Transformation Groups
Vol. 28, No. 2, 2023, pp. 733–768

INTERNAL NATURAL TRANSFORMATIONS AND FROBENIUS ALGEBRAS IN THE DRINFELD CENTER

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Abstract. For $\mathcal{M}$ and $\mathcal{N}$ finite module categories over a finite tensor category $\mathcal{C}$, the category $\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ of right exact module functors is a finite module category over the Drinfeld center $\mathcal{Z}(\mathcal{C})$. We study the internal Homs of this module category, which we call internal natural transformations. With the help of certain integration functors that map $\mathcal{C}$-$\mathcal{C}$-bimodule functors to objects of $\mathcal{Z}(\mathcal{C})$, we express them as ends over internal Homs and define horizontal and vertical compositions. We show that if $\mathcal{M}$ and $\mathcal{N}$ are exact $\mathcal{C}$-modules and $\mathcal{C}$ is pivotal, then the $\mathcal{Z}(\mathcal{C})$-module $\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ is exact. We compute its relative Serre functor and show that if $\mathcal{M}$ and $\mathcal{N}$ are even pivotal module categories, then $\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ is pivotal as well. Its internal Ends are then a rich source for Frobenius algebras in $\mathcal{Z}(\mathcal{C})$.

1. Introduction

Module categories over monoidal categories have been a prominent topic in representation theory in the past two decades. The theory is particularly well developed for finite tensor categories and their finite module and bimodule categories. Indeed, many notions and results in the theory of finite-dimensional representations over finite-dimensional Hopf algebras have found their natural conceptual home in this setting. Examples of such notions include the unimodularity of a finite tensor category and factorizability of a braided finite tensor category. Results include Radford's S^4-formula [ENO], including its generalization to bimodule categories [FSS1], the equivalence of various characterizations of the non-degeneracy of a braiding on a finite tensor category [Sh2], and the theory of 'reflections' of Hopf algebras [BLS]. Moreover, module and bimodule categories have been used

DOI: 10.1007/s00031-021-09678-5

* JF is supported by VR under project no. 2017-03836.

** CS is partially supported by the RTG 1670 "Mathematics inspired by String theory and Quantum Field Theory" and by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy–EXC 2121 "Quantum Universe"–390833306.

Received August 31, 2020. Accepted February 22, 2021.
Published online December 22, 2021.
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intensively in the study of subfactors, of two-dimensional conformal field theory, and of three-dimensional topological field theory.

The following fact about module categories is well known. Let $\mathcal{C}$ be a finite tensor category and $\mathcal{M}$ and $\mathcal{N}$ be finite $\mathcal{C}$-modules. Then the category $\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ of right exact module functors is a finite module category over the Drinfeld center $\mathcal{Z}(\mathcal{C})$ (which is a finite tensor category). In this paper we study the internal Homs $\underline{\mathrm{Hom}}_{\mathcal{Z}(\mathcal{C})}(G,H)$ for $G,H\in\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$. We denote these internal Homs by $\underline{\mathrm{Nat}}(G,H)\in\mathcal{Z}(\mathcal{C})$ and call them *internal natural transformations*.

For the vector space of ordinary natural transformation between two linear functors, the Yoneda lemma implies a useful formula in terms of an end over morphism spaces:

$$\mathrm{Nat}(G,H) \;=\int_{m\in\mathcal{M}} \mathrm{Hom}_{\mathcal{N}}(G(m),H(m)).$$

The structure morphisms $\mathrm{Nat}(G,H)\to\mathrm{Hom}_{\mathcal{N}}(G(m),H(m))$ of this end just give the components of the natural transformation. One of the main results of this paper, Theorem 9, is a similar expression

$$\underline{\mathrm{Nat}}(F,G) \;=\int_{m\in\mathcal{M}} \underline{\mathrm{Hom}}_{\mathcal{N}}(F(m),G(m))$$

for the internal natural transformations as objects in $\mathcal{Z}(\mathcal{C})$. In particular, we show that the end on the right-hand side has a natural structure of an object in the Drinfeld center.

A crucial ingredient that allows us to obtain this result are two functors

$$\begin{aligned}\textstyle\int_\bullet &:\mathcal{F}un_{\mathcal{C}|\mathcal{C}}({}^{\#}\!\mathcal{M}\boxtimes\mathcal{M},\mathcal{C})\to\mathcal{Z}(\mathcal{C}) \quad\text{and}\\ \textstyle\int^\bullet &:\mathcal{F}un_{\mathcal{C}|\mathcal{C}}(\mathcal{M}^{\#}\boxtimes\mathcal{M},\mathcal{C})\to\mathcal{Z}(\mathcal{C})\end{aligned}$$

given by

$$\textstyle\int_\bullet:\;\; G\mapsto \displaystyle\int_{m\in\mathcal{M}} G(m,m) \quad\text{and}\quad \textstyle\int^\bullet:\;\; H\mapsto \displaystyle\int^{m\in\mathcal{M}} H(m,m) \qquad (1.1)$$

for $G\in\mathcal{F}un_{\mathcal{C}|\mathcal{C}}({}^{\#}\!\mathcal{M}\boxtimes\mathcal{M},\mathcal{C})$ and $H\in\mathcal{F}un_{\mathcal{C}|\mathcal{C}}(\mathcal{M}^{\#}\boxtimes\mathcal{M},\mathcal{C})$, respectively, where ${}^{\#}\!\mathcal{M}$ and $\mathcal{M}^{\#}$ are two right $\mathcal{C}$-module structures on the opposite category $\mathcal{M}^{\mathrm{opp}}$. The existence of these functors, which we call *central integration functors*, is shown in Theorem 8.

Since the internal natural transformations are internal Homs, they come with associative compositions. It follows in particular that for any module functor F the object $\underline{\mathrm{Nat}}(F,F)$ has a natural structure of a unital associative algebra in $\mathcal{Z}(\mathcal{C})$. We show that the structure morphisms $\underline{\mathrm{Nat}}(G,H)\to\underline{\mathrm{Hom}}_{\mathcal{N}}(G(m),H(m))$ of the end behave in the same way as the component maps of an ordinary natural transformation. This allows us to define horizontal and vertical compositions which obey the Eckmann—Hilton relation. As a consequence, the object $\underline{\mathrm{Nat}}(\mathrm{Id}_{\mathcal{M}},\mathrm{Id}_{\mathcal{M}})$

of internal natural endotransformations of the identity functor is a commutative algebra in the braided category $\mathcal{Z}(\mathcal{C})$.

We also study the situation that the monoidal category $\mathcal{C}$ has the additional structure of a *pivotal* tensor category. (This endows its Drinfeld center $\mathcal{Z}(\mathcal{C})$ with a pivotal structure as well.) Moreover, we assume that the module categories under investigation are now *exact* $\mathcal{C}$-modules. As we show in Proposition 16, in this case the two central integration functors (1.1) are related by the Nakayama functor $\mathrm{N}^{\mathrm{r}}_{\mathcal{M}} \in \mathcal{R}ex(\mathcal{M},\mathcal{M})$ according to

$$\int^{\bullet} = \int_{\bullet} \circ \big(\mathrm{Id}_{\mathcal{M}^{\mathrm{opp}}} \boxtimes \mathrm{N}^{\mathrm{r}}_{\mathcal{M}}\big).$$

Based on this result we show in Theorem 18 that for any pair $\mathcal{M}_1, \mathcal{M}_2$ of exact module categories over a pivotal finite tensor category $\mathcal{C}$, the functor category $\mathcal{R}ex_{\mathcal{C}}(\mathcal{M}_1,\mathcal{M}_2)$ is an *exact* $\mathcal{Z}(\mathcal{C})$-module. Specifically, we compute its relative Serre functor to be

$$\mathrm{S}^{\mathrm{r}}_{\mathcal{R}ex} = \mathrm{N}^{\mathrm{r}}_{\mathcal{N}} \circ (D\,.\,-) \circ \mathrm{N}^{\mathrm{r}}_{\mathcal{M}},$$

with D the distinguished invertible object of $\mathcal{C}$. In Corollary 19 we then conclude that in case $\mathcal{C}$ is unimodular, this exact module category is *pivotal* (in the sense of Definition 3). It follows that in this case $\underline{\mathrm{Nat}}(F,F)$ has the structure of a Frobenius algebra, and in particular $\underline{\mathrm{Nat}}(\mathrm{Id}_{\mathcal{M}},\mathrm{Id}_{\mathcal{M}})$ has the structure of a commutative Frobenius algebra. In this way, $\mathcal{C}$-module categories become a rich source of Frobenius algebras in $\mathcal{Z}(\mathcal{C})$.

This paper is organized as follows. After setting the stage in Section 2, in Section 3 we study relations between bimodule functors with codomain $\mathcal{C}$ and the Drinfeld center of $\mathcal{C}$, which leads us to the notion of central integration functors. Section 4 deals with internal natural transformations. In particular, in Section 4.2 we explain how they can be expressed as an end, and in Section 4.3 we introduce and study their horizontal and vertical compositions. Finally, in Section 5 we combine these results with the theory of relative Serre functors and pivotal module categories to examine exactness and pivotality of the functor category $\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ as a module category over $\mathcal{Z}(\mathcal{C})$.

A direct application of our results (and, in fact, also a major motivation for our investigations) is in the description of bulk fields in rigid logarithmic two-dimensional conformal field theories, i.e., conformal field theories whose chiral data are described by a modular finite tensor category $\mathcal{C}$. This application will be discussed in detail elsewhere. Here we content ourselves with mentioning the basic idea. When $\mathcal{C}$ is modular, then we have a braided equivalence $\mathcal{Z}(\mathcal{C}) \simeq \mathcal{C}^{\mathrm{rev}} \boxtimes \mathcal{C}$. The algebra of bulk fields (or, more generally, disorder and defect fields) in full local conformal field theory can therefore be regarded as an object in $\mathcal{Z}(\mathcal{C})$.

The field algebras in local conformal field theories should be Frobenius algebras; this has, e.g., been demonstrated for bulk algebras of rigid logarithmic conformal field theories in [FuS]. It is also well known that there are different full local conformal field theories that share the same chiral data based on a given modular tensor category $\mathcal{C}$. It has been established almost two decades ago that in case $\mathcal{C}$ is semisimple, the datum that in addition to the chiral data is needed to characterize a local conformal field theory is a (semisimple, indecomposable) $\mathcal{C}$-module category

[FFFS], [FFRS]. The results of Section 5 show that for $\mathcal{C}$ not semisimple, a *pivotal* indecomposable module category $\mathcal{M}$ is a natural candidate for such an additional datum. Boundary conditions of the full conformal field theory are then described by objects $m \in \mathcal{M}$ and boundary fields by internal Homs $\underline{\mathrm{Hom}}(m, m') \in \mathcal{C}$. By Theorem 3.15 of [Sh4], the algebra $\underline{\mathrm{Hom}}(m, m)$ is a symmetric Frobenius algebra for any $m \in \mathcal{M}$. A right exact module functor $G \in \mathcal{R}ex_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ describes a topological defect line between local conformal field theories characterized by $\mathcal{M}_1$ and by $\mathcal{M}_2$, respectively. It is then natural to propose that the defect fields that change a defect line labeled by G to a defect line labeled by H are given by $\underline{\mathrm{Nat}}(G, H) \in \mathcal{Z}(\mathcal{C})$. In particular, $\underline{\mathrm{Nat}}(G, G) \in \mathcal{Z}(\mathcal{C})$ is a symmetric Frobenius algebra; as a special case, $\underline{\mathrm{Nat}}(\mathrm{Id}_M, \mathrm{Id}_{\mathcal{M}}) \in \mathcal{Z}(\mathcal{C})$ a commutative symmetric Frobenius algebra, as befits the space of bulk fields. This proposal also leads to natural candidates for operator product expansions and passes non-trivial consistency checks.

2. Background

In this section we fix our notation and mention some pertinent structures and concepts.

2.1. Basic concepts

Monoidal categories

We denote the tensor product of a monoidal category by $\otimes$ and the monoidal unit by $\mathbf{1}$, and the associativity and unit constraints by α, l and r, i.e., a monoidal category is a quintuple $\mathcal{C} = (\mathcal{C}, \otimes, \mathbf{1}, \alpha, l, r)$. For better readability of various formulas, we sometimes take, without loss of generality, the tensor product to be strict, i.e., take the associator α and unit constraints l and r to be identities.

Module categories

The notion of a (left) *module category* $\mathcal{M}$ over a monoidal category $\mathcal{C}$, or $\mathcal{C}$-module, for short, categorifies the notion of module over a ring: There is an *action functor* $\mathcal{C} \times \mathcal{M} \to \mathcal{M}$, exact in its first variable, together with a mixed associator and a mixed unitor that obey mixed pentagon and triangle relations. For background on module categories, as well as module functors and module natural transformations, see, e.g., [EGNO, Chap. 7] or [Sh3, Sect. 2.3]. In the present paper, module categories will be left modules unless stated otherwise. We denote the action morphism by a dot and the mixed associator by a, i.e., a has components $a_{c,c',m} \colon (c \otimes c') \,.\, m \to c \,.\, (c' \,.\, m)$ with $c, c' \in \mathcal{C}$ and $m \in \mathcal{M}$. The natural isomorphism that defines the structure of a $\mathcal{C}$-module functor G is denoted by ϕ^G, i.e., ϕ^G has components $\phi^G_{c,m} \colon G(c.m) \to c \,.\, (G(m))$ with $c \in \mathcal{C}$ and $m \in \mathcal{M}$. Analogous notations are used for right module categories and for bimodule categories. In this paper we study module and bimodule categories over monoidal categories which in addition satisfy suitable finiteness conditions.

Finite categories

We fix an algebraically closed field $\Bbbk$. A *finite* $\Bbbk$-linear category is an abelian category that is equivalent as abelian category to the category of finite-dimensional modules over a finite-dimensional $\Bbbk$-algebra. A *finite tensor category* is a finite $\Bbbk$-linear category which is rigid monoidal with appropriate compatibility conditions among the structures, see, e.g., [EO] or [Sh3, Sect. 2.5]. Since a finite tensor category is rigid, its tensor product functor is exact. Our conventions concerning dualities of a rigid category $\mathcal{C}$ are as follows. The right dual of an object c is denoted by $c^\vee$, and the right evaluation and coevaluation are morphisms

$$\mathrm{ev}_c^{\mathrm{r}} \in \mathrm{Hom}_{\mathcal{C}}(c^\vee \otimes c, \mathbf{1}) \quad \text{and} \quad \mathrm{coev}_c^{\mathrm{r}} \in \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, c \otimes c^\vee),$$

while the left evaluation and coevaluation are

$$\mathrm{ev}_c^{\mathrm{l}} \in \mathrm{Hom}_{\mathcal{C}}(c \otimes {}^\vee c, \mathbf{1}) \quad \text{and} \quad \mathrm{coev}_c^{\mathrm{l}} \in \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, {}^\vee c \otimes c)$$

with ${}^\vee c$ the left dual of c.

A module category $\mathcal{M}$ over a finite tensor category $\mathcal{C}$ is called *finite* iff $\mathcal{M}$ is a finite $\Bbbk$-linear abelian category and the action of $\mathcal{C}$ on $\mathcal{M}$ is linear and right exact in both variables. The category of right exact module endofunctors of a finite module category $\mathcal{M}$ over a finite tensor category $\mathcal{C}$ is again a finite tensor category [EGNO, Prop. 7.11.6.]; we denote it by $\mathcal{C}_{\mathcal{M}}^\star$. A finite $\mathcal{C}$-module is called *exact* iff $p\,.\,m$ is projective in $\mathcal{M}$ for each projective $p \in \mathcal{C}$ and each $m \in \mathcal{M}$. In particular, $\mathcal{C}$ is an exact module category over itself [EO, Def. 3.1]. Indecomposable exact module categories over H-mod, for H a finite-dimensional Hopf algebra, are classified in [AM, Sect. 3.2]. For recent results see also [Sh4]. All tensor categories and all module and bimodule categories over them that we will consider in the rest of this paper are finite.

Drinfeld centers

For $\mathcal{C}$ a monoidal category and $\mathcal{M}$ a $\mathcal{C}$-$\mathcal{C}$-bimodule category, a *half-braiding* for an object $m \in \mathcal{M}$ is a natural family $\sigma = (\sigma_c)_{c \in \mathcal{C}}$ of morphisms $\sigma_c\colon c\,.\,m \to m\,.\,c$ such that

$$(\sigma_c \otimes \mathrm{id}_d) \circ a_{c,m,d}^{-1} \circ (\mathrm{id}_c \otimes \sigma_d) = a_{c,m,d}^{-1} \circ \sigma_{c \otimes d} \circ a_{c,d,m}^{-1}$$

for all $c, d \in \mathcal{C}$ and all $m \in \mathcal{M}$. Here we use the same symbol a for the three mixed associativity constraints of the bimodule category $\mathcal{M}$. The *Drinfeld center* $\mathcal{Z}(\mathcal{M}) \equiv \mathcal{Z}_{\mathcal{C}}(\mathcal{M})$ of the $\mathcal{C}$-$\mathcal{C}$-bimodule $\mathcal{M}$ is the category that has as objects pairs (m, σ) consisting of an object of $\mathcal{M}$ and a half-braiding on it. The morphisms $\mathrm{Hom}_{\mathcal{Z}(\mathcal{M})}((m,\sigma),(m',\sigma'))$ are those morphisms $m \xrightarrow{f} m'$ in $\mathcal{M}$ that satisfy he condition $(f \otimes \mathrm{id}_n) \circ \sigma_n = \sigma'_n \circ (\mathrm{id}_n \otimes f)$ for all $n \in \mathcal{M}$ [GNN, Def. 2.1]. In the special case that $\mathcal{M}$ is $\mathcal{C}$ regarded as a bimodule category over itself, this definition reproduces the standard notion of the Drinfeld center of a monoidal category [EGNO, Def. 7.13.1]. For any monoidal category $\mathcal{C}$, the Drinfeld center $\mathcal{Z}(\mathcal{C})$ carries a natural braided monoidal structure obtained from the half-braiding.

Unimodular categories

In any finite tensor category there is (uniquely up to isomorphism) a *distinguished invertible object* D, an invertible object that comes [ENO, Thm. 3.3] with coherent isomorphisms $D \otimes x \cong x^{\vee\vee\vee\vee} \otimes D$. A *unimodular* finite tensor category is a finite tensor category $\mathcal{A}$ for which the distinguished invertible object is the monoidal unit. There are several equivalent characterizations of unimodularity [Sh1], e.g., the forgetful functor $U\colon \mathcal{Z}(\mathcal{A}) \to \mathcal{A}$ from the Drinfeld center is a Frobenius functor.

Modular categories

For $\mathcal{C}$ a braided finite tensor category, we denote by $\mathcal{C}^{\text{rev}}$ its *reverse*, i.e., the same monoidal category, but with inverse braiding. There is a canonical braided functor

$$\Xi_{\mathcal{C}}: \quad \mathcal{C}^{\text{rev}} \boxtimes \mathcal{C} \to \mathcal{Z}(\mathcal{C}) \tag{2.1}$$

from the *enveloping category* of $\mathcal{C}$, i.e., the Deligne product of $\mathcal{C}^{\text{rev}}$ with $\mathcal{C}$ (which exists, as $\mathcal{C}$ is finite abelian), to the Drinfeld center of $\mathcal{C}$. As a functor, $\Xi_{\mathcal{C}}$ maps the object $u \boxtimes v \in \mathcal{C}^{\text{rev}} \boxtimes \mathcal{C}$ to the tensor product $u \otimes v \in \mathcal{C}$ endowed with the half-braiding $\gamma_{u\otimes v}$ that has components $\gamma_{c;u\otimes v} := (\mathrm{id}_u \otimes \beta_{v,c}^{-1}) \circ (\beta_{c,u} \otimes \mathrm{id}_v)$ for $c \in \mathcal{C}$, with β the braiding in $\mathcal{C}$. The braided monoidal structure on the functor $\Xi_{\mathcal{C}}$ is given by the coherent family $\mathrm{id}_u \otimes \beta_{v,x} \otimes \mathrm{id}_y$ of isomorphisms from $u \otimes v \otimes x \otimes y$ to $u \otimes x \otimes v \otimes y$.

A finite tensor category $\mathcal{C}$ is *non-degenerate* iff the functor $\Xi_{\mathcal{C}}$ is an equivalence. If $\mathcal{C}$ is even a ribbon category, then $\mathcal{C}^{\text{rev}}$ is a ribbon category with the inverse twist. A non-degenerate finite ribbon category is a *modular tensor category*, or modular category, for short. Traditionally, the term modular category has been used under the additional assumption that the finite tensor category $\mathcal{C}$ is semisimple, i.e., a fusion category; in our context, such a restriction is not natural. A modular category is in particular unimodular [ENO, Prop. 4.5].

The central monad and comonad

The Drinfeld center comes with a forgetful functor $U: \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ that omits the half-braiding. U is exact and hence has a left and a right adjoint. These adjunctions are (co)monadic and thus give rise to a monad

$$\begin{aligned} \mathrm{Z}: \quad & \mathcal{C} \to \mathcal{C}, \\ & c \mapsto \int^{x\in\mathcal{C}} x \otimes c \otimes {}^\vee x \end{aligned}$$

on $\mathcal{C}$ and to a comonad

$$\begin{aligned} \mathrm{Z}: \quad & \mathcal{C} \to \mathcal{C} \\ & c \mapsto \int_{x\in\mathcal{C}} {}^\vee x \otimes c \otimes x \end{aligned} \tag{2.2}$$

(or, equivalently, $c \mapsto \int_{x\in\mathcal{C}} x \otimes c \otimes x^\vee$). These are called the *central monad* and *central comonad*, respectively. Since the adjunctions are monadic and comonadic respectively, we have canonical equivalences of the categories of Z-modules, Z-comodules and the Drinfeld center $\mathcal{Z}(\mathcal{C})$. For more details about the central monad and comonad see, e.g., [BV] and [TV, Chap. 9].

When dealing with coends, such as the one defining the central monad, a convenient tool is the following form of the Yoneda lemma (see, e.g., [FSS1, Prop. 2.7]):

Lemma 1. *Let $\mathcal{A}$ and $\mathcal{B}$ be finite linear categories and $F: \mathcal{A} \to \mathcal{B}$ a linear functor. Then there is a natural isomorphism*

$$\int^{a\in\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(a,-) \otimes F(a) \cong F \tag{2.3}$$

of linear functors.

An analogous co-Yoneda lemma holds for ends: $\int_{a\in\mathcal{A}}\mathrm{Hom}_{\mathcal{A}}(-,a)^*\otimes F(a)\cong F$. The isomorphisms in these formulas are uniquely determined by universal properties; accordingly, from now on we will write them as equalities.

Eilenberg–Watts calculus

For any pair of finite linear categories $\mathcal{A}$ and $\mathcal{B}$ there are two pairs of two-sided adjoint equivalences

$$\mathcal{L}ex(\mathcal{A},\mathcal{B}) \underset{\Phi^{\mathrm{l}}}{\overset{\Psi^{\mathrm{l}}}{\rightleftarrows}} \mathcal{A}^{\mathrm{opp}}\boxtimes\mathcal{B} \underset{\Phi^{\mathrm{r}}}{\overset{\Psi^{\mathrm{r}}}{\leftrightarrows}} \mathcal{R}ex(\mathcal{A},\mathcal{B}) \tag{2.4}$$

given by [Sh1], [FSS1]

$$\begin{aligned}\Phi^{\mathrm{l}}(\overline{a}\boxtimes b) &:= \mathrm{Hom}_{\mathcal{A}}(a,-)\otimes b, \quad \Psi^{\mathrm{l}}(F) := \int^{a\in\mathcal{A}}\overline{a}\boxtimes F(a) \quad \text{and}\\ \Phi^{\mathrm{r}}(\overline{a}\boxtimes b) &:= \mathrm{Hom}_{\mathcal{A}}(-,a)^*\otimes b, \quad \Psi^{\mathrm{r}}(G) := \int_{a\in\mathcal{A}}\overline{a}\boxtimes G(a).\end{aligned} \tag{2.5}$$

This provides a Morita invariant version of the classical Eilenberg–Watts description of right or left exact functors between the categories R-mod and S-mod of modules over unital rings in terms of S-R-bimodules. The functors (2.5) are therefore called Eilenberg–Watts equivalences.

Applying these equivalences to the identity functor on $\mathcal{A}$, regarded as a left exact functor, yields a right exact endofunctor

$$\mathrm{N}^{\mathrm{r}}_{\mathcal{A}} := \Phi^{\mathrm{r}}\circ\Psi^{\mathrm{l}}(\mathrm{Id}_{\mathcal{A}}) = \int^{a\in\mathcal{A}}\mathrm{Hom}_{\mathcal{A}}(-,a)^*\otimes a \ \in \mathcal{R}ex(\mathcal{A},\mathcal{A}), \tag{2.6}$$

which is called the *Nakayama functor* of the finite linear category $\mathcal{A}$ [FSS1, Def. 3.14]. Analogously, by applying $\Phi^{\mathrm{l}}\circ\Psi^{\mathrm{r}}$ to $\mathrm{Id}_{\mathcal{A}}$ regarded as a right exact functor we obtain a left exact analogue $\mathrm{N}^{\mathrm{l}}_{\mathcal{A}} = \int_{a\in\mathcal{A}}\mathrm{Hom}_{\mathcal{A}}(a,-)\otimes a\in\mathcal{L}ex(\mathcal{A},\mathcal{A})$. The functor $\mathrm{N}^{\mathrm{l}}_{\mathcal{A}}$ is left adjoint to $\mathrm{N}^{\mathrm{r}}_{\mathcal{A}}$. For $\mathcal{A}$ and $\mathcal{B}$ finite tensor categories, the Nakayama functor of a finite $\mathcal{A}$-$\mathcal{B}$-bimodule $\mathcal{M}$ has a natural structure of a *twisted* bimodule functor, in the sense that there are coherent isomorphisms

$$\mathrm{N}^{\mathrm{r}}_{\mathcal{M}}(a.m.b)\cong {}^{\vee\vee}a.\,\mathrm{N}^{\mathrm{r}}_{\mathcal{M}}(m).b^{\vee\vee}$$

for all $m\in\mathcal{M}$, $a\in\mathcal{A}$ and $b\in\mathcal{B}$ [FSS1, Thm. 4.5].

2.2. Internal Hom

For any $m\in\mathcal{M}$ we denote the *action functor* by $H_m = -.m\colon\ \mathcal{C}\to\mathcal{M}$. As H_m is (right) exact, the following functors exist:

Definition 1. Let $\mathcal{C}$ be a monoidal category and $\mathcal{M}$ be a $\mathcal{C}$-module. An *internal Hom* of $\mathcal{M}$ in $\mathcal{C}$ is a functor

$$\underline{\mathrm{Hom}}_{\mathcal{M}}(¿;?):\quad \mathcal{M}^{\mathrm{opp}}\boxtimes\mathcal{M}\to\mathcal{C}$$

such that for every $m\in\mathcal{M}$ the functor $\underline{\mathrm{Hom}}_{\mathcal{M}}(m,-)\colon\mathcal{M}\to\mathcal{C}$ is right adjoint to the action functor H_m, i.e., such that for any two objects $m,m'\in\mathcal{M}$ there is a natural family

$$\mathrm{Hom}_{\mathcal{C}}(c,\underline{\mathrm{Hom}}_{\mathcal{M}}(m,m'))\cong\mathrm{Hom}_{\mathcal{M}}(c\,.\,m,m') \tag{2.7}$$

of isomorphisms (see, e.g., [Os]).

Being a right adjoint, the internal Hom is left exact. When it is clear from the context which module category $\mathcal{M}$ is concerned, we simply write $\underline{\mathrm{Hom}}$ in place of $\underline{\mathrm{Hom}}_{\mathcal{M}}$.

We note that there is no separate notion of a 'relative adjoint' functor:

Lemma 2. *Let $\mathcal{C}$ be a monoidal category and $\mathcal{M}, \mathcal{N}$ be $\mathcal{C}$-modules. Let $F\colon \mathcal{M} \to \mathcal{N}$ and $G\colon \mathcal{N} \to \mathcal{M}$ be module functors. Then F and G are adjoint functors if and only if there are functorial isomorphisms*

$$\underline{\mathrm{Hom}}_{\mathcal{N}}(F(m), n) \,\cong\, \underline{\mathrm{Hom}}_{\mathcal{M}}(m, G(n)). \tag{2.8}$$

Proof. By the definition of the internal Hom and the fact that F is a module functor, we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{M}}(c\,.\,m, G(n)) &\cong \mathrm{Hom}_{\mathcal{C}}\big(c, \underline{\mathrm{Hom}}_{\mathcal{M}}(m, G(n))\big) \quad \text{and} \\ \mathrm{Hom}_{\mathcal{N}}(F(c.m), n) \cong \mathrm{Hom}_{\mathcal{N}}(c\,.\,F(m), n) &\cong \mathrm{Hom}_{\mathcal{C}}\big(c, \underline{\mathrm{Hom}}_{\mathcal{N}}(F(m), n)\big) \end{aligned}$$

for every $c \in \mathcal{C}$, $m \in \mathcal{M}$ and $n \in \mathcal{N}$. Thus (2.8) implies that F and G are adjoint. The converse holds by the Yoneda lemma. □

Algebra structure on internal Ends

The counits of the adjunctions $H_m \dashv \underline{\mathrm{Hom}}(m, -)$ provide for any two objects $m, m' \in \mathcal{M}$ a canonical morphism

$$\underline{\mathrm{ev}}_{m,m'}\colon \underline{\mathrm{Hom}}(m, m')\,.\,m \to m', \tag{2.9}$$

in $\mathcal{M}$ given by the image of the identity morphism in $\mathrm{End}_{\mathcal{C}}(\underline{\mathrm{Hom}}(m, m'))$ under the defining isomorphism (2.7). The composition $\underline{\mathrm{ev}}_{m'.m''} \circ (\mathrm{id}_{\underline{\mathrm{Hom}}(m',m'')} \otimes \underline{\mathrm{ev}}_{m.m'}) \circ a_{\underline{\mathrm{Hom}}(m',m''),\underline{\mathrm{Hom}}(m,m'),m}$ furnishes an associative multiplication

$$\underline{\mu}_{m,m',m''}\colon \underline{\mathrm{Hom}}(m', m'') \otimes \underline{\mathrm{Hom}}(m, m') \;\to\; \underline{\mathrm{Hom}}(m, m'') \tag{2.10}$$

on internal Homs. Moreover, the component

$$\mathbf{1} \to \underline{\mathrm{Hom}}(m, \mathbf{1}.m) = \underline{\mathrm{Hom}}(m, m) \tag{2.11}$$

of the unit of the adjunction $H_m \dashv \underline{\mathrm{Hom}}(m, -)$ is a unit for the multiplication (2.10).

Compatibility with module functors

A module functor $G\colon \mathcal{M} \to \mathcal{N}$ induces a natural morphism $\mathrm{Hom}_{\mathcal{M}}(c.m, m') \to \mathrm{Hom}_{\mathcal{N}}(c.G(m), G(m'))$ for any $m, m' \in \mathcal{M}$ and $c \in \mathcal{C}$, and thus $\mathrm{Hom}_{\mathcal{C}}(c, \underline{\mathrm{Hom}}(m, m')) \to \mathrm{Hom}_{\mathcal{C}}(c, \underline{\mathrm{Hom}}(G(m), G(m')))$. By the Yoneda lemma this induces a morphism

$$\underline{G}\colon \underline{\mathrm{Hom}}(m, m') \to \underline{\mathrm{Hom}}(G(m), G(m')) \tag{2.12}$$

of internal Hom objects in $\mathcal{C}$. It is easy to check that $\underline{G \circ G'} = \underline{G} \circ \underline{G'}$, where on the left-hand side one deals with composition of functors and on the right-hand side with composition of morphisms in $\mathcal{C}$.

Internal Hom as a bimodule functor

For a left $\mathcal{C}$-module $\mathcal{M}$ the opposite category $\mathcal{M}^{\rm opp}$ can be endowed in many ways with the structure of a right $\mathcal{C}$-module, which are related by the monoidal functor of taking biduals. Of relevance to us are the following two choices of the right $\mathcal{C}$-action: either $\overline{m}\,.\,c := \overline{{}^\vee c\,.\,m}$ for $m \in \mathcal{M}$, or else $\overline{m}\,.\,c := \overline{c^\vee.\,m}$. We denote the former right $\mathcal{C}$-module by ${}^\#\mathcal{M}$ and the latter by $\mathcal{M}^\#$. Then in particular both ${}^\#\mathcal{M} \boxtimes \mathcal{M}$ and $\mathcal{M}^\# \boxtimes \mathcal{M}$ have a natural structure of a $\mathcal{C}$-bimodule. It follows that $\underline{\rm Hom}$ is naturally a bimodule functor, with $\mathcal{C}$ regarded as a bimodule over itself:

Lemma 3 ([Sh4, Lem. 2.7])**.** *The functor* $\underline{\rm Hom}(¿;?) \colon {}^\#\mathcal{M} \boxtimes \mathcal{M} \to \mathcal{C}$ *is a bimodule functor.*

Proof. For any $c \in \mathcal{C}$ the endofunctor F_c of $\mathcal{M}$ defined by $F_c(m) = c\,.\,m$ is left exact and thus has a left adjoint. Indeed we have

$$\begin{aligned} {\rm Hom}_\mathcal{C}(\gamma, \underline{\rm Hom}(m, c.m')) &\cong {\rm Hom}_\mathcal{M}(\gamma\,.\,m, c\,.\,m') \\ &\cong {\rm Hom}_\mathcal{M}(c^\vee.\,(\gamma.m), m') \\ &\cong {\rm Hom}_\mathcal{C}(c^\vee \otimes \gamma, \underline{\rm Hom}(m, m')) \\ &\cong {\rm Hom}_\mathcal{C}(\gamma, c\,.\,\underline{\rm Hom}(m, m')) \end{aligned}$$

for any $\gamma \in \mathcal{C}$. Similarly, ${\rm Hom}_\mathcal{C}(\gamma, \underline{\rm Hom}(c.m, m')) \cong {\rm Hom}_\mathcal{M}(\gamma, {\rm Hom}(m, m') \otimes c^\vee)$ for $\gamma \in \mathcal{C}$. Thus there are isomorphisms

$$\begin{aligned} \underline{\rm Hom}(m, c'\,.\,m') &\cong c' \otimes \underline{\rm Hom}(m, m') \quad \text{and} \\ \underline{\rm Hom}(c\,.\,m, m') &\cong {\rm Hom}(m, m') \otimes c^\vee. \end{aligned}$$

Taken together, these imply the natural isomorphisms

$$\underline{\rm Hom}(c\,.\,m, c'\,.\,m') \cong c' \otimes \underline{\rm Hom}(m, m') \otimes c^\vee \tag{2.13}$$

that are required for $\underline{\rm Hom}(¿;?)$ to be a bimodule functor. □

Internal coHom

There is an obvious dual notion to the internal Hom: For $\mathcal{C}$ a monoidal category and $\mathcal{M}$ a $\mathcal{C}$-module, an *internal coHom* of $\mathcal{M}$ in $\mathcal{C}$ is a functor $\underline{\rm coHom}(¿;?) \colon \mathcal{M}^{\rm opp} \boxtimes \mathcal{M} \to \mathcal{C}$ such that for any $m, m' \in \mathcal{M}$ there is a natural family

$${\rm Hom}_\mathcal{C}(\underline{\rm coHom}(m', m), c) \cong {\rm Hom}_\mathcal{M}(m, c\,.\,m') \tag{2.14}$$

of isomorphisms. By exactness of the action functors H_m, also internal coHoms exist. Being a left adjoint, the internal coHom is right exact.

On the left-hand side of the isomorphism (2.14) we have ${\rm Hom}_\mathcal{C}(\underline{\rm coHom}(m', m), c) \cong {\rm Hom}_\mathcal{C}(c^\vee, \underline{\rm coHom}(m', m)^\vee)$, while the morphism space on the right-hand side is ${\rm Hom}_\mathcal{M}(m, c\,.\,m') \cong {\rm Hom}_\mathcal{C}(c^\vee, \underline{\rm Hom}(m, m'))$. Thus the internal Hom and coHom are are indeed dual to each other:

$$\begin{aligned} \underline{\rm coHom}(m', m)^\vee &\cong \underline{\rm Hom}(m, m') \quad \text{and} \\ \underline{\rm coHom}(m', m) &\cong {}^\vee\underline{\rm Hom}(m, m'). \end{aligned} \tag{2.15}$$

By taking left duals in (2.10) we obtain a coassociative comultiplication

$$\begin{aligned}\underline{\mathrm{coHom}}(m'',m) = {}^\vee\underline{\mathrm{Hom}}(m,m'') &\to {}^\vee\big(\underline{\mathrm{Hom}}(m',m'')\otimes\underline{\mathrm{Hom}}(m,m')\big)\\ &\cong \underline{\mathrm{coHom}}(m',m)\otimes\underline{\mathrm{coHom}}(m'',m').\end{aligned}\tag{2.16}$$

A counit for this comultiplication is given by the left dual of the unit for the multiplication (2.10)

Analogously to the morphism (2.12), for any module functor $G\colon \mathcal{M}\to\mathcal{N}$ we get a morphism $\underline{G}\colon \underline{\mathrm{coHom}}(G(m),G(m'))\longrightarrow\underline{\mathrm{coHom}}(m,m')$ of internal coHom objects in $\mathcal{C}$. And analogously to Lemma 3 one shows that $\underline{\mathrm{coHom}}(¿;?)\colon \mathcal{M}^\#\boxtimes\mathcal{M}\to\mathcal{C}$ is naturally a bimodule functor.

Remark 1. For $\mathcal{C}$ as a module over itself, the internal Hom is $\underline{\mathrm{Hom}}(c,c') = c'\otimes c^\vee$, and the family (2.7) reduces to the natural isomorphism that is furnished by the right duality. Similarly we then have $\underline{\mathrm{coHom}}(c,c') = c'\otimes {}^\vee c$.

2.3. Pivotal module categories

An additional structure that a finite tensor category $\mathcal{C}$ may admit is a *pivotal* structure, i.e., a monoidal isomorphism $\pi\colon \mathrm{Id}_\mathcal{C}\to -^{\vee\vee}$ from the identity functor to the right double-dual functor. The presence of a pivotal structure has important consequences; for instance, while the notion of a Frobenius algebra makes sense in any monoidal category $\mathcal{C}$, the one of a symmetric Frobenius algebra does so only if $\mathcal{C}$ is pivotal (and even depends on the choice of pivotal structure). Pivotality will be used extensively in Section 5; we therefore discuss it here in some detail.

If $\mathcal{M}$ and $\mathcal{N}$ are left modules over a pivotal finite tensor category $\mathcal{C}$, then the Eilenberg–Watts equivalences (2.4) of linear categories induce adjoint equivalences involving categories of left and right exact module functors: we have [FSS3, Prop. 4.1]

$$\begin{aligned}\mathcal{L}ex_\mathcal{C}(\mathcal{M},\mathcal{N}) \;\underset{\Phi^{\mathrm{l}}}{\overset{\Psi^{\mathrm{l}}}{\rightleftarrows}}\; \mathcal{Z}(\mathcal{M}^\#\boxtimes\mathcal{N}) \quad\text{and}\\ \mathcal{Z}({}^\#\mathcal{M}\boxtimes\mathcal{N}) \;\underset{\Phi^{\mathrm{r}}}{\overset{\Psi^{\mathrm{r}}}{\leftrightarrows}}\; \mathcal{R}ex_\mathcal{C}(\mathcal{M},\mathcal{N}).\end{aligned}\tag{2.17}$$

Here $\mathcal{M}^\#$ and ${}^\#\mathcal{M}$ are the right modules with underlying linear category $\mathcal{M}^{\mathrm{opp}}$ described above whereby, as already noted, $\mathcal{M}^\#\boxtimes\mathcal{N}$ and ${}^\#\mathcal{M}\boxtimes\mathcal{N}$ naturally become $\mathcal{C}$-bimodules, with associated Drinfeld centers $\mathcal{Z}(\mathcal{M}^\#\boxtimes\mathcal{N})$ and $\mathcal{Z}({}^\#\mathcal{M}\boxtimes\mathcal{N})$ as described in Section 2.1.

Remark 2. Any pivotal tensor category is equivalent, as a pivotal category, to a strict pivotal category [NgS, Thm. 2.2]. Thus in case the finite tensor category $\mathcal{C}$ of our interest has a pivotal structure, for many purposes we may replace it by a strict pivotal one in which $c^\vee = {}^\vee c$ holds for every $c\in\mathcal{C}$. When doing so, ${}^\#\mathcal{M}$ and $\mathcal{M}^\#$ are the same $\mathcal{C}$-module; we denote it by the symbol $\overline{\mathcal{M}}$. The equivalences (2.17) then combine to

$$\mathcal{L}ex_\mathcal{C}(\mathcal{M},\mathcal{N}) \;\underset{\Phi^{\mathrm{l}}}{\overset{\Psi^{\mathrm{l}}}{\rightleftarrows}}\; \mathcal{Z}(\overline{\mathcal{M}}\boxtimes\mathcal{N}) \;\underset{\Phi^{\mathrm{r}}}{\overset{\Psi^{\mathrm{r}}}{\leftrightarrows}}\; \mathcal{R}ex_\mathcal{C}(\mathcal{M},\mathcal{N}).\tag{2.18}$$

Furthermore, in this case the Nakayama functor $\mathrm{N}^{\mathrm{r}}_{\mathcal{M}}$ of $\mathcal{M}$ becomes an ordinary module functor, rather than a module functor twisted by a double dual.

Next we note that the Drinfeld center $\mathcal{Z}(\mathcal{C})$ of a rigid category $\mathcal{C}$ is rigid as well. Moreover, the components $\pi_c\colon c \to c^{\vee\vee}$ of a pivotal structure on $\mathcal{C}$ are even morphisms in $\mathcal{Z}(\mathcal{C})$. Thus if $\mathcal{C}$ is pivotal, then $\mathcal{Z}(\mathcal{C})$ inherits a distinguished pivotal structure [EGNO, Exc. 7.13.6]. For pivotal $\mathcal{C}$ we will consider $\mathcal{Z}(\mathcal{C})$ with this pivotal structure.

Exact module categories over a pivotal tensor category turn out to have a particularly interesting theory. Let us first recall

Definition 2 ([FSS1, Def. 4.22])**.** Let $\mathcal{M}$ be a left $\mathcal{C}$-module. A *right relative Serre functor* on $\mathcal{M}$ is an endofunctor $\mathrm{S}^{\mathrm{r}}_{\mathcal{M}}$ of $\mathcal{M}$ equipped with a family

$$\underline{\mathrm{Hom}}(m,n)^{\vee} \xrightarrow{\cong} \underline{\mathrm{Hom}}(n, \mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(m)) \tag{2.19}$$

of isomorphisms natural in $m, n \in \mathcal{M}$. A *left relative Serre functor* $\mathrm{S}^{\mathrm{l}}_{\mathcal{M}}$ on $\mathcal{M}$ comes analogously with a family

$${}^{\vee}\underline{\mathrm{Hom}}(m,n) \xrightarrow{\cong} \underline{\mathrm{Hom}}(\mathrm{S}^{\mathrm{l}}_{\mathcal{M}}(n), m) \tag{2.20}$$

of natural isomorphisms.

Remark 3. According to Theorem 4.26 of [FSS1] the Nakayama and relative Serre functors of an exact module $\mathcal{M}$ over a finite tensor category $\mathcal{C}$ are related by $\mathrm{N}^{\mathrm{l}}_{\mathcal{M}} \cong D_{\mathcal{C}} \,.\, \mathrm{S}^{\mathrm{l}}_{\mathcal{M}}$ and $\mathrm{N}^{\mathrm{r}}_{\mathcal{M}} \cong D_{\mathcal{C}}^{-1} \,.\, \mathrm{S}^{\mathrm{r}}_{\mathcal{M}}$. In particular the Nakayama and relative Serre functors coincide iff $\mathcal{C}$ is unimodular.

It is known [FSS1, Prop. 4.24] that a finite left $\mathcal{C}$-module admits a relative Serre functor if and only if it is an exact module category. In this case the relative Serre functor is an equivalence of categories. A right relative Serre functor on $\mathcal{M}$ is a twisted module functor [FSS1, Lem. 4.23] in the sense that there are coherent natural isomorphisms

$$\phi^{\mathrm{S}^{\mathrm{r}}}_{c,m}\colon\ \mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(c.m) \xrightarrow{\cong} c^{\vee\vee} .\, \mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(m). \tag{2.21}$$

Similarly there are coherent natural isomorphisms $\mathrm{S}^{\mathrm{l}}_{\mathcal{M}}(c.m) \cong {}^{\vee\vee}c \,.\, \mathrm{S}^{\mathrm{l}}_{\mathcal{M}}(m)$. These results allow one to give

Definition 3 ([Sc2, Def. 5.2] and [Sh4, Def. 3.11])**.** A *pivotal structure*, or *inner-product structure*, on an exact module category $\mathcal{M}$ over a pivotal finite tensor category $(\mathcal{C}, \pi)$ is an isomorphism $\pi^{\mathcal{M}}\colon \mathrm{Id}_{\mathcal{M}} \xrightarrow{\cong} \mathrm{S}^{\mathrm{r}}_{\mathcal{M}}$ of functors such that the equality $\phi^{\mathrm{S}^{\mathrm{r}}}_{c,m} \circ \pi^{\mathcal{M}}_{c.m} = \pi_c \,.\, \pi^{\mathcal{M}}_m$ of morphisms from $c \,.\, m$ to $c^{\vee\vee} .\, \mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(m)$ holds for every $c \in \mathcal{C}$ and every $m \in \mathcal{M}$.

In short, a pivotal structure is an isomorphism, as module functors, from the identity functor to the Serre functor, where the pivotal structure on $\mathcal{C}$ has been used to turn them into module functors of the same type. If the module category $\mathcal{M}$ is indecomposable, then the identity functor $\mathrm{Id}_{\mathcal{M}}$ is a simple object in the category of right exact module endofunctors. Thus Schur's lemma implies

Lemma 4 ([Sh4, Lem. 3.12]). *Let $\mathcal{M}$ be an indecomposable exact module category over a pivotal finite tensor category. A pivotal structure on $\mathcal{M}$, if it exists, is unique up to a scalar multiple.*

Remark 4. (i) As a special case of [FSS1, Lem. 4.23], consider a pivotal finite tensor category $\mathcal{C}$ as a module category over itself. We have

$$\mathrm{S}^{\mathrm{l}}_{\mathcal{C}}(c.\mathbf{1}) = {}^{\vee\vee}c\,.\,\mathrm{S}^{\mathrm{l}}_{\mathcal{C}}(\mathbf{1}) = {}^{\vee\vee}c,$$

so in this case $\mathrm{S}^{\mathrm{l}}_{\mathcal{C}}$ coincides with the bidual functor. The pivotal structure of $\mathcal{C}$ then endows the module category ${}_{\mathcal{C}}\mathcal{C}$ with the structure of a pivotal module category.
(ii) It follows [Sh4, Thm. 3.13] that for $\mathcal{M}$ an indecomposable pivotal exact $\mathcal{C}$-module, the finite tensor category $\mathcal{C}^{\star}_{\mathcal{M}}$ of right exact $\mathcal{C}$-module endofunctors is a pivotal tensor category.

We are now in a position to introduce further structure on an exact $\mathcal{C}$-module $\mathcal{M}$: Denote by $\underline{\mathrm{coev}}_{c,m} : c \to \underline{\mathrm{Hom}}(m, c.m)$ the unit of the adjunction $H_m \vdash \underline{\mathrm{Hom}}(m,-)$. Let S^{r} be a relative Serre functor on $\mathcal{M}$. The *internal trace* [Sh4, Def. 3.7] is the composition

$$\underline{\mathrm{tr}}_m : \underline{\mathrm{Hom}}(m, \mathrm{S}^{\mathrm{r}}(m)) \xrightarrow{\cong} \underline{\mathrm{Hom}}(m,m)^{\vee} \xrightarrow{\underline{\mathrm{coev}}^{\vee}} \mathbf{1}^{\vee} \cong \mathbf{1},$$

where the first isomorphism is a component of the inverse of the defining structural morphism of S^{r}. The internal trace is related to a non-degenerate pairing

$$\underline{\mathrm{Hom}}(n, \mathrm{S}^{\mathrm{r}}(m)) \otimes \underline{\mathrm{Hom}}(m,n) \to \mathbf{1};$$

indeed this pairing factors into the composition of the internal Hom and the internal trace. Based on these structures the following has been shown recently:

Theorem 5 ([Sh4, Thm. 3.15]). *Let $(\mathcal{M}, \pi^{\mathcal{M}})$ be a pivotal exact module category over a pivotal finite tensor category $(\mathcal{C}, \pi)$. Then the algebra $\underline{\mathrm{Hom}}(m,m)$ in $\mathcal{C}$ has the structure of a symmetric Frobenius algebra with Frobenius form*

$$\lambda_m : \underline{\mathrm{Hom}}(m,m) \xrightarrow{(\underline{\pi^{\mathcal{M}}_m})_*} \underline{\mathrm{Hom}}(m, \mathrm{S}^{\mathrm{r}}(m)) \xrightarrow{\underline{\mathrm{tr}}_m} \mathbf{1}$$

with $(\underline{\pi^{\mathcal{M}}_m})_* = \underline{\mathrm{Hom}}(m, \pi^{\mathcal{M}}_m)$, for any $m \in \mathcal{M}$.

3. Integrating bimodule functors to objects in the Drinfeld center

Recall from the paragraph before Lemma 3 the right $\mathcal{C}$-module structures ${}^{\#}\mathcal{M}$ and $\mathcal{M}^{\#}$ that are defined on the opposite category $\mathcal{M}^{\mathrm{opp}}$ of the abelian category underlying a $\mathcal{C}$-module $\mathcal{M}$. We have

Lemma 6. *Let $\mathcal{M}$ be a $\mathcal{C}$-module.*

(i) *Let $G \colon {}^{\#}\mathcal{M} \boxtimes \mathcal{M} \to \mathcal{C}$ be a bimodule functor. Then the end $\int_{m\in\mathcal{M}} G(m,m)$ has a natural structure of a comodule over the central comonad Z of $\mathcal{C}$, and can thus be seen as an object in the Drinfeld center $\mathcal{Z}(\mathcal{C})$.*

(ii) *Let $H \colon \mathcal{M}^{\#} \boxtimes \mathcal{M} \to \mathcal{C}$ be a bimodule functor. Then the coend $\int^{m\in\mathcal{M}} H(m,m)$ has a natural structure of a module over the central monad Z of $\mathcal{C}$, and can thus be seen as an object in the Drinfeld center $\mathcal{Z}(\mathcal{C})$.*

Proof. Abbreviate $\int_{m\in\mathcal{M}} G(m,m) =: g$. To obtain a candidate δ_g for the coaction of Σ on g we first concatenate the structure morphisms j^g of the end and the bimodule functor structure of G (which will be suppressed in the considerations below), which gives us a family

$$g \xrightarrow{j^g_{c.m}} G(c.m, c.m) \xrightarrow[\cong]{\phi^G} c\,.\,G(m,m)\,.\,c^\vee \tag{3.1}$$

of morphisms. Since j^g is dinatural, this constitutes a dinatural transformation from the object g to the functor

$$\begin{aligned} (\mathcal{C}\boxtimes\mathcal{M})^{\text{opp}} \boxtimes (\mathcal{C}\boxtimes\mathcal{M}) &\to \mathcal{C}, \\ \overline{(c\boxtimes m)} \boxtimes (c'\boxtimes m') &\mapsto c'\,.\,G(m,m')\,.\,c^\vee. \end{aligned}$$

Invoking the behavior of (co)ends over Deligne products [FSS1, Sect. 3.4] this, in turn, gives rise to a morphism

$$\begin{aligned} \delta_g: \quad g \longrightarrow \int_{c\boxtimes m\in\mathcal{C}\boxtimes\mathcal{M}} c\,.\,G(m,m)\,.\,c^\vee &\cong \int_{c\in\mathcal{C}}\int_{m\in\mathcal{M}} c\,.\,G(m,m)\,.\,c^\vee \\ &\cong \int_{c\in\mathcal{C}} c\,.\int_{m\in\mathcal{M}} G(m,m)\,.\,c^\vee \\ &= \Sigma(g), \end{aligned} \tag{3.2}$$

such that

$$(c\,.\,j^g_m\,.\,c^\vee) \circ \varsigma^g_c \circ \delta_g = \phi^G \circ j^g_{c.m} \colon\; g \to c\,.\,G(m,m)\,.\,c^\vee \tag{3.3}$$

with $\varsigma^x_c\colon \Sigma(x) \to c\otimes x\otimes c^\vee$ the structure morphism of the central comonad. Writing $\hat\delta_c := \varsigma_c \circ \delta_g$ for $c\in\mathcal{C}$, together with (3.1) this gives the commutative diagram

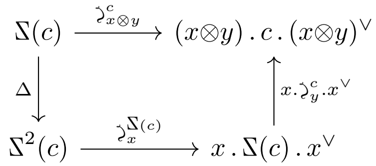
$$\begin{array}{ccccc} G(c.m,c.m) & & \xrightarrow{\quad\cong\quad} & & c\,.\,G(m,m)\,.\,c^\vee \\ & \nwarrow^{j^g_{c.m}} & & \nearrow^{c.j^g_m.c^\vee} & \\ & g & \xrightarrow{\hat\delta_c} & c\,.\,g\,.\,c^\vee & \\ & & \searrow_{\delta_g} \quad \nearrow_{\varsigma_c} & & \\ & & \Sigma(g) & & \end{array} \tag{3.4}$$

(We do not directly use this diagram here, but it will be instrumental in the proof of Lemma 7 below.)

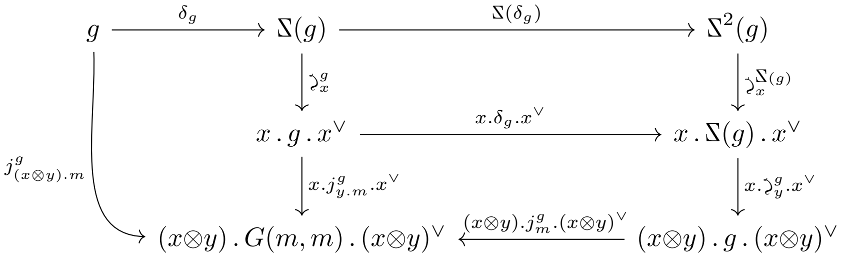
Now denote by $\Delta\colon \Sigma(c)\to\Sigma^2(c)$ the comultiplication of Σ. To see the coaction property of δ_g we compare two commutative diagrams. The first of these is

$$\begin{array}{ccccccc} g & \xrightarrow{\delta_g} & \Sigma(g) & \xrightarrow{\Delta} & \Sigma^2(g) & \xrightarrow{\varsigma^{\Sigma(g)}_x} & x\,.\,\Sigma(g)\,.\,x^\vee \\ & & & & & & \big\downarrow{\scriptstyle x.\varsigma^g_y.x^\vee} \\ & & \Sigma(g) & \xrightarrow{\qquad\varsigma^g_{x\otimes y}\qquad} & & & (x\otimes y)\,.\,g\,.\,(x\otimes y)^\vee \\ & & & & & & \big\downarrow{\scriptstyle (x\otimes y).j^g_m.(x\otimes y)^\vee} \\ g & & \xrightarrow{\qquad\qquad j^g_{(x\otimes y).m}\qquad\qquad} & & & & (x\otimes y)\,.\,G(m,m)\,.\,(x\otimes y)^\vee \end{array} \tag{3.5}$$

Here the upper square commutes by the defining property



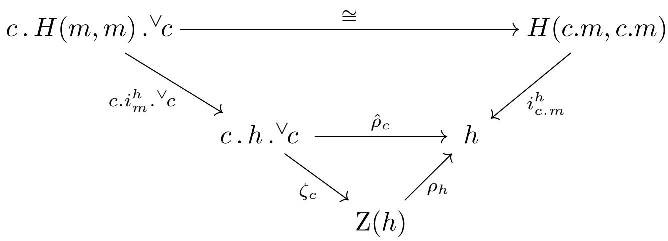
of the comultiplication, while the lower square commutes owing to the relation (3.3). The second diagram is



(3.6)

Here the left and the lower right square commute again due to (3.3), while the upper right square commutes by the definition of Σ. Comparing the outer hexagons of the diagrams (3.5) and (3.6) establishes the comodule property of the morphism δ_g and thus proves claim (i).

Claim (ii) follows by applying claim (i) to the opposite category. For instance, the commutative diagram analogous to (3.4) reads



(3.7)

where $\zeta_c^x\colon c\otimes x\otimes {}^\vee c \to \mathrm{Z}(x)$ is the structure morphism of the central monad and we have set $h := \int^{m\in\mathcal{M}} H(m,m)$ and $\hat\rho_c := \rho_h \circ \zeta_c$. $\square$

An analogous result holds for natural transformations:

Lemma 7. *Let $\mathcal{M}$ be a $\mathcal{C}$-module.*

(i) *Let $G_1, G_2\colon {}^\#\mathcal{M}\boxtimes\mathcal{M} \to \mathcal{C}$ be bimodule functors and $\nu\colon G_1 \to G_2$ be a bimodule natural transformation. Then the morphism*

$$\overline{\nu} := \int_{m\in\mathcal{M}} \nu_{m,m}\colon \int_{m\in\mathcal{M}} G_1(m,m) \to \int_{m\in\mathcal{M}} G_2(m,m)$$

in $\mathcal{C}$ that is induced by the functoriality of the end is even a morphism of Σ-comodules.

(ii) *Let $H_1, H_2\colon \mathcal{M}^{\#} \boxtimes \mathcal{M} \to \mathcal{C}$ be bimodule functors and $\nu\colon H_1 \to H_2$ be a bimodule natural transformation. Then the morphism*

$$\int^{m\in\mathcal{M}} \nu_{m,m}\colon \int^{m\in\mathcal{M}} H_1(m,m) \to \int^{m\in\mathcal{M}} H_2(m,m)$$

in $\mathcal{C}$ induced by the functoriality of the coend is even a morphism of Z-*modules.*

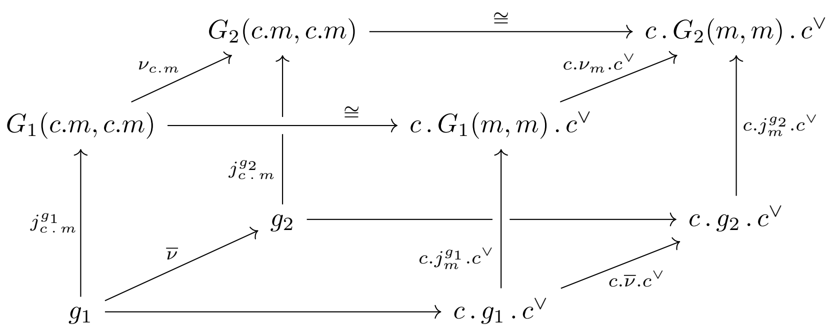
Proof. We prove claim (i); the proof of (ii) is dual. Consider two copies of the diagram (3.4), one for G_1 and one for G_2. We can connect the top lines of these two diagrams by the morphisms

$$\nu_{c.m,c.m}\colon G_1(c.m,c.m) \to G_2(c.m,c.m) \quad \text{and}$$
$$c\,.\,\nu_{m,m}\,.\,c^{\vee}\colon c\,.\,G_1(m,m)\,.\,c^{\vee} \to c\,.\,G_2(m,m)\,.\,c^{\vee}.$$

The resulting square commutes because ν is required to be a bimodule natural transformation. Similarly, we can connect the second line of the diagram for G_1 to the second line of the diagram for G_2 by the morphisms

$$\overline{\nu}\colon g_1 \to g_2 \quad \text{and}$$
$$c\,.\,\overline{\nu}\,.\,c^{\vee}\colon c\,.\,g_1\,.\,c^{\vee} \to c\,.\,g_2\,.\,c^{\vee}.$$

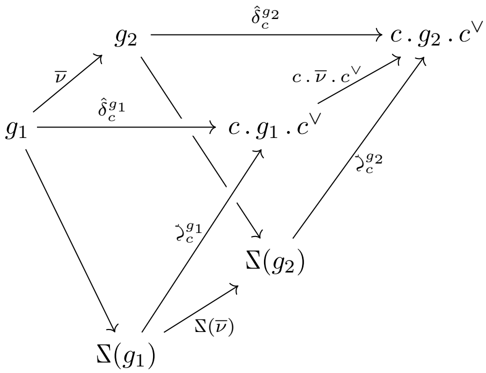
The two resulting squares that involve the dinatural structure morphisms for g_1 and g_2, respectively, commute, owing to the definition of $\overline{\nu}$ and the functoriality of the $\mathcal{C}$-actions. Thus in short, in the diagram



(3.8)

the left, right, front, back and top squares commute for every $m \in \mathcal{M}$. As a consequence, the square at the bottom of (3.8) commutes as well.

Proceeding in the same way as above, the lower triangles in the two diagrams of type (3.4) for G_1 and G_2 can be combined to



The two triangles in this diagram commute by construction, the top square is just the bottom square of (3.8), and the square on the right commutes by the functoriality of the end in the central comonad Σ. Thus the square on the left commutes as well. This is the desired result: it states that $\overline{\nu}$ is a morphism of Σ-comodules. $\square$

We combine Lemma 6 and Lemma 7 to

Theorem 8. *Let $\mathcal{C}$ be a finite tensor category and $\mathcal{M}$ be a $\mathcal{C}$-module. Then the assignments*

$$\int_\bullet : G \mapsto \int_{m\in\mathcal{M}} G(m,m) \quad \text{and} \quad \int^\bullet : H \mapsto \int^{m\in\mathcal{M}} H(m,m)$$

for $G \in \mathcal{F}un_{\mathcal{C}|\mathcal{C}}({}^{\#}\!\mathcal{M} \boxtimes \mathcal{M}, \mathcal{C})$ and $H \in \mathcal{F}un_{\mathcal{C}|\mathcal{C}}(\mathcal{M}^{\#} \boxtimes \mathcal{M}, \mathcal{C})$ provide functors

$$\begin{aligned} \int_\bullet :&\quad \mathcal{F}un_{\mathcal{C}|\mathcal{C}}({}^{\#}\!\mathcal{M} \boxtimes \mathcal{M}, \mathcal{C}) \to \mathcal{Z}(\mathcal{C}) \quad \text{and} \\ \int^\bullet :&\quad \mathcal{F}un_{\mathcal{C}|\mathcal{C}}(\mathcal{M}^{\#} \boxtimes \mathcal{M}, \mathcal{C}) \to \mathcal{Z}(\mathcal{C}), \end{aligned} \tag{3.9}$$

respectively.

We call the functors (3.9) the *central integration functors* for the $\mathcal{C}$-module $\mathcal{M}$.

4. Internal natural transformations

4.1. Definition

Let $\mathcal{C}$ be a finite tensor category and $\mathcal{M}$ and $\mathcal{N}$ be left $\mathcal{C}$-modules. It is well known that the category $\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ of right exact module functors is a finite category, and in fact has a natural structure of a finite $\mathcal{Z}(\mathcal{C})$-module as follows: For $(c_0,\beta_0) \in \mathcal{Z}(\mathcal{C})$ and $F \in \mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ the functor

$$\begin{aligned} \mathcal{M} &\to \mathcal{N}, \\ m &\mapsto c_0 \,.\, F(m) \end{aligned}$$

is again right exact. Also, it acquires the structure of a $\mathcal{C}$-module functor by the composition

$$\begin{aligned} c_0 \,.\, F(c\,.\,m) &\xrightarrow{\cong} \,.\,(c\,.\,F(m)) \xrightarrow{\cong} (c_0 \otimes c)\,.\,F(m) \\ &\xrightarrow[\cong]{(\beta_0)_c\,.\,F(m)} (c \otimes c_0)\,.\,F(m) \xrightarrow{\cong} c\,.\,(c_0\,.\,F(m)), \end{aligned} \tag{4.1}$$

where the first isomorphism is furnished by the module functor structure on F, and the second and forth use the mixed associativity constraint for $\mathcal{N}$. We write $(c_0, \beta_0)\,.\,F \in \mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ for the module functor obtained this way from the object $(c_0, \beta_0) \in \mathcal{Z}(\mathcal{C})$ and the functor $F \in \mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$. It is straightforward to check that this prescription endows the finite functor category $\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ with the structure of a module over the finite tensor category $\mathcal{Z}(\mathcal{C})$.

Definition 4. Let $\mathcal{C}$ be a finite tensor category and let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{C}$-modules. Endow the functor category $\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ with the structure of a module over the finite tensor category $\mathcal{Z}(\mathcal{C})$ as described above. Given $G, H \in \mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$, we call the internal Hom

$$\underline{\mathrm{Nat}}(G,H) := \underline{\mathrm{Hom}}_{\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})}(G,H) \;\in \mathcal{Z}(\mathcal{C}) \tag{4.2}$$

the object of *internal natural transformations* from G to H.

Dually we set

$$\underline{\mathrm{coNat}}(G,H) := \underline{\mathrm{coHom}}_{\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})}(G,H) \;\in \mathcal{Z}(\mathcal{C}). \tag{4.3}$$

Remark 5. The Yoneda lemma in the form of formula (2.3) allows one to express internal natural transformations as a coend:

$$\underline{\mathrm{Nat}}(F,G) \,=\, \int^{z\in\mathcal{Z}(\mathcal{C})} \mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}(z, \underline{\mathrm{Nat}}(F,G)) \otimes_{\Bbbk} z \,\in \mathcal{Z}(\mathcal{C}) \tag{4.4}$$

or, equivalently,

$$\underline{\mathrm{Nat}}(F,G) \,=\, \int^{z\in\mathcal{Z}(\mathcal{C})} \mathrm{Hom}_{\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})}(z.F, G) \otimes_{\Bbbk} z \,\in \mathcal{Z}(\mathcal{C}) \tag{4.5}$$

by the adjunction defining the internal Hom (4.2). We denote the dinatural structure morphisms of the coend (4.5) by

$$\begin{aligned} \imath_z^{F,G} \colon \mathrm{Hom}_{\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})}(z.F, G) \otimes_{\Bbbk} z& \\ \to \int^{z'\in\mathcal{Z}(\mathcal{C})} &\mathrm{Hom}_{\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})}(z'.F, G) \otimes_{\Bbbk} z'. \end{aligned} \tag{4.6}$$

4.2. Description as an end: component morphisms for relative natural transformations

An ordinary natural transformation is a family of morphisms. As a consequence the set of ordinary natural transformations between any two functors $G,H\colon \mathcal{C}\to\mathcal{D}$ can (for $\mathcal{C}$ essentially small) be expressed as an end:

$$\mathrm{Nat}(G,H) \;=\int_{c\in\mathcal{C}} \mathrm{Hom}_{\mathcal{D}}(G(c),H(c)). \tag{4.7}$$

The structure morphisms

$$j_c^{G,H}\colon \int_{c'\in\mathcal{C}} \mathrm{Hom}_{\mathcal{D}}(G(c'),H(c')) \to \mathrm{Hom}_{\mathcal{D}}(G(c),H(c)) \tag{4.8}$$

of this end are just the projections to the components of the natural transformation. As a special case, the natural transformations of the identity functor of a category $\mathcal{C}$ give the *center* $\mathrm{End}(\mathrm{Id}_{\mathcal{C}}) = \int_{c\in\mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(c,c)$ of $\mathcal{C}$. The latter provides a Morita invariant formulation of the center $Z(A)$ of an algebra A, according to the isomorphism $\mathrm{End}(\mathrm{Id}_{A\text{-mod}}) \cong Z(A)$.

We are now going to show that the results of the previous subsection allow us to express internal natural transformations as an end as well. We start by noticing that in situations which involve module categories, it can be rewarding to replace morphism sets (or rather, morphism spaces) by internal Homs — the relative Serre functors introduced in Definition 2 provide an illustrative example. It is thus natural to consider for any pair $G,H\colon \mathcal{M}\to\mathcal{N}$ of module functors the end

$$\underline{\mathrm{Nat}}'(G,H) := \int_{m\in\mathcal{M}} \underline{\mathrm{Hom}}_{\mathcal{N}}(G(m),H(m)). \tag{4.9}$$

We denote the members of the dinatural family for this end by

$$j_m^{G,H}\colon \int_{m'\in\mathcal{M}} \underline{\mathrm{Hom}}_{\mathcal{N}}(G(m'),H(m')) \to \underline{\mathrm{Hom}}_{\mathcal{N}}(G(m),H(m)). \tag{4.10}$$

Similarly to the situation for ordinary natural transformations, the morphism $j_m^{G,H}$ in $\mathcal{C}$ plays the role of projecting to the mth 'component' $\underline{\mathrm{Hom}}_{\mathcal{N}}(G(m),H(m))$ of the object $\underline{\mathrm{Nat}}'(G,H)$.

Dually, we have for $G,H\colon \mathcal{M}\to\mathcal{N}$ the coend

$$\underline{\mathrm{coNat}}'(G,H) = \int^{m\in\mathcal{M}} \underline{\mathrm{coHom}}_{\mathcal{N}}(G(m),H(m)). \tag{4.11}$$

We denote its dinatural family by

$$\widehat{\jmath}_m^{G,H}\colon \underline{\mathrm{coHom}}_{\mathcal{N}}(G(m),H(m)) \to \int^{m'\in\mathcal{M}} \underline{\mathrm{coHom}}_{\mathcal{N}}(G(m'),H(m')).$$

A crucial observation is now that, since by Lemma 3 the internal Hom and internal coHom are bimodule functors, by Theorem 8 we can (and will) regard the objects (4.9) and (4.11), for $G,H\in\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$, as objects in $\mathcal{Z}(\mathcal{C})$. (It should be appreciated, though, that the structure morphisms $j_m^{G,H}$ and $\widehat{\jmath}_m^{G,H}$ are morphisms in $\mathcal{C}$ and not in $\mathcal{Z}(\mathcal{C})$.) We are then ready to state

Theorem 9. *Let $\mathcal{C}$ be a finite tensor category and $\mathcal{M}$ and $\mathcal{N}$ be finite $\mathcal{C}$-modules.*

(i) *The end $\underline{\mathrm{Nat}}'(G,H) \in \mathcal{Z}(\mathcal{C})$ is canonically isomorphic to the internal natural transformations:*

$$\underline{\mathrm{Nat}}(F,G) = \int_{m\in\mathcal{M}} \underline{\mathrm{Hom}}(F(m),G(m)).$$

(ii) *Analogously, the internal coHom* (4.3) *is canonically isomorphic to a coend:*

$$\underline{\mathrm{coNat}}(F,G) = \int^{m\in\mathcal{M}} \underline{\mathrm{coHom}}(F(m),G(m)).$$

Proof. We prove (i); the proof of (ii) is dual. By the adjunction that defines the internal Hom $\underline{\mathrm{Nat}}(F,G)$, proving (i) is equivalent to showing the adjunction

$$\mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}(z, \underline{\mathrm{Nat}}'_{\mathcal{Z}(\mathcal{C})}(G,H)) \cong \mathrm{Hom}_{\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})}(z.G,H) \tag{4.12}$$

for $G,H \in \mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ and $z \in \mathcal{Z}(\mathcal{C})$. Writing $z=(c_0,\beta_0)$ with $c_0 \in \mathcal{C}$ and β_0 a half-braiding on c_0, the right-hand side of (4.12) can be described as follows. The data characterizing a module natural transformation from $z.G$ to H are a family

$$\text{(D)}: \qquad \big(\eta_m\colon\ c_0.G(m) \to H(m)\big)_{m\in\mathcal{M}}$$

of morphisms in $\mathcal{N}$, indexed by elements in $m \in \mathcal{M}$, and subject to two types of conditions:

(C1) Naturality: For every morphism $m \xrightarrow{f} m'$ in $\mathcal{M}$ the diagram

$$\begin{array}{ccc} c_0 \,.\, G(m) & \xrightarrow{\eta_m} & H(m) \\ {\scriptstyle c_0\,.\,G(f)}\big\downarrow & & \big\downarrow{\scriptstyle H(f)} \\ c_0 \,.\, G(m') & \xrightarrow{\eta_{m'}} & H(m') \end{array}$$

in $\mathcal{N}$ commutes.

(C2) Module natural transformation: With ϕ^H the datum turning H into a module functor, for every $c \in \mathcal{C}$ and $m \in \mathcal{M}$ the diagram

$$\begin{array}{ccc} c_0 \,.\, G(c\,.\,m) & \xrightarrow{\eta_{c.m}} & H(c\,.\,m) \\ {\scriptstyle \Phi_{c_0.G}}\big\downarrow & & \big\downarrow{\scriptstyle \phi^H} \\ (c\otimes c_0)\,.\,G(m) & \xrightarrow{c.\eta_m} & c\,.\,H(m) \end{array}$$

in $\mathcal{N}$ commutes, where the morphism $\Phi_{c_0.G}$ is the module functor datum ϕ^G for G, followed by a half-braiding.

An element of the morphism space on the left-hand side of (4.12) is a morphism

$$\widetilde{\eta}: \quad (c_0,\beta_0) \to \underline{\mathrm{Nat}}'_{\mathcal{Z}(\mathcal{C})}(G,H)$$

to an end in $\mathcal{C}$. After post-composing with the structure morphisms $j_m^{G,H}$ of that end, this amounts to a dinatural family of morphisms in $\mathcal{C}$, labeled by objects $m \in \mathcal{M}$. The data of this family are

$$(\mathrm{D}'): \qquad \big(\widetilde{\eta}_m := j_m^{G,H} \circ \widetilde{\eta}\colon\ c_0 \to \underline{\mathrm{Hom}}(G(m), H(m))\big)_{m \in \mathcal{M}},$$

and they are subject to two constraints:

(C1′) Dinaturality: For every morphism $m \xrightarrow{f} m'$ in $\mathcal{M}$ the diagram

$$\begin{array}{ccc} c_0 & \xrightarrow{\quad \widetilde{\eta}_m \quad} & \underline{\mathrm{Hom}}(G(m), H(m)) \\ {\scriptstyle \widetilde{\eta}_{m'}} \big\downarrow & & \big\downarrow {\scriptstyle \underline{\mathrm{Hom}}(G(m), H(f))} \\ \underline{\mathrm{Hom}}(G(m'), H(m')) & \xrightarrow{\underline{\mathrm{Hom}}(G(f), H(m'))} & \underline{\mathrm{Hom}}(G(m), H(m')) \end{array}$$

in $\mathcal{C}$ commutes.

(C2′) Compatibility with the half-braiding: For every $c \in \mathcal{C}$, the diagram

$$\begin{array}{ccc} c_0 \otimes c & \xrightarrow{\tilde{\eta}_{c_0} \otimes c} & \underline{\mathrm{Nat}}'(G, H) \otimes c \\ {\scriptstyle \beta_0^{-1}} \big\downarrow & & \big\downarrow \\ c \otimes c_0 & \xrightarrow{c \otimes \tilde{\eta}_{c_0}} & c \otimes \underline{\mathrm{Nat}}'(G, H) \end{array} \tag{4.13}$$

commutes, where the right downwards arrow is the component at c of the distinguished half-braiding on $\underline{\mathrm{Nat}}'(G, H) \in \mathcal{Z}(\mathcal{C})$.

To compare the two sides of (4.12), first notice that the adjunction defining the internal Hom for the $\mathcal{C}$-module $\mathcal{N}$ gives natural isomorphisms

$$\mathrm{Hom}_{\mathcal{N}}(c_0 \,.\, G(m), H(m)) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(c_0, \underline{\mathrm{Hom}}(G(m), H(m))) \tag{4.14}$$

for all $m \in \mathcal{M}$. This adjunction maps data of type (D) to data of type (D′). We are going to show that also the respective conditions on these data are mapped to each other.

Thus consider condition (C1), i.e., the equality $H(f) \circ \eta_m = \eta_{m'} \circ c_0 \,.\, G(f)$ of morphisms in $\mathrm{Hom}_{\mathcal{N}}(c_0 \,.\, G(m), H(m'))$ for all morphisms $m \xrightarrow{f} m'$. By definition of $\underline{\mathrm{Hom}}(G(m), H(f))$, the internal Hom adjunction maps the morphism $G(f) \circ \eta_m$ to $\underline{\mathrm{Hom}}(G(m), H(f)) \circ \widetilde{\eta}_m$ with $\widetilde{\eta}_m$ is the image of η_m under the adjunction (4.14). A similar argument applies to pre-composition, showing that $\eta_{m'} \circ c_0 \,.\, G(f)$ is mapped by (4.14) to $\underline{\mathrm{Hom}}(G(f), H(m') \circ \widetilde{\eta}_m$. Together it follows that indeed condition (C1) is mapped to condition (C1′), so that (C1) $\Leftrightarrow$ (C1′).

Next we pick an object $m \in \mathcal{M}$ and post-compose the two composite morphisms in the commuting diagram (C2′) with the canonical morphism $c \otimes j_m^{G,H}$ of the end, thereby obtaining morphisms in $\mathrm{Hom}_{\mathcal{C}}(c_0 \otimes c, c \otimes \underline{\mathrm{Hom}}(G(m), H(m))$. Now take the upper-right composite morphism in (4.13). We can use the right dual of c to consider equivalently a morphism

$$c_0 \xrightarrow{\widetilde{\eta}} \underline{\mathrm{Nat}}'(G,H) \to c \otimes \underline{\mathrm{Hom}}(G(m),H(m)) \otimes c^\vee . \tag{4.15}$$

Here the second morphism can be recognized as the one we used to get the structure of a comodule over the central comonad on $\underline{\mathrm{Nat}}'(G,H)$. Hence the morphism (4.15) can be written as

$$\begin{aligned} c_0 \xrightarrow{\widetilde{\eta}} \underline{\mathrm{Nat}}'(G,H) &\xrightarrow{j^{G,H}_{c.m}} \underline{\mathrm{Hom}}(G(c\,.\,m),H(c\,.\,m)) \\ &\xrightarrow{\underline{\mathrm{Hom}}((\phi^G)^{-1},\phi^H)} c \otimes \underline{\mathrm{Hom}}(G(m),H(m)) \otimes c^\vee \end{aligned}$$

(here we suppress the bimodule functor structure of $\underline{\mathrm{Hom}}$). By the definition of $\widetilde{\eta}_m$, this morphism is nothing but

$$\begin{aligned} c_0 \xrightarrow{\widetilde{\eta}_{c.m}} &\ \underline{\mathrm{Hom}}(G(c\,.\,m),H(c\,.\,m)) \\ &\xrightarrow{\underline{\mathrm{Hom}}((\phi^G)^{-1},\phi^H)} c \otimes \underline{\mathrm{Hom}}(G(m),H(m)) \otimes c^\vee . \end{aligned}$$

Under the internal-Hom adjunction this morphism is mapped to

$$c_0\,.\,(c\,.\,G(m)) \xrightarrow{c_0.(\phi^G)^{-1}} c_0\,.\,G(c\,.\,m) \xrightarrow{\eta_{c.m}} H(c\,.\,m) \xrightarrow{\phi^H} c\,.\,H(m). \tag{4.16}$$

Applying similar arguments to the lower-left composite morphism in (4.13) gives the morphism

$$\begin{aligned} c_0 \to c \otimes c_0 \otimes c^\vee &\xrightarrow{c\otimes\widetilde{\eta}\otimes c^\vee} c \otimes \underline{\mathrm{Nat}}'(G,H) \otimes c^\vee \\ &\to c \otimes \underline{\mathrm{Hom}}(G(m),H(m)) \otimes c^\vee , \end{aligned}$$

where the first morphism is obtained by combining the half-braiding β_0 with the coevaluation of c. This is nothing but

$$c_0 \to c \otimes c_0 \otimes c^\vee \xrightarrow{c\otimes\widetilde{\eta}_m\otimes c^\vee} c \otimes \underline{\mathrm{Hom}}(G(m),H(m)) \otimes c^\vee ,$$

and is thus under the internal-Hom adjunction mapped to the morphism

$$\begin{aligned} c_0\,.\,(c\,.\,G(m)) \cong (c_0 \otimes c)\,.\,G(m) &\xrightarrow{\beta_0.G(m)} (c \otimes c_0)\,.\,G(m) \\ &\xrightarrow{c.\eta_m} c\,.\,H(m). \end{aligned} \tag{4.17}$$

Recalling the definition of $\Phi_{c_0.G}$ in terms of ϕ^G and the half-braiding β_0, we see that equality of the morphisms (4.16) and (4.17) is precisely the commuting diagram (C2). Thus we have established also the equivalence (C2) $\Leftrightarrow$ (C2$'$). $\square$

It is instructive to express the situation considered in the proof of Theorem 9 schematically: We have a commuting diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}((c_0,\beta_0),\underline{\mathrm{Nat}}'_{\mathcal{Z}(\mathcal{C})}(F,G)) & \xrightarrow[\cong]{(4.12)} & \mathrm{Hom}_{\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})}((c_0,\beta_0).F,G) \\ \big\downarrow & & \big\downarrow \\ \prod_m \mathrm{Hom}_{\mathcal{C}}(c_0,\underline{\mathrm{Hom}}(F(m),G(m))) & \xrightarrow{\cong} & \prod_m \mathrm{Hom}_{\mathcal{N}}(c_0.F(m),G(m)) \end{array} \tag{4.18}$$

Here the left downwards arrow is post-composition by the structure morphisms of the end (4.9) i.e., maps f to $j_m^{F,G}\circ f$ for some $m\in\mathcal{M}$. The right downwards arrow comes from the fact that a natural transformation is a family of morphisms. The lower horizontal arrow is component-wise the internal Hom adjunction.

Example 1. For $\mathcal{N}=\mathcal{M}$ and $G=H=\mathrm{Id}_{\mathcal{M}}$, the object of internal natural transformations is

$$\mathrm{F}_{\mathcal{M}} := \underline{\mathrm{Nat}}(\mathrm{Id}_{\mathcal{M}},\mathrm{Id}_{\mathcal{M}}) = \int_{m\in\mathcal{M}} \underline{\mathrm{Hom}}_{\mathcal{M}}(m,m) \,\in \mathcal{Z}(\mathcal{C}).$$

In particular, for $\mathcal{C}$ as a module category over itself, this is

$$\mathrm{F}_{\mathcal{C}} = \int_{c\in\mathcal{C}} \underline{\mathrm{Hom}}_{\mathcal{C}}(c,c) = \int_{c\in\mathcal{C}} c\otimes c^\vee \,\in \mathcal{Z}(\mathcal{C})\,. \tag{4.19}$$

If $\mathcal{C}$ is semisimple, this is the object $\bigoplus_i x_i\otimes x_i^\vee$, with the summation being over the finitely many isomorphism classes of simple objects, and with half-braiding as given, e.g., in [BK, Thm. 2.3]. It is natural to refer to the object $\mathrm{F}_{\mathcal{M}}$ in $\mathcal{Z}(\mathcal{C})$ as the *center* of the $\mathcal{C}$-module $\mathcal{M}$.

Remark 6. Given a functor $F\in\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$, define a functor

$$\begin{aligned} L^F\colon \mathcal{Z}(\mathcal{C}) &\to \mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N}), \\ z &\mapsto z\,.\,F. \end{aligned}$$

In the special case that $\mathcal{M}=\mathcal{N}={}_{\mathcal{C}}\mathcal{C}$ is $\mathcal{C}$ seen as a module over itself, we have a canonical identification $\mathcal{R}ex_{\mathcal{C}}(\mathcal{C},\mathcal{C})\cong\mathcal{C}$, under which $L^{\mathrm{Id}_{\mathcal{C}}}:\mathcal{Z}(\mathcal{C})\to\mathcal{C}$ is the forgetful functor. It follows from the proof of Theorem 9 that the right adjoint of L^F is the functor

$$\begin{aligned} R^F\colon \mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N}) &\to \mathcal{Z}(\mathcal{C}), \\ G &\mapsto \textstyle\int_{m\in\mathcal{M}} \underline{\mathrm{Hom}}_{\mathcal{N}}(F(m),G(m)) = \underline{\mathrm{Nat}}(F,G). \end{aligned} \tag{4.20}$$

In the special case $\mathcal{M}=\mathcal{N}={}_{\mathcal{C}}\mathcal{C}$ as well as $F=\mathrm{Id}_{\mathcal{C}}$ the functor (4.20) is given by

$$c_0 \mapsto \int_{c\in\mathcal{C}} \underline{\mathrm{Hom}}(c,c\otimes c_0) \cong \int_{c\in\mathcal{C}} c\otimes c_0\otimes c^\vee,$$

where we use Remark 1 and the central comonad (2.2). In other words, the adjunction (4.12) can be regarded as a generalization of the adjunction that defines the central comonad.

An interesting consequence of Theorem 9 is obtained when combining it with the fact [Sc1] that, for $\mathcal{C}$ a finite tensor category and $\mathcal{M}$ a $\mathcal{C}$-module, there is an explicit braided equivalence

$$\theta_{\mathcal{M}} \colon \mathcal{Z}(\mathcal{C}) \to \mathcal{Z}(\mathcal{C}^\star_{\mathcal{M}}).$$

Let us compute the object $\theta_{\mathcal{M}}(\underline{\mathrm{Nat}}(F,G)) \in \mathcal{Z}(\mathcal{C}^\star_{\mathcal{M}}))$ for $F,G \in \mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{M})$. To this end we use the fact that a right exact functor between finite abelian categories admits a right adjoint, and that the right adjoint of the forgetful functor $U_{\mathcal{M}} \colon \mathcal{Z}(\mathcal{C}^\star_{\mathcal{M}}) \to \mathcal{C}^\star_{\mathcal{M}}$, which is exact, is the coinduction functor associated with the central comonad on $\mathcal{C}^\star_{\mathcal{M}}$.

Proposition 10. *Let $\mathcal{C}$ be a finite tensor category and $\mathcal{M}$ a $\mathcal{C}$-module. For $F,G \in \mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{M})$ we have*

$$\theta_{\mathcal{M}}(\underline{\mathrm{Nat}}(F,G)) \cong \widetilde{I}(G \circ F^{\mathrm{r.a.}}) \in \mathcal{Z}(\mathcal{C}^\star_{\mathcal{M}}),$$

where $F^{\mathrm{r.a.}}$ is the right adjoint of F and

$$\begin{aligned} \widetilde{I} \colon \mathcal{C}^\star_{\mathcal{M}} &\to \mathcal{Z}(\mathcal{C}^\star_{\mathcal{M}}), \\ \varphi &\mapsto \int_{\psi \in \mathcal{C}^\star_{\mathcal{M}}} \psi^{\mathrm{r.a.}} \circ \varphi \circ \psi \end{aligned}$$

is the right adjoint of the forgetful functor $U_{\mathcal{M}}$.

Proof. Application of the composition of $\theta_{\mathcal{M}}$ with the forgetful functor $U_{\mathcal{M}} \colon \mathcal{Z}(\mathcal{C}^\star_{\mathcal{M}}) \to \mathcal{C}^\star_{\mathcal{M}}$ to the object $(c_0,\beta_0) \in \mathcal{Z}(\mathcal{C})$ of the Drinfeld center gives the $\mathcal{C}$-module endofunctor $(c_0,\beta_0) \,.\, \mathrm{Id}_{\mathcal{M}}$, which as a functor is given by acting with c_0 and has a module functor structure given by β_0. Pre-composing with $F \in \mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{M})$ yields

$$(U_{\mathcal{M}} \circ \theta_{\mathcal{M}})(c_0,\beta_0) \circ F = L^F(c_0,\beta_0),$$

with L^F as introduced in Remark 6. By the adjunction in Remark 6 we thus have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}^\star_{\mathcal{M}}}((U_{\mathcal{M}} \circ \theta_{\mathcal{M}})(c_0,\beta_0) \circ F, G) &\cong \mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}((c_0,\beta_0), R^F(G)) \\ &= \mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}((c_0,\beta_0), \underline{\mathrm{Nat}}(F,G)) \\ &\cong \mathrm{Hom}_{\mathcal{Z}(\mathcal{C}^\star_{\mathcal{M}})}(\theta_{\mathcal{M}}(c_0,\beta_0), \theta_{\mathcal{M}}(\underline{\mathrm{Nat}}(F,G))), \end{aligned}$$

where the last isomorphism holds because $\theta_{\mathcal{M}}$ is an equivalence. It follows that for all $\varphi \in \mathcal{Z}(\mathcal{C}^\star_{\mathcal{M}})$ we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{Z}(\mathcal{C}^\star_{\mathcal{M}})}(\varphi, \theta_{\mathcal{M}}(\underline{\mathrm{Nat}}(F,G))) &\cong \mathrm{Hom}_{\mathcal{C}^\star_{\mathcal{M}}}(U_{\mathcal{M}}(\varphi) \circ F, G) \\ &\cong \mathrm{Hom}_{\mathcal{C}^\star_{\mathcal{M}}}(U_{\mathcal{M}}(\varphi), G \circ F^{\mathrm{r.a.}}) \\ &\cong \mathrm{Hom}_{\mathcal{Z}(\mathcal{C}^\star_{\mathcal{M}})}(\varphi, \widetilde{I}(G \circ F^{\mathrm{r.a.}})). \end{aligned}$$

Here the first isomorphism uses the definition (4.20) of R^F together with the adjunction we just derived (and again the fact that $\theta_{\mathcal{M}}$ is an equivalence). □

Remark 7. Specifically for the case that $F = G = \mathrm{Id}_{\mathcal{M}}$, we find that

$$\theta_{\mathcal{M}}(\underline{\mathrm{Nat}}(\mathrm{Id}_{\mathcal{M}}, \mathrm{Id}_{\mathcal{M}})) \cong \int_{\psi \in \mathcal{C}_{\mathcal{M}}^{\star}} \psi^{\mathrm{r.a.}} \circ \psi .$$

Comparing this formula with (4.19), we can rephrase this by saying that *after application of Schauenburg's equivalence $\theta_{\mathcal{M}}$, the center of any module category is diagonal.* (In the application to two-dimensional conformal field theory alluded to at the end of the Introduction, this implies that the bulk state space of any full conformal field theory becomes diagonal when regarded not as an object of $\mathcal{Z}(\mathcal{C})$, but as an object in the equivalent category $\mathcal{Z}(\mathcal{C}_{\mathcal{M}}^{\star})$.)

4.3. Compositions

Ordinary natural transformations can be composed horizontally as well as vertically. Both compositions are conveniently described component-wise. A *vertical* composition of *internal* natural transformations clearly exists, being just a particular instance of the multiplication of internal Homs. In this subsection we introduce in addition a *horizontal* composition of internal natural transformations. We also describe their vertical composition from a different perspective. As we will see, these compositions are again naturally formulated in terms of components. Indeed, the constructions can largely be performed in analogy with those for ordinary natural transformations, including an Eckmann–Hilton argument.

We start by observing that for the $\mathcal{Z}(\mathcal{C})$-module $\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ the natural evaluation $\underline{\mathrm{ev}}$ (2.9) of internal Homs, which is used to obtain their multiplication, is a natural transformation

$$\underline{\mathrm{ev}}_{F,G} \colon \underline{\mathrm{Nat}}(F,G) \,.\, F \to G \tag{4.21}$$

between module functors in $\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$. Under the defining adjunction isomorphism of the internal Hom, the evaluation $\underline{\mathrm{ev}}_{F,G}$ is induced from the identity morphism on $\underline{\mathrm{Nat}}(F,G)$. Owing to the isomorphism $\underline{\mathrm{Nat}} \cong \underline{\mathrm{Nat}}'$ the latter is a morphism whose codomain is an end, so that it can be described as a dinatural family $\big(\underline{\mathrm{Nat}}(F,G) \to \underline{\mathrm{Hom}}(F(m),G(m))\big)_{m\in\mathcal{M}}$, and this family is just the structure morphism of the end. Thus the components of the evaluation $\underline{\mathrm{ev}}_{F,G}$ are just the images in

$$\big(\underline{\mathrm{Nat}}(F,G).F(m) \to G(m)\big)_{m\in\mathcal{M}} \tag{4.22}$$

of the structure morphisms of the end under the internal Hom adjunction for the module category $\mathcal{M}$ over $\mathcal{C}$. The associative multiplication of internal Homs $\underline{\mathrm{Nat}}(-,-)$ is a family of morphisms

$$\underline{\mu}_{\mathrm{ver}} \equiv \underline{\mu}_{\mathrm{ver}}(G_1,G_2,G_3) \colon \underline{\mathrm{Nat}}(G_2,G_3) \otimes \underline{\mathrm{Nat}}(G_1,G_2) \longrightarrow \underline{\mathrm{Nat}}(G_1,G_3) \tag{4.23}$$

in $\mathcal{Z}(\mathcal{C})$ obeying the standard associativity condition, and as described in (2.11) we have units

$$\underline{\mathrm{id}}_F \in \mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}(\mathbf{1}_{\mathcal{Z}(\mathcal{C})}, \underline{\mathrm{Nat}}(F,F)). \tag{4.24}$$

Definition 5. Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{C}$-modules and $G_1, G_2, G_3 \in \mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$. The morphism $\underline{\mu}_{\mathrm{ver}}$ from $\underline{\mathrm{Nat}}(G_2,G_3) \otimes \underline{\mathrm{Nat}}(G_1,G_2)$ to $\underline{\mathrm{Nat}}(G_1,G_3)$ that is introduced in (4.23) is called the *vertical composition* of internal natural transformations.

The following result justifies this terminology:

Proposition 11. *Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{C}$-modules and $G_1, G_2, G_3 \in \mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$. Consider for $m \in \mathcal{M}$ the composition*

$$\begin{aligned} \alpha_m \colon & \underline{\mathrm{Nat}}(G_2,G_3) \otimes \underline{\mathrm{Nat}}(G_1,G_2) \\ & \xrightarrow{j_m^{G_2,G_3} \otimes j_m^{G_1,G_2}} \underline{\mathrm{Hom}}_{\mathcal{N}}(G_2(m),G_3(m)) \otimes \underline{\mathrm{Hom}}_{\mathcal{N}}(G_1(m),G_2(m)) \\ & \xrightarrow{\underline{\mu}_{G_1(m),G_2(m),G_3(m)}} \underline{\mathrm{Hom}}_{\mathcal{N}}(G_1(m),G_3(m)). \end{aligned} \tag{4.25}$$

We have

$$\alpha_m = j_m^{G_1,G_3} \circ \underline{\mu}_{\mathrm{ver}}$$

for any $m \in \mathcal{M}$.

Moreover, for any $F \in \mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ the unit $\underline{\mathrm{id}}_F \in \mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}(\mathbf{1}_{\mathcal{Z}(\mathcal{C})}, \underline{\mathrm{Nat}}(F,F))$ satisfies

$$j_m^{F,F} \circ \underline{\mathrm{id}}_F = \underline{\mathrm{id}}_{F(m)} \in \mathrm{Hom}_{\mathcal{C}}\big(\mathbf{1}, \underline{\mathrm{Hom}}_{\mathcal{N}}(F(m),F(m))\big).$$

for all $m \in \mathcal{M}$.

Put differently, when thinking about the structure morphisms $j_m^{G,G'}$ (4.10) of the end (4.9) as projections to components, the vertical composition $\underline{\mu}_{\mathrm{ver}}$ of internal natural transformations is reduced to the multiplication (2.10) of internal Homs in the same way as the vertical composition of ordinary natural transformations is reduced in components to the composition of ordinary Homs.

Proof. The image in $\mathrm{Hom}_{\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})}(\underline{\mathrm{Nat}}(G_2,G_3) \otimes \underline{\mathrm{Nat}}(G_1,G_2) \,.\, G_1, G_3)$ of $\underline{\mu}_{\mathrm{ver}}$ $(G_1,G_2,G_3) \in \mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}(\underline{\mathrm{Nat}}(G_2,G_3) \otimes \underline{\mathrm{Nat}}(G_1,G_2), \underline{\mathrm{Nat}}(G_1,G_3))$ under the adjunction is given, by the definition of composition of internal Homs, by the composite

$$\underline{\mathrm{Nat}}(G_2,G_3) \otimes \underline{\mathrm{Nat}}(G_1,G_2) \,.\, G_1 \xrightarrow{\mathrm{id} \otimes \underline{\mathrm{ev}}} \underline{\mathrm{Nat}}(G_2,G_3) \,.\, G_2 \xrightarrow{\underline{\mathrm{ev}}} G_3$$

(we suppress the mixed associator of the module category $\mathcal{N}$). The components of this module natural transformation are of the form appearing in the lower right corner of the diagram (4.18). On the other hand, the morphism α_m is of the type that appears in the lower left corner of that diagram. We must show that they are related by applying the internal-Hom adjunction to each component in the direct product. But this is indeed the case because, as noted above (see (4.22)), the evaluation is related to the structure maps of the end by the internal-Hom adjunction. Hence the claim follows.

The assertion about the unit $\underline{\mathrm{id}}_F$ follows directly from the definitions: By the adjunction (4.12) it is related to $\mathrm{id}_F \in \mathrm{Hom}_{\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})}(\mathbf{1} \,.\, F, F)$. Expressed in terms of the diagram (4.18), we have the identity morphism in the component $\mathrm{Hom}_{\mathcal{N}}(\mathbf{1} \,.\, F(m), F(m))$ of the natural isomorphism for every $m \in \mathcal{M}$. □

Next we note that a $\Bbbk$-linear category $\mathcal{D}$ can be seen as a module category over the monoidal category $\mathrm{vect}_\Bbbk$ of finite-dimensional $\Bbbk$-vector spaces. For a $\mathrm{vect}_\Bbbk$-module the Hom and internal Hom coincide, so that the internal-Hom adjunction gives, for each pair $d, d' \in \mathcal{D}$ of objects, a natural evaluation

$$\underline{\mathrm{ev}}^\Bbbk_{d,d'} : \mathrm{Hom}_\mathcal{D}(d,d') \otimes_\Bbbk d \to d'$$

as the image of the identity map under the linear isomorphism

$$\mathrm{Hom}_\mathcal{D}(\mathrm{Hom}_\mathcal{D}(d,d') \otimes_\Bbbk d, d') \xrightarrow{\cong} \mathrm{Hom}_\mathcal{D}(d,d')^* \otimes_\Bbbk \mathrm{Hom}_\mathcal{D}(d,d') .$$

It should be appreciated that the composition of $\mathrm{Hom}_\mathcal{D}$ as an internal Hom for $\mathcal{D}$ as a $\mathrm{vect}_\Bbbk$-module and the ordinary composition of morphisms in $\mathcal{D}$ coincide.

This observation allows us to give the following convenient description of the evaluation $\underline{\mathrm{ev}}_{F,G} : \underline{\mathrm{Nat}}(F,G) \,.\, F \to G$ in (4.21): Invoking from Remark 5 the expression (4.5) for $\underline{\mathrm{Nat}}(F,G)$, we get

$$\begin{aligned} \underline{\mathrm{ev}}_{F,G} = \int^{z\in\mathcal{Z}(\mathcal{C})} \underline{\mathrm{ev}}^\Bbbk_{z.F,G} : \; & \Big(\int^{z\in\mathcal{Z}(\mathcal{C})} \mathrm{Hom}_{\mathcal{R}ex_\mathcal{C}(\mathcal{M},\mathcal{N})}(z.F,G) \otimes_\Bbbk z \Big) \,.\, F \\ & = \int^{z\in\mathcal{Z}(\mathcal{C})} \mathrm{Hom}_{\mathcal{R}ex_\mathcal{C}(\mathcal{M},\mathcal{N})}(z.F,G) \otimes_\Bbbk z.F \to G . \end{aligned}$$

Here in the equality we use that the action functor is exact, and by $\int^{z\in\mathcal{Z}(\mathcal{C})} \underline{\mathrm{ev}}^\Bbbk_{z.F,G}$ we denote the morphism out of the coend $\int^{z\in\mathcal{Z}(\mathcal{C})} \mathrm{Hom}_{\mathcal{R}ex_\mathcal{C}(\mathcal{M},\mathcal{N})}(z.F,G) \otimes_\Bbbk z.F$ that is defined by the family $\big(\underline{\mathrm{ev}}^\Bbbk_{z.F,G} : \mathrm{Hom}_{\mathcal{R}ex_\mathcal{C}(\mathcal{M},\mathcal{N})}(z.F,G) \otimes_\Bbbk z.F \to G\big)_{z\in\mathcal{Z}(\mathcal{C})}$ which is dinatural in $z \in \mathcal{Z}(\mathcal{C})$.

Next we define modified vertical and horizontal compositions of relative natural transformations.

To set the stage for the horizontal composition, we formulate

Lemma 12. *Let $\mathcal{M}$ and $\mathcal{N}$ be module categories over a finite tensor category $\mathcal{C}$, and let $F, G \in \mathcal{R}ex_\mathcal{C}(\mathcal{M},\mathcal{N})$. Then for any pair z, z' of objects in $\mathcal{Z}(\mathcal{C})$ the module functor constraint of G induces an isomorphism*

$$(z \,.\, G) \circ (z' \,.\, F) \xrightarrow{\cong} (z \otimes_{\mathcal{Z}(\mathcal{C})} z') \,.\, (G \circ F)$$

of $\mathcal{C}$-module functors.

Proof. For any $m \in \mathcal{M}$ there is an isomorphism

$$(z \,.\, G) \circ (z' \,.\, F)(m) = z \,.\, G(z'.F(m)) \xrightarrow{\cong} (z \otimes_{\mathcal{Z}(\mathcal{C})} z') \,.\, (G \circ F)(m)$$

of functors. Due to the fact that the braiding is natural and thus compatible with the module functor datum ϕ^G, this is even an isomorphism of $\mathcal{C}$-module functors. □

This result allows us to give

Definition 6.

(i) Let $\mathcal{M}$ and $\mathcal{N}$ be finite module categories over a finite tensor category $\mathcal{C}$. For any triple $F, G, H \in \mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ the *modified vertical composition*

$$\widetilde{\mu_{\text{ver}}} : \text{Hom}_{\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})}(z.G, H) \otimes_{\Bbbk} \text{Hom}_{\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})}(z'.F, G) \\ \to \text{Hom}_{\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})}((z\otimes z') \,.\, F, H) \tag{4.26}$$

of natural transformations is defined by

$$\widetilde{\mu_{\text{ver}}}(\beta,\alpha) \colon (z\otimes z') \,.\, F \xrightarrow{\cong} z \,.\, (z' \,.\, F) \xrightarrow{z.\alpha} z \,.\, G \xrightarrow{\beta} H$$

for $z, z' \in \mathcal{Z}(\mathcal{C})$.

(ii) Let $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{M}_3$ be finite module categories over a finite tensor category $\mathcal{C}$. For $z, z' \in \mathcal{Z}(\mathcal{C})$, $F, F' \in \mathcal{R}ex_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ and $G, G' \in \mathcal{R}ex_{\mathcal{C}}(\mathcal{M}_2, \mathcal{M}_3)$ the *modified horizontal composition*

$$\widetilde{\mu_{\text{hor}}} : \text{Hom}_{\mathcal{R}ex_{\mathcal{C}}(\mathcal{M}_2,\mathcal{M}_3)}(z.G, G') \otimes_{\Bbbk} \text{Hom}_{\mathcal{R}ex_{\mathcal{C}}(\mathcal{M}_1,\mathcal{M}_2)}(z'.F, F') \\ \to \text{Hom}_{\mathcal{R}ex_{\mathcal{C}}(\mathcal{M}_1,\mathcal{M}_3)}((z\otimes z') \,.\, G\circ F, G'\circ F') \tag{4.27}$$

of natural transformations is defined to be the composition of the ordinary horizonal composition with the isomorphism given in Lemma 12.

Remark 8. Admittedly, the modified vertical composition $\widetilde{\mu_{\text{ver}}}$ looks somewhat unnatural. But it should be appreciated that by using the duality we have identifications

$$\psi_{R,R';w} : \text{Hom}_{\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})}(w.R, R') \xrightarrow{\cong} \text{Hom}_{\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})}(R, {}^{\vee}w.R').$$

Using these identification maps, the composition $\widetilde{\mu_{\text{ver}}}$ reduces to the ordinary vertical composition: $(\widetilde{\mu_{\text{ver}}})_{F,G,H} = \psi^{-1}_{z'.F,H;z} \circ (\underline{\mu}_{\text{ver}})_{z'.F,G,{}^{\vee}z.H} \circ (\psi_{G,H;z} \otimes \text{id})$.

Remark 9. In the special case $G = F$ and $z' = \mathbf{1}_{\mathcal{Z}(\mathcal{C})}$, the identity natural transformation in $\text{Hom}_{\mathcal{R}ex_{\mathcal{C}}(\mathcal{M}_1,\mathcal{M}_2)}(\mathbf{1}_{\mathcal{Z}(\mathcal{C})} \,.\, F, F)$ is a unit $\widetilde{\eta}^{\text{ver}}_F$ for the modified vertical composition $\widetilde{\mu_{\text{ver}}}$.

Similarly, for $\mathcal{M}_3 = \mathcal{M}_2$, $G = G' = \text{Id}_{\mathcal{M}_2}$ and $z = \mathbf{1}_{\mathcal{Z}(\mathcal{C})}$, the identity natural transformation in $\text{Hom}_{\mathcal{R}ex_{\mathcal{C}}(\mathcal{M}_2,\mathcal{M}_2)}(\mathbf{1}_{\mathcal{Z}(\mathcal{C})} \,.\, \text{Id}_{\mathcal{M}_2}, \text{Id}_{\mathcal{M}_2})$ is a unit $\widetilde{\eta}^{\text{hor}}_{\mathcal{M}_2}$ for the modified horizontal composition $\widetilde{\mu_{\text{hor}}}$. It should be appreciated that whenever both units are defined, they are the same, $\widetilde{\eta}^{\text{hor}}_{\mathcal{M}} = \widetilde{\eta}^{\text{ver}}_{\text{id}_{\mathcal{M}}}$.

Proposition 13. *Let $\mathcal{M}$ and $\mathcal{N}$ be finite module categories over a finite tensor category $\mathcal{C}$, and let $F, G, H \in \mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$.*

The vertical composition $\underline{\mu}_{\rm ver} \colon \underline{\rm Nat}(G,H) \otimes \underline{\rm Nat}(F,G) \to \underline{\rm Nat}(F,H)$ *defined in* (4.23) *is the morphism out of the coend*

$$\begin{aligned} &\underline{\rm Nat}(G,H) \otimes \underline{\rm Nat}(F,G) \\ &\quad = \int^{z\in\mathcal{Z}(\mathcal{C})} {\rm Hom}_{\mathcal{R}ex_\mathcal{C}(\mathcal{M},\mathcal{N})}(z.G,H) \otimes_\Bbbk z \\ &\qquad\qquad \otimes_{\mathcal{Z}(\mathcal{C})} \int^{z'\in\mathcal{Z}(\mathcal{C})} {\rm Hom}_{\mathcal{R}ex_\mathcal{C}(\mathcal{M},\mathcal{N})}(z'.F,G) \otimes_\Bbbk z' \\ &\quad \cong \int^{z,z'\in\mathcal{Z}(\mathcal{C})} {\rm Hom}_{\mathcal{R}ex_\mathcal{C}(\mathcal{M},\mathcal{N})}(z.G,H) \otimes_\Bbbk {\rm Hom}_{\mathcal{R}ex_\mathcal{C}(\mathcal{M},\mathcal{N})}(z'.F,G) \\ &\qquad\qquad \otimes_\Bbbk (z \otimes_{\mathcal{Z}(\mathcal{C})} z') \end{aligned} \tag{4.28}$$

that is given by the family

$$\begin{aligned} &{\rm Hom}_{\mathcal{R}ex_\mathcal{C}(\mathcal{M},\mathcal{N})}(z.G,H) \otimes_\Bbbk {\rm Hom}_{\mathcal{R}ex_\mathcal{C}(\mathcal{M},\mathcal{N})}(z'.F,G) \otimes_\Bbbk (z \otimes_{\mathcal{Z}(\mathcal{C})} z') \\ &\quad \xrightarrow{\widetilde{\mu_{\rm ver}}} {\rm Hom}_{\mathcal{R}ex_\mathcal{C}(\mathcal{M},\mathcal{N})}((z \otimes_{\mathcal{Z}(\mathcal{C})} z') \,.\, F,H) \otimes_\Bbbk (z \otimes_{\mathcal{Z}(\mathcal{C})} z') \\ &\quad \xrightarrow{\imath^{F,H}_{z\otimes z'}} \int^{\zeta\in\mathcal{Z}(\mathcal{C})} {\rm Hom}_{\mathcal{R}ex_\mathcal{C}(\mathcal{M},\mathcal{N})}(\zeta \,.\, F,H) \otimes_\Bbbk \zeta \;=\; \underline{\rm Nat}(F,H) \end{aligned} \tag{4.29}$$

of morphisms in $\mathcal{Z}(\mathcal{C})$ *(with* $\imath^{F,H}$ *the dinatural family defined in* (4.6)*) which is dinatural in* $z,z' \in \mathcal{Z}(\mathcal{C})$.

Proof. We write $\widehat{\mu}_{\rm ver}$ for the morphism out of the coend (4.28) that is defined by (4.29). We have to compare the morphisms

$$\big(\underline{\rm Nat}(G,H) \otimes \underline{\rm Nat}(F,G)\big) \,.\, F \xrightarrow{\widehat{\mu}_{\rm ver}.{\rm id}_F} \underline{\rm Nat}(F,H) \,.\, F \xrightarrow{\underline{\rm ev}} H \tag{4.30}$$

and

$$\underline{\rm Nat}(G,H) \,.\, \big(\underline{\rm Nat}(F,G) \,.\, F\big) \xrightarrow{{\rm id}\,.\,\underline{\rm ev}} \underline{\rm Nat}(G,H) \,.\, G \xrightarrow{\underline{\rm ev}} H. \tag{4.31}$$

Now the morphism (4.30) can be expressed in terms of the family

$$\begin{aligned} &{\rm Hom}_{\mathcal{R}ex_\mathcal{C}(\mathcal{M},\mathcal{N})}(z.G,H) \otimes_\Bbbk {\rm Hom}_{\mathcal{R}ex_\mathcal{C}(\mathcal{M},\mathcal{N})}(z'.F,G) \otimes_\Bbbk (z \otimes_{\mathcal{Z}(\mathcal{C})} z') \,.\, F \\ &\quad \xrightarrow{\widetilde{\mu_{\rm ver}}.{\rm id}_F} {\rm Hom}_{\mathcal{R}ex_\mathcal{C}(\mathcal{M},\mathcal{N})}((z \otimes_{\mathcal{Z}(\mathcal{C})} z') \,.\, F,H) \otimes_\Bbbk (z \otimes_{\mathcal{Z}(\mathcal{C})} z' \,.\, F) \\ &\quad \xrightarrow{{\rm ev}_{(z\otimes z').F,H}} H \end{aligned} \tag{4.32}$$

that is dinatural in $z,z' \in \mathcal{Z}(\mathcal{C})$, while the morphism (4.31) is described by the family

$$\begin{aligned} &{\rm Hom}_{\mathcal{R}ex_\mathcal{C}(\mathcal{M},\mathcal{N})}(z.G,H) \otimes_\Bbbk z \,\otimes_{\mathcal{Z}(\mathcal{C})} {\rm Hom}_{\mathcal{R}ex_\mathcal{C}(\mathcal{M},\mathcal{N})}(z'.F,G) \otimes_\Bbbk z' \,.\, F \\ &\quad \xrightarrow{{\rm id} \otimes \underline{\rm ev}_{z'.F,G}} {\rm Hom}_{\mathcal{R}ex_\mathcal{C}(\mathcal{M},\mathcal{N})}(z.G,H) \otimes_\Bbbk z \,.\, G \xrightarrow{\underline{\rm ev}_{z.G,H}} H. \end{aligned} \tag{4.33}$$

Since upon use of dualities, $\widetilde{\mu_{\rm ver}}$ boils down to the ordinary vertical composition (see Remark 8), the two composites (4.32) and (4.33) coincide. It thus follows that $\widehat{\mu}_{\rm ver}$ indeed describes the vertical composition $\underline{\mu}_{\rm ver}$ of internal natural transformations. □

Remark 10. We describe again the unit of the vertical composition. It is the morphism in $\mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}(\mathbf{1}_{\mathcal{Z}(\mathcal{C})}, \underline{\mathrm{Nat}}(F,F))$ that is given by the composite

$$\begin{aligned}\mathbf{1}_{\mathcal{Z}(\mathcal{C})} &\longrightarrow \mathrm{Hom}_{\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})}(\mathbf{1}_{\mathcal{Z}(\mathcal{C})} . F, F) \otimes_{\Bbbk} \mathbf{1}_{\mathcal{Z}(\mathcal{C})} \\ &\xrightarrow{\imath^{F,F}_{\mathbf{1}_{\mathcal{Z}(\mathcal{C})}}} \int^{z \in \mathcal{Z}(\mathcal{C})} \mathrm{Hom}_{\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})}(z . F, F) \otimes_{\Bbbk} z,\end{aligned}$$

where the first morphism is determined by the unit $\widetilde{\eta}^{\mathrm{ver}}_F \in \mathrm{Hom}_{\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})}(\mathbf{1} . F, F)$, while the second is the structure morphism of the coend.

We are now in a position to introduce also a *horizontal* composition of internal natural transformations:

Definition 7. Let $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{M}_3$ be finite module categories over a finite tensor category $\mathcal{C}$, and let $F, F' \in \mathcal{R}ex_{\mathcal{C}}(\mathcal{M}_1, \mathcal{M}_2)$ and $G, G' \in \mathcal{R}ex_{\mathcal{C}}(\mathcal{M}_2, \mathcal{M}_3)$. The *horizontal composition*

$$\underline{\mathrm{Nat}}_{\mathcal{R}ex_{\mathcal{C}}(\mathcal{M}_2,\mathcal{M}_3)}(G, G') \otimes \underline{\mathrm{Nat}}_{\mathcal{R}ex_{\mathcal{C}}(\mathcal{M}_1,\mathcal{M}_2)}(F, F') \longrightarrow \underline{\mathrm{Nat}}_{\mathcal{R}ex_{\mathcal{C}}(\mathcal{M}_1,\mathcal{M}_3)}(G \circ F, G' \circ F')$$

is the morphism $\underline{\mu}_{\mathrm{hor}}$ out of the coend

$$\begin{aligned}&\underline{\mathrm{Nat}}(G, G') \otimes \underline{\mathrm{Nat}}(F, F') \\ &\quad = \int^{z \in \mathcal{Z}(\mathcal{C})} \mathrm{Hom}_{\mathcal{R}ex_{\mathcal{C}}(\mathcal{M}_2,\mathcal{M}_3)}(z.G, G') \otimes_{\Bbbk} z \\ &\qquad\qquad \otimes_{\mathcal{Z}(\mathcal{C})} \int^{z' \in \mathcal{Z}(\mathcal{C})} \mathrm{Hom}_{\mathcal{R}ex_{\mathcal{C}}(\mathcal{M}_1,\mathcal{M}_2)}(z'.F, F') \otimes_{\Bbbk} z' \\ &\quad \cong \int^{z, z' \in \mathcal{Z}(\mathcal{C})} \mathrm{Hom}_{\mathcal{R}ex_{\mathcal{C}}(\mathcal{M}_2,\mathcal{M}_3)}(z.G, G') \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{R}ex_{\mathcal{C}}(\mathcal{M}_1,\mathcal{M}_2)}(z'.F, F') \\ &\qquad\qquad \otimes_{\Bbbk} (z \otimes_{\mathcal{Z}(\mathcal{C})} z')\end{aligned}$$

that is given by the family

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{R}ex_{\mathcal{C}}(\mathcal{M}_2,\mathcal{M}_3)}(z.G, G') \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{R}ex_{\mathcal{C}}(\mathcal{M}_1,\mathcal{M}_2)}(z'.F, F') \otimes_{\Bbbk} (z \otimes_{\mathcal{Z}(\mathcal{C})} z') \\ &\xrightarrow{\widetilde{\mu_{\mathrm{hor}}}} \mathrm{Hom}_{\mathcal{R}ex_{\mathcal{C}}(\mathcal{M}_1,\mathcal{M}_3)}\big((z \otimes_{\mathcal{Z}(\mathcal{C})} z') . (G \circ F), G' \circ F'\big) \otimes_{\Bbbk} (z \otimes_{\mathcal{Z}(\mathcal{C})} z') \\ &\xrightarrow{\imath^{G \circ F, G' \circ F'}_{z \otimes z'}} \int^{\zeta \in \mathcal{Z}(\mathcal{C})} \mathrm{Hom}_{\mathcal{R}ex_{\mathcal{C}}(\mathcal{M}_1,\mathcal{M}_3)}(\zeta . (G \circ F), G' \circ F') \otimes_{\Bbbk} \zeta \\ &\qquad = \underline{\mathrm{Nat}}(G \circ F, G' \circ F')\end{aligned} \tag{4.34}$$

of morphisms in $\mathcal{Z}(\mathcal{C})$, which is dinatural in $z, z' \in \mathcal{Z}(\mathcal{C})$.

The so defined horizontal composition $\underline{\mu}_{\mathrm{hor}}$ and the vertical composition $\underline{\mu}_{\mathrm{ver}}$ satisfy the Eckmann–Hilton property:

Proposition 14. *Let $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{M}_3$ be finite module categories over a finite tensor category $\mathcal{C}$. Then for any two triples $F,G,H \in \mathcal{R}ex_{\mathcal{C}}(\mathcal{M}_1,\mathcal{M}_2)$ and $F',G',H' \in \mathcal{R}ex_{\mathcal{C}}(\mathcal{M}_2,\mathcal{M}_3)$, the diagram*

$$\begin{array}{ccc}
\begin{array}{l}\underline{\mathrm{Nat}}(G',H') \otimes \underline{\mathrm{Nat}}(G,H) \\ \quad \otimes\, \underline{\mathrm{Nat}}(F',G') \otimes \underline{\mathrm{Nat}}(F,G)\end{array} & \longrightarrow & \begin{array}{l}\underline{\mathrm{Nat}}(G',H') \otimes \underline{\mathrm{Nat}}(F',G') \\ \quad \otimes\, \underline{\mathrm{Nat}}(G,H) \otimes \underline{\mathrm{Nat}}(F,G)\end{array} \\
\Big\downarrow {\scriptstyle \underline{\mu}_{\mathrm{hor}} \otimes \underline{\mu}_{\mathrm{hor}}} & & \Big\downarrow {\scriptstyle \underline{\mu}_{\mathrm{ver}} \otimes \underline{\mu}_{\mathrm{ver}}} \\
\underline{\mathrm{Nat}}(G' \circ G, H' \circ H) \otimes \underline{\mathrm{Nat}}(F' \circ F, G' \circ G) & & \underline{\mathrm{Nat}}(F',H') \otimes \underline{\mathrm{Nat}}(F,H) \\
\searrow {\scriptstyle \underline{\mu}_{\mathrm{ver}}} & & \swarrow {\scriptstyle \underline{\mu}_{\mathrm{hor}}} \\
& \underline{\mathrm{Nat}}(F' \circ F, H' \circ H) &
\end{array}$$

commutes. Here the horizontal arrow is the braiding in $\mathcal{Z}(\mathcal{C})$.

Proof. When describing all horizontal and vertical compositions in the diagram in terms of dinatural families analogous to (4.29) and (4.34), the statement follows directly from the standard properties of vertical and horizontal compositions of module natural transformations. □

As usual, the Eckmann–Hilton property allows one to derive commutativity.

Corollary 15. *Let $\mathcal{M}$ be a finite module category over a finite tensor category $\mathcal{C}$. Then the algebra $\underline{\mathrm{Nat}}(\mathrm{Id}_{\mathcal{M}}, \mathrm{Id}_{\mathcal{M}})$ in $\mathcal{Z}(\mathcal{C})$ is braided commutative.*

Proof. Put all six functors in Proposition 14 to be the identity functor of $\mathcal{M}$. Then comparison with Remark 9 tells us that the horizontal and vertical composition have the same unit. □

5. Pivotal module categories and Frobenius algebras

So far we have been dealing with finite tensor categories and their (bi)module categories. In terms of modular functors, these structures are naturally related to a *framed* modular functor [DSS], [FSS2]. To have a relation with an *oriented* modular functor, additional algebraic structure is required, in particular the finite tensor categories should come with a pivotal structure. As a consequence, the algebras arising as internal Ends will have the additional structure of a Frobenius algebra. In this section we study relative natural transformations in such a setting.

Thus we now suppose that $\mathcal{C}$ is a *pivotal* finite tensor category and that $\mathcal{M}$ is an *exact* $\mathcal{C}$-module. Without loss of generality we then further assume that $\mathcal{C}$ is strict pivotal, so that the Nakayama functor of $\mathcal{M}$ is an ordinary module functor and ${}^{\#}\mathcal{M} = \mathcal{M}^{\#} =: \overline{\mathcal{M}}$, see Remark 2. In this situation, the central integration functors $\int_\bullet$ and $\int^\bullet$ appearing in Theorem 8 are both functors from $\mathcal{F}un_{\mathcal{C}|\mathcal{C}}(\overline{\mathcal{M}} \boxtimes \mathcal{M}, \mathcal{C})$ to $\mathcal{Z}(\mathcal{C})$. We now show that they are actually related by the module Eilenberg–Watts equivalences (2.18):

Proposition 16. *Let $\mathcal{C}$ be a pivotal finite tensor category and $\mathcal{M}$ be an exact $\mathcal{C}$-module. Then the two functors $\int_\bullet, \int^\bullet \colon \mathcal{F}un_{\mathcal{C}|\mathcal{C}}(\overline{\mathcal{M}} \boxtimes \mathcal{M}, \mathcal{C}) \to \mathcal{Z}(\mathcal{C})$ satisfy*

$$\int^\bullet = \int_\bullet \circ \left(\mathrm{Id}_{\overline{\mathcal{M}}} \boxtimes \mathrm{N}^{\mathrm{r}}_{\mathcal{M}}\right)$$

with $\mathrm{N}^{\mathrm{r}}_{\mathcal{M}}$ the right exact Nakayama functor of $\mathcal{M}$.

Proof. Since $\mathcal{M}$ is exact, all module and bimodule functors with domain $\overline{\mathcal{M}} \boxtimes \mathcal{M}$ are exact. For $G \in \mathcal{F}un_{\mathcal{C}|\mathcal{C}}(\overline{\mathcal{M}} \boxtimes \mathcal{M}, \mathcal{C})$ we therefore have

$$\begin{aligned} \int_{m\in\mathcal{M}} G(\overline{m} \boxtimes m) \;&\cong\; G\big(\textstyle\int_{m\in\mathcal{M}} \overline{m} \boxtimes m\big) \quad \text{and} \\ \int^{m\in\mathcal{M}} G(\overline{m} \boxtimes m) \;&\cong\; G\big(\textstyle\int^{m\in\mathcal{M}} \overline{m} \boxtimes m\big) \end{aligned} \tag{5.1}$$

as isomorphisms of objects in $\mathcal{C}$. Now for any object $\mu \in \mathcal{Z}(\overline{\mathcal{M}} \boxtimes \mathcal{M})$ with underlying object $\dot{\mu} \in \overline{\mathcal{M}} \boxtimes \mathcal{M}$, the object $G(\dot{\mu})$ comes with a canonical half-braiding, given by $G(\dot{\mu}) \otimes c \cong G(\dot{\mu} \,.\, c) \cong G(c \,.\, \dot{\mu}) \cong c \otimes G(\dot{\mu})$ for $c \in \mathcal{C}$. Moreover, the objects $\int_{m\in\mathcal{M}} \overline{m} \boxtimes m$ and $\int^{m\in\mathcal{M}} \overline{m} \boxtimes m$ of $\mathcal{C}$ are endowed with natural balancings [FSS1, Cor. 4.3], i.e., they are in fact objects of $\mathcal{Z}(\overline{\mathcal{M}} \boxtimes \mathcal{M})$. It follows that the isomorphisms in (5.1) are even isomorphisms in $\mathcal{Z}(\mathcal{C})$. Analogous isomorphisms in $\mathcal{Z}(\mathcal{C})$ are valid when $\overline{m} \boxtimes m$ is replaced by $\overline{m} \boxtimes H(m)$ for any module endofuctor H. Hence it remains to show that there is an isomorphism

$$\int^{m\in\mathcal{M}} \overline{m} \boxtimes m \;\cong\; \int_{m\in\mathcal{M}} \overline{m} \boxtimes \mathrm{N}^{\mathrm{r}}_{\mathcal{M}}(m) \tag{5.2}$$

of objects in $\mathcal{Z}(\overline{\mathcal{M}} \boxtimes \mathcal{M})$. That this is indeed the case follows from the two-sided adjoint equivalences (2.18). Indeed, when considering the identity functor on $\mathcal{M}$ as a left exact functor with trivial module structure, one has $\Phi^{\mathrm{l}}(\mathrm{Id}_{\mathcal{M}}) = \int^{m\in\mathcal{M}} \overline{m} \boxtimes m \in \mathcal{Z}(\overline{\mathcal{M}} \boxtimes \mathcal{M})$; the isomorphism (5.2) is then obtained by recalling the definition (2.6) of the Nakayama functor together with the fact that the Eilenberg–Watts functors Φ^{r} and Ψ^{r} are quasi-inverses. □

Of particular interest to us is a statement that follows from Proposition 16 together with the following result (for which $\mathcal{C}$ is not required to be pivotal):

Lemma 17. *Let $\mathcal{C}$ be a finite tensor category and $\mathcal{M}$ and $\mathcal{N}$ be exact $\mathcal{C}$-module categories. Let $F, G \colon \mathcal{M} \to \mathcal{N}$ be module functors. Then there is an isomorphism*

$$\int^{m\in\mathcal{M}} \underline{\mathrm{coHom}}(F(m), G(m)) \;\cong\; \int^{m\in\mathcal{M}} \underline{\mathrm{Hom}}(\mathrm{S}^{\mathrm{l}}_{\mathcal{N}} \circ F(m), G(m)) \tag{5.3}$$

as objects in $\mathcal{Z}(\mathcal{C})$. In particular, the coends on both sides of this equality exist.

Proof. For any pair of module functors $F, G\colon \mathcal{M} \to \mathcal{N}$, the functors $\underline{\mathrm{coHom}}(F,G)$ and $\underline{\mathrm{Hom}}(\mathrm{S}^{\mathrm{l}}_{\mathcal{N}} \circ F, G)$ are bimodule functors from $\mathcal{M}^{\#} \boxtimes \mathcal{M}$ to $\mathcal{C}$. That they are actually bimodule functors follows from the properties of the internal Hom and coHom together with the twisted bimodule property of the left relative Serre functor [FSS1, Lem. 4.23] which is analogous to (2.21). By combining the defining property (2.20) of a left relative Serre functor and the relation (2.15) between internal Hom and coHom, we obtain, for any pair F, G of module functors, an isomorphism

$$\underline{\mathrm{coHom}}(F,G) \cong \underline{\mathrm{Hom}}(\mathrm{S}^{\mathrm{l}}_{\mathcal{N}} \circ F, G) \tag{5.4}$$

of bimodule functors. By Theorem 8, this isomorphism of functors implies an isomorphism of their coends (5.3) as objects in $\mathcal{Z}(\mathcal{C})$. □

Theorem 18. *Let $\mathcal{C}$ be a pivotal finite tensor category and $\mathcal{M}$ and $\mathcal{N}$ be exact $\mathcal{C}$-modules. Then the functor category $\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ is an exact module category over $\mathcal{Z}(\mathcal{C})$.*

Proof. Applying Proposition 16 to the bimodule functor $\underline{\mathrm{Hom}}(\mathrm{S}^{\mathrm{l}}_{\mathcal{N}} \circ F, G)$ we obtain an isomorphism

$$\int^{m\in\mathcal{M}} \underline{\mathrm{Hom}}(\mathrm{S}^{\mathrm{l}}_{\mathcal{N}} \circ F(m), G(m)) \;\cong \int_{m\in\mathcal{M}} \underline{\mathrm{Hom}}(\mathrm{S}^{\mathrm{l}}_{\mathcal{N}} \circ F(m), G \circ \mathrm{N}^{\mathrm{r}}_{\mathcal{M}}(m))$$

of objects in $\mathcal{Z}(\mathcal{C})$. The left and right Nakayama functors of a finite linear category form an adjoint pair [FSS1, Lem. 3.16]. Using that $\mathrm{S}^{\mathrm{l}}_{\mathcal{N}} = D^{-1} .\, \mathrm{N}^{\mathrm{l}}_{\mathcal{N}} = D^{\vee} .\, \mathrm{N}^{\mathrm{l}}_{\mathcal{N}}$ with D the distinguished invertible object of $\mathcal{C}$ (see Remark 3), it follows that there are isomorphisms

$$\begin{aligned}\underline{\mathrm{Hom}}(\mathrm{S}^{\mathrm{l}}_{\mathcal{N}} \circ F, G \circ \mathrm{N}^{\mathrm{r}}_{\mathcal{M}}) &\cong \underline{\mathrm{Hom}}(\mathrm{N}^{\mathrm{l}}_{\mathcal{N}} \circ F, D \,.\, G \circ \mathrm{N}^{\mathrm{r}}_{\mathcal{M}}) \\ &\cong \underline{\mathrm{Hom}}(F, \mathrm{N}^{\mathrm{r}}_{\mathcal{N}} \circ D \,.\, G \circ \mathrm{N}^{\mathrm{r}}_{\mathcal{M}})\end{aligned}$$

of functors. Combining this statement with Lemma 17 we find an isomorphism

$$\int^{m\in\mathcal{M}} \underline{\mathrm{coHom}}(F(m), G(m)) \cong \int_{m\in\mathcal{M}} \underline{\mathrm{Hom}}(F(m), \mathrm{N}^{\mathrm{r}}_{\mathcal{N}} \circ D \,.\, G \circ \mathrm{N}^{\mathrm{r}}_{\mathcal{M}}(m)) \tag{5.5}$$

as objects in $\mathcal{Z}(\mathcal{C})$. This means that the functor $\mathrm{N}^{\mathrm{r}}_{\mathcal{N}} \circ (D \,.\, -) \circ \mathrm{N}^{\mathrm{r}}_{\mathcal{M}}$ is a relative Serre functor for $\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$. By Proposition 4.24 of [FSS1], this shows in particular that $\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ is an exact module category over $\mathcal{Z}(\mathcal{C})$. □

We will use the notation

$$\mathrm{S}^{\mathrm{r}}_{\mathcal{R}ex} := \mathrm{N}^{\mathrm{r}}_{\mathcal{N}} \circ (D \,.\, -) \circ \mathrm{N}^{\mathrm{r}}_{\mathcal{M}} \tag{5.6}$$

for the relative Serre functor for $\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ obtained in the proof.

Remark 11. If $\mathcal{C}$ is unimodular, then $\mathrm{S}^{\mathrm{r}}_{\mathcal{R}ex} = \mathrm{N}^{\mathrm{r}}_{\mathcal{N}} \circ (-) \circ \mathrm{N}^{\mathrm{r}}_{\mathcal{M}}$ is the right Nakayama functor $\mathrm{N}^{\mathrm{r}}_{\mathcal{R}ex(\mathcal{M},\mathcal{N})}$ of the category $\mathcal{R}ex(\mathcal{M},\mathcal{N})$ of all functors from $\mathcal{M}$ to $\mathcal{N}$ [FSS1, Lem. 3.21].

It is instructive to check explicitly that the functor (5.6) satisfies the following two properties which befit the relative Serre functor of $\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ (taking, for perspicuity, $\mathcal{C}$ just to be pivotal, rather than strict pivotal):

(1) $\mathrm{S}^{\mathrm{r}}_{\mathcal{R}ex}$ is a twisted $\mathcal{Z}(\mathcal{C})$-module functor, i.e., there are coherent natural isomorphisms

$$\phi^{\mathcal{R}ex}_{z,F} : \mathrm{S}^{\mathrm{r}}_{\mathcal{R}ex}(z.F) \xrightarrow{\cong} z^{\vee\vee}.\,\mathrm{S}^{\mathrm{r}}_{\mathcal{R}ex}(F) \tag{5.7}$$

for $z \in \mathcal{Z}(\mathcal{C})$ and $F \in \mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$.

(2) $\mathrm{S}^{\mathrm{r}}_{\mathcal{R}ex}$ is an endofunctor of $\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$. That is, for $F \in \mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$, the functor $\mathrm{S}^{\mathrm{r}}_{\mathcal{R}ex}(F)$ is again in $\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$, i.e., is a (non-twisted) module functor: there are coherent natural isomorphisms

$$\mathrm{S}^{\mathrm{r}}_{\mathcal{R}ex}(F)(c\,.\,m) \xrightarrow{\cong} c\,.\,\big(\mathrm{S}^{\mathrm{r}}_{\mathcal{R}ex}(F)(m)\big)$$

for $c \in \mathcal{C}$ and $m \in \mathcal{M}$.

Let us first check that the functor $\mathrm{S}^{\mathrm{r}}_{\mathcal{R}ex}$ sends $\mathcal{C}$-module functors to $\mathcal{C}$-module functors. For $F \in \mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$, $c \in \mathcal{C}$ and $m \in \mathcal{M}$ we have

$$\begin{aligned}
\mathrm{S}^{\mathrm{r}}_{\mathcal{R}ex}(F)(c\,.\,m) \equiv\;& \mathrm{N}^{\mathrm{r}}_{\mathcal{N}} \circ (D\,.\,F) \circ \mathrm{N}^{\mathrm{r}}_{\mathcal{M}}(c\,.\,m) \\
&\xrightarrow{\cong} \mathrm{N}^{\mathrm{r}}_{\mathcal{N}} \circ (D\,.\,F)\big({}^{\vee\vee}c\,.\,\mathrm{N}^{\mathrm{r}}_{\mathcal{M}}(m)\big) \\
&\xrightarrow{\cong} \mathrm{N}^{\mathrm{r}}_{\mathcal{N}} \circ \big(D\,.\,({}^{\vee\vee}c\,.\,F \circ \mathrm{N}^{\mathrm{r}}_{\mathcal{M}}(m)\big) \\
&\xrightarrow{\cong} \mathrm{N}^{\mathrm{r}}_{\mathcal{N}}\big((D \otimes {}^{\vee\vee}c)\,.\,F \circ \mathrm{N}^{\mathrm{r}}_{\mathcal{M}}(m)\big) \\
&\xrightarrow{\cong} \mathrm{N}^{\mathrm{r}}_{\mathcal{N}}\big((c^{\vee\vee} \otimes D)\,.\,F \circ \mathrm{N}^{\mathrm{r}}_{\mathcal{M}}(m)\big) \\
&\xrightarrow{\cong} {}^{\vee\vee}c^{\vee\vee}\,.\,\big(\mathrm{N}^{\mathrm{r}}_{\mathcal{N}} \circ (D\,.\,F) \circ \mathrm{N}^{\mathrm{r}}_{\mathcal{M}}(m)\big) \\
&\;=\; c\,.\,\big(\mathrm{N}^{\mathrm{r}}_{\mathcal{N}} \circ (D\,.\,F) \circ \mathrm{N}^{\mathrm{r}}_{\mathcal{M}}(m)\big) \;\equiv\; c\,.\,\big(\mathrm{S}^{\mathrm{r}}_{\mathcal{R}ex}(F)(m)\big).
\end{aligned}$$

Here we first use that $\mathrm{N}^{\mathrm{r}}_{\mathcal{M}}$ is a twisted module functor, then that F is a module functor, then the module constraint, then the fact that by the Radford S^4-theorem the quadruple right dual of $\mathcal{C}$ is naturally isomorphic to conjugation by D, and finally that $\mathrm{N}^{\mathrm{r}}_{\mathcal{N}}$ is a twisted module functor. The same type of calculation, only one step shorter, shows that $\mathrm{S}^{\mathrm{r}}_{\mathcal{R}ex}$ is a properly twisted $\mathcal{Z}(\mathcal{C})$-module functor: For $F \in \mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ and $z \in \mathcal{Z}(\mathcal{C})$ we have (writing $z = (\dot z, \beta)$)

$$\begin{aligned}
\mathrm{S}^{\mathrm{r}}_{\mathcal{R}ex}(z\,.\,F) \equiv\;& \mathrm{N}^{\mathrm{r}}_{\mathcal{N}} \circ (D\,.\,(z\,.\,F)) \circ \mathrm{N}^{\mathrm{r}}_{\mathcal{M}} \xrightarrow{\cong} \mathrm{N}^{\mathrm{r}}_{\mathcal{N}} \circ \big((D \otimes \dot z)\,.\,F\big) \circ \mathrm{N}^{\mathrm{r}}_{\mathcal{M}} \\
&\xrightarrow{\cong} \mathrm{N}^{\mathrm{r}}_{\mathcal{N}} \circ (\dot z^{\vee\vee\vee\vee} \otimes D)\,.\,F) \circ \mathrm{N}^{\mathrm{r}}_{\mathcal{M}} \\
&\xrightarrow{\cong} \dot z^{\vee\vee}\,.\,\big(\mathrm{N}^{\mathrm{r}}_{\mathcal{N}} \circ (D\,.\,F) \circ \mathrm{N}^{\mathrm{r}}_{\mathcal{M}}\big) \;\equiv\; z^{\vee\vee}\,.\,\big(\mathrm{S}^{\mathrm{r}}_{\mathcal{R}ex}(F)\big).
\end{aligned}$$

So far, we have imposed the requirement of being pivotal on the finite tensor category $\mathcal{C}$. Now assume further that the $\mathcal{C}$-modules $\mathcal{M}$ and $\mathcal{N}$ are pivotal as well, with respective pivotal structures $\pi^{\mathcal{M}}$ and $\pi^{\mathcal{N}}$. The pivotal structure $\pi^{\mathcal{M}}$ gives a family of isomorphisms $\pi^{\mathcal{M}}_m \colon m \longrightarrow \mathrm{S}^{\mathrm{r}}_{\mathcal{M}}(m) = D\,.\,\mathrm{N}^{\mathrm{r}}_{\mathcal{M}}(m)$ which are twisted module natural transformations, and analogously for $\pi^{\mathcal{N}}$. In particular, for any functor $F \in \mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ and any object $m \in \mathcal{M}$ we can form the composite

$$\begin{aligned} F(m) &\xrightarrow{F(\pi^{\mathcal{M}}_m)} F(D\,.\,\mathrm{N}^{\mathrm{r}}_{\mathcal{M}}(m)) \\ &\xrightarrow{\pi^{\mathcal{N}}_{F(D.\mathrm{N}^{\mathrm{r}}_{\mathcal{M}}(m))}} D\,.\,\mathrm{N}^{\mathrm{r}}_{\mathcal{N}} \circ F \circ (D\,.\,\mathrm{N}^{\mathrm{r}}_{\mathcal{M}})(m) \\ &\cong D \otimes {}^{\vee\vee}D.\,\mathrm{N}^{\mathrm{r}}_{\mathcal{N}} \circ F \circ \mathrm{N}^{\mathrm{r}}_{\mathcal{M}}(m). \end{aligned}$$

In case $\mathcal{C}$ is *unimodular*, i.e., $D = \mathbf{1}$, this gives us a family of isomorphisms

$$\pi^{\mathcal{N}}_{F\circ\mathrm{N}^{\mathrm{r}}_{\mathcal{M}}(m)} \circ F(\pi^{\mathcal{M}}_m) \colon F(m) \xrightarrow{\cong} \mathrm{N}^{\mathrm{r}}_{\mathcal{N}} \circ F \circ \mathrm{N}^{\mathrm{r}}_{\mathcal{M}}(m) = \mathrm{S}^{\mathrm{r}}_{\mathcal{R}ex}(F)(m)$$

which provides an isomorphism

$$\mathrm{Id}_{\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})} \xrightarrow{\cong} \mathrm{S}^{\mathrm{r}}_{\mathcal{R}ex} \tag{5.8}$$

of endofunctors of $\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$. We then arrive at

Corollary 19. *Let $\mathcal{C}$ be a unimodular pivotal finite tensor category and $\mathcal{M}$ and $\mathcal{N}$ be pivotal module categories over $\mathcal{C}$. Then the functor category $\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ has a structure of a pivotal module category over the pivotal finite tensor category $\mathcal{Z}(\mathcal{C})$.*

Proof. By unimodularity of $\mathcal{C}$ we have $\mathrm{N}^{\mathrm{r}}_{\mathcal{M}} = \mathrm{S}^{\mathrm{r}}_{\mathcal{M}}$ and $\mathrm{N}^{\mathrm{r}}_{\mathcal{N}} = \mathrm{S}^{\mathrm{r}}_{\mathcal{N}}$. As in the proof of Proposition 16 let us take, without loss of generality, the pivotal structure of $\mathcal{C}$ to be strict. Then upon using the isomorphisms $\mathrm{S}^{\mathrm{r}}_{\mathcal{M}} \to \mathrm{Id}_{\mathcal{M}}$ and $\mathrm{S}^{\mathrm{r}}_{\mathcal{N}} \to \mathrm{Id}_{\mathcal{N}}$ that come from the pivotal structure on $\mathcal{M}$ and $\mathcal{N}$, respectively, to trivialize the relative Serre and thus Nakayama functors of $\mathcal{M}$ and $\mathcal{N}$, the consistency condition that according to Definition 3 has to be met in order for the isomorphism (5.8) to be a pivotal structure on $\mathcal{R}ex_{\mathcal{C}}(\mathcal{M},\mathcal{N})$ reduces to a combination of identities and is thus trivially satisfied. We refrain from spelling out the explicit form that the consistency condition takes after inserting the unique isomorphisms involved in those trivializations, as it does not add any further insight. □

By combining Corollary 19 with Theorem 5 we further get

Corollary 20. *Let $\mathcal{C}$ be a unimodular pivotal finite tensor category, let $\mathcal{M}$ and $\mathcal{N}$ be exact $\mathcal{C}$-modules with pivotal structures, and let $F \colon \mathcal{M} \to \mathcal{N}$ a module functor. Then the algebra $\underline{\mathrm{Nat}}(F,F)$ has a natural structure of a symmetric Frobenius algebra in the Drinfeld center $\mathcal{Z}(\mathcal{C})$. In particular, $\underline{\mathrm{Nat}}(\mathrm{Id}_{\mathcal{M}},\mathrm{Id}_{\mathcal{M}})$ has a natural structure of a commutative symmetric Frobenius algebra.*

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Funding Information Open Access funding provided by Karlstad University.