfrak{g} \rightarrow \mathfrak{gl}_\epsilon(V)$ and the adjoint representation $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}_\epsilon(\mathfrak{g})$ is defined by $\text{ad}(x) := \{x, \ \}$ for all $x \in \mathfrak{g}$. An ϵ -orthogonal representation V of $\mathfrak{g}$ is a Γ -graded vector space V together with a non-degenerate ϵ -symmetric bilinear form $(\ , \)$ and a morphism of colour Lie algebras $\pi : \mathfrak{g} \rightarrow \mathfrak{so}_\epsilon(V, (\ , \))$. We say that $\mathfrak{g}$ is ϵ -quadratic if there is a bilinear form B on $\mathfrak{g}$ which is ϵ -symmetric, non-degenerate and ad-invariant in the sense that

$$B(\{x, y\}, z) = -\epsilon(x, y)B(y, \{x, z\}) \quad \forall x, y, z \in \mathfrak{g}.$$

The ϵ -Killing form K is the ϵ -symmetric ad-invariant bilinear form on $\mathfrak{g}$ given by

$$K(x, y) := \text{Tr}_\epsilon(\text{ad}(x)\text{ad}(y)) \quad \forall x, y \in \mathfrak{g}.$$

Remark 2.10.

a) If $\pi_1 : \mathfrak{g} \rightarrow \text{End}(V_1)$ and $\pi_2 : \mathfrak{g} \rightarrow \text{End}(V_2)$ are representations of a colour Lie algebra $\mathfrak{g}$, then $\pi : \mathfrak{g} \rightarrow \text{End}(V_1 \otimes V_2)$ given by

$$\pi(x)(v_1 \otimes v_2) := \pi_1(x)(v_1) \otimes v_2 + \epsilon(x, v_1)v_1 \otimes \pi_2(x)(v_2),$$

for all $x \in \mathfrak{g}$ and for all $v_1 \otimes v_2 \in V_1 \otimes V_2$, is also a representation of $\mathfrak{g}$. In particular, if V is a finite-dimensional representation of $\mathfrak{g}$, then $T(V)$ and $\Lambda_\epsilon(V)$ are representations of $\mathfrak{g}$. If $(V, (\ , \))$ is an ϵ -orthogonal representation of $\mathfrak{g}$, then $C_\epsilon(V, (\ , \))$ is also a representation of $\mathfrak{g}$.

b) If $\mathfrak{g}$ is a finite-dimensional ϵ -quadratic colour Lie algebra, the isomorphism between $\text{End}(\mathfrak{g})$ and $\mathfrak{g} \otimes \mathfrak{g}$ is $\mathfrak{g}$ -equivariant.

Unless otherwise stated, we suppose all ϵ -orthogonal representations of dimension at least two.

Definition 2.11. Let $\mathfrak{g}$ be a finite-dimensional colour Lie algebra.

a) The ϵ -universal enveloping algebra $U_\epsilon(\mathfrak{g})$ of $\mathfrak{g}$ is defined by

$$U_\epsilon(\mathfrak{g}) := T(\mathfrak{g})/I(\mathfrak{g}),$$

where $T(\mathfrak{g})$ is the tensor algebra over $\mathfrak{g}$ and $I(\mathfrak{g})$ is the two-sided ideal of $T(\mathfrak{g})$ generated by elements of the form

$$x \otimes y - \epsilon(x, y)y \otimes x - \{x, y\} \quad \forall x, y \in \mathfrak{g}.$$

- b) The ϵ -symmetric algebra $S_\epsilon(\mathfrak{g})$ of $\mathfrak{g}$ is defined to be $U_\epsilon(\mathfrak{g})$ where the bracket of $\mathfrak{g}$ is trivial.
- c) If $\mathfrak{g}$ is ϵ -quadratic, then the Casimir element $\Omega(\mathfrak{g}) \in U_\epsilon(\mathfrak{g})$ is defined by

$$\Omega(\mathfrak{g}) := \sum_i x^i x_i \in U_\epsilon(\mathfrak{g})$$

where $\{x_i\}$ is a basis of $\mathfrak{g}$ and $\{x^i\}$ its dual basis.

The action of $\mathfrak{g}$ on $T(\mathfrak{g})$ of Remark 2.10 defines an action of $\mathfrak{g}$ on $U_\epsilon(\mathfrak{g})$. For the universal property of the ϵ -universal enveloping algebra, see [25]. Since $\Omega(\mathfrak{g}) \in U_\epsilon(\mathfrak{g})$ corresponds to $\text{Id}_{\mathfrak{g}} \in \text{End}(\mathfrak{g})$, then $\Omega(\mathfrak{g})$ is in the ϵ -centre $\mathcal{Z}_\epsilon(\mathfrak{g})$ of $U_\epsilon(\mathfrak{g})$ where the ϵ -centre is defined by

$$\mathcal{Z}_\epsilon(\mathfrak{g}) := \{x \in U_\epsilon(\mathfrak{g}) \mid xy = \epsilon(x, y)yx \quad \forall y \in U_\epsilon(\mathfrak{g})\}.$$

We now define and study the moment map associated to an ϵ -orthogonal representation.

Definition 2.12. Let $\pi : \mathfrak{g} \rightarrow \mathfrak{so}_\epsilon(V, (\ , \))$ be a finite-dimensional ϵ -orthogonal representation of a finite-dimensional ϵ -quadratic colour Lie algebra $(\mathfrak{g}, B_{\mathfrak{g}})$. We define the moment map $\mu : V \times V \rightarrow \mathfrak{g}$ to be the bilinear map given by

$$B_{\mathfrak{g}}(x, \mu(v, w)) = (\pi(x)(v), w) \quad \forall v, w \in V, \forall x \in \mathfrak{g}.$$

The moment map μ is ϵ -antisymmetric and of degree 0. The classical example of a moment map is the moment map of the fundamental representation of $\mathfrak{so}_\epsilon(V, (\ , \))$.

Proposition 2.13. *Let V be a finite-dimensional Γ -graded vector space together with a non-degenerate ϵ -symmetric bilinear form $(\ , \)$.*

- a) *Consider the ϵ -orthogonal representation of the ϵ -quadratic colour Lie algebra $(\mathfrak{so}_\epsilon(V, (\ , \)), B)$ where $B(f, g) = -\frac{1}{2}\text{Tr}_\epsilon(fg)$ for all $f, g \in \mathfrak{so}_\epsilon(V, (\ , \))$. Then, the corresponding moment map μ_{can} satisfies*

$$\mu_{\text{can}}(u, v)(w) = \epsilon(v, w)(u, w)v - (v, w)u \quad \forall u, v, w \in V.$$

- b) *The moment map $\mu_{\text{can}} : \Lambda_\epsilon^2(V) \rightarrow \mathfrak{so}_\epsilon(V, (\ , \))$ is an isomorphism of Γ -graded vector spaces.*
- c) *For $f \in \mathfrak{so}_\epsilon(V, (\ , \))$, we have*

$$\mu_{\text{can}}^{-1}(f) = -\frac{1}{2} \sum_i f(e^i) \wedge e_i, \quad Q_2(\mu_{\text{can}}^{-1}(f)) = -\frac{1}{2} \sum_i f(e^i) e_i$$

where $\{e_i\}$ is a basis of V and $\{e^i\}$ its dual basis.

Proof. A straightforward calculation shows a) (or see [18]). For b), see [4] if $\text{char}(k) = 0$ or chapter 3 of [17]. We now prove c). Let $v \in V$. We have

$$\mu_{\text{can}}\left(-\frac{1}{2}\sum_i f(e^i) \wedge e_i\right)(v) = -\frac{1}{2}\sum_{i,j} (\epsilon(e_i, v)(f(e^i), v)e_i - (e_i, v)f(e^i)) = f(v).$$

Since $\sum_i (f(e^i), e_i) = 0$ by Remark 2.3 and $Q_2(u \wedge v) = uv - (u, v)$ for all $u, v \in V$, we have

$$Q_2(\mu_{\text{can}}^{-1}(f)) = -\frac{1}{2}\sum_i f(e^i)e_i + \frac{1}{2}\sum_i (f(e^i), e_i) = -\frac{1}{2}\sum_i f(e^i)e_i. \quad \square$$

2.4. Basic colour Lie algebras

Suppose that $\text{char}(k) = 0$. Let $(\mathfrak{g}, B_{\mathfrak{g}})$ be a finite-dimensional ϵ -quadratic colour Lie algebra and let $\mathfrak{h}$ be a Cartan subalgebra of the Lie algebra $\mathfrak{g}_0$. For $\alpha \in \mathfrak{h}^*$, define

$$\mathfrak{g}^{\alpha} = \{x \in \mathfrak{g} \mid \{h, x\} = \alpha(h)x \quad \forall h \in \mathfrak{h}\}.$$

We have

$$\{\mathfrak{g}^{\alpha}, \mathfrak{g}^{\beta}\} \subseteq \mathfrak{g}^{\alpha+\beta} \quad \forall \alpha, \beta \in \mathfrak{h}^*.$$

Suppose that $\dim(\mathfrak{g}^{\alpha}) \leq 1$ for all non-zero $\alpha \in \mathfrak{h}^*$, $\text{ad}(\mathfrak{h})$ is diagonalisable in the sense that $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}^{\alpha}$ and $\mathfrak{h}$ is self-normalising in the sense that $\mathfrak{g}^0 = \mathfrak{h}$. A non-zero element $\alpha \in \mathfrak{h}^*$ with $\mathfrak{g}^{\alpha} \neq \{0\}$ is called a root of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and the set of all roots is denoted Δ .

Proposition 2.14. *We have the following:*

- a) *If $\alpha, \beta \in \mathfrak{h}^*$ are such that $\alpha + \beta \neq 0$, then $B_{\mathfrak{g}}(\mathfrak{g}^{\alpha}, \mathfrak{g}^{\beta}) = 0$.*
- b) *The restriction of $B_{\mathfrak{g}}$ to $\mathfrak{g}^{\alpha} \times \mathfrak{g}^{-\alpha}$ is non-degenerate.*

Proof. Since $\alpha + \beta \neq 0$, there exists $h \in \mathfrak{h}$ such that $(\alpha + \beta)(h) \neq 0$. Let $x \in \mathfrak{g}^{\alpha}$ and let $y \in \mathfrak{g}^{\beta}$. We have

$$\alpha(h)B_{\mathfrak{g}}(x, y) = B_{\mathfrak{g}}(\{h, x\}, y) = -B_{\mathfrak{g}}(x, \{h, y\}) = -\beta(h)B_{\mathfrak{g}}(x, y),$$

then $(\alpha + \beta)(h)B_{\mathfrak{g}}(x, y) = 0$ which implies $B_{\mathfrak{g}}(x, y) = 0$. This shows a) and b) follows since $B_{\mathfrak{g}}$ is non-degenerate. $\square$

Thus, if α is a root, $-\alpha$ is also a root and so one can define Δ^+ and Δ^- such that $\Delta = \Delta^+ \cup \Delta^-$ and such that $\alpha \in \Delta^{\pm}$ implies $-\alpha \in \Delta^{\mp}$. Suppose that the vanishing of a linear combination $\sum_{\alpha \in \Delta^{\pm}} c_{\alpha} \alpha$ with integral coefficients $c_{\alpha} \geq 0$ implies that $c_{\alpha} = 0$ for all $\alpha \in \Delta^{\pm}$. The elements in Δ^+ (resp. Δ^-) are called the positive (resp. negative) roots. For $\alpha \in \Delta$ we set $\epsilon(\alpha) := \epsilon(x)$ where x is a non-zero element of $\mathfrak{g}^{\alpha}$ and we define the Weyl vector ρ by

$$\rho := \frac{1}{2} \sum_{\alpha \in \Delta^+} \epsilon(\alpha) \alpha.$$

If we define $\mathfrak{g}^+ = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}^{\alpha}$ and $\mathfrak{g}^- = \bigoplus_{\alpha \in \Delta^-} \mathfrak{g}^{\alpha}$, we have that $\mathfrak{g}^+$ and $\mathfrak{g}^-$ are nilpotent colour Lie algebras and a decomposition

$$\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+.$$

By Proposition 2.14 $(\mathfrak{h}, B_{\mathfrak{g}}|_{\mathfrak{h}})$ is a non-degenerate quadratic vector space. For $\alpha \in \mathfrak{h}^*$ define $h_{\alpha} \in \mathfrak{h}$ to be the unique element in $\mathfrak{h}$ such that $\alpha = B_{\mathfrak{g}}(h_{\alpha}, \cdot) \in \mathfrak{h}^*$ and define a non-degenerate symmetric bilinear form $B_{\mathfrak{g}^*}(\cdot, \cdot)$ on $\mathfrak{h}^*$ by

$$B_{\mathfrak{g}^*}(\alpha, \beta) = B_{\mathfrak{g}}(h_{\alpha}, h_{\beta}) \quad \forall \alpha, \beta \in \mathfrak{h}^*.$$

Proposition 2.15. *We have the following.*

a) *Let $\{h_i\}$ be a basis of $\mathfrak{h}$ and let $\{h^i\}$ its dual basis. We have*

$$B_{\mathfrak{g}^*}(\alpha, \beta) = \sum_i \alpha(h^i) \beta(h_i) \quad \forall \alpha, \beta \in \mathfrak{h}^*.$$

b) *For a root α and $x \in \mathfrak{g}^\alpha, y \in \mathfrak{g}^{-\alpha}$ we have $\{x, y\} = B_{\mathfrak{g}}(x, y)h_\alpha$.*

Proof. a) We have

$$B_{\mathfrak{g}^*}(\alpha, \beta) = B_{\mathfrak{g}}(h_\alpha, h_\beta) = \sum_i B_{\mathfrak{g}}(B_{\mathfrak{g}}(h_\alpha, h^i)h_i, \beta) = \sum_i \alpha(h^i) \beta(h_i).$$

b) For $z \in \mathfrak{g}^\gamma$, where $\gamma \in \mathfrak{h}^* \setminus \{0\}$, by Proposition 2.14 we have

$$B_{\mathfrak{g}}(z, \{x, y\}) = B_{\mathfrak{g}}(z, B_{\mathfrak{g}}(x, y)h_\alpha) = 0.$$

For $h \in \mathfrak{h}$ we have

$$\begin{aligned} B_{\mathfrak{g}}(h, \{x, y\}) &= B_{\mathfrak{g}}(\{h, x\}, y) \\ &= \alpha(h) B_{\mathfrak{g}}(x, y) \\ &= B_{\mathfrak{g}}(h_\alpha, h) B_{\mathfrak{g}}(x, y) \\ &= B_{\mathfrak{g}}(h, B_{\mathfrak{g}}(x, y)h_\alpha) \end{aligned}$$

and so $\{x, y\} = B_{\mathfrak{g}}(x, y)h_\alpha$. $\square$

Definition 2.16. Let $\pi : \mathfrak{g} \rightarrow \text{End}(V)$ be a representation of $\mathfrak{g}$. We say that V is a representation of $\mathfrak{g}$ of highest weight λ if there exist a homogeneous non-zero element $v \in V$ and a linear form $\lambda \in \mathfrak{h}^*$ such that

- v generates V as $\mathfrak{g}$ -module;
- v is annihilated by $\mathfrak{g}^+$;
- $\pi(h)(v) = \lambda(h)v$ for all $h \in \mathfrak{h}$;
- $\dim(\{u \in V \mid \pi(h)(u) = \lambda(h)u \quad \forall h \in \mathfrak{h}\}) = 1$.

The action of the Casimir element on a highest weight representation satisfies the following proposition.

Proposition 2.17. *The Casimir element $\Omega(\mathfrak{g}) \in U_\epsilon(\mathfrak{g})$ satisfies to*

$$\pi(\Omega(\mathfrak{g})) = B_{\mathfrak{g}^*}(\lambda + 2\rho, \lambda) \text{Id}_V.$$

Proof. Since $\Omega(\mathfrak{g}) \in \mathcal{Z}_\epsilon(\mathfrak{g})$, clearly it acts on V as a multiple of the identity. Let $\{h_i\}$ be a basis of $\mathfrak{h}$ and let $\{h^i\}$ be its dual basis. For each $\alpha \in \Delta^+$ let $e_\alpha \in \mathfrak{g}^\alpha$ and $e_{-\alpha} \in \mathfrak{g}^{-\alpha}$ be such that $B_{\mathfrak{g}}(e_\alpha, e_{-\alpha}) = 1$. The set

$$\{h_i\} \cup \{e_\alpha, \forall \alpha \in \Delta^+\} \cup \{e_{-\alpha}, \forall \alpha \in \Delta^+\}$$

is a basis of $\mathfrak{g}$ and its dual basis is

$$\{h^i\} \cup \{e_{-\alpha}, \forall \alpha \in \Delta^+\} \cup \{\epsilon(\alpha)e_\alpha, \forall \alpha \in \Delta^+\}.$$

Hence we have

$$\Omega(\mathfrak{g}) = \sum_i h^i h_i + \sum_{\alpha \in \Delta^+} e_{-\alpha} e_{\alpha} + \sum_{\alpha \in \Delta^+} \epsilon(\alpha) e_{\alpha} e_{-\alpha}.$$

By Proposition 2.15 we have $\{e_{\alpha}, e_{-\alpha}\} = h_{\alpha}$ and so

$$\Omega(\mathfrak{g}) = \sum_i h^i h_i + 2 \sum_{\alpha \in \Delta^+} e_{-\alpha} e_{\alpha} + \sum_{\alpha \in \Delta^+} \epsilon(\alpha) h_{\alpha}. \quad (3)$$

We have

$$\pi\left(\sum_i h^i h_i\right)(v) = \sum_i \lambda(h^i) \lambda(h_i) v = B_{\mathfrak{g}^*}(\lambda, \lambda) v. \quad (4)$$

Since $e_{\alpha} \in \mathfrak{g}^+$ annihilates v , we have

$$\pi\left(2 \sum_{\alpha \in \Delta^+} e_{-\alpha} e_{\alpha}\right)(v) = 0. \quad (5)$$

Finally, we have

$$\pi\left(\sum_{\alpha \in \Delta^+} \epsilon(\alpha) h_{\alpha}\right)(v) = \sum_{\alpha \in \Delta^+} \epsilon(\alpha) \lambda(h_{\alpha}) v$$

and since

$$\begin{aligned} 2B_{\mathfrak{g}^*}(\rho, \lambda) &= \sum_i \sum_{\alpha \in \Delta^+} \epsilon(\alpha) \alpha(h^i) \lambda(h_i) = \sum_i \sum_{\alpha \in \Delta^+} \epsilon(\alpha) B_{\mathfrak{g}}(h_{\alpha}, h^i) B_{\mathfrak{g}}(h_{\lambda}, h_i) \\ &= \sum_{\alpha \in \Delta^+} \epsilon(\alpha) B_{\mathfrak{g}}(h_{\lambda}, \sum_i B_{\mathfrak{g}}(h_{\alpha}, h^i) h_i) = \sum_{\alpha \in \Delta^+} \epsilon(\alpha) B_{\mathfrak{g}}(h_{\lambda}, h_{\alpha}) \\ &= \sum_{\alpha \in \Delta^+} \epsilon(\alpha) \lambda(h_{\alpha}) \end{aligned}$$

we obtain that

$$\pi\left(\sum_{\alpha \in \Delta^+} \epsilon(\alpha) h_{\alpha}\right)(v) = 2B_{\mathfrak{g}^*}(\rho, \lambda) v. \quad (6)$$

From Equations (3), (4), (5) and (6) we obtain that

$$\pi(\Omega(\mathfrak{g}))(v) = B_{\mathfrak{g}^*}(\lambda + 2\rho, \lambda) v. \quad \square$$

3. Representations of colour Lie type and the strange Freudenthal–de Vries formula

In this section, from a finite-dimensional ϵ -orthogonal representation $\pi : \mathfrak{g} \rightarrow \mathfrak{so}_{\epsilon}(V, (\ , \))$ of a finite-dimensional ϵ -quadratic colour Lie algebra $\mathfrak{g}$, we show how to construct an ϵ -quadratic colour Lie algebra structure on $\mathfrak{g} \oplus V$ using a slightly different point of view than in [18]. Using this characterisation we state a strange Freudenthal and de Vries formula ([8]) for basic colour Lie algebras.

The ϵ -Clifford algebra $C_\epsilon(V, (,))$ is $\mathbb{Z}_2 \times \Gamma$ -graded and so consider the commutation factor $\tilde{\epsilon}$ of $\mathbb{Z}_2 \times \Gamma$ given by

$$\tilde{\epsilon}((a_1, \gamma_1), (a_2, \gamma_2)) := (-1)^{a_1 a_2} \epsilon(\gamma_1, \gamma_2) \quad \forall (a_1, \gamma_1), (a_2, \gamma_2) \in \mathbb{Z}_2 \times \Gamma.$$

Then $C_\epsilon(V, (,))$ is a $(\mathbb{Z}_2 \times \Gamma, \tilde{\epsilon})$ -colour Lie algebra for the bracket

$$\{x, y\} := x \cdot y - \tilde{\epsilon}(x, y) y \cdot x \quad \forall x, y \in C_\epsilon(V, (,)).$$

Define $\pi_* : \mathfrak{g} \rightarrow C_\epsilon(V, (,))$ by

$$\pi_* = -\frac{1}{2} Q_2 \circ \mu_{\text{can}}^{-1} \circ \pi,$$

where $Q_2 : \Lambda_\epsilon^2(V) \rightarrow C_\epsilon(V, (,))$ is the quantisation map of Definition 2.6 and where $\mu_{\text{can}} : \Lambda_\epsilon^2(V) \rightarrow \mathfrak{so}_\epsilon(V, (,))$ is the isomorphism of Proposition 2.13. We have

$$\pi_*(x) = \frac{1}{4} \sum_i \pi(x)(e^i) e_i \quad \forall x \in \mathfrak{g} \quad (7)$$

where $\{e_i\}$ is a basis of V and $\{e^i\}$ its dual basis. The map π_* is a colour Lie algebra morphism and so we extend it to $\pi_* : U_\epsilon(\mathfrak{g}) \rightarrow C_\epsilon(V, (,))$ using the universal property of the ϵ -universal enveloping algebra $U_\epsilon(\mathfrak{g})$.

Theorem 3.1. *Let $\pi : \mathfrak{g} \rightarrow \mathfrak{so}_\epsilon(V, (,))$ be a finite-dimensional ϵ -orthogonal representation of a finite-dimensional ϵ -quadratic colour Lie algebra $(\mathfrak{g}, B_\mathfrak{g})$ and let $\mu : \Lambda_\epsilon^2(V) \rightarrow \mathfrak{g}$ be its moment map. Let $\phi : \Lambda_\epsilon^2(V) \rightarrow V$ be of degree 0 and satisfy*

$$\pi(x)(\phi(v, w)) = \phi(\pi(x)(v), w) + \epsilon(x, v) \phi(v, \pi(x)(w)) \quad \forall x \in \mathfrak{g}, \forall v, w \in V, \quad (8)$$

$$(\phi(u, v), w) = -\epsilon(u, v)(v, \phi(u, w)) \quad \forall u, v, w \in V. \quad (9)$$

Let $\tilde{\mathfrak{g}} := \mathfrak{g} \oplus V$, let $B_{\tilde{\mathfrak{g}}} := B_\mathfrak{g} \perp (,)$ and let $\{ , \} \in \Lambda_\epsilon^2(\tilde{\mathfrak{g}}) \rightarrow \tilde{\mathfrak{g}}$ be the unique map which extends the bracket of $\mathfrak{g}$, the action of $\mathfrak{g}$ on V and such that

$$\{v, w\} = \mu(v, w) + \phi(v, w) \quad \forall v, w \in V.$$

Define the cubic term $\psi \in C_\epsilon(V, (,))$ to be the element corresponding to the ϵ -alternating trilinear map $(u, v, w) \mapsto \frac{1}{2}(u, \phi(v, w))$ under the isomorphism of vector spaces between $\Lambda_\epsilon^3(V)^*$ and $\Lambda_\epsilon^3(V)$ of Proposition 2.7 and under the quantisation map $Q_3 : \Lambda_\epsilon^3(V) \rightarrow C_\epsilon(V, (,))$.

The following are equivalent:

- a) $(\tilde{\mathfrak{g}}, B_{\tilde{\mathfrak{g}}}, \{ , \})$ is an ϵ -quadratic colour Lie algebra.
- b) $\pi_*(\Omega(\mathfrak{g})) + \psi^2 \in k$.

Proof. For $u, v, w \in \tilde{\mathfrak{g}}$, if u, v or w is an element of $\mathfrak{g}$ then a straightforward calculation shows that

$$\epsilon(w, u)\{\{u, v\}, w\} + \epsilon(u, v)\{\{v, w\}, u\} + \epsilon(v, w)\{\{w, u\}, v\} = 0.$$

Let $u, v, w \in V$ and $x \in \mathfrak{g}$. We have

$$B_{\mathfrak{g}}(\{u, \{v, w\}\}, x) = -\epsilon(u, v + w)\epsilon(u + v + w, x)(\pi(x)(\phi(v, w)), u)$$

and by Equation (8) we obtain

$$\begin{aligned} B_{\mathfrak{g}}(\{u, \{v, w\}\}, x) \\ = -\epsilon(u, v + w)B_{\mathfrak{g}}(\{v, \{w, u\}\}, x) + \epsilon(u, v + w)\epsilon(v, w)B_{\mathfrak{g}}(\{w, \{v, u\}\}, x). \end{aligned}$$

Hence,

$$B_{\mathfrak{g}}(\epsilon(w, u)\{\{u, v\}, w\} + \epsilon(u, v)\{\{v, w\}, u\} + \epsilon(v, w)\{\{w, u\}, v\}, x) = 0.$$

It follows that $\tilde{\mathfrak{g}}$ is a colour Lie algebra if and only if

$$(\epsilon(v_3, v_1)\{\{v_1, v_2\}, v_3\} + \epsilon(v_1, v_2)\{\{v_2, v_3\}, v_1\} + \epsilon(v_2, v_3)\{\{v_3, v_1\}, v_2\}, v_4) = 0$$

for all $v_1, v_2, v_3, v_4 \in V$. Define the map $J \in \Lambda_{\epsilon}(V)^*$ by

$$\begin{aligned} J(v_1, v_2, v_3, v_4) &= \epsilon(v_1, v_3)(\epsilon(v_3, v_1)\{\{v_1, v_2\}, v_3\} \\ &\quad + \epsilon(v_1, v_2)\{\{v_2, v_3\}, v_1\} + \epsilon(v_2, v_3)\{\{v_3, v_1\}, v_2\}, v_4) \end{aligned}$$

for all $v_1, v_2, v_3, v_4 \in V$ and $\tilde{J} \in \Lambda_{\epsilon}(V)$ to be the corresponding element of J under the isomorphism of Proposition 2.7. By Proposition 2.7, we have

$$Q_4(\tilde{J}) = \frac{1}{4!} \sum_{i,j,n,m} J(e^i, e^j, e^n, e^m) e_m e_n e_j e_i$$

and so $Q_4(\tilde{J})$ is equal to

$$\begin{aligned} \frac{1}{4!} \sum_{i,j,n,m} \Big(\{\{e^i, e^j\}, e^n\} + \epsilon(e_i, e_j + e_n)\{\{e^j, e^n\}, e^i\} \\ + \epsilon(e_i + e_j, e_n)\{\{e^n, e^i\}, e^j\}, e^m \Big) e_m e_n e_j e_i. \end{aligned}$$

Since

$$e_n e_j e_i = \epsilon(e_j + e_n, e_i) e_i e_n e_j - 2\epsilon(e_j, e_i)(e_n, e_i) e_j + 2(e_j, e_i) e_n,$$

we have

$$\begin{aligned} \sum_{i,j,n,m} \epsilon(e_i, e_j + e_n)(\{\{e^j, e^n\}, e^i\}, e^m) e_m e_n e_j e_i \\ = \sum_{i,j,n,m} (\{\{e^i, e^j\}, e^n\}, e^m) e_m e_n e_j e_i \\ - 4 \sum_{j,n} (\mu(e^j, e^n)(e_n) + \phi(\phi(e^j, e^n), e_n)) e_j \end{aligned}$$

and since $e_n e_j = -\epsilon(e_n, e_j) e_j e_n + 2(e_n, e_j)$ we also have

$$\begin{aligned}
 & \sum_{i,j,n,m} \epsilon(e_i + e_j, e_n) (\{\{e^n, e^i\}, e^j\}, e^m) e_m e_n e_j e_i \\
 &= - \sum_{i,j,n,m} \epsilon(e_j, e_n) (\{\{e^i, e^n\}, e^j\}, e^m) e_m e_n e_j e_i \\
 &= \sum_{i,j,n,m} (\{\{e^i, e^j\}, e^n\}, e^m) e_m e_n e_j e_i \\
 &\quad - 2 \sum_{j,n} (\mu(e^j, e^n)(e_n) + \phi(\phi(e^j, e^n), e_n)) e_j.
 \end{aligned}$$

Hence we have

$$\begin{aligned}
 Q_4(\tilde{J}) &= \frac{1}{8} \sum_{i,j,n,m} (\{\{e^i, e^j\}, e^n\}, e^m) e_m e_n e_j e_i \\
 &\quad - \frac{1}{4} \sum_{j,n} (\mu(e^j, e^n)(e_n) + \phi(\phi(e^j, e^n), e_n)) e_j.
 \end{aligned}$$

We have

$$\begin{aligned}
 & \sum_{j,n} (\mu(e^j, e^n)(e_n) + \phi(\phi(e^j, e^n), e_n)) e_j \\
 &= Q_2 \left(\sum_{j,n} (\mu(e^j, e^n)(e_n) + \phi(\phi(e^j, e^n), e_n)) \wedge e_j \right) \\
 &\quad + \sum_{j,n} ((\mu(e^j, e^n)(e_n) + \phi(\phi(e^j, e^n), e_n)), e_j).
 \end{aligned}$$

Since

$$\begin{aligned}
 \sum_{j,n} \mu(e^j, e^n)(e_n) \wedge e_j &= \sum_{i,j,n} (\mu(e^j, e^n)(e_n), e^i) e_i \wedge e_j \\
 &= \sum_{i,j,n} B_{\mathfrak{g}}(\mu(e^j, e^n), \mu(e_n, e^i)) e_i \wedge e_j \\
 &= - \sum_{i,j,n} \epsilon(e_i) \epsilon(e_n) \epsilon(e_n, e_i) e_i \wedge (\mu(e_n, e^i)(e^n), e^j) e_j \\
 &= - \sum_{i,n} \epsilon(e_i) \epsilon(e_n) \epsilon(e_n, e_i) e_i \wedge \mu(e_n, e^i)(e^n) \\
 &= - \sum_{i,n} \mu(e^i, e^n)(e_n) \wedge e_i
 \end{aligned}$$

we obtain that

$$\sum_{j,n} \mu(e^j, e^n)(e_n) \wedge e_j = 0$$

and a similar calculation, using (9), shows

$$\sum_{j,n} \phi(\phi(e^j, e^n), e_n) \wedge e_j = 0.$$

Hence, we have

$$\begin{aligned} \sum_{j,n} (\mu(e^j, e^n)(e_n) + \phi(\phi(e^j, e^n), e_n))e_j \\ = \sum_{j,n} ((\mu(e^j, e^n)(e_n) + \phi(\phi(e^j, e^n), e_n)), e_j) \end{aligned}$$

and

$$\begin{aligned} Q_4(\tilde{J}) &= \frac{1}{8} \sum_{i,j,n,m} (\{\{e^i, e^j\}, e^n\}, e^m) e_m e_n e_j e_i \\ &\quad - \frac{1}{4} \sum_{j,n} (\mu(e^j, e^n)(e_n) + \phi(\phi(e^j, e^n), e_n), e_j). \end{aligned} \quad (10)$$

On the other hand, we have

$$\begin{aligned} \pi_*(\Omega(\mathfrak{g})) &= \frac{1}{16} \sum_k \sum_{i,j,n,m} (\pi(x^k)(e^j), e^i) (\pi(x_k)(e^m), e^n) e_i e_j e_n e_m \\ &= \frac{1}{16} \sum_k \sum_{i,j,n,m} B_{\mathfrak{g}}(x^k, \mu(e^j, e^i)) B_{\mathfrak{g}}(x_k, \mu(e^m, e^n)) e_i e_j e_n e_m \\ &= \frac{1}{16} \sum_{i,j,n,m} B_{\mathfrak{g}} \left(\sum_k B_{\mathfrak{g}}(x_k, \mu(e^m, e^n)) x^k, \mu(e^j, e^i) \right) e_i e_j e_n e_m \\ &= \frac{1}{16} \sum_{i,j,n,m} B_{\mathfrak{g}}(\mu(e^m, e^n), \mu(e^j, e^i)) e_i e_j e_n e_m. \end{aligned}$$

Finally, since

$$\psi = \frac{1}{12} \sum_{i,j} \phi(e^i, e^j) e_j e_i,$$

one can show that

$$\psi^2 = \frac{1}{16} \sum_{i,j,k} (\phi(\phi(e^i, e^j), e^k), e^l) e_l e_k e_j e_i - \frac{1}{12} \sum_{k,l} (\phi(\phi(e^k, e^l), e_l), e_k)$$

and so

$$\begin{aligned} \pi_*(\Omega(\mathfrak{g})) + \psi^2 &= \frac{1}{16} \sum_{i,j,n,m} (B_{\mathfrak{g}}(\mu(e^m, e^n)(e^j), e^i) + (\phi(\phi(e^m, e^n), e^j), e^i)) e_i e_j e_n e_m \\ &\quad - \frac{1}{12} \sum_{k,l} (\phi(\phi(e^k, e^l), e_l), e_k). \end{aligned}$$

Using (10) we obtain

$$\begin{aligned}\pi_*(\Omega(\mathfrak{g})) + \psi^2 &= \frac{1}{2}Q_4(\tilde{J}) + \frac{1}{8} \sum_{j,n} (\mu(e^j, e^n)(e_n), e_j) \\ &\quad + \frac{1}{24} \sum_{j,n} (\phi(\phi(e^j, e^n), e_n), e_j).\end{aligned}\tag{11}$$

Since $\tilde{\mathfrak{g}}$ is a colour Lie algebra if and only if $J \equiv 0$, it follows from (11) that this is equivalent to

$$\pi_*(\Omega(\mathfrak{g})) + \psi^2 \in k. \quad \square$$

If there exist a cubic term ψ such that one of the equivalent conditions of this Theorem is satisfied, we say that the representation $\pi : \mathfrak{g} \rightarrow \mathfrak{so}_\epsilon(V, (\ , \))$ is of colour Lie type. If one of these conditions is satisfied for $\psi = 0$, we say that the representation is of colour $\mathbb{Z}_2$ -Lie type. The scalar involved in this Theorem can be more explicitly characterised.

Corollary 3.2. *If $\pi : \mathfrak{g} \rightarrow \mathfrak{so}_\epsilon(V, (\ , \))$ is of colour Lie type for a cubic term ψ , then*

$$\pi_*(\Omega(\mathfrak{g})) + \psi^2 = \frac{1}{24} \left(\text{Tr}_\epsilon(\text{ad}_{\tilde{\mathfrak{g}}}(\Omega(\tilde{\mathfrak{g}}))) - \text{Tr}_\epsilon(\text{ad}_{\mathfrak{g}}(\Omega(\mathfrak{g}))) \right).$$

Proof. We have

$$\begin{aligned}\text{Tr}_\epsilon(\text{ad}_{\tilde{\mathfrak{g}}}(\Omega(\tilde{\mathfrak{g}}))) - \text{Tr}_\epsilon(\text{ad}_{\mathfrak{g}}(\Omega(\mathfrak{g}))) &= \sum_i B_{\tilde{\mathfrak{g}}}(\text{ad}(\Omega(\tilde{\mathfrak{g}}))(x^i), x_i) + \sum_i (\text{ad}(\Omega(\tilde{\mathfrak{g}}))(e^i), e_i) - \sum_i B_{\mathfrak{g}}(\text{ad}(\Omega(\mathfrak{g}))(x^i), x_i) \\ &= \sum_{i,j} B_{\tilde{\mathfrak{g}}}(\{e^j, \{e_j, x^i\}\}, x_i) + \sum_{i,j} (\{x^j, \{x_j, e^i\}\}, e_i) + \sum_{i,j} (\{e^j, \{e_j, e^i\}\}, e_i) \\ &= 2 \sum_{i,j} B_{\tilde{\mathfrak{g}}}(\{e^j, \{e_j, x^i\}\}, x_i) + \sum_{i,j} (\{e^j, \{e_j, e^i\}\}, e_i).\end{aligned}$$

Since

$$\begin{aligned}\sum_{i,j} B_{\tilde{\mathfrak{g}}}(\{e^j, \{e_j, x^i\}\}, x_i) &= \sum_{i,j} B_{\tilde{\mathfrak{g}}}(\epsilon(e_j)\mu(x^i(e_j), e^j), x_i) \\ &= \sum_{i,j,k} \epsilon(e_j)B_{\tilde{\mathfrak{g}}}(\mu((e_k, x^i(e_j))e^k, e^j), x_i) \\ &= \sum_{i,j,k} \epsilon(e_j)\epsilon(e_k)B_{\tilde{\mathfrak{g}}}(\mu(e^k, e^j), B_{\tilde{\mathfrak{g}}}(x^i, \mu(e_j, e_k))x_i) \\ &= \sum_{j,k} B_{\tilde{\mathfrak{g}}}(\mu(e^k, e^j), \mu(e_j, e_k)) \\ &= \sum_{j,k} (\mu(e^k, e^j)(e_j), e_k)\end{aligned}$$

and

$$\sum_{i,j}(\{e^j, \{e_j, e^i\}\}, e_i) = \sum_{i,j}(\mu(e^j, e^i)(e_i), e_j) + \sum_{i,j}(\phi(e^j, \phi(e_j, e^i)), e_i)$$

we have

$$\mathrm{Tr}_\epsilon(\mathrm{ad}_{\tilde{\mathfrak{g}}}(\Omega(\tilde{\mathfrak{g}}))) - \mathrm{Tr}_\epsilon(\mathrm{ad}_{\mathfrak{g}}(\Omega(\mathfrak{g}))) = 3 \sum_{i,j}(\mu(e^j, e^i)(e_i), e_j) + \sum_{i,j}(\phi(e^j, \phi(e_j, e^i)), e_i).$$

Using (11), we obtain

$$\pi_*(\Omega(\mathfrak{g})) + \psi^2 = \frac{1}{24} \left(\mathrm{Tr}_\epsilon(\mathrm{ad}_{\tilde{\mathfrak{g}}}(\Omega(\tilde{\mathfrak{g}}))) - \mathrm{Tr}_\epsilon(\mathrm{ad}_{\mathfrak{g}}(\Omega(\mathfrak{g}))) \right). \quad \square$$

Here is a particular family of examples of representations of colour $\mathbb{Z}_2$ -Lie type.

Proposition 3.3. *Let $\mathfrak{g}$ be a finite-dimensional colour Lie algebra such that its ϵ -Killing form $B_{\mathfrak{g}}$ is non-degenerate. The ϵ -orthogonal adjoint representation $\mathrm{ad} : \mathfrak{g} \rightarrow \mathfrak{so}_\epsilon(\mathfrak{g}, B_{\mathfrak{g}})$ is of colour $\mathbb{Z}_2$ -Lie type and*

$$\mathrm{ad}_*(\Omega(\mathfrak{g})) = \frac{1}{8} \dim_\epsilon(\mathfrak{g}).$$

Proof. The moment map μ of $\mathrm{ad} : \mathfrak{g} \rightarrow \mathfrak{so}_\epsilon(\mathfrak{g}, B_{\mathfrak{g}})$ satisfies to $\mu(u, v) = \{u, v\}$ for $u, v \in \mathfrak{g}$ and then $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}$ is a $\mathbb{Z}_2$ -graded colour Lie algebra. From Theorem 3.1 and Corollary 3.2 we have

$$\mathrm{ad}_*(\Omega(\mathfrak{g})) = \frac{1}{24} \left(\mathrm{Tr}_\epsilon(\mathrm{ad}_{\tilde{\mathfrak{g}}}(\Omega(\tilde{\mathfrak{g}}))) - \mathrm{Tr}_\epsilon(\mathrm{ad}_{\mathfrak{g}}(\Omega(\mathfrak{g}))) \right).$$

The ϵ -quadratic form $B_{\mathfrak{g}}$ is the ϵ -Killing form of $\mathfrak{g}$ and then the ϵ -quadratic form $B_{\tilde{\mathfrak{g}}} = B_{\mathfrak{g}} \perp B_{\mathfrak{g}}$ on $\tilde{\mathfrak{g}}$ satisfies to

$$B_{\tilde{\mathfrak{g}}} = \frac{1}{2} K_{\tilde{\mathfrak{g}}}$$

where $K_{\tilde{\mathfrak{g}}}$ is the ϵ -Killing form of $\tilde{\mathfrak{g}}$. Hence, if $\Omega_K(\tilde{\mathfrak{g}}) \in U_\epsilon(\tilde{\mathfrak{g}})$ is the Casimir element corresponding to $K_{\tilde{\mathfrak{g}}}$, we have

$$\Omega(\tilde{\mathfrak{g}}) = 2\Omega_K(\tilde{\mathfrak{g}}).$$

Thus,

$$\mathrm{Tr}_\epsilon(\mathrm{ad}_{\tilde{\mathfrak{g}}}(\Omega(\tilde{\mathfrak{g}}))) = 2\mathrm{Tr}_\epsilon(\mathrm{ad}_{\tilde{\mathfrak{g}}}(\Omega_K(\tilde{\mathfrak{g}}))) = 2\dim_\epsilon(\tilde{\mathfrak{g}}) = 4\dim_\epsilon(\mathfrak{g})$$

and so

$$\mathrm{ad}_*(\Omega(\mathfrak{g})) = \frac{1}{8} \dim_\epsilon(\mathfrak{g}). \quad \square$$

Comparing the scalar coming from the previous example with the action of the Casimir element on an oscillator representation we obtain a strange Freudenthal and de Vries formula for basic colour Lie algebras.

Theorem 3.4. *Suppose that $\text{char}(k) = 0$. Let $\mathfrak{g}$ be a finite-dimensional basic colour Lie algebra in the sense of Subsection 2.4 and such that its ϵ -Killing form $B_{\mathfrak{g}}$ is non-degenerate. We have*

$$\dim_{\epsilon}(\mathfrak{g}) = 24B_{\mathfrak{g}}^*(\rho, \rho),$$

where ρ is the Weyl vector of $\mathfrak{g}$ with respect to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a choice of a full set of positive roots for the action of $\text{ad}(\mathfrak{h})$.

Proof. We use here the same notation as in Subsection 2.4 and we can suppose without loss of generality that k is algebraically closed.

From Proposition 2.15 we know that $\mathfrak{g}^+$ and $\mathfrak{g}^-$ are isotropic for $B_{\mathfrak{g}}$. Let $\widetilde{W}, \widetilde{W}^*$ be maximal isotropic subspaces of $\mathfrak{h}$ and let U be $\{0\}$ or a one-dimensional vector space of degree 0 orthogonal to W and W' such that $\mathfrak{h} = \widetilde{W}^* \oplus \widetilde{W} \oplus U$. If we define $W = \widetilde{W} \oplus \mathfrak{g}^-$ and $W^* = \widetilde{W}^* \oplus \mathfrak{g}^+$ then W and W^* are maximal isotropic subspaces of $\mathfrak{g}$ and we can consider the oscillator representation $m : C_{\epsilon}(\mathfrak{g}, B_{\mathfrak{g}}) \rightarrow \text{End}(\Lambda_{\epsilon}(W))$ (see Definition 2.5). Composing m with $\text{ad}_* : \mathfrak{g} \rightarrow C_{\epsilon}(\mathfrak{g}, B_{\mathfrak{g}})$ we obtain a representation $m_* : \mathfrak{g} \rightarrow \text{End}(\Lambda_{\epsilon}(W))$.

For $h \in \mathfrak{h}$ we have that

$$\begin{aligned} \text{ad}_*(h) &= \frac{1}{4} \sum_{\alpha \in \Delta^+} (\{h, e_{-\alpha}\}e_{\alpha} + \epsilon(\alpha)\{h, e_{\alpha}\}e_{-\alpha}) \\ &= \frac{1}{4} \sum_{\alpha \in \Delta^+} (-\alpha(h)e_{-\alpha}e_{\alpha} + \epsilon(\alpha)\alpha(h)e_{\alpha}e_{-\alpha}) \\ &= \frac{1}{2} \sum_{\alpha \in \Delta^+} (-\alpha(h)e_{-\alpha}e_{\alpha} + \epsilon(\alpha)\alpha(h)) \end{aligned}$$

and since $e_{\alpha} \in W^*$ we have

$$m_*(h)(1) = \rho(h)1. \quad (12)$$

Let $\gamma \in \Delta^+$. Let $\{h_i\}$ be a basis of $\mathfrak{h}$ and let $\{h^i\}$ be its dual basis. We have

$$\text{ad}_*(e_{\gamma}) = \frac{1}{4} \left(\sum_i \{e_{\gamma}, h^i\}h_i + \sum_{\alpha \in \Delta^+} (\{e_{\gamma}, e_{-\alpha}\}e_{\alpha} + \epsilon(\alpha)\{e_{\gamma}, e_{\alpha}\}e_{-\alpha}) \right)$$

and since $e_{\alpha} \in W^*$ we have

$$m_*(e_{\gamma})(1) = \frac{1}{4} \left(\sum_i m(\{e_{\gamma}, h^i\}h_i)(1) + \sum_{\alpha \in \Delta^+} \epsilon(\alpha)m(\{e_{\gamma}, e_{\alpha}\}e_{-\alpha})(1) \right).$$

Using Proposition 2.14, we have

$$\sum_{\alpha \in \Delta^+} \epsilon(\alpha)m(\{e_{\gamma}, e_{\alpha}\}e_{-\alpha})(1) = \frac{1}{2} \sum_{\alpha \in \Delta^+} \epsilon(\alpha)B_{\mathfrak{g}}(\{e_{\gamma}, e_{\alpha}\}, e_{-\alpha}) = 0$$

and so

$$m_*(e_\gamma)(1) = \frac{1}{4} \sum_i m(\{e_\gamma, h^i\} h_i)(1).$$

Let $\{H_i\}$ be a basis of $\widetilde{W}^*$ and let $\{H^i\}$ be the basis of $\widetilde{W}$ such that $B_{\mathfrak{g}}(H_i, H^j) = \delta_{ij}$. If $U \neq \{0\}$ let u be a basis of U such that $B_{\mathfrak{g}}(u, u) = 1$. The basis $\{h_i\}$ can be chosen such that $\{h_i\} = \{H_i\} \cup \{H^i\}$ if $U = \{0\}$ or such that $\{h_i\} = \{H_i\} \cup \{H^i\} \cup \{u\}$ if $U \neq \{0\}$. Since $H_i \in W^*$ we have

$$\begin{aligned} \frac{1}{4} \sum_i m(\{e_\gamma, H_i\} H^i + \{e_\gamma, H^i\} H_i)(1) &= \frac{1}{4} \sum_i m(\{e_\gamma, H_i\} H^i)(1) \\ &= -\frac{1}{4} \sum_i \gamma(H_i) m(e_\gamma H^i)(1) \\ &= -\frac{1}{2} \sum_i \gamma(H_i) B_{\mathfrak{g}}(e_\gamma, H^i) \\ &= 0, \end{aligned}$$

and if $U \neq \{0\}$ we also have

$$\frac{1}{4} m(\{e_\gamma, u\} u)(1) = \frac{-\gamma(u)}{4} m(e_\gamma u)(1) = \frac{-\gamma(u)}{4} m(e_\gamma)(1) = 0$$

and so

$$m_*(e_\gamma)(1) = 0. \quad (13)$$

Hence, from Equations (12) and (13), we have that $m_*(\mathfrak{g})(1)$ is a representation of $\mathfrak{g}$ of highest weight ρ . By Proposition 2.17 we have

$$m_*(\Omega(\mathfrak{g}))|_{m_*(\mathfrak{g})(1)} = 3B_{\mathfrak{g}^*}(\rho, \rho).$$

Since $m_* = m \circ \text{ad}_*$, by Proposition 3.3, we also have

$$m_*(\Omega(\mathfrak{g}))|_{m_*(\mathfrak{g})(1)} = \frac{1}{8} \dim_\epsilon(\mathfrak{g})$$

and so

$$\dim_\epsilon(\mathfrak{g}) = 24B_{\mathfrak{g}^*}(\rho, \rho). \quad \square$$

4. The cubic Dirac operator

In this section, we define cubic Dirac operators for ϵ -orthogonal representations of ϵ -quadratic colour Lie algebras and we prove a Parthasarathy formula ([20]).

Let $\pi : \mathfrak{g} \rightarrow \mathfrak{so}_\epsilon(V, (\ , \))$ be a finite-dimensional ϵ -orthogonal representation of colour Lie type of a finite-dimensional ϵ -quadratic colour Lie algebra $(\mathfrak{g}, B_{\mathfrak{g}})$ and consider the associated ϵ -quadratic colour Lie algebra $(\widetilde{\mathfrak{g}}, B_{\widetilde{\mathfrak{g}}})$ where $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus V$ and $B_{\widetilde{\mathfrak{g}}} = B_{\mathfrak{g}} \perp (\ , \)$. The ϵ -enveloping algebra $U_\epsilon(\widetilde{\mathfrak{g}})$ is Γ -graded, the ϵ -Clifford

algebra $C_\epsilon(V, (\ , \))$ is $\mathbb{Z}_2 \times \Gamma$ -graded, hence $U_\epsilon(\tilde{\mathfrak{g}}) \otimes C_\epsilon(V, (\ , \))$ is $\mathbb{Z}_2 \times \Gamma$ -graded and we consider the product

$$a \otimes b \cdot c \otimes d := \tilde{\epsilon}(b, c)ac \otimes bd \quad \forall a \otimes b, c \otimes d \in U_\epsilon(\tilde{\mathfrak{g}}) \otimes C_\epsilon(V, (\ , \)).$$

Then $U_\epsilon(\tilde{\mathfrak{g}}) \otimes C_\epsilon(V, (\ , \))$ is a $(\mathbb{Z}_2 \times \Gamma, \tilde{\epsilon})$ -colour Lie algebra for the bracket

$$\{x, y\} := x \cdot y - \tilde{\epsilon}(x, y)y \cdot x \quad \forall x, y \in U_\epsilon(\tilde{\mathfrak{g}}) \otimes C_\epsilon(V, (\ , \)).$$

We now define the cubic Dirac operator of the representation $\pi : \mathfrak{g} \rightarrow \mathfrak{so}_\epsilon(V, (\ , \))$ of colour Lie type.

Definition 4.1. Let $\{e_i\}$ be a basis of V and let $\{e^i\}$ be its dual basis. Define $D \in U_\epsilon(\tilde{\mathfrak{g}}) \otimes C_\epsilon(V, (\ , \))$ by

$$D = \sum_i e^i \otimes e_i,$$

and the cubic Dirac operator $\tilde{D} \in U_\epsilon(\tilde{\mathfrak{g}}) \otimes C_\epsilon(V, (\ , \))$ by

$$\tilde{D} = D + 1 \otimes \psi,$$

where ψ is the cubic term of the colour Lie type representation $\pi : \mathfrak{g} \rightarrow \mathfrak{so}_\epsilon(V, (\ , \))$.

Define the diagonal embedding $\Delta : \mathfrak{g} \rightarrow U_\epsilon(\tilde{\mathfrak{g}}) \otimes C_\epsilon(V, (\ , \))$ by

$$\Delta(x) = x \otimes 1 + 1 \otimes \pi_*(x) \quad \forall x \in \mathfrak{g}.$$

This is a colour Lie algebra morphism and we extend it to the homomorphism $\Delta : U_\epsilon(\mathfrak{g}) \rightarrow U_\epsilon(\tilde{\mathfrak{g}}) \otimes C_\epsilon(V, (\ , \))$.

We calculate the square of the cubic Dirac operator of a representation of colour Lie type analogously to the Parthasarathy formula for the square of the Dirac operator of a Cartan decomposition of a reductive Lie algebra.

Theorem 4.2. Let $\pi : \mathfrak{g} \rightarrow \mathfrak{so}_\epsilon(V, (\ , \))$ be a finite-dimensional ϵ -orthogonal representation of colour Lie type of a finite-dimensional ϵ -quadratic colour Lie algebra $(\mathfrak{g}, B_{\mathfrak{g}})$. Let $\tilde{\mathfrak{g}}$ be the associated colour Lie algebra and let $\tilde{D}$ be the cubic Dirac operator. We have

$$\tilde{D}^2 = \Omega(\tilde{\mathfrak{g}}) \otimes 1 - \Delta(\Omega(\mathfrak{g})) + \frac{1}{24} (\text{Tr}_\epsilon(\text{ad}_{\tilde{\mathfrak{g}}}(\Omega(\tilde{\mathfrak{g}}))) - \text{Tr}_\epsilon(\text{ad}_{\mathfrak{g}}(\Omega(\mathfrak{g})))) 1 \otimes 1,$$

where $\Omega(\tilde{\mathfrak{g}})$ (resp. $\Omega(\mathfrak{g})$) is the Casimir element of $\tilde{\mathfrak{g}}$ (resp. $\mathfrak{g}$).

Proof. Let ψ be the cubic term of the representation of colour Lie type $\pi : \mathfrak{g} \rightarrow \mathfrak{so}_\epsilon(V, (\ , \))$. We have

$$\begin{aligned} D^2 &= \frac{1}{2} \sum_{i,j} (e^i \otimes e_i \cdot e^j \otimes e_j + e^j \otimes e_j \cdot e^i \otimes e_i) \\ &= \frac{1}{2} \sum_{i,j} (\epsilon(e_j, e_i) e^i e^j \otimes e_i e_j + \epsilon(e_i, e_j) e^j e^i \otimes e_j e_i) \\ &= \frac{1}{2} \sum_{i,j} (\epsilon(e_j, e_i) e^i e^j \otimes e_i e_j - e^j e^i \otimes e_i e_j) + \sum_{i,j} \epsilon(e_i, e_j) e^j e^i \otimes (e_j, e_i) \\ &= -\frac{1}{2} \sum_{i,j} \{e^j, e^i\} \otimes e_i e_j + \sum_{i,j} \epsilon(e_i, e_j) e^j e^i \otimes (e_j, e_i) \\ &= -\frac{1}{2} \sum_{i,j} \mu(e^j, e^i) \otimes e_i e_j - \frac{1}{2} \sum_{i,j} \phi(e^j, e^i) \otimes e_i e_j + \sum_{i,j} \epsilon(e_i, e_j) e^j e^i \otimes (e_j, e_i). \end{aligned}$$

Firstly, using Equation (1), we have

$$\begin{aligned}\sum_{i,j} \epsilon(e_i, e_j) e^j e^i \otimes (e_j, e_i) &= \sum_{i,j} e^j(e_i, e_j) e^i \otimes 1 \\ &= (\Omega(\widetilde{\mathfrak{g}}) - \Omega(\mathfrak{g})) \otimes 1,\end{aligned}$$

using Equations (1) and (7), we also have

$$\begin{aligned}-\frac{1}{2} \sum_{i,j} \mu(e^j, e^i) \otimes e_i e_j &= -\frac{1}{2} \sum_{i,j} \sum_k B_{\mathfrak{g}}(x_k, \mu(e^j, e^i)) x^k \otimes e_i e_j \\ &= -\frac{1}{2} \sum_{i,j} \sum_k (\pi(x_k)(e^j), e^i) x^k \otimes e_i e_j \\ &= -\frac{1}{2} \sum_k x^k \otimes \sum_j \pi(x_k)(e^j) e_j \\ &= -2 \sum_k x^k \otimes \pi_*(x_k)\end{aligned}$$

and hence

$$D^2 = -2 \sum_k x^k \otimes \pi_*(x_k) - \frac{1}{2} \sum_{i,j} \phi(e^j, e^i) \otimes e_i e_j + (\Omega(\widetilde{\mathfrak{g}}) - \Omega(\mathfrak{g})) \otimes 1. \quad (14)$$

We have

$$1 \otimes \psi \cdot D = \frac{1}{12} \sum_{i,j,k,l} \epsilon(e_l) e_l \otimes \psi(e^k, e^j, e^i) e_i e_j e_k e^l.$$

Since

$$\begin{aligned}e_i e_j e_k e^l &= -\epsilon(e_l, e_i + e_j + e_k) e^l e_i e_j e_k + 2\epsilon(e_l, e_j + e_k) \delta_{il} e_j e_k \\ &\quad - 2\epsilon(e_l, e_k) \delta_{jl} e_i e_k + 2\delta_{kl} e_i e_j\end{aligned}$$

and

$$\begin{aligned}\frac{1}{12} \sum_{i,j,k,l} \epsilon(e_l) e_l \otimes \psi(e^k, e^j, e^i) (2\epsilon(e_l, e_j + e_k) \delta_{il} e_j e_k - 2\epsilon(e_l, e_k) \delta_{jl} e_i e_k + 2\delta_{kl} e_i e_j) \\ = \frac{1}{2} \sum_{j,k} \phi(e^k, e^j) \otimes e_j e_k\end{aligned}$$

we obtain

$$1 \otimes \psi \cdot D = -\frac{1}{12} \sum_{i,j,k,l} \epsilon(e_l) e_l \otimes \psi(e^k, e^j, e^i) e^l e_i e_j e_k + \frac{1}{2} \sum_{j,k} \phi(e^k, e^j) \otimes e_j e_k.$$

Since

$$D \cdot 1 \otimes \psi = \frac{1}{12} \sum_{i,j,k,l} \epsilon(e_l) e_l \otimes \psi(e^k, e^j, e^i) e^l e_i e_j e_k$$

we have

$$\begin{aligned}
 D \cdot 1 \otimes \psi + 1 \otimes \psi \cdot D &= \sum_l e^l \otimes e_l \cdot 1 \otimes \frac{1}{12} \sum_{i,j,k} \psi(e^k, e^j, e^i) e_i e_j e_k \\
 &\quad + 1 \otimes \frac{1}{12} \sum_{i,j,k} \psi(e^k, e^j, e^i) e_i e_j e_k \cdot \sum_l e^l \otimes e_l \\
 &= \frac{1}{2} \sum_{i,j} \phi(e^j, e^i) \otimes e_i e_j
 \end{aligned}$$

and so from Equation (14) we obtain

$$D^2 = -2 \sum_k x^k \otimes \pi_*(x_k) + (\Omega(\tilde{\mathfrak{g}}) - \Omega(\mathfrak{g})) \otimes 1 + 1 \otimes \psi^2.$$

We have

$$\begin{aligned}
 \Delta(\Omega(\mathfrak{g})) &= \sum_k \Delta(x^k) \Delta(x_k) \\
 &= \sum_k (x^k \otimes 1 + 1 \otimes \pi_*(x^k)) (x_k \otimes 1 + 1 \otimes \pi_*(x_k)) \\
 &= \sum_k x^k x_k \otimes 1 + \sum_k x^k \otimes \pi_*(x_k) \\
 &\quad + \sum_k \epsilon(x_k) x_k \otimes \pi_*(x^k) + \sum_k 1 \otimes \pi_*(x^k) \pi_*(x_k) \\
 &= \Omega(\mathfrak{g}) \otimes 1 + 2 \sum_k x^k \otimes \pi_*(x_k) + 1 \otimes \pi_*(\Omega(\mathfrak{g})),
 \end{aligned}$$

hence

$$(D + 1 \otimes \psi)^2 = \Omega(\tilde{\mathfrak{g}}) \otimes 1 - \Delta(\Omega(\mathfrak{g})) + (1 \otimes \pi_*(\Omega(\mathfrak{g})) + 1 \otimes \psi^2)$$

and from Corollary 3.2 we obtain

$$\tilde{D}^2 = \Omega(\tilde{\mathfrak{g}}) \otimes 1 - \Delta(\Omega(\mathfrak{g})) + \frac{1}{24} (\mathrm{Tr}_\epsilon(\mathrm{ad}_{\tilde{\mathfrak{g}}}(\Omega(\tilde{\mathfrak{g}}))) - \mathrm{Tr}_\epsilon(\mathrm{ad}_{\mathfrak{g}}(\Omega(\mathfrak{g})))) 1 \otimes 1. \quad \square$$

Corollary 4.3. *Suppose that $\mathrm{char}(k) = 0$, the colour Lie algebras $\tilde{\mathfrak{g}}$ and $\mathfrak{g}$ are basic in the sense of Subsection 2.4 and $B_{\tilde{\mathfrak{g}}}$ (resp. $B_{\mathfrak{g}}$) is a multiple of the ϵ -Killing form of $\tilde{\mathfrak{g}}$ (resp. $\mathfrak{g}$). We have*

$$\tilde{D}^2 = \Omega(\tilde{\mathfrak{g}}) \otimes 1 - \Delta(\Omega(\mathfrak{g})) + (B_{\tilde{\mathfrak{g}}^*}(\rho_{\tilde{\mathfrak{g}}}, \rho_{\tilde{\mathfrak{g}}}) - B_{\mathfrak{g}^*}(\rho_{\mathfrak{g}}, \rho_{\mathfrak{g}})) 1 \otimes 1,$$

where $\rho_{\tilde{\mathfrak{g}}}$ (resp. $\rho_{\mathfrak{g}}$) is the Weyl vector of $\tilde{\mathfrak{g}}$ (resp. $\mathfrak{g}$) with respect to a Cartan subalgebra $\mathfrak{h}_{\tilde{\mathfrak{g}}}$ (resp. $\mathfrak{h}_{\mathfrak{g}}$) of $\tilde{\mathfrak{g}}$ (resp. $\mathfrak{g}$) and a choice of a full set of positive roots for the action of $\mathrm{ad}(\mathfrak{h}_{\tilde{\mathfrak{g}}})$ (resp. $\mathrm{ad}(\mathfrak{h}_{\mathfrak{g}})$).

Proof. Let $c \in k^*$ be such that $B_{\mathfrak{g}} = cK_{\mathfrak{g}}$, where $K_{\mathfrak{g}}$ is the ϵ -Killing form of $\mathfrak{g}$. Hence, if $\Omega_K(\mathfrak{g}) \in U_{\epsilon}(\mathfrak{g})$ is the Casimir element corresponding to $K_{\mathfrak{g}}$, we have

$$\Omega(\mathfrak{g}) = \frac{1}{c}\Omega_K(\mathfrak{g}).$$

Thus,

$$\mathrm{Tr}_{\epsilon}(\mathrm{ad}_{\mathfrak{g}}(\Omega(\mathfrak{g}))) = \frac{1}{c}\mathrm{Tr}_{\epsilon}(\mathrm{ad}_{\mathfrak{g}}(\Omega_K(\mathfrak{g}))) = \frac{1}{c}\dim_{\epsilon}(\mathfrak{g})$$

and by the strange Freudenthal–de Vries formula (Theorem 3.4) we obtain

$$\mathrm{Tr}_{\epsilon}(\mathrm{ad}_{\mathfrak{g}}(\Omega(\mathfrak{g}))) = \frac{24}{c}K_{\mathfrak{g}^*}(\rho_{\mathfrak{g}}, \rho_{\mathfrak{g}}).$$

Since $B_{\mathfrak{g}^*} = \frac{1}{c}K_{\mathfrak{g}^*}$, we have

$$\mathrm{Tr}_{\epsilon}(\mathrm{ad}_{\mathfrak{g}}(\Omega(\mathfrak{g}))) = 24B_{\mathfrak{g}^*}(\rho_{\mathfrak{g}}, \rho_{\mathfrak{g}}). \quad \square$$

5. Differential complex induced by the cubic Dirac operator

In this section we show that the cubic Dirac operator of a representation of colour Lie type induces a differential complex and we calculate the associated cohomology, which is the analogue of the algebraic Vogan conjecture proved by Huang and Pandžić ([9]) for colour Lie algebras.

Definition 5.1. Let $\pi : \mathfrak{g} \rightarrow \mathfrak{so}_{\epsilon}(V, (\ , \))$ be a finite-dimensional ϵ -orthogonal representation of colour Lie type of a finite-dimensional ϵ -quadratic colour Lie algebra $(\mathfrak{g}, B_{\mathfrak{g}})$. Let $\tilde{\mathfrak{g}}$ be the associated colour Lie algebra and let $\tilde{D}$ be the cubic Dirac operator. Define the induced differential $d \in \mathrm{End}(U_{\epsilon}(\tilde{\mathfrak{g}}) \otimes C_{\epsilon}(V, (\ , \)))$ by

$$d(a) := \{\tilde{D}, a\} \quad \forall a \in U_{\epsilon}(\tilde{\mathfrak{g}}) \otimes C_{\epsilon}(V, (\ , \)).$$

Consider the $\mathfrak{g}$ -invariant subspace of $U_{\epsilon}(\tilde{\mathfrak{g}}) \otimes C_{\epsilon}(V, (\ , \))$ defined by

$$(U_{\epsilon}(\tilde{\mathfrak{g}}) \otimes C_{\epsilon}(V, (\ , \)))^{\mathfrak{g}} = \{a \in U_{\epsilon}(\tilde{\mathfrak{g}}) \otimes C_{\epsilon}(V, (\ , \)) \mid \{\Delta(x), a\} = 0 \quad \forall x \in \mathfrak{g}\},$$

and define $\mathcal{Z}_{\epsilon}(\Delta(\mathfrak{g})) \subset U_{\epsilon}(\tilde{\mathfrak{g}}) \otimes C_{\epsilon}(V, (\ , \))$ by $\mathcal{Z}_{\epsilon}(\Delta(\mathfrak{g})) := \Delta(\mathcal{Z}_{\epsilon}(\mathfrak{g}))$.

Proposition 5.2. In $(U_{\epsilon}(\tilde{\mathfrak{g}}) \otimes C_{\epsilon}(V, (\ , \)))^{\mathfrak{g}}$, we have $d^2 = 0$ and $\mathcal{Z}_{\epsilon}(\Delta(\mathfrak{g})) \subset \mathrm{Ker}(d)$.

Proof. We first prove that $d^2 = 0$. Let $a \in U_{\epsilon}(\tilde{\mathfrak{g}}) \otimes C_{\epsilon}(V, (\ , \))$. We have

$$\begin{aligned} d^2(a) &= d(\tilde{D}a - \tilde{\epsilon}(\tilde{D}, a)a\tilde{D}) \\ &= \tilde{D}^2a - \tilde{\epsilon}(\tilde{D}, \tilde{D}a)\tilde{D}a\tilde{D} - \tilde{\epsilon}(\tilde{D}, a)\tilde{D}a\tilde{D} + \tilde{\epsilon}(\tilde{D}, a + a\tilde{D})a\tilde{D}^2 \end{aligned}$$

and since $\tilde{\epsilon}(\tilde{D}, a) = (-1)^{|a|}$ and $\tilde{\epsilon}(\tilde{D}, \tilde{D}a) = -(-1)^{|a|}$ we obtain

$$d^2(a) = \tilde{D}^2a - a\tilde{D}^2.$$

From Theorem 4.2, we have $\tilde{D}^2 = \Omega(\tilde{\mathfrak{g}}) \otimes 1 - \Delta(\Omega(\mathfrak{g})) + c1 \otimes 1$ where $c \in k$ and hence

$$d^2(a) = \Omega(\tilde{\mathfrak{g}}) \otimes 1 \cdot a - a \cdot \Omega(\tilde{\mathfrak{g}}) \otimes 1 - \Delta(\Omega(\mathfrak{g})) \cdot a + a \cdot \Delta(\Omega(\mathfrak{g})).$$

Since $\Omega(\tilde{\mathfrak{g}}) \in \mathcal{Z}_\epsilon(\tilde{\mathfrak{g}})$, we obtain

$$d^2(a) = -\Delta(\Omega(\mathfrak{g})) \cdot a + a \cdot \Delta(\Omega(\mathfrak{g})).$$

And hence, if $a \in (U_\epsilon(\tilde{\mathfrak{g}}) \otimes C_\epsilon(V, (\ , \)))^{\mathfrak{g}}$, then we obtain that $d^2(a) = 0$.

The assertion $\mathcal{Z}_\epsilon(\Delta(\mathfrak{g})) \subset \text{Ker}(d)$ follows from a straightforward calculation since D corresponds to Id_V and since the cubic term ψ is invariant by the action of $\mathfrak{g}$. $\square$

We now prove the analogue of the Vogan conjecture for colour Lie algebras.

Theorem 5.3. *Suppose $\text{char}(k) = 0$. Let $\pi : \mathfrak{g} \rightarrow \mathfrak{so}_\epsilon(V, (\ , \))$ be a finite-dimensional ϵ -orthogonal representation of colour Lie type of a finite-dimensional ϵ -quadratic colour Lie algebra $(\mathfrak{g}, B_{\mathfrak{g}})$. Let $\tilde{\mathfrak{g}}$ be the associated colour Lie algebra, let $\tilde{D}$ be the cubic Dirac operator and let d be its induced differential on the algebra $(U_\epsilon(\tilde{\mathfrak{g}}) \otimes C_\epsilon(V, (\ , \)))^{\mathfrak{g}}$. Then we have*

$$\text{Ker}(d) = \mathcal{Z}_\epsilon(\Delta(\mathfrak{g})) \oplus \text{Im}(d).$$

Proof. We first need the following lemma.

Lemma 5.4. *Suppose that $\epsilon(v) = 1$ for all $v \in V$ or suppose $\text{char}(k) = 0$. Let V be a finite-dimensional Γ -graded vector space. Let $X_n(V) := S_\epsilon(V) \otimes \Lambda_\epsilon^n(V)$ and $d_{n+1} : X_{n+1}(V) \rightarrow X_n(V)$ be defined by*

$$d_{n+1}(x \otimes y_0 \wedge \cdots \wedge y_n) = \sum_{i=0}^n (-1)^i \epsilon(y_0 + \cdots + y_{i-1}, y_i) x y_i \otimes y_0 \wedge \cdots \wedge \hat{y}_i \wedge \cdots \wedge y_n.$$

Then

$$\cdots \xrightarrow{d_{n+1}} X_n(V) \xrightarrow{d_n} \cdots \xrightarrow{d_1} X_0(V) \xrightarrow{d_0} k \rightarrow \{0\}, \quad (15)$$

where $d_0 : S_\epsilon(V) \otimes k \rightarrow k$ is the augmentation map, is an exact chain complex. That is to say that $(X_\bullet(V), d)$ is a standard resolution of the trivial $S_\epsilon(V)$ -module k .

Proof. One can check that $d_n \circ d_{n+1} = 0$ and that (15) is exact in $X_{-1}(V) = k$. We now show by induction on $\dim(V)$ that this chain complex is exact.

• **Initialisation:** Suppose that $\dim(V) = 1$ and let v be a basis of V . If $\epsilon(v) = 1$, we have $\Lambda_\epsilon^n(V) = \{0\}$ for all $n \geq 2$ and then the chain complex $(X_\bullet(V), d)$ is of the form

$$\{0\} \rightarrow S(V) \otimes V \xrightarrow{d_1} S(V) \otimes k \xrightarrow{d_0} k \rightarrow \{0\}. \quad (16)$$

The map d_1 is clearly injective. We have $\text{Ker}(d_0) = \sum_{i \geq 1} S_\epsilon^i(V) \otimes k$ and since for $x \otimes v \in S_\epsilon(V) \otimes V$ we have $d_1(x \otimes v) = xv \otimes 1$ we obtain that $\text{Ker}(d_0) = \text{Im}(d_1)$. Hence (16) is a short exact sequence.

If $\epsilon(v) = -1$, every y in $X_n(V) = \Lambda(V) \otimes S^n(V)$ is of the form $y = (\sum_i \alpha_i + \beta_i v) \otimes v^n$ and then, since $\text{char}(k) = 0$, $d_n(y) = \sum_i n\alpha_i v \otimes v^{n-1}$ is equal to 0 if and only if $y = \sum_i \beta_i v \otimes v^n$ that is to say if and only if $y \in \text{Im}(d_{n+1})$ and so the chain complex (15) is exact.

• Induction: Let $n \in \mathbb{N}$ and suppose that the chain complex (15) is exact for every Γ -graded vector space V of dimension n . Let V be a Γ -graded vector space of dimension $n+1$. Let v be an homogeneous element of V and let $W \subset V$ be a Γ -graded vector space such that $V = W \oplus kv$.

Since $\Lambda_\epsilon(V) \cong \Lambda_\epsilon(W) \otimes \Lambda_\epsilon(kv)$ and since the family $\{v^{\wedge i}, \forall i \in \mathbb{N}\}$ generates $\Lambda_\epsilon(kv)$, we define a filtration of $X_n(V)$ by

$$F_p(X_n(V)) := S_\epsilon(V) \otimes \bigoplus_{i=0}^p \Lambda_\epsilon^{n-i}(W) \wedge v^{\wedge i}.$$

The multiplication on the right by $v^{\wedge p}$ defines an isomorphism between the chain complex $(\text{gr}(X)_\bullet, \text{gr}(d))$ and the chain complex $(S_\epsilon(V) \otimes_{S_\epsilon(W)} X_{\bullet-p}(W), 1 \otimes d)$.

The chain complex $(X_\bullet(W), d)$ is exact by assumption. Moreover, $S_\epsilon(V)$ is a free $S_\epsilon(W)$ -module, so the chain complex $(S_\epsilon(V) \otimes_{S_\epsilon(W)} X_\bullet(W), 1 \otimes d)$ is exact and hence

$(X_\bullet(V), d)$ is exact. $\square$

Denote by $\alpha(x)$ the $\mathbb{Z}_2$ -degree of $x \in U_\epsilon(\tilde{\mathfrak{g}}) \otimes C_\epsilon(V, (\cdot, \cdot))$. The standard filtration of $U_\epsilon(\tilde{\mathfrak{g}})$ induces a filtration of $U_\epsilon(\tilde{\mathfrak{g}}) \otimes C_\epsilon(V, (\cdot, \cdot))$ and since

$$\mathfrak{g} \cdot F_n(U_\epsilon(\tilde{\mathfrak{g}}) \otimes C_\epsilon(V, (\cdot, \cdot))) \subseteq F_n(U_\epsilon(\tilde{\mathfrak{g}}) \otimes C_\epsilon(V, (\cdot, \cdot))),$$

then we have a filtration of $(U_\epsilon(\tilde{\mathfrak{g}}) \otimes C_\epsilon(V, (\cdot, \cdot)))^{\mathfrak{g}}$ by

$$F_n((U_\epsilon(\tilde{\mathfrak{g}}) \otimes C_\epsilon(V, (\cdot, \cdot)))^{\mathfrak{g}}) := F_n(U_\epsilon(\tilde{\mathfrak{g}}) \otimes C_\epsilon(V, (\cdot, \cdot)))^{\mathfrak{g}}.$$

Since $\text{gr}(U_\epsilon(\tilde{\mathfrak{g}})) \cong S_\epsilon(\tilde{\mathfrak{g}})$ (see [25]) and $S_\epsilon(\mathfrak{g} \oplus V) \cong S_\epsilon(\mathfrak{g}) \otimes S_\epsilon(V)$ the map

$$d : U_\epsilon(\tilde{\mathfrak{g}}) \otimes C_\epsilon(V, (\cdot, \cdot)) \rightarrow U_\epsilon(\tilde{\mathfrak{g}}) \otimes C_\epsilon(V, (\cdot, \cdot))$$

induces

$$\text{gr}(d) : S_\epsilon(\mathfrak{g}) \otimes S_\epsilon(V) \otimes C_\epsilon(V, (\cdot, \cdot)) \rightarrow S_\epsilon(\mathfrak{g}) \otimes S_\epsilon(V) \otimes C_\epsilon(V, (\cdot, \cdot)).$$

For $a \otimes x \otimes y \in S_\epsilon(\mathfrak{g}) \otimes S_\epsilon(V) \otimes C_\epsilon(V, (\cdot, \cdot))$ we have

$$\text{gr}(d)(a \otimes x \otimes y) = a \otimes \sum_i x e^i \otimes (e_i y - (-1)^{\alpha(y)} \epsilon(e_i, y) y e_i),$$

where $\{e_i\}$ is a basis of V and $\{e^i\}$ its dual basis. The quantisation map $Q : \Lambda_\epsilon(V) \rightarrow C_\epsilon(V, (\cdot, \cdot))$ is a canonical isomorphism of Γ -graded vector spaces. If we consider $d' : S_\epsilon(\mathfrak{g}) \otimes S_\epsilon(V) \otimes \Lambda_\epsilon(V) \rightarrow S_\epsilon(\mathfrak{g}) \otimes S_\epsilon(V) \otimes \Lambda_\epsilon(V)$ given by

$$d'(a \otimes x \otimes y_0 \wedge \cdots \wedge y_n) = 2a \otimes \sum_{i=0}^n (-1)^i \epsilon(y_0 + \cdots + y_{i-1}, y_i) x y_i \otimes y_0 \wedge \cdots \wedge \hat{y}_i \wedge \cdots \wedge y_n,$$

using

$$Q(y_0 \wedge \cdots \wedge y_n) = \frac{1}{n+1} \sum_{\sigma \in S(\llbracket 1, n \rrbracket, \{n+1\})} p(\sigma, y_0 + \cdots + y_n) Q(y_{\sigma(0)} \wedge \cdots \wedge y_{\sigma(n-1)}) y_{\sigma(n)},$$

where $S(\llbracket 1, n \rrbracket, \{n+1\})$ denotes the shuffle permutations of $\llbracket 1, n \rrbracket$ and $\{n+1\}$ in $\llbracket 1, n+1 \rrbracket$, then one can show by induction on n that $\text{gr}(d) \circ Q = Q \circ d'$.

Hence, by Lemma 5.4, the kernel $\text{Ker}(d') \subseteq S_\epsilon(V) \otimes \Lambda_\epsilon(V)$ satisfies

$$\text{Ker}(d') = \text{Im}(d) \oplus k \cdot 1 \otimes 1,$$

and so the kernel $\text{Ker}(d') \subseteq (S_\epsilon(\mathfrak{g}) \otimes S_\epsilon(V) \otimes \Lambda_\epsilon(V))^{\mathfrak{g}}$ satisfies

$$\text{Ker}(d') = \text{Im}(d) \oplus (S_\epsilon(\mathfrak{g}))^{\mathfrak{g}} \otimes 1 \otimes 1. \quad (17)$$

Now we go back to the study of the kernel of d . We have

$$\mathcal{Z}_\epsilon(\Delta(\mathfrak{g})) \oplus \text{Im}(d) \subseteq \text{Ker}(d),$$

from Proposition 5.2 and we show the other inclusion by induction on the degree of the filtration of $(U_\epsilon(\tilde{\mathfrak{g}}) \otimes C_\epsilon(V, (\cdot, \cdot)))^{\mathfrak{g}}$.

- Initialisation: Since $F_{-1}((U_\epsilon(\tilde{\mathfrak{g}}) \otimes C_\epsilon(V, (\cdot, \cdot)))^{\mathfrak{g}}) = \{0\}$ we clearly have

$$\text{Ker}(d|_{F_{-1}((U_\epsilon(\tilde{\mathfrak{g}}) \otimes C_\epsilon(V, (\cdot, \cdot)))^{\mathfrak{g}})}) \subseteq \mathcal{Z}_\epsilon(\Delta(\mathfrak{g})) \oplus \text{Im}(d).$$

- Induction: Let $n \in \mathbb{N}$ and suppose that

$$\text{Ker}(d|_{F_n((U_\epsilon(\tilde{\mathfrak{g}}) \otimes C_\epsilon(V, (\cdot, \cdot)))^{\mathfrak{g}})}) \subseteq \mathcal{Z}_\epsilon(\Delta(\mathfrak{g})) \oplus \text{Im}(d).$$

Let $x \in \text{Ker}(d|_{F_{n+1}((U_\epsilon(\tilde{\mathfrak{g}}) \otimes C_\epsilon(V, (\cdot, \cdot)))^{\mathfrak{g}})})$. We have $\text{gr}(d)(\text{gr}(x)) = 0$ and by (17), there exist $y \in \text{gr}_n((U_\epsilon(\tilde{\mathfrak{g}}) \otimes C_\epsilon(V, (\cdot, \cdot)))^{\mathfrak{g}})$ and $z \in S_\epsilon(\mathfrak{g})^{\mathfrak{g}}$ such that

$$\text{gr}(x) = \text{gr}(d)(y) + z \otimes 1.$$

Consider the symmetrisation $s : S_\epsilon(\mathfrak{g}) \rightarrow U_\epsilon(\mathfrak{g})$ (see [26]). We have $x - d(s(y)) - \Delta(s(z)) \in \text{Ker}(d)$ and

$$x - d(s(y)) - \Delta(s(z)) \in F_n((U_\epsilon(\tilde{\mathfrak{g}}) \otimes C_\epsilon(V, (\cdot, \cdot)))^{\mathfrak{g}})$$

since

$\text{gr}(x - d(s(y)) - \Delta(s(z))) = \text{gr}(x) - \text{gr}(d(s(y))) - \text{gr}(\Delta(s(z))) = \text{gr}(x) - \text{gr}(d)(y) - z \otimes 1$ is equal to zero. Hence, by assumption, there exist $y' \in (U_\epsilon(\tilde{\mathfrak{g}}) \otimes C_\epsilon(V, (\cdot, \cdot)))^{\mathfrak{g}}$ and $z' \in \mathcal{Z}_\epsilon(\Delta(\mathfrak{g}))$ such that

$$x - d(s(y)) - \Delta(s(z)) = d(y') + z'$$

and so

$$x = d(y' + s(y)) + z' + \Delta(s(z))$$

which proves that

$$\text{Ker}(d) = \mathcal{Z}_\epsilon(\Delta(\mathfrak{g})) \oplus \text{Im}(d). \quad \square$$

Corollary 5.5. *Let $z \in \mathcal{Z}_\epsilon(\mathfrak{g})$. There exist a unique $a \in \mathcal{Z}_\epsilon(\Delta(\mathfrak{g}))$ and a unique $b \in (U_\epsilon(\tilde{\mathfrak{g}}) \otimes C_\epsilon(V, (\cdot, \cdot)))^{\mathfrak{g}}$ such that*

$$z \otimes 1 = a + \tilde{D}b + b\tilde{D}.$$

Proof. Clearly, $z \otimes 1$ is $\mathfrak{g}$ -invariant and in the kernel of d . Hence, by Theorem 5.3, there exist an unique $a \in \mathcal{Z}_\epsilon(\Delta(\mathfrak{g}))$ and an unique $b \in (U_\epsilon(\tilde{\mathfrak{g}}) \otimes C_\epsilon(V, (\cdot, \cdot)))^{\mathfrak{g}}$ such that $z \otimes 1 = a + d(b)$. We have $\alpha(z \otimes 1) = 0$, then $\alpha(d(b)) = 0$ and so $\alpha(b) = 1$. Hence we have $d(b) = \tilde{D}b + b\tilde{D}$ and we obtain $z \otimes 1 = a + \tilde{D}b + b\tilde{D}$. $\square$

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