DIAGRAMMATIC CONSTRUCTION OF REPRESENTATIONS OF SMALL QUANTUM $\mathfrak{sl}_2$

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Abstract. We provide a combinatorial description of the monoidal category generated by the fundamental representation of the small quantum group of $\mathfrak{sl}_2$ at a root of unity q of odd order. Our approach is diagrammatic, and it relies on an extension of the Temperley–Lieb category specialized at $\delta = -q - q^{-1}$.

1. Introduction

The main goal of this paper is, very roughly speaking, to find a diagrammatic description of the category $\bar{U}$ -mod of finite-dimensional left $\bar{U}$ -modules, where $\bar{U} = \bar{U}_q \mathfrak{sl}_2$ is a finite-dimensional Hopf algebra called the *small quantum group of*

DOI: 10.1007/s00031-021-09670-z

*Supported by JSPS Postdoctoral Fellowship for Research in Japan ID PE18705.

Received December 13, 2019. Accepted May 21, 2021.

Published online September 30, 2021.

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$\mathfrak{sl}_2$. First introduced by Lusztig in [Lu90], $\bar{U}$ arises from a quantum deformation of the universal enveloping algebra of $\mathfrak{sl}_2$ with deformation parameter $q = e^{2\pi i/r}$, where $3 \leq r \in \mathbb{Z}$ is an odd integer called the *level*. The motivation behind this work is topological: indeed, $\bar{U}$ is a factorizable ribbon Hopf algebra, which means it can be used as an algebraic tool for building nonsemisimple TQFTs [DGP17]. At present, constructions rely either on the structure of Hopf algebras, or on some general categorical machinery [DGGPR19]. Although this theory produces interesting new phenomena, which differ strikingly from the semisimple case [DGGPR20], it exploits a rather elaborate technical setup that can require some time to fully digest. However, in the case of the small quantum group $\bar{U}$, we can look for a more combinatorial approach by figuring out an explicit description of a large category of $\bar{U}$ -modules in diagrammatic terms. This idea is pursued in [DM20], where a skein theoretic reformulation of the topological invariants of closed 3-manifold constructed from $\bar{U}$ is obtained, based on results provided here.

The starting point for our work is a diagrammatic description of the category of representations of a different Hopf algebra $U = U_q \mathfrak{sl}_2$ called the *full quantum group*, or the *divided power quantum group* of $\mathfrak{sl}_2$. This version of quantum $\mathfrak{sl}_2$ is infinite-dimensional, and its definition is also due to Lusztig. In fact, the small quantum group $\bar{U}$ was originally introduced as a finite-dimensional Hopf subalgebra of U . It was first shown in [MM92] that $\text{End}_U(X^{\otimes m})$ is isomorphic to the *mth Temperley–Lieb algebra* $\text{TL}(m)$ of [TL71] with parameter $\delta = -A^2 - A^{-2}$ specialized at $A = q^{(r+1)/2}$, where X is the fundamental simple U -module of dimension 2; see also [FK97], [AST15]. Since the small quantum group $\bar{U}$ is a Hopf subalgebra of U , we have a natural restriction functor from U -mod to $\bar{U}$ -mod. However, there exist indecomposable U -modules that become decomposable when considered as $\bar{U}$ -modules, and $\text{End}_{\bar{U}}(X^{\otimes m})$ is not isomorphic to $\text{TL}(m)$ in general.

The question of understanding $\text{End}_{\bar{U}}(X^{\otimes m})$ has been studied in greater detail in the case of even roots of unity, when the deformation parameter is $q = e^{2\pi i/r}$ for some even integer $4 \leq r \in \mathbb{Z}$. Indeed, in this case U contains a finite-dimensional Hopf subalgebra $\bar{U}$ that is called the *restricted quantum group* of $\mathfrak{sl}_2$. In [GST12], it is shown that an extension of the *mth Temperley–Lieb algebra* $\text{TL}(m)$, called the *lattice W -algebra*, is isomorphic to $\text{End}_{\bar{U}}(X^{\otimes m})$ for every $m \geq 0$ when $r = 4$, and it is conjectured that the isomorphism holds in general for every even $r \geq 4$. In [Mo17], generators and relations for a description of $\text{End}_{\bar{U}}(X^{\otimes m})$ in terms of planar diagrams are provided for every $m \geq 0$ and every even $r \geq 4$, and it is conjectured that these give a complete presentation. However, for even values of r , the Hopf algebra $\bar{U}$ is not ribbon, and although it admits a ribbon extension, the latter is not factorizable. In particular, the machinery of [DGP17] only produces quantum invariants of closed 3-manifolds, not TQFTs. This is why we turn our attention to the less-discussed odd level case. In order to do this, we consider an extension $\tilde{\text{TL}}$ of the Temperley–Lieb category TL , which is obtained by introducing four additional generating morphisms, as well as several relations between them. These additional generators correspond to the nontrivial splitting, in $\bar{U}$ -mod, of a certain indecomposable U -module. The definition of $\tilde{\text{TL}}$, which is given in Section 5, allows us to establish our main result.

Theorem 1. *There exists a full monoidal linear functor*

$$F_{\tilde{\mathrm{TL}}} : \tilde{\mathrm{TL}} \rightarrow \bar{U}\text{-mod}.$$

This allows us to describe in diagrammatic terms every morphism in the full monoidal subcategory of $\bar{U}\text{-mod}$ generated by X .

Structure of the paper

In Section 2, we recall the definition of the small quantum group $\bar{U}$ and the main properties of its category of finite-dimensional representations $\bar{U}\text{-mod}$. In particular, we collect definitions of simple and indecomposable projective $\bar{U}$ -modules, and we give explicit formulas for $\bar{U}$ -module morphisms between them. In Section 3, we recall the definition of the Temperley–Lieb category TL_A of indeterminate A , and within it we highlight a special family of idempotent endomorphisms $g_m \in \mathrm{TL}_A(m) = \mathrm{TL}_A(m, m)$ for $r \leq m \leq 2r - 2$. Under the specialization $A = q^{(r+1)/2}$, these planar diagrams, which were first considered in [GW93], extend the standard family of (*simple*) *Jones–Wenzl idempotents* $f_m \in \mathrm{TL}_A(m)$ of [Jo83], [We87], which are well defined for $0 \leq m \leq r - 1$. Since the endomorphisms g_m can be interpreted in terms of indecomposable projective $\bar{U}$ -modules, we refer to them as *nonsemisimple Jones–Wenzl idempotents*. Our explicit recursive formulas provide an odd level counterpart to the work of Ibanez [Ib15] and Moore [Mo18], who independently came up with similar ones for the restricted quantum group of $\mathfrak{sl}_2$ at even roots of unity. In Section 4, we recall the definition of a well-known monoidal linear functor $F_{\mathrm{TL}} : \mathrm{TL} \rightarrow \bar{U}\text{-mod}$, where TL denotes the category obtained from TL_A under the specialization $A = q^{(r+1)/2}$, and we explicitly compute the image of simple and nonsemisimple Jones–Wenzl idempotents. In doing so, we show that the functor F_{TL} is not full. In Section 5, we define the extended Temperley–Lieb category $\tilde{\mathrm{TL}}$ by introducing four additional generating morphisms to TL , and we extend F_{TL} to a functor $F_{\tilde{\mathrm{TL}}} : \tilde{\mathrm{TL}} \rightarrow \bar{U}\text{-mod}$. This allows us to prove our main result in Section 6. We do so in two steps. First, we show that the union of simple and nonsemisimple Jones–Wenzl idempotents dominates the category $\tilde{\mathrm{TL}}$. Then, we exhibit explicit morphisms that ensure fullness of $F_{\tilde{\mathrm{TL}}}$. In Section 7, we list additional relations satisfied by generating morphisms of $\tilde{\mathrm{TL}}$ in the quotient $\bar{\mathrm{TL}} = \tilde{\mathrm{TL}}/\ker F_{\tilde{\mathrm{TL}}}$, which will be useful in future works. It remains an open question to understand whether these relations provide a complete presentation of $\bar{\mathrm{TL}}$, in the sense that they generate the kernel of $F_{\tilde{\mathrm{TL}}}$. However, it seems likely that further relations, generalizing equation (57) to the case $r \leq m \leq 2r - 2$, will be needed.

Acknowledgments. This work was supported by JSPS Fellowship for Research in Japan, ID PE18705.

2. Small quantum group

In this section, we recall the definition, due to Lusztig, of the small quantum group of $\mathfrak{sl}_2$ at odd roots of unity. To start, let us fix an odd integer $3 \leq r \in \mathbb{Z}$ and let us consider the primitive r th root of unity $q = e^{2\pi i/r}$. For every integer $k \geq 0$

we introduce the notation

$$\{k\} := q^k - q^{-k}, \quad \{k\}' := q^k + q^{-k}, \quad [k] := \frac{\{k\}}{\{1\}}, \quad [k]! := \prod_{j=1}^k [j].$$

We denote with $\bar{U}$ the algebra over $\mathbb{C}$ with generators $\{E, F, K\}$ and relations

$$\begin{aligned} E^r = F^r = 0, \quad K^r = 1, \\ KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}. \end{aligned}$$

This algebra was first introduced in [Lu90, Sect. 6.5] under the notation $\tilde{\mathfrak{u}}$. We can make $\bar{U}$ into a Hopf algebra, called the *small quantum group of $\mathfrak{sl}_2$* , by setting

$$\begin{aligned} \Delta(E) &= E \otimes K + 1 \otimes E, & \varepsilon(E) &= 0, & S(E) &= -EK^{-1}, \\ \Delta(F) &= K^{-1} \otimes F + F \otimes 1, & \varepsilon(F) &= 0, & S(F) &= -KF, \\ \Delta(K) &= K \otimes K, & \varepsilon(K) &= 1, & S(K) &= K^{-1}. \end{aligned}$$

Lusztig actually considers the opposite coproduct, while we use the one of Kassel [Ka95, Sect. VII.1] and Majid [Ma95, Example 3.4.3]. The Hopf algebra $\bar{U}$ admits a ribbon structure [Ro93, Sect. 3.5] that is factorizable [Ly94, Cor. A.3.3], although we will need neither result in this paper. Moving on to the representation theory of $\bar{U}$, we recall the definition of simple and indecomposable projective $\bar{U}$ -modules. The reader can find more details in [Su94, Sect. 3], although our notation is closer to the one of [FGST05, Appendix C] and [Ar08, Sect. 4]. Note however that all these references actually focus on the restricted quantum group of $\mathfrak{sl}_2$, which corresponds to even values of r instead of odd ones. For every integer $0 \leq m \leq r-1$ we denote with X_m the simple $\bar{U}$ -module with basis

$$\{a_j^m \mid 0 \leq j \leq m\}$$

and action given, for all integers $0 \leq j \leq m$, by

$$\begin{aligned} K \cdot a_j^m &= q^{m-2j} a_j^m, \\ E \cdot a_j^m &= [j][m-j+1] a_{j-1}^m, \\ F \cdot a_j^m &= a_{j+1}^m, \end{aligned}$$

where $a_{-1}^m := a_{m+1}^m := 0$. As a $\bar{U}$ -module, X_m is generated by the *highest weight vector* a_0^m . Next, for every integer $r \leq m \leq 2r-2$ we denote with P_m the indecomposable projective $\bar{U}$ -module with basis

$$\{a_j^m, x_k^m, y_k^m, b_j^m \mid 0 \leq j \leq 2r-m-2, 0 \leq k \leq m-r\}$$

and action given, for all integers $0 \leq j \leq 2r-m-2$ and $0 \leq k \leq m-r$, by

$$\begin{aligned}
 K \cdot a_j^m &= q^{-m-2j-2} a_j^m, \\
 E \cdot a_j^m &= -[j][m+j+1] a_{j-1}^m, \\
 F \cdot a_j^m &= a_{j+1}^m, \\
 K \cdot x_k^m &= q^{m-2k} x_k^m, \\
 E \cdot x_k^m &= [k][m-k+1] x_{k-1}^m, \\
 F \cdot x_k^m &= \begin{cases} x_{k+1}^m & 0 \leq k < m-r, \\ a_0^m & k = m-r, \end{cases} \\
 K \cdot y_k^m &= q^{m-2k} y_k^m, \\
 E \cdot y_k^m &= \begin{cases} a_{2r-m-2}^m & k = 0, \\ [k][m-k+1] y_{k-1}^m & 0 < k \leq m-r, \end{cases} \\
 F \cdot y_k^m &= y_{k+1}^m, \\
 K \cdot b_j^m &= q^{-m-2j-2} b_j^m, \\
 E \cdot b_j^m &= \begin{cases} x_{m-r}^m & j = 0, \\ a_{j-1}^m - [j][m+j+1] b_{j-1}^m & 0 < j \leq 2r-m-2, \end{cases} \\
 F \cdot b_j^m &= \begin{cases} b_{j+1}^m & 0 \leq j < 2r-m-2, \\ y_0^m & j = 2r-m-2, \end{cases}
 \end{aligned}$$

where $a_{-1}^m := a_{2r-m-1}^m := x_{-1}^m := y_{m-r+1}^m := 0$. As a $\bar{U}$ -module, P_m is generated by the *dominant vector* b_0^m . Let us mention that P_m is usually denoted P_{2r-m-2} , because it is the projective cover of X_{2r-m-2} for every integer $r \leq m \leq 2r-2$. Our change of notation is motivated by later convenience. Next, we need to specify identifications for relevant submodules of iterated tensor products of the fundamental simple $\bar{U}$ -module $X := X_1$. Many details about decompositions of tensor products can be found in [Su94, Sect. 3], and an even more general treatment is given in [KS09, Sect. 3]. Again, these references discuss the restricted quantum group of $\mathfrak{sl}_2$, which corresponds to even values of r rather than odd ones, but we will not actually need any of their results, as all the formulas we will list in here can be easily checked by hand. For every integer $0 \leq m \leq r-1$, there is a unique isomorphic copy of X_m among the submodules of $X^{\otimes m}$, which we still denote X_m by abuse of notation. We construct it recursively. If $m = 0$, a standard basis of $X_0 \subset X^{\otimes 0}$ is obtained by setting $a_0^0 := 1$. If $0 < m \leq r-1$, a standard basis of $X_m \subset X^{\otimes m}$ is obtained by setting

$$a_j^m := F^j \cdot (a_0^{m-1} \otimes a_0^1). \quad (1)$$

Similarly, for every integer $r \leq m \leq 2r-2$, there is an isomorphic copy of P_m of multiplicity one among the submodules of $X^{\otimes m}$, which we still denote P_m by abuse of notation. However, this submodule is no longer unique (for instance, although $X_{r-1} \otimes X_1$ and $X_1 \otimes X_{r-1}$ are isomorphic, they are not the same submodule of

$X^{\otimes r}$). Therefore, we fix a choice recursively. If $m = r$, a standard basis of $P_r \subset X^{\otimes r}$ is obtained by setting

$$\begin{aligned} a_j^r &:= F^j \cdot (qa_0^{r-1} \otimes a_1^1 + a_1^{r-1} \otimes a_0^1), \\ x_0^r &:= a_0^{r-1} \otimes a_0^1, \\ b_j^r &:= F^j \cdot (a_0^{r-1} \otimes a_1^1), \\ y_0^r &:= a_{r-1}^{r-1} \otimes a_1^1. \end{aligned} \quad (2)$$

If $r < m \leq 2r - 2$, a standard basis of $P_m \subset X^{\otimes m}$ is obtained by setting

$$\begin{aligned} a_j^m &:= F^j \cdot ([m+1]qa_0^{m-1} \otimes a_1^1 + a_1^{m-1} \otimes a_0^1), \\ x_k^m &:= F^k \cdot (x_0^{m-1} \otimes a_0^1), \\ b_j^m &:= F^j \cdot \left(\frac{q^{-m}}{[m+1]} a_0^{m-1} \otimes a_1^1 + [m]qb_0^{m-1} \otimes a_1^1 + \frac{[m]}{[m+1]} b_1^{m-1} \otimes a_0^1 \right), \\ y_k^m &:= F^k \cdot \left(\frac{q^{-m}}{[m+1]} a_{2r-m-1}^{m-1} \otimes a_1^1 + \frac{[m]}{[m+1]} y_0^{m-1} \otimes a_0^1 \right). \end{aligned} \quad (3)$$

For $m = 2r + 1$, we denote with X_{r-1}^+ and X_{r-1}^- the two submodules of $X^{\otimes 2r-1}$ isomorphic to X_{r-1} with standard bases obtained by setting

$$\begin{aligned} a_j^{r-1,+} &:= F^j \cdot (x_0^{2r-2} \otimes a_0^1), \\ a_j^{r-1,-} &:= F^j \cdot (qa_0^{2r-2} \otimes a_1^1 - y_0^{2r-2} \otimes a_0^1). \end{aligned} \quad (4)$$

The direct sum $X_{2r-1} := X_{r-1}^+ \oplus X_{r-1}^-$ can be lifted to a simple representation of the divided power quantum group, and it is this nontrivial splitting, which will motivate the introduction of four additional generating morphisms in the Temperley–Lieb category. It will also be convenient to fix isomorphisms for supplements of these submodules. If $m = 2$, we denote with X'_0 the subspace of $X_1 \otimes X_1$ isomorphic to X_0 with standard basis obtained by setting

$$a_0'^0 := qa_0^1 \otimes a_1^1 - a_1^1 \otimes a_0^1. \quad (5)$$

If $2 < m \leq r - 1$, we denote with X'_{m-2} the subspace of $X_{m-1} \otimes X_1$ isomorphic to X_{m-2} with standard basis obtained by setting

$$a_j'^{m-2} := F^j \cdot ([m-1]qa_0^{m-1} \otimes a_1^1 - a_1^{m-1} \otimes a_0^1). \quad (6)$$

If $m = r + 1$, we denote with X'_{r-1}^+ and X'_{r-1}^- the submodules of $X_{r-1} \otimes X_1 \otimes X_1$ isomorphic to X_{r-1} with standard bases obtained by setting

$$\begin{aligned} a_j'^{r-1,+} &:= F^j \cdot (qa_0^{r-1} \otimes a_0^1 \otimes a_1^1 - a_0^{r-1} \otimes a_1^1 \otimes a_0^1), \\ a_j'^{r-1,-} &:= F^j \cdot (qa_0^{r-1} \otimes a_1^1 \otimes a_0^1 + a_1^{r-1} \otimes a_0^1 \otimes a_0^1). \end{aligned} \quad (7)$$

If $r \leq m \leq 2r - 1$, we denote with P'_{m-2} the subspace of $P_{m-1} \otimes X_1$ isomorphic to P_{m-2} with standard basis obtained by setting

$$\begin{aligned} a_j^{m-2} &:= F^j \cdot (-a_0^{m-1} \otimes a_0^1), \\ x_k^{m-2} &:= F^k \cdot ([m-1]qx_0^{m-1} \otimes a_1^1 - x_1^{m-1} \otimes a_0^1), \\ b_j^{m-2} &:= F^j \cdot \left(\frac{q^m}{[m-1]}x_{m-r-1}^{m-1} \otimes a_1^1 - \frac{[m]}{[m-1]}b_0^{m-1} \otimes a_0^1 \right), \\ y_k^{m-2} &:= F^k \cdot \left([m]qy_0^{m-1} \otimes a_1^1 - \frac{[m]}{[m-1]}y_1^{m-1} \otimes a_0^1 \right). \end{aligned} \quad (8)$$

Finally, let us fix our notation for $\bar{U}$ -module morphisms. Dimensions of nonzero vector spaces of morphisms between simple and indecomposable projective objects of $\bar{U}$ -mod are given by

$$\begin{aligned} \dim_{\mathbb{C}} \text{End}_{\bar{U}}(X_m) &= 1 & 0 \leq m \leq r-1, \\ \dim_{\mathbb{C}} \text{End}_{\bar{U}}(P_m) &= 2 & r \leq m \leq 2r-2, \\ \dim_{\mathbb{C}} \text{End}_{\bar{U}}(X_{2r-1}) &= 4, \\ \dim_{\mathbb{C}} \text{Hom}_{\bar{U}}(P_m, X_{2r-m-1}) &= \dim_{\mathbb{C}} \text{Hom}_{\bar{U}}(X_{2r-m-1}, P_m) = 1 & r \leq m \leq 2r-2, \\ \dim_{\mathbb{C}} \text{Hom}_{\bar{U}}(P_m, P_{3r-m-2}) &= 2 & r \leq m \leq 2r-2, \\ \dim_{\mathbb{C}} \text{Hom}_{\bar{U}}(X_{r-1}, X_{2r-1}) &= \dim_{\mathbb{C}} \text{Hom}_{\bar{U}}(X_{2r-1}, X_{r-1}) = 2. \end{aligned}$$

Nontrivial morphisms between simple and indecomposable projective objects of $\bar{U}$ -mod are:

- $\varepsilon_m \in \text{End}_{\bar{U}}(P_m)$ determined by

$$\varepsilon_m(a_j^m) := \varepsilon_m(x_k^m) := \varepsilon_m(y_k^m) := 0, \quad \varepsilon_m(b_j^m) := a_j^m \quad (9)$$

for all integers $r \leq m \leq 2r-2$, $0 \leq j \leq 2r-m-2$, and $0 \leq k \leq m-r$;

- $\pi_m \in \text{Hom}_{\bar{U}}(P_m, X_{2r-m-2})$ determined by

$$\pi_m(a_j^m) := \pi_m(x_k^m) := \pi_m(y_k^m) := 0, \quad \pi_m(b_j^m) := a_j^{2r-m-2} \quad (10)$$

for all integers $r \leq m \leq 2r-2$, $0 \leq j \leq 2r-m-2$, and $0 \leq k \leq m-r$;

- $\iota_m \in \text{Hom}_{\bar{U}}(X_{2r-m-2}, P_m)$ determined by

$$\iota_m(a_j^{2r-m-2}) := a_j^m \quad (11)$$

for all integers $r \leq m \leq 2r-2$ and $0 \leq j \leq 2r-m-2$;

- $\gamma_m^+ \in \text{Hom}_{\bar{U}}(P_m, P_{3r-m-2})$ determined by

$$\gamma_m^+(a_j^m) := \gamma_m^+(x_k^m) := 0, \quad \gamma_m^+(y_k^m) := a_k^{3r-m-2}, \quad \gamma_m^+(b_j^m) := x_j^{3r-m-2} \quad (12)$$

for all integers $r \leq m \leq 2r-2$, $0 \leq j \leq 2r-m-2$, and $0 \leq k \leq m-r$;

- $\gamma_m^- \in \text{Hom}_{\bar{U}}(P_m, P_{3r-m-2})$ determined by

$$\gamma_m^-(a_j^m) := \gamma_m^-(y_k^m) := 0, \quad \gamma_m^-(x_k^m) := a_k^{3r-m-2}, \quad \gamma_m^-(b_j^m) := y_j^{3r-m-2} \quad (13)$$

- for all integers $r \leq m \leq 2r - 2$, $0 \leq j \leq 2r - m - 2$, and $0 \leq k \leq m - r$;
- $\pi_{2r-1}^\varepsilon \in \text{Hom}_{\bar{U}}(X_{2r-1}, X_{r-1})$ determined by

$$\pi_{2r-1}^\varepsilon(a_j^{r-1, \varepsilon'}) := \delta_{\varepsilon, \varepsilon'} a_j^{r-1} \quad (14)$$

- for all $\varepsilon, \varepsilon' \in \{+, -\}$ and every integer $0 \leq j \leq r - 1$;
- $\iota_{2r-1}^\varepsilon \in \text{Hom}_{\bar{U}}(X_{r-1}, X_{2r-1})$ determined by

$$\iota_{2r-1}^\varepsilon(a_j^{r-1}) := a_j^{r-1, \varepsilon} \quad (15)$$

for every $\varepsilon \in \{+, -\}$ and every integer $0 \leq j \leq r - 1$.

These morphisms satisfy the following relations:

$$\varepsilon_m = \iota_m \pi_m = \gamma_{3r-m-2}^+ \gamma_m^- = \gamma_{3r-m-2}^- \gamma_m^+, \quad (16)$$

$$\varepsilon_m^2 = \gamma_{3r-m-2}^+ \gamma_m^+ = \gamma_{3r-m-2}^- \gamma_m^- = 0, \quad (17)$$

$$\pi_{2r-1}^{\varepsilon'} \iota_{2r-1}^\varepsilon = \delta_{\varepsilon, \varepsilon'} \text{id}_{X_{r-1}}, \quad (18)$$

$$\sum_{\varepsilon, \varepsilon'} \iota_{2r-1}^\varepsilon \pi_{2r-1}^\varepsilon = \text{id}_{X_{2r-1}}. \quad (19)$$

3. Temperley–Lieb category

In this section, we discuss a generalization of Jones–Wenzl idempotents at odd roots of unity. Nonsemisimple Temperley–Lieb algebras at roots of unity and evaluable idempotents were first studied by Goodman and Wenzl in [GW93]. Recursive formulas for these idempotents already appeared in the work of Ibanez [Ib15] and Moore [Mo18], who independently treated the case of even roots of unity, which is complementary to ours.

For an indeterminate A and for every integer $k \geq 0$ we introduce the notation

$$\{k\}_A := A^{2k} - A^{-2k}, \quad \{k\}'_A := A^{2k} + A^{-2k}, \quad [k]_A := \frac{\{k\}_A}{\{1\}_A}, \quad [k]_A! := \prod_{j=1}^k [j]_A.$$

The *Temperley–Lieb category* TL_A is the $\mathbb{C}(A)$ -linear category with set of objects given by $\mathbb{N}$, and with vector space of morphisms from $m \in \text{TL}_A$ to $m' \in \text{TL}_A$ denoted $\text{TL}_A(m, m')$ and linearly generated by planar (m, m') -tangles modulo the subspace linearly generated by planar (m, m') -tangles of the form

$$u \cup t + [2]_A \cdot t,$$

where t is a planar (m, m') -tangle, and where u is the unknot. Composition in TL_A is defined by vertical gluing, and denoted like multiplication, while tensor product is defined by horizontal juxtaposition. Equipped with this monoidal structure, it is easy to see that TL_A is a rigid category: for every $m \in \text{TL}_A$ the dual $m^* \in \text{TL}_A$ is given by m itself, while for every $t \in \text{TL}_A(m, m')$ the dual $t^* \in \text{TL}_A(m', m)$ is obtained by applying a rotation of angle π to t . For every $m \in \text{TL}_A$, the m th *Temperley–Lieb algebra* is the $\mathbb{C}(A)$ -algebra $\text{TL}_A(m) := \text{TL}_A(m, m)$.

Let us recall now the definition of *simple Jones–Wenzl idempotents*. These endomorphisms of TL_A were first discovered by Jones [Jo83], although the recursive definition given here is due to Wenzl [We87]; see also Lickorish [Li97]. For every integer $m \geq 0$ the m th simple Jones–Wenzl idempotent $f_m \in \mathrm{TL}_A(m)$ is recursively defined as

$$f_0 := \boxed{f_1} := \left| \begin{array}{c} \cdots \\ \cdots \end{array} \right| \quad \boxed{f_m} := \boxed{f_{m-1}} + \frac{[m-1]_A}{[m]_A} \cdot \left| \begin{array}{c} \cdots \\ \cdots \\ \boxed{f_{m-1}} \\ \cdots \\ \boxed{f_{m-1}} \\ \cdots \end{array} \right|. \quad (20)$$

These endomorphisms satisfy

$$\boxed{f_m} = \boxed{f_m} = 0 \quad (21)$$

for all integers $1 \leq j \leq m-1$. This implies

$$\boxed{f_m} = \boxed{f_n} \cdots \boxed{f_m} = \boxed{f_m} \quad (22)$$

for all integers $0 \leq n \leq m$. It can be shown that $f_m^* = f_m$ for every integer $m \geq 0$. Furthermore, for all integers $0 \leq k \leq m$ we have

$$\boxed{f_m} = (-1)^k \frac{[m+1]_A}{[m-k+1]_A} \cdot \boxed{f_{m-k}}. \quad (23)$$

Now we can define a new family of endomorphisms $g_m, h_m \in \mathrm{TL}_A(m)$ for every integer $r \leq m \leq 2r-2$ by setting

$$g_m := f_m + \frac{1}{[r]_A} \cdot h_m \quad (24)$$

where

(25)

These endomorphisms satisfy

(26)

(27)

for all integers $r \leq m \leq 2r - 2$ and $1 \leq j \leq m - 1$, provided j is not equal to $r - 1$. We clearly have

(28)

(29)

for all integers $r \leq n \leq m \leq 2r - 2$. Furthermore, a direct computation involving equation (23) shows

(30)

$$\begin{array}{c} \cdots \\ \cdots \\ \boxed{g_m} \\ \cdots \\ \boxed{h_n} \\ \cdots \end{array} = \begin{array}{c} \cdots \\ \cdots \\ \boxed{h_n} \\ \cdots \\ \boxed{g_m} \\ \cdots \end{array} = \begin{array}{c} \cdots \\ \boxed{h_m} \\ \cdots \end{array} \quad (31)$$

$$\begin{array}{c} \cdots \\ \cdots \\ \boxed{h_m} \\ \cdots \\ \boxed{h_n} \\ \cdots \end{array} = \begin{array}{c} \cdots \\ \cdots \\ \boxed{h_n} \\ \cdots \\ \boxed{h_m} \\ \cdots \end{array} = [r]_A \cdot \begin{array}{c} \cdots \\ \boxed{h_m} \\ \cdots \end{array} \quad (32)$$

for all integers $r \leq n \leq m \leq 2r-2$. Observe that $g_{2r-2}^* = g_{2r-2}$ and $h_{2r-2}^* = h_{2r-2}$.

Lemma 2. For every integer $r \leq m \leq 2r-2$, if $0 \leq k \leq m-r$ we have

$$\begin{array}{c} \overbrace{\quad\quad\quad}^{m-k} \\ \cdots \\ \boxed{g_m} \\ \cdots \\ \underbrace{\quad\quad\quad}_{m-k} \end{array} \text{ (with a loop on the right) } = (-1)^k \frac{[m+1]_A}{[m-k+1]_A} \cdot \begin{array}{c} \cdots \\ \boxed{g_{m-k}} \\ \cdots \end{array} \\
 - (-1)^k \frac{\{r\}'_A[k]_A}{[2r-m+k-1]_A[m-k+1]_A} \cdot \begin{array}{c} \cdots \\ \boxed{h_{m-k}} \\ \cdots \end{array} \quad (33)$$

$$\begin{array}{c} \overbrace{\quad\quad\quad}^{m-k} \\ \cdots \\ \boxed{h_m} \\ \cdots \\ \underbrace{\quad\quad\quad}_{m-k} \end{array} \text{ (with a loop on the right) } = (-1)^k \frac{[2r-m-1]_A}{[2r-m+k-1]_A} \cdot \begin{array}{c} \cdots \\ \boxed{h_{m-k}} \\ \cdots \end{array} \quad (34)$$

while if $m-r < k \leq m$ we have

$$\begin{array}{c} \overbrace{\quad\quad\quad}^{m-k} \\ \cdots \\ \boxed{g_m} \\ \cdots \\ \underbrace{\quad\quad\quad}_{m-k} \end{array} \text{ (with a loop on the right) } = (-1)^k \frac{[r]_A \{m-r+1\}'_A}{[m-k+1]_A} \cdot \begin{array}{c} \cdots \\ \boxed{f_{m-k}} \\ \cdots \end{array} \quad (35)$$

$$\begin{array}{c} \overbrace{\quad\quad\quad}^{m-k} \\ \vdots \\ \boxed{h_m} \quad \cdots \\ \vdots \\ \underbrace{\quad\quad\quad}_{m-k} \end{array} = (-1)^k \frac{[r]_A [2r-m-1]_A}{[m-k+1]_A} \cdot \begin{array}{c} \vdots \\ \boxed{f_{m-k}} \\ \vdots \end{array}. \quad (36)$$

Proof. Equation (34) follows directly from equation (25), and this implies equation (33) using equation (23) together with the equality

$$\frac{\{r\}'_A [k]_A}{[2r-m+k-1]_A [m-k+1]_A} = \frac{1}{[r]_A} \left(\frac{[m+1]_A}{[m-k+1]_A} - \frac{[2r-m-1]_A}{[2r-m+k-1]_A} \right).$$

Similarly, equation (36) follows directly from equation (23), and this implies equation (35) using equation (23) together with the equality

$$[r]_A \{m-r+1\}'_A = [m+1]_A + [2r-m-1]_A. \quad \square$$

Lemma 3. *The idempotent g_m satisfies*

$$\begin{array}{c} \vdots \\ \boxed{g_r} \\ \vdots \end{array} = \begin{array}{c} \vdots \\ \boxed{f_{r-1}} \\ \vdots \end{array} \Bigg|, \quad (37)$$

$$\begin{array}{c} \vdots \\ \boxed{g_{r+1}} \\ \vdots \end{array} = \begin{array}{c} \vdots \\ \boxed{f_{r-1}} \\ \vdots \end{array} \Bigg| + \frac{[r-1]_A}{[r+1]_A} \left(\begin{array}{c} \vdots \\ \boxed{f_{r-1}} \\ \vdots \end{array} \begin{array}{c} \vdots \\ \vdots \end{array} + \begin{array}{c} \vdots \\ \boxed{f_{r-1}} \\ \vdots \end{array} \begin{array}{c} \vdots \\ \vdots \end{array} \right) \\ + \frac{[r-1]_A [2]_A}{[r+1]_A} \cdot \begin{array}{c} \vdots \\ \boxed{f_{r-1}} \\ \vdots \end{array} \begin{array}{c} \vdots \\ \vdots \end{array} + \frac{[r]_A}{[r+1]_A} \cdot \begin{array}{c} \vdots \\ \boxed{f_{r-1}} \\ \vdots \end{array} \begin{array}{c} \vdots \\ \vdots \end{array}, \quad (38)$$

$$\begin{array}{c} \vdots \\ \boxed{g_m} \\ \vdots \end{array} = \begin{array}{c} \vdots \\ \boxed{g_{m-1}} \\ \vdots \end{array} \Bigg| + \frac{[m-1]_A}{[m]_A} \cdot \begin{array}{c} \vdots \\ \boxed{g_{m-1}} \\ \vdots \end{array} \begin{array}{c} \vdots \\ \vdots \end{array} + \frac{\{r\}'_A}{[2r-m]_A [m]_A} \cdot \begin{array}{c} \vdots \\ \boxed{g_{m-1}} \\ \vdots \end{array} \begin{array}{c} \vdots \\ \vdots \end{array} \quad (39)$$

for every integer $r+2 \leq m \leq 2r-2$.

Proof. The statement is proved by direct computation. For $m = r$, equation (37) follows from the recurrence formula (20) defining f_r . For $m = r + 1$, equation (38) follows from a double application of equation (20). For $r + 1 < m \leq 2r - 2$, let us look at the right-hand side of equation (39). For what concerns the first term, equation (24) yields

$$\left| \begin{array}{c} \cdots \\ g_{m-1} \\ \cdots \end{array} \right| = \left| \begin{array}{c} \cdots \\ f_{m-1} \\ \cdots \end{array} \right| + \frac{1}{[r]_A} \cdot \left| \begin{array}{c} \cdots \\ h_{m-1} \\ \cdots \end{array} \right|.$$

For what concerns the second term, equations (29) and (31) yield

$$\begin{aligned} \left| \begin{array}{c} \cdots \\ g_{m-1} \\ \cdots \\ g_{m-1} \\ \cdots \end{array} \right| &= \left| \begin{array}{c} \cdots \\ f_{m-1} \\ \cdots \\ f_{m-1} \\ \cdots \end{array} \right| + \frac{1}{[r]_A} \cdot \left| \begin{array}{c} \cdots \\ f_{m-1} \\ \cdots \\ h_{m-1} \\ \cdots \end{array} \right| + \frac{1}{[r]_A} \cdot \left| \begin{array}{c} \cdots \\ h_{m-1} \\ \cdots \\ f_{m-1} \\ \cdots \end{array} \right| + \frac{1}{[r]_A^2} \cdot \left| \begin{array}{c} \cdots \\ h_{m-1} \\ \cdots \\ h_{m-1} \\ \cdots \end{array} \right| \\ &= \left| \begin{array}{c} \cdots \\ f_{m-1} \\ \cdots \\ f_{m-1} \\ \cdots \end{array} \right| + \frac{1}{[r]_A^2} \cdot \left| \begin{array}{c} \cdots \\ h_{m-1} \\ \cdots \\ h_{m-1} \\ \cdots \end{array} \right|. \end{aligned}$$

For what concerns the third term, equations (31) and (32) yield

$$\left| \begin{array}{c} \cdots \\ g_{m-1} \\ \cdots \\ h_{m-2} \\ \cdots \\ g_{m-1} \\ \cdots \end{array} \right| = \frac{1}{[r]_A} \cdot \left| \begin{array}{c} \cdots \\ h_{m-1} \\ \cdots \\ h_{m-1} \\ \cdots \end{array} \right|.$$

This means that, using the equality

$$\frac{\{r\}'_A}{[2r-m]_A[m]_A} = \frac{1}{[r]_A} \left(\frac{[2r-m+1]_A}{[2r-m]_A} - \frac{[m-1]_A}{[m]_A} \right),$$

we can rewrite the right-hand side of equation (39) as

$$\left| \begin{array}{c} \cdots \\ f_{m-1} \\ \cdots \end{array} \right| + \frac{[m-1]_A}{[m]_A} \cdot \left| \begin{array}{c} \cdots \\ f_{m-1} \\ \cdots \\ f_{m-1} \\ \cdots \end{array} \right| + \frac{1}{[r]_A} \cdot \left| \begin{array}{c} \cdots \\ h_{m-1} \\ \cdots \end{array} \right| + \frac{[2r-m+1]_A}{[r]_A^2[2r-m]_A} \cdot \left| \begin{array}{c} \cdots \\ h_{m-1} \\ \cdots \\ h_{m-1} \\ \cdots \end{array} \right|.$$

But now we have

$= [2r - m]_A^2 \cdot$

$= (-1)^m \frac{[r]_A [2r - m]_A^2}{[2r - m + 1]_A} \cdot$

$= \frac{[r]_A [2r - m]_A}{[2r - m + 1]_A} \left(- \right.$

$+ \right.$

$\left. \right)$

where the first equality follows from equation (25), the second equality follows from equation (23), and the third equality follows from equation (20). This means that the right-hand side of equation (39) can be further rewritten as

$$\left| \begin{array}{c} \cdots \\ f_{m-1} \\ \cdots \end{array} \right| + \frac{[m-1]_A}{[m]_A} \cdot \left| \begin{array}{c} \cdots \\ f_{m-1} \\ \cdots \\ f_{m-1} \\ \cdots \end{array} \right| + \frac{1}{[r]_A} \cdot \left| \begin{array}{c} \cdots \\ h_m \\ \cdots \end{array} \right| = \left| \begin{array}{c} \cdots \\ f_m \\ \cdots \end{array} \right| + \frac{1}{[r]_A} \cdot \left| \begin{array}{c} \cdots \\ h_m \\ \cdots \end{array} \right|.$$

Now the statement follows from equation (24). $\square$

Lemma 4. *The idempotent f_{2r-1} satisfies*

$$\left| \begin{array}{c} \cdots \\ f_{2r-1} \\ \cdots \end{array} \right| = \left| \begin{array}{c} \cdots \\ g_{2r-2} \\ \cdots \end{array} \right| + \frac{[2r-2]_A}{[2r-1]_A} \cdot \left| \begin{array}{c} \cdots \\ g_{2r-2} \\ \cdots \\ g_{2r-2} \\ \cdots \end{array} \right| + \frac{\{r\}'_A}{[2r-1]_A} \cdot \left| \begin{array}{c} \cdots \\ g_{2r-2} \\ \cdots \\ h_{2r-3} \\ \cdots \\ g_{2r-2} \\ \cdots \end{array} \right|. \quad (40)$$

Proof. We follow the proof of equation (39) in Lemma 3, but this time we suppose $m = 2r-1$. Then, the same argument implies that the right-hand side of equation (40) can be rewritten as

$$\left| \begin{array}{c} \cdots \\ f_{2r-2} \\ \cdots \end{array} \right| + \frac{[2r-2]_A}{[2r-1]_A} \cdot \left| \begin{array}{c} \cdots \\ f_{2r-2} \\ \cdots \\ f_{2r-2} \\ \cdots \end{array} \right| + \frac{1}{[r]_A} \cdot \left| \begin{array}{c} \cdots \\ h_{2r-2} \\ \cdots \end{array} \right| + \frac{[2]_A}{[r]_A^2} \cdot \left| \begin{array}{c} \cdots \\ h_{2r-2} \\ \cdots \\ h_{2r-2} \\ \cdots \end{array} \right|.$$

But now we have

$$\begin{aligned}
 & \text{Diagram 1} = \text{Diagram 2} = -\frac{[r]_A}{[2]_A} \cdot \text{Diagram 3} = -\frac{[r]_A}{[2]_A} \cdot \text{Diagram 4}
 \end{aligned}$$

where the first equality follows from equation (25), the second equality follows from equation (23), and the third equality follows from equation (20). Now the statement follows from equation (20). $\square$

From now on, we will focus on the specialization

$$\text{TL} := \text{TL}_{q^{(r+1)/2}}.$$

We observe that f_m can be specialized to TL only for $0 \leq m \leq r-1$, because the formula defining f_r has a pole at $A = q^{(r+1)/2}$. On the other hand, thanks to Lemma 3, g_m can be specialized to TL for all $r \leq m \leq 2r-2$. Similarly, thanks to Lemma 4, f_{2r-1} can be specialized to TL. We call g_m the m th nonsemisimple Jones–Wenzl idempotent.

Remark 1. It can be proven by induction that in TL the idempotent f_{r-1} satisfies

$$\begin{aligned}
 & \text{Diagram 1} = - \text{Diagram 2} \quad \text{Diagram 3} = - \text{Diagram 4} \quad (41)
 \end{aligned}$$

In particular, in TL we have $h_m^* = h_m$ for every integer $r \leq m \leq 2r-2$, because a rotation of angle π fixes h_m . This is true, however, only once we specialize A to $q^{(r+1)/2}$, so the same property does not hold for g_m .

4. Monoidal linear functor

In this section, we study a monoidal linear functor $F_{\text{TL}} : \text{TL} \rightarrow \bar{U}\text{-mod}$, compared with [FK97]. By abuse of notation, we still denote by TL the idempotent completion of TL . This means we promote idempotent endomorphisms $p_m \in \text{TL}(m)$ to objects of TL , and we set

$$\text{TL}(p_m, p_{m'}) = \{t \in \text{TL}(m, m') \mid tp_m = t = p_{m'}t\}$$

with the morphism p_m being the identity of the object p_m . As a consequence, we will occasionally confuse direct summands of objects of $\bar{U}\text{-mod}$ with the corresponding idempotent endomorphisms. We also start adopting a graphical notation featuring labels given by integers $m \geq 0$ placed next to endpoints of edges, standing for the number of parallel strands contained in the plane.

Let us consider the monoidal linear functor

$$F_{\text{TL}} : \text{TL} \rightarrow \bar{U}\text{-mod}$$

sending the object $1 \in \text{TL}$ to $X \in \bar{U}\text{-mod}$, and sending the morphism $\cup \in \text{TL}(0, 2)$ to $c \in \text{Hom}_{\bar{U}}(\mathbb{C}, X \otimes X)$ defined by

$$c(1) := qa_0^1 \otimes a_1^1 - a_1^1 \otimes a_0^1,$$

and the morphism $\cap \in \text{TL}(2, 0)$ to $e \in \text{Hom}_{\bar{U}}(X \otimes X, \mathbb{C})$ defined by

$$e(a_0^1 \otimes a_0^1) = 0, \quad e(a_0^1 \otimes a_1^1) = -1, \quad e(a_1^1 \otimes a_0^1) = q^{-1}, \quad e(a_1^1 \otimes a_1^1) = 0.$$

In order to study F_{TL} , let us define projection morphisms $p_m \in \text{TL}(g_m, f_{2r-m-2})$ and injection morphisms $i_m \in \text{TL}(f_{2r-m-2}, g_m)$ as

$$\begin{array}{ccc} \begin{array}{c} 2r-m-2 \\ | \\ \boxed{p_m} \\ | \\ m \end{array} & := & \begin{array}{c} 2r-m-2 \\ | \\ \boxed{f_{r-1}} \\ | \quad \cup \\ r-1 \quad r-m-1 \end{array} \end{array} \quad \begin{array}{ccc} \begin{array}{c} m \\ | \\ \boxed{i_m} \\ | \\ 2r-m-2 \end{array} & := & \begin{array}{c} r-1 \quad r-m-1 \\ | \quad \cup \\ \boxed{f_{r-1}} \\ | \\ 2r-m-2 \end{array} \end{array} \quad (42)$$

for every integer $r \leq m \leq 2r-2$, which immediately gives

$$\begin{array}{ccc} \begin{array}{c} 2r-m-2 \\ | \\ \boxed{p_m} \\ | \\ m \end{array} & = & \begin{array}{c} 2r-m-2 \\ | \\ \boxed{p_{m-1}} \\ | \quad \cup \\ m-1 \quad 1 \end{array} \end{array} \quad \begin{array}{ccc} \begin{array}{c} m \\ | \\ \boxed{i_m} \\ | \\ 2r-m-2 \end{array} & = & \begin{array}{c} m-1 \quad 1 \\ | \quad \cup \\ \boxed{i_{m-1}} \\ | \\ 2r-m-2 \end{array} \end{array} \quad (43)$$

The fact that $p_m g_m = p_m$ and that $g_m i_m = i_m$ follows from a direct computation.

Lemma 5. *For every integer $0 \leq m \leq r-1$, the object f_m satisfies*

$$F_{\text{TL}}(f_m) = X_m, \quad (44)$$

for $r \leq m \leq 2r-2$, the object g_m and the morphisms h_m , p_m , and i_m satisfy

$$F_{\text{TL}}(g_m) = P_m, \quad (45)$$

$$F_{\text{TL}}(h_m) = -\frac{1}{[m+1]}\varepsilon_m, \quad (46)$$

$$F_{\text{TL}}(p_m) = (-1)^m \frac{[m-r]!}{[m+1]}\pi_m, \quad (47)$$

$$F_{\text{TL}}(i_m) = \frac{1}{[m-r+1]!}\iota_m, \quad (48)$$

and for $m = 2r-1$ the object f_{2r-1} satisfies

$$F_{\text{TL}}(f_{2r-1}) = X_{2r-1}, \quad (49)$$

where the morphisms ε_m , π_m , and ι_m are defined by equations (9) to (11).

The proof of Lemma 5 is a lengthy computation, which will occupy the remainder of this section.

Proof. In order to discuss the proof, we first need to fix some notation. Let us denote with Y_m the unique submodule of $X^{\otimes m}$ satisfying

$$X^{\otimes m} \cong X_m \oplus Y_m \quad (50)$$

for every integer $0 \leq m \leq r-1$, and let us denote with $\varphi_m \in \text{End}_{\bar{U}}(X^{\otimes m})$ the idempotent endomorphism with image X_m and kernel Y_m . Next, let us denote with Q_m the unique submodule of $X^{\otimes m}$ satisfying

$$X^{\otimes m} \cong P_m \oplus Q_m \quad (51)$$

for every integer $r \leq m \leq 2r-2$, and let us denote with $\psi_m \in \text{End}_{\bar{U}}(X^{\otimes m})$ the idempotent endomorphism with image P_m and kernel Q_m . Similarly, let us denote with Y_{2r-1} the unique submodule of $X^{\otimes 2r-1}$ satisfying

$$X^{\otimes 2r-1} \cong X_{r-1}^+ \oplus X_{r-1}^- \oplus Y_{2r-1}, \quad (52)$$

and for every $\varepsilon \in \{+, -\}$ let us denote with $\varphi_{2r-1}^\varepsilon \in \text{End}_{\bar{U}}(X^{\otimes 2r-1})$ the idempotent endomorphism with image X_{2r-1}^ε and kernel $X_{r-1}^{\bar{\varepsilon}} \oplus Y_{2r-1}$, where $\bar{\varepsilon} \in \{+, -\}$ denotes the opposite of ε .

The strategy will be to prove by induction on $0 \leq m \leq 2r-1$ that $F_{\text{TL}}(f_m) = \varphi_m$, if $0 \leq m \leq r-1$, that $F_{\text{TL}}(g_m) = \psi_m$, if $r \leq m \leq 2r-2$, and finally that $F_{\text{TL}}(f_{2r-1}) = \varphi_{2r-1}^+ + \varphi_{2r-1}^-$. This is precisely what equations (44), (45), and (49) stand for. This will allow us to establish equations (47) and (48), which immediately imply equation (46) by restriction to the submodule P_m of $X^{\otimes m}$.

An important observation is that, since the functor F_{TL} sends all morphisms of TL to intertwiners, it is sufficient to check these equalities only for highest-weight vectors in the case of simple $\bar{U}$ -modules, and dominant vectors in the case of indecomposable projective $\bar{U}$ -modules.

For $m = 0$ and for $m = 1$, equation (44) follows directly from the definitions. For $1 < m \leq r - 1$, specializing equation (20) at $A = q^{(r+1)/2}$ gives

$$f_m = f_{m-1} \otimes f_1 - f'_{m-2},$$

where

$$\begin{array}{c} m \\ | \\ \boxed{f'_{m-2}} \\ | \\ m \end{array} := -\frac{[m-1]}{[m]} \cdot \begin{array}{c} m-1 \quad 1 \\ | \quad | \\ \boxed{f_{m-1}} \quad \text{cup} \\ | \quad | \\ \boxed{f_{m-1}} \quad \text{cap} \\ | \quad | \\ m-1 \quad 1 \end{array}. \quad (53)$$

We know $F_{\text{TL}}(f_{m-1} \otimes f_1)$ restricts to the identity on $X_{m-1} \otimes X_1$ and to zero on $Y_{m-1} \otimes X_1$ thanks to the induction hypothesis. Then, using equation (1), we have

$$a_0^m = a_0^{m-1} \otimes a_0^1 = a_0^{m-2} \otimes a_0^1 \otimes a_0^1.$$

This means that using the definition of F_{TL} and equation (53) we get

$$F_{\text{TL}}(f'_{m-2})(a_0^m) = 0, \quad F_{\text{TL}}(f_m)(a_0^m) = a_0^m.$$

On the other hand, using equation (6), we have

$$\begin{aligned} a_0^{m-2} &= [m-1]qa_0^{m-1} \otimes a_1^1 - a_1^{m-1} \otimes a_0^1 \\ &= [m-1]qa_0^{m-2} \otimes a_0^1 \otimes a_1^1 - q^{-m+2}a_0^{m-2} \otimes a_1^1 \otimes a_0^1 - a_1^{m-2} \otimes a_0^1 \otimes a_0^1, \end{aligned}$$

and, using equations (1) and (6), we have

$$a_0^{m-2} \otimes a_0^1 = a_0^{m-1}, \quad a_0^{m-2} \otimes a_1^1 = \frac{1}{[m-1]}a_1^{m-1} + \frac{1}{[m-1]}a_0^{m-3}.$$

This means that, using the definition of F_{TL} and equation (53), we get

$$F_{\text{TL}}(f'_{m-2})(a_0^{m-2}) = a_0^{m-2}, \quad F_{\text{TL}}(f_m)(a_0^{m-2}) = 0.$$

Therefore, $F_{\text{TL}}(f_m) = \varphi_m$. This proves equation (44). For $m = r$, equation (37) gives

$$g_r = f_{r-1} \otimes f_1.$$

We know $F_{\text{TL}}(f_{r-1} \otimes f_1)$ restricts to the identity on $X_{r-1} \otimes X_1$ and to zero on $Y_{r-1} \otimes X_1$ thanks to equation (44), and so $F_{\text{TL}}(g_r) = \psi_r$. This proves equation (45). For what concerns equation (47), using equation (2), we have

$$b_0^r = a_0^{r-1} \otimes a_1^1 = a_0^{r-2} \otimes a_0^1 \otimes a_1^1.$$

Then, using the definition of F_{TL} and equation (43), this means

$$F_{\text{TL}}(p_r)(b_0^r) = -a_0^{r-2}.$$

For what concerns equation (48), using equations (1) and (6) we have

$$a_0^{r-2} \otimes a_0^1 = a_0^{r-1}, \quad a_0^{r-2} \otimes a_1^1 = -a_1^{r-1} - a_0'^{r-3},$$

and using equation (2) we have

$$a_0^r = qa_0^{r-1} \otimes a_1^1 + a_1^{r-1} \otimes a_0^1.$$

Then, using the definition of F_{TL} and equation (43), this means

$$F_{\text{TL}}(i_r)(a_0^{r-2}) = a_0^r.$$

Now equation (46) follows from equations (47) and (48). For $m = r+1$, specializing equation (38) at $A = q^{(r+1)/2}$ gives

$$g_{r+1} = f_{r-1} \otimes f_1 \otimes f_1 - f_{r-1}'^+ - f_{r-1}'^-$$

where

$$\begin{array}{c} r+1 \\ | \\ \boxed{f_{r-1}'^+} \\ | \\ r+1 \end{array} := \begin{array}{c} r-1 \quad 1 \quad 1 \\ | \quad | \quad | \\ \boxed{f_{r-1}} \\ | \quad | \quad | \\ \boxed{f_{r-1}} \\ | \quad | \quad | \\ r-1 \quad 1 \quad 1 \end{array} \quad \begin{array}{c} r+1 \\ | \\ \boxed{f_{r-1}'^-} \\ | \\ r+1 \end{array} := \begin{array}{c} r-1 \quad 1 \quad 1 \\ | \quad | \quad | \\ \boxed{f_{r-1}} \\ | \quad | \quad | \\ \boxed{f_{r-1}} \\ | \quad | \quad | \\ r-1 \quad 1 \quad 1 \end{array} + [2] \cdot \begin{array}{c} r-1 \quad 1 \quad 1 \\ | \quad | \quad | \\ \boxed{f_{r-1}} \\ | \quad | \quad | \\ \boxed{f_{r-1}} \\ | \quad | \quad | \\ r-1 \quad 1 \quad 1 \end{array}. \quad (54)$$

We know $F_{\text{TL}}(f_{r-1} \otimes f_1 \otimes f_1)$ restricts to the identity on $X_{r-1} \otimes X_1 \otimes X_1$ and to zero on $Y_{r-1} \otimes X_1 \otimes X_1$. Then using equation (3) we have

$$\begin{aligned} b_0^{r+1} &= \left(\frac{1}{[2]} + q \right) a_0^{r-1} \otimes a_1^1 \otimes a_1^1 + \frac{q^{-1}}{[2]} a_1^{r-1} \otimes a_0^1 \otimes a_1^1 + \frac{1}{[2]} a_1^{r-1} \otimes a_1^1 \otimes a_0^1 \\ &= \left(\frac{1}{[2]} + q \right) a_0^{r-2} \otimes a_0^1 \otimes a_1^1 \otimes a_1^1 + \frac{q}{[2]} a_0^{r-2} \otimes a_1^1 \otimes a_0^1 \otimes a_1^1 \\ &\quad + \frac{q^2}{[2]} a_0^{r-2} \otimes a_1^1 \otimes a_1^1 \otimes a_0^1 + \frac{q^{-1}}{[2]} a_1^{r-2} \otimes a_0^1 \otimes a_0^1 \otimes a_1^1 \\ &\quad + \frac{1}{[2]} a_1^{r-2} \otimes a_0^1 \otimes a_1^1 \otimes a_0^1. \end{aligned}$$

and using equation (6) we have

$$a_0^{r-2} \otimes a_1^1 = -a_1^{r-1} - a_0'^{r-3}, \quad a_1^{r-2} \otimes a_0^1 = [2]qa_1^{r-1} + q^2a_0'^{r-3}.$$

This means that using the definition of F_{TL} and equation (54) we get

$$F_{\text{TL}}(f_{r-1}^{'+}) (b_0^{r+1}) = 0, \quad F_{\text{TL}}(f_{r-1}'^-) (b_0^{r+1}) = 0, \quad F_{\text{TL}}(g_{r+1}) (b_0^{r+1}) = b_0^{r+1}.$$

On the other hand, using equation (7) we have

$$\begin{aligned} a_0'^{r-1,+} &= qa_0^{r-1} \otimes a_0^1 \otimes a_1^1 - a_0^{r-1} \otimes a_1^1 \otimes a_0^1 \\ &= qa_0'^{r-2} \otimes a_0^1 \otimes a_0^1 \otimes a_1^1 - a_0'^{r-2} \otimes a_0^1 \otimes a_1^1 \otimes a_0^1, \\ a_0'^{r-1,-} &= qa_0^{r-1} \otimes a_1^1 \otimes a_0^1 + a_1^{r-1} \otimes a_0^1 \otimes a_0^1 \\ &= qa_0'^{r-2} \otimes a_0^1 \otimes a_1^1 \otimes a_0^1 + q^2a_0'^{r-2} \otimes a_1^1 \otimes a_0^1 \otimes a_0^1 + a_1^{r-2} \otimes a_0^1 \otimes a_0^1 \otimes a_0^1. \end{aligned}$$

This means that using the definition of F_{TL} and equation (54) we get

$$\begin{aligned} F_{\text{TL}}(f_{r-1}^{'+}) (a_0'^{r-1,+}) &= a_0'^{r-1,+}, & F_{\text{TL}}(f_{r-1}^{'+}) (a_0'^{r-1,-}) &= 0, \\ F_{\text{TL}}(f_{r-1}'^-) (a_0'^{r-1,+}) &= 0, & F_{\text{TL}}(f_{r-1}'^-) (a_0'^{r-1,-}) &= a_0'^{r-1,-}, \\ F_{\text{TL}}(g_{r+1}) (a_0'^{r-1,+}) &= 0, & F_{\text{TL}}(g_{r+1}) (a_0'^{r-1,-}) &= 0. \end{aligned}$$

Therefore, $F_{\text{TL}}(g_{r+1}) = \psi_{r+1}$. This proves equation (45). For what concerns equation (47), using equation (3) we have

$$b_0^{r+1} = \frac{q^{-1}}{[2]}a_0^r \otimes a_1^1 + qb_0^r \otimes a_1^1 + \frac{1}{[2]}b_1^r \otimes a_0^1,$$

and using equation (1) we have

$$a_0^{r-2} = a_0^{r-3} \otimes a_0^1, \quad a_1^{r-2} = q^3a_0^{r-3} \otimes a_1^1 + a_1^{r-3} \otimes a_0^1.$$

Then, using the definition of F_{TL} and equation (43), this means

$$F_{\text{TL}}(p_{r+1}) (b_0^{r+1}) = \frac{1}{[2]}a_0^{r-3}.$$

For what concerns equation (48), using equations (1) and (6) we have

$$a_0^{r-3} \otimes a_0^1 = a_0^{r-2}, \quad a_0^{r-3} \otimes a_1^1 = -\frac{1}{[2]}a_1^{r-2} - \frac{1}{[2]}a_0'^{r-4},$$

and using equation (3) we have

$$a_0^{r+1} = [2]qa_0^r \otimes a_1^1 + a_1^r \otimes a_0^1.$$

Then, using the definition of F_{TL} and equation (43), this means

$$F_{\text{TL}}(i_{r+1})(a_0^{r-3}) = \frac{1}{[2]} a_0^{r+1}.$$

Now equation (46) follows from equations (47) and (48). For $r+1 < m \leq 2r-2$, specializing equation (39) at $A = q^{(r+1)/2}$ gives

$$g_m = g_{m-1} \otimes f_1 - g'_{m-2}$$

where

$$\begin{array}{c} m \\ | \\ \boxed{g'_{m-2}} \\ | \\ m \end{array} := -\frac{[m-1]}{[m]} \cdot \begin{array}{c} m-1 \quad 1 \\ | \quad | \\ \boxed{g_{m-1}} \\ | \quad | \\ \boxed{g_{m-1}} \\ | \quad | \\ m-1 \quad 1 \end{array} + \frac{2}{[m]^2} \cdot \begin{array}{c} m-1 \quad 1 \\ | \quad | \\ \boxed{g_{m-1}} \\ | \quad | \\ \boxed{h_{m-2}} \\ | \quad | \\ \boxed{g_{m-1}} \\ | \quad | \\ m-1 \quad 1 \end{array}. \quad (55)$$

We know $F_{\text{TL}}(g_{m-1} \otimes f_1)$ restricts to the identity on $P_{m-1} \otimes X_1$ and to zero on $Q_{m-1} \otimes X_1$, thanks to the induction hypothesis. Then, using equation (3) we have

$$\begin{aligned} b_0^m &= \frac{q^{-m}}{[m+1]} a_0^{m-1} \otimes a_1^1 + [m] q b_0^{m-1} \otimes a_1^1 + \frac{[m]}{[m+1]} b_1^{m-1} \otimes a_0^1 \\ &= \left(\frac{[m] q^{-m+1}}{[m+1]} + q^{-m+2} \right) a_0^{m-2} \otimes a_1^1 \otimes a_1^1 + [m][m-1] q^2 b_0^{m-2} \otimes a_1^1 \otimes a_1^1 \\ &\quad + \frac{q^{-m}}{[m+1]} a_1^{m-2} \otimes a_0^1 \otimes a_1^1 + [m-1] q b_1^{m-2} \otimes a_0^1 \otimes a_1^1 \\ &\quad + \frac{q^{-m+1}}{[m+1]} a_1^{m-2} \otimes a_1^1 \otimes a_0^1 + [m-1] q^2 b_1^{m-2} \otimes a_1^1 \otimes a_0^1 \\ &\quad + \frac{[m-1]}{[m+1]} b_2^{m-2} \otimes a_0^1 \otimes a_0^1. \end{aligned}$$

This means that using the definition of F_{TL} and equation (55) we get

$$F_{\text{TL}}(g'_{m-2})(b_0^m) = 0, \quad F_{\text{TL}}(g_m)(b_0^m) = b_0^m.$$

On the other hand, using equation (8) we have

$$\begin{aligned} b_0^{m-2} &= \frac{q^m}{[m-1]} x_{m-r-1}^{m-1} \otimes a_1^1 - \frac{[m]}{[m-1]} b_0^{m-1} \otimes a_0^1 \\ &= q^m x_{m-r-2}^{m-2} \otimes a_1^1 \otimes a_1^1 + \frac{q^m}{[m-1]} a_0^{m-2} \otimes a_0^1 \otimes a_1^1 \\ &\quad - \frac{q^{-m+1}}{[m-1]} a_0^{m-2} \otimes a_1^1 \otimes a_0^1 - [m] q b_0^{m-2} \otimes a_1^1 \otimes a_0^1 \\ &\quad - b_1^{m-2} \otimes a_0^1 \otimes a_0^1, \end{aligned}$$

and using equations (3) and (8) we have

$$\begin{aligned} a_0^{m-2} \otimes a_0^1 &= -a_0'^{m-3}, \\ b_0^{m-2} \otimes a_0^1 &= \frac{q^{m-1}}{[m-1]^2} x_{m-r-1}^{m-1} + \frac{q^{m-1}}{[m-1]^2} a_0'^{m-3} - \frac{[m-2]}{[m-1]} b_0'^{m-3}, \\ a_0^{m-2} \otimes a_1^1 &= \frac{1}{[m-1]} a_0^{m-1} + \frac{1}{[m-1]} a_1'^{m-3}, \\ b_0^{m-2} \otimes a_1^1 &= -\frac{\{m-1\}'}{[m-1]^3} a_0^{m-1} + \frac{[m]}{[m-1]^2} b_0^{m-1} - \frac{\{m-1\}'}{[m-1]^3} a_1'^{m-3} \\ &\quad + \frac{[m-2]}{[m-1]^2} b_1'^{m-3}. \end{aligned}$$

This means that using the definition of F_{TL} and equation (55) we get

$$F_{\text{TL}}(g'_{m-2})(b_0'^{m-2}) = b_0'^{m-2}, \quad F_{\text{TL}}(g_m)(b_0'^{m-2}) = 0.$$

Therefore, $F_{\text{TL}}(g_m) = \psi_m$. This proves equation (45). For what concerns equation (47), using equation (3) we have

$$b_0^m = \frac{q^{-m}}{[m+1]} a_0^{m-1} \otimes a_1^1 + [m] q b_0^{m-1} \otimes a_1^1 + \frac{[m]}{[m+1]} b_1^{m-1} \otimes a_0^1,$$

and using equation (1) we have

$$a_0^{2r-m-1} = a_0^{2r-m-2} \otimes a_0^1, \quad a_1^{2r-m-1} = q^{m+2} a_0^{2r-m-2} \otimes a_1^1 + a_1^{2r-m-2} \otimes a_0^1.$$

Then, using the definition of F_{TL} and equation (43), this means

$$F_{\text{TL}}(p_m)(b_0^m) = (-1)^m \frac{[m-r]!}{[m+1]} a_0^{2r-m-2}.$$

For what concerns equation (48), using equations (1) and (6) we have

$$\begin{aligned} a_0^{2r-m-2} \otimes a_0^1 &= a_0^{2r-m-1}, \\ a_0^{2r-m-2} \otimes a_1^1 &= -\frac{1}{[m+1]} a_1^{2r-m-1} - \frac{1}{[m+1]} a_0'^{2r-m-3}, \end{aligned}$$

and using equation (3) we have

$$a_0^m = [m+1] q a_0^{m-1} \otimes a_1^1 + a_1^{m-1} \otimes a_0^1.$$

Then, using the definition of F_{TL} and equation (43), this means

$$F_{\text{TL}}(i_m)(a_0^{2r-m-2}) = \frac{1}{[m-r+1]!} a_0^m.$$

Now equation (46) is a consequence of equations (47) and (48). Next, let us denote with Y_{2r-1} the unique submodule of $X^{\otimes 2r-1}$ satisfying $X^{\otimes 2r-1} \cong X_{2r-1} \oplus Y_{2r-1}$. For $m = 2r - 1$, specializing equation (40) at $A = q^{(r+1)/2}$ gives

$$f_{2r-1} = g_{2r-2} \otimes f_1 - g'_{2r-3}$$

where

$$\begin{array}{c} 2r-1 \\ | \\ \boxed{g'_{2r-3}} \\ | \\ 2r-1 \end{array} := -[2] \cdot \begin{array}{c} 2r-2 \quad 1 \\ | \quad | \\ \boxed{g_{2r-2}} \\ | \quad | \\ \boxed{g_{2r-2}} \\ | \quad | \\ 2r-2 \quad 1 \end{array} + 2 \cdot \begin{array}{c} 2r-2 \quad 1 \\ | \quad | \\ \boxed{g_{2r-2}} \\ | \quad | \\ \boxed{h_{2r-3}} \\ | \quad | \\ \boxed{g_{2r-2}} \\ | \quad | \\ 2r-2 \quad 1 \end{array}. \quad (56)$$

We know $F_{\text{TL}}(g_{2r-2} \otimes f_1)$ restricts to the identity on $P_{2r-2} \otimes X_1$ and to zero on $Q_{2r-2} \otimes X_1$, thanks to equation (45). Then, using equation (4) we have

$$\begin{aligned} a_0^{r-1,+} &= x_0^{2r-2} \otimes a_0^1 = x_0^{2r-3} \otimes a_0^1 \otimes a_0^1, \\ a_0^{r-1,-} &= qa_0^{2r-2} \otimes a_1^1 - y_0^{2r-2} \otimes a_0^1 \\ &= -q^2 a_0^{2r-3} \otimes a_1^1 \otimes a_1^1 + qa_1^{2r-3} \otimes a_0^1 \otimes a_1^1 \\ &\quad + q^2 a_1^{2r-3} \otimes a_1^1 \otimes a_0^1 - [2]y_0^{2r-3} \otimes a_0^1 \otimes a_0^1. \end{aligned}$$

This means that, using the definition of F_{TL} and equation (56), we get

$$\begin{aligned} F_{\text{TL}}(g'_{2r-3})(a_0^{r-1,+}) &= 0, & F_{\text{TL}}(f_{2r-1})(a_0^{r-1,+}) &= a_0^{r-1,+}, \\ F_{\text{TL}}(g'_{2r-3})(a_0^{r-1,-}) &= 0, & F_{\text{TL}}(f_{2r-1})(a_0^{r-1,-}) &= a_0^{r-1,-}. \end{aligned}$$

On the other hand, using equation (8) we have

$$\begin{aligned} b_0^{2r-3} &= -\frac{q^{-1}}{[2]} x_{r-2}^{2r-2} \otimes a_1^1 - \frac{1}{[2]} b_0^{2r-2} \otimes a_0^1 \\ &= q^{-1} x_{r-3}^{2r-3} \otimes a_1^1 \otimes a_1^1 - \frac{q^{-1}}{[2]} a_0^{2r-3} \otimes a_0^1 \otimes a_1^1 \\ &\quad + \frac{q^2}{[2]} a_0^{2r-3} \otimes a_1^1 \otimes a_0^1 + qb_0^{2r-3} \otimes a_1^1 \otimes a_0^1 \\ &\quad - b_1^{2r-3} \otimes a_0^1 \otimes a_0^1, \end{aligned}$$

and using equations (3) and (8) we have

$$\begin{aligned} a_0^{2r-3} \otimes a_0^1 &= -a_0'^{2r-4}, \\ b_0^{2r-3} \otimes a_0^1 &= \frac{q^{-2}}{[2]^2} x_{r-2}^{2r-2} + \frac{q^{-2}}{[2]^2} a_0'^{2r-4} - \frac{[3]}{[2]} b_0'^{2r-4}, \\ a_0^{2r-3} \otimes a_1^1 &= -\frac{1}{[2]} a_0^{2r-2} - \frac{1}{[2]} a_1'^{2r-4}, \\ b_0^{2r-3} \otimes a_1^1 &= \frac{\{2\}'}{[2]^3} a_0^{2r-2} - \frac{1}{[2]^2} b_0^{2r-2} + \frac{\{2\}'}{[2]^3} a_1'^{2r-4} - \frac{[3]}{[2]^2} b_1'^{2r-4}. \end{aligned}$$

This means that, using the definition of F_{TL} and equation (56), we get

$$F_{\text{TL}}(g'_{2r-3})(b_0'^{2r-3}) = b_0'^{2r-3}, \quad F_{\text{TL}}(f_{2r-1})(b_0'^{2r-3}) = 0.$$

Therefore, $F_{\text{TL}}(f_{2r-1}) = \varphi_{2r-1}^+ + \varphi_{2r-1}^-$. This proves equation (49). $\square$

The functor $F_{\text{TL}} : \text{TL} \rightarrow \bar{U}\text{-mod}$ is not full. Indeed, we have

$$\dim_{\mathbb{C}} \text{TL}(g_m, g_{3r-m-2}) = 0$$

for every integer $r \leq m \leq 2r - 2$, as follows directly from a parity argument, $3r - 2m - 2$ being odd. This does not match the dimension of $\text{Hom}_{\bar{U}}(P_m, P_{3r-m-2})$, as recalled in Section 2.

5. Extended Temperley–Lieb category

In this section, we introduce an extended version of the category TL over which we define a monoidal linear functor with target $\bar{U}\text{-mod}$.

Definition 1. The *extended Temperley–Lieb category* $\tilde{\text{TL}}$ is the smallest linear category containing TL as a linear subcategory, as well as morphisms

$$p_{2r-1}^+, p_{2r-1}^- \in \tilde{\text{TL}}(f_{2r-1}, f_{r-1}), \quad i_{2r-1}^+, i_{2r-1}^- \in \tilde{\text{TL}}(f_{r-1}, f_{2r-1})$$

satisfying, for every integer $0 \leq m \leq r - 1$ and all $\varepsilon, \varepsilon' \in \{+, -\}$, equations

$$\begin{array}{ccc} \begin{array}{c} m \quad r-1 \\ | \quad | \\ \boxed{p_{2r-1}^{\varepsilon'}} \\ | \quad | \\ \boxed{i_{2r-1}^{\varepsilon}} \\ | \quad | \\ r-1 \quad m \end{array} & = \delta_{\varepsilon, \varepsilon'} \cdot & \begin{array}{c} m \quad r-1 \\ | \quad | \\ \boxed{f_{r-1}} \\ | \quad | \\ r-1 \quad m \end{array} \end{array} \quad \begin{array}{ccc} \begin{array}{c} r-1 \quad m \\ | \quad | \\ \boxed{p_{2r-1}^{\varepsilon'}} \\ | \quad | \\ \boxed{i_{2r-1}^{\varepsilon}} \\ | \quad | \\ m \quad r-1 \end{array} & = \delta_{\varepsilon, \varepsilon'} \cdot & \begin{array}{c} r-1 \quad m \\ | \quad | \\ \boxed{f_{r-1}} \\ | \quad | \\ m \quad r-1 \end{array} \end{array} \quad (57)$$

$$\sum_{\varepsilon \in \{+, -\}} \begin{array}{ccc} \begin{array}{c} m \quad 2r-1 \\ | \quad | \\ \boxed{i_{2r-1}^{\varepsilon}} \\ | \quad | \\ \boxed{p_{2r-1}^{\varepsilon}} \\ | \quad | \\ 2r-1 \quad m \end{array} & = & \begin{array}{c} m \quad 2r-1 \\ | \quad | \\ \boxed{f_{2r-1}} \\ | \quad | \\ 2r-1 \quad m \end{array} \end{array} \quad \sum_{\varepsilon \in \{+, -\}} \begin{array}{ccc} \begin{array}{c} 2r-1 \quad m \\ | \quad | \\ \boxed{i_{2r-1}^{\varepsilon}} \\ | \quad | \\ \boxed{p_{2r-1}^{\varepsilon}} \\ | \quad | \\ m \quad 2r-1 \end{array} & = & \begin{array}{c} 2r-1 \quad m \\ | \quad | \\ \boxed{f_{2r-1}} \\ | \quad | \\ m \quad 2r-1 \end{array} \end{array} \quad (58)$$

$$\begin{array}{ccc} \begin{array}{c} 2r-2 \\ | \quad | \\ \boxed{i_{2r-1}^{\varepsilon}} \\ | \quad | \\ r-1 \quad 1 \end{array} & = - & \begin{array}{c} 2r-2 \\ | \quad | \\ \boxed{i_{2r-1}^{\varepsilon}} \\ | \quad | \\ 1 \quad r-1 \end{array} \end{array} \quad \begin{array}{ccc} \begin{array}{c} r-1 \quad 1 \\ | \quad | \\ \boxed{p_{2r-1}^{\varepsilon}} \\ | \quad | \\ 2r-2 \end{array} & = - & \begin{array}{c} 1 \quad r-1 \\ | \quad | \\ \boxed{p_{2r-1}^{\varepsilon}} \\ | \quad | \\ 2r-2 \end{array} \end{array} \quad (59)$$

Remark 2. As an immediate consequence of equation (59), for every $\varepsilon \in \{+, -\}$ we have

$$\begin{array}{ccc}
 \begin{array}{c} 2r-1 \quad 1 \\ \boxed{i_{2r-1}^\varepsilon} \\ r-2 \quad r-2 \end{array} & = - & \begin{array}{c} 1 \quad 2r-1 \\ \boxed{i_{2r-1}^\varepsilon} \\ r-2 \end{array} \\
 \begin{array}{c} r-2 \\ \boxed{p_{2r-1}^\varepsilon} \\ 2r-2 \end{array} & = 0 &
 \end{array}
 \quad (60)$$

$$\begin{array}{ccc}
 \begin{array}{c} r-2 \\ \boxed{p_{2r-1}^\varepsilon} \\ 2r-2 \end{array} & = 0 &
 \begin{array}{c} r-2 \\ \boxed{i_{2r-1}^\varepsilon} \\ r-2 \end{array} = 0.
 \end{array}
 \quad (61)$$

We define a monoidal linear functor

$$F_{\tilde{\text{TL}}} : \tilde{\text{TL}} \rightarrow \bar{U}\text{-mod}$$

extending $F_{\text{TL}} : \text{TL} \rightarrow \bar{U}\text{-mod}$ by setting

$$F_{\tilde{\text{TL}}}(p_{2r-1}^\varepsilon) := \pi_{2r-1}^\varepsilon, \quad F_{\tilde{\text{TL}}}(i_{2r-1}^\varepsilon) := \iota_{2r-1}^\varepsilon$$

for all $\varepsilon, \varepsilon' \in \{+, -\}$.

Lemma 6. *The monoidal linear functor $F_{\tilde{\text{TL}}} : \tilde{\text{TL}} \rightarrow \bar{U}\text{-mod}$ is well defined.*

The proof of Lemma 6 is a lengthy computation that will occupy the remainder of this section.

Proof. In order to discuss the proof, let us first choose names for the morphisms appearing in equations (57) and (58), which we rewrite as

$$\begin{array}{ll}
 t_{r-1,m}^{\varepsilon,\varepsilon'} = \delta_{\varepsilon,\varepsilon'} \cdot t_{r-1,m}, & u_{r-1,m}^{\varepsilon,\varepsilon'} = \delta_{\varepsilon,\varepsilon'} \cdot u_{r-1,m}, \\
 \sum_{\varepsilon \in \{+,-\}} t_{2r-1,m}^\varepsilon = t_{2r-1,m}, & \sum_{\varepsilon \in \{+,-\}} u_{2r-1,m}^\varepsilon = u_{2r-1,m},
 \end{array}$$

respectively. We also give names to the images of these morphisms under the functor $F_{\tilde{\text{TL}}}$ by setting

$$\begin{array}{ll}
 \tau_{r-1,m}^{\varepsilon,\varepsilon'} := F_{\tilde{\text{TL}}}(t_{r-1,m}^{\varepsilon,\varepsilon'}), & \tau_{r-1,m} := F_{\tilde{\text{TL}}}(t_{r-1,m}), \\
 v_{r-1,m}^{\varepsilon,\varepsilon'} := F_{\tilde{\text{TL}}}(u_{r-1,m}^{\varepsilon,\varepsilon'}), & v_{r-1,m} := F_{\tilde{\text{TL}}}(u_{r-1,m}), \\
 \tau_{2r-1,m}^\varepsilon := F_{\tilde{\text{TL}}}(t_{2r-1,m}^\varepsilon), & \tau_{2r-1,m} := F_{\tilde{\text{TL}}}(t_{2r-1,m}), \\
 v_{2r-1,m}^\varepsilon := F_{\tilde{\text{TL}}}(u_{2r-1,m}^\varepsilon), & v_{2r-1,m} := F_{\tilde{\text{TL}}}(u_{2r-1,m}).
 \end{array}$$

Therefore, what we need to show is

$$\tau_{r-1,m}^{\varepsilon,\varepsilon'} = \delta_{\varepsilon,\varepsilon'} \tau_{r-1,m}, \quad v_{r-1,m}^{\varepsilon,\varepsilon'} = \delta_{\varepsilon,\varepsilon'} v_{r-1,m},$$

$$\sum_{\varepsilon \in \{+, -\}} \tau_{2r-1, m}^\varepsilon = \tau_{2r-1, m}, \quad \sum_{\varepsilon \in \{+, -\}} v_{2r-1, m}^\varepsilon = v_{2r-1, m}.$$

When $m = 0$, the relations reduce to equations (18) and (19). Then let us suppose $0 < m \leq r - 1$. Thanks to equation (86), we have

$$\begin{aligned} a_0^{r-1} &= a_0^m \otimes a_0^{r-m-1} = a_0^{r-m-1} \otimes a_0^m, \\ a_0^{r-1, +} &= a_0^m \otimes x_0^{2r-m-1} = x_0^{2r-m-1} \otimes a_0^m, \\ a_{r-1}^{r-1} &= (-1)^m a_m^m \otimes a_{r-m-1}^{r-m-1} = (-1)^m a_{r-m-1}^{r-m-1} \otimes a_m^m, \\ a_{r-1}^{r-1, -} &= (-1)^{m+1} [m] a_m^m \otimes y_{r-m-1}^{2r-m-1} = (-1)^{m+1} [m] y_{r-m-1}^{2r-m-1} \otimes a_m^m. \end{aligned}$$

Then, for all integers $0 \leq j \leq m$ and $0 \leq k \leq r - 1$, thanks to equations (80) to (82) and (86), we have

$$\begin{aligned} &\tau_{r-1, m}^{+, +}(a_k^{r-1} \otimes a_j^m) \\ &= \sum_{\ell=\max\{m+k-r+1, 0\}}^{\min\{m, k\}} \frac{[k]!}{[\ell]![k-\ell]!} q^{-(k-\ell)(m-\ell)} a_\ell^m \otimes \pi_{2r-1}^+(x_{k-\ell}^{2r-m-1} \otimes a_j^m) \\ &= \sum_{\ell=\max\{m+k-r+1, 0\}}^{\min\{m, k\}} \frac{[k]!}{[\ell]![k-\ell]!} q^{-(k-\ell)(m-\ell)} a_\ell^m \otimes \varphi_{r-1}(a_{k-\ell}^{r-m-1} \otimes a_j^m) \\ &= \tau_{r-1, m}(a_k^{r-1} \otimes a_j^m), \end{aligned}$$

we have

$$\begin{aligned} &\tau_{r-1, m}^{-, -}(a_k^{r-1} \otimes a_j^m) \\ &= -[m] \sum_{\ell=\max\{m+k-r+1, 0\}}^{\min\{m, k\}} \frac{[k]!}{[\ell]![k-\ell]!} q^{-(k-\ell)(m-\ell)} a_\ell^m \otimes \pi_{2r-1}^-(y_{k-\ell}^{2r-m-1} \otimes a_j^m) \\ &= \sum_{\ell=\max\{m+k-r+1, 0\}}^{\min\{m, k\}} \frac{[k]!}{[\ell]![k-\ell]!} q^{-(k-\ell)(m-\ell)} a_\ell^m \otimes \varphi_{r-1}(a_{k-\ell}^{r-m-1} \otimes a_j^m) \\ &= \tau_{r-1, m}(a_k^{r-1} \otimes a_j^m), \end{aligned}$$

we have

$$\begin{aligned} &v_{r-1, m}^{+, +}(a_j^m \otimes a_k^{r-1}) \\ &= \sum_{\ell=\max\{m+k-r+1, 0\}}^{\min\{m, k\}} \frac{[k]!}{[\ell]![k-\ell]!} q^{(m+k-\ell+1)\ell} \pi_{2r-1}^+(a_j^m \otimes x_{k-\ell}^{2r-m-1}) \otimes a_\ell^m \\ &= \sum_{\ell=\max\{m+k-r+1, 0\}}^{\min\{m, k\}} \frac{[k]!}{[\ell]![k-\ell]!} q^{(m+k-\ell+1)\ell} \varphi_{r-1}(a_j^m \otimes a_{k-\ell}^{r-m-1}) \otimes a_\ell^m \\ &= v_{r-1, m}(a_j^m \otimes a_k^{r-1}), \end{aligned}$$

we have

$$\begin{aligned}
 & v_{r-1,m}^{-,-}(a_j^m \otimes a_k^{r-1}) \\
 &= -[m] \sum_{\ell=\max\{m+k-r+1,0\}}^{\min\{m,k\}} \frac{[k]!}{[\ell]![k-\ell]!} q^{(m+k-\ell+1)\ell} \pi_{2r-1}^{-}(a_j^m \otimes y_{k-\ell}^{2r-m-1}) \otimes a_\ell^m \\
 &= \sum_{\ell=\max\{m+k-r+1,0\}}^{\min\{m,k\}} \frac{[k]!}{[\ell]![k-\ell]!} q^{(m+k-\ell+1)\ell} \varphi_{r-1}(a_j^m \otimes a_{k-\ell}^{r-m-1}) \otimes a_\ell^m \\
 &= v_{r-1,m}(a_j^m \otimes a_k^{r-1}),
 \end{aligned}$$

we have

$$\begin{aligned}
 \tau_{r-1,m}^{+,-}(a_k^{r-1} \otimes a_j^m) &= \tau_{r-1,m}^{-,+}(a_k^{r-1} \otimes a_j^m) = 0, \\
 v_{r-1,m}^{+,-}(a_j^m \otimes a_k^{r-1}) &= v_{r-1,m}^{-,+}(a_j^m \otimes a_k^{r-1}) = 0,
 \end{aligned}$$

we have

$$\begin{aligned}
 & \tau_{2r-1,m}^{+}(a_k^{r-1,+} \otimes a_j^m) \\
 &= \sum_{\ell=\max\{m+k-r+1,0\}}^{\min\{m,k\}} \frac{[k]!}{[\ell]![k-\ell]!} q^{-(k-\ell)(m-\ell)} a_\ell^m \otimes \iota_{2r-1}^{+}(a_{k-\ell}^{r-m-1} \otimes a_j^m) \\
 &= \sum_{\ell=\max\{m+k-r+1,0\}}^{\min\{m,k\}} \frac{[k]!}{[\ell]![k-\ell]!} q^{-(k-\ell)(m-\ell)} a_\ell^m \otimes \varphi_{2r-1}^{+}(x_{k-\ell}^{2r-m-1} \otimes a_j^m) \\
 &= \tau_{2r-1,m}(a_k^{r-1,+} \otimes a_j^m),
 \end{aligned}$$

we have

$$\begin{aligned}
 & \tau_{2r-1,m}^{-}(a_k^{r-1,-} \otimes a_j^m) \\
 &= \sum_{\ell=\max\{m+k-r+1,0\}}^{\min\{m,k\}} \frac{[k]!}{[\ell]![k-\ell]!} q^{-(k-\ell)(m-\ell)} a_\ell^m \otimes \iota_{2r-1}^{-}(a_{k-\ell}^{r-m-1} \otimes a_j^m) \\
 &= -[m] \sum_{\ell=\max\{m+k-r+1,0\}}^{\min\{m,k\}} \frac{[k]!}{[\ell]![k-\ell]!} q^{-(k-\ell)(m-\ell)} a_\ell^m \otimes \varphi_{2r-1}^{-}(y_{k-\ell}^{2r-m-1} \otimes a_j^m) \\
 &= \tau_{2r-1,m}(a_k^{r-1,-} \otimes a_j^m),
 \end{aligned}$$

we have

$$\begin{aligned}
 & v_{r-1,m}^{+}(a_j^m \otimes a_k^{r-1,+}) \\
 &= \sum_{\ell=\max\{m+k-r+1,0\}}^{\min\{m,k\}} \frac{[k]!}{[\ell]![k-\ell]!} q^{(m+k-\ell+1)\ell} \iota_{2r-1}^{+}(a_j^m \otimes a_{k-\ell}^{r-m-1}) \otimes a_\ell^m \\
 &= \sum_{\ell=\max\{m+k-r+1,0\}}^{\min\{m,k\}} \frac{[k]!}{[\ell]![k-\ell]!} q^{(m+k-\ell+1)\ell} \varphi_{2r-1}^{+}(a_j^m \otimes x_{k-\ell}^{2r-m-1}) \otimes a_\ell^m \\
 &= v_{2r-1,m}(a_j^m \otimes a_k^{r-1,+}),
 \end{aligned}$$

we have

$$\begin{aligned}
 & v_{2r-1,m}^-(a_j^m \otimes a_k^{r-1,-}) \\
 &= \sum_{\ell=\max\{m+k-r+1,0\}}^{\min\{m,k\}} \frac{[k]!}{[\ell]![k-\ell]!} q^{(m+k-\ell+1)\ell} \iota_{2r-1}^-(a_j^m \otimes a_{k-\ell}^{r-m-1}) \otimes a_\ell^m \\
 &= -[m] \sum_{\ell=\max\{m+k-r+1,0\}}^{\min\{m,k\}} \frac{[k]!}{[\ell]![k-\ell]!} q^{(m+k-\ell+1)\ell} \varphi_{2r-1}^-(a_j^m \otimes y_{k-\ell}^{2r-m-1}) \otimes a_\ell^m \\
 &= v_{2r-1,m}(a_j^m \otimes a_k^{r-1,-}),
 \end{aligned}$$

and we have

$$\begin{aligned}
 \tau_{2r-1,m}^+(a_k^{r-1,-} \otimes a_j^m) &= \tau_{2r-1,m}^-(a_k^{r-1,+} \otimes a_j^m) = 0, \\
 v_{r-1,m}^+(a_j^m \otimes a_k^{r-1,-}) &= v_{r-1,m}^-(a_j^m \otimes a_k^{r-1,+}) = 0.
 \end{aligned}$$

Similarly, let us choose names for the morphisms appearing in equation (59), which we rewrite as

$$c_r^\varepsilon = -\tilde{c}_r^\varepsilon, \quad d_r^\varepsilon = -\tilde{d}_r^\varepsilon.$$

Equations (1), (2), and (86) give

$$\begin{aligned}
 b_0^r &= a_0^{r-1} \otimes a_1^1 = a_0^1 \otimes a_0^{r-1} \otimes a_1^1, \\
 a_0^{r-1,+} &= x_0^{2r-2} \otimes a_0^1, \\
 a_0^{r-1,-} &= q a_0^{2r-2} \otimes a_1^1 - y_0^{2r-2} \otimes a_0^1, \\
 a_1^{r-1,+} &= q^{-1} a_0^1 \otimes x_1^{2r-2} + a_1^1 \otimes x_0^{2r-2}, \\
 a_1^{r-1,-} &= -q^{-1} a_0^1 \otimes y_1^{2r-2} - a_1^1 \otimes y_0^{2r-2},
 \end{aligned}$$

and so equation (80) gives

$$\begin{aligned}
 F_{\tilde{\text{TL}}}(c_r^+)(b_0^r) &= -x_0^{2r-2}, & F_{\tilde{\text{TL}}}(\tilde{c}_r^+)(b_0^r) &= x_0^{2r-2}, \\
 F_{\tilde{\text{TL}}}(c_r^-)(b_0^r) &= y_0^{2r-2}, & F_{\tilde{\text{TL}}}(\tilde{c}_r^-)(b_0^r) &= -y_0^{2r-2}.
 \end{aligned}$$

Furthermore, equations (74) and (75) give

$$x_0^r = a_0^{r-1} \otimes a_0^1 = a_0^1 \otimes a_0^{r-1}, \quad y_0^r = a_{r-1}^{r-1} \otimes a_1^1 = a_1^1 \otimes a_{r-1}^{r-1},$$

and so equations (81) and (82) give

$$\begin{aligned}
 F_{\tilde{\text{TL}}}(d_r^+)(b_0^{2r-2}) &= y_0^r, & F_{\tilde{\text{TL}}}(\tilde{d}_r^+)(b_0^{2r-2}) &= -y_0^r, \\
 F_{\tilde{\text{TL}}}(d_r^-)(b_0^{2r-2}) &= -x_0^r, & F_{\tilde{\text{TL}}}(\tilde{d}_r^-)(b_0^{2r-2}) &= x_0^r.
 \end{aligned}$$

□

6. Proof of Theorem 1

In this section, we establish our main result. We do this in two main steps. First of all, for convenience of notation, let us set

$$t_m := \begin{cases} f_m & 0 \leq m \leq r-1 \\ g_m & r \leq m \leq 2r-2 \end{cases} \in \tilde{\text{TL}}$$

for all integers $0 \leq m \leq 2r-2$. Then let us show the family of objects

$$\{t_m \in \tilde{\text{TL}} \mid 0 \leq m \leq 2r-2\}$$

dominates $\tilde{\text{TL}}$.

Proposition 7. *For every $m \in \tilde{\text{TL}}$, there exists a finite set $J_{m,n}$ for every integer $0 \leq n \leq 2r-2$ and morphisms $u_m^{n,j} \in \tilde{\text{TL}}(m,n)$ and $v_{n,j}^m \in \tilde{\text{TL}}(n,m)$ for every $j \in J_{m,n}$ satisfying*

$$\text{id}_m = \sum_{n=0}^{2r-2} \sum_{j \in J_{m,n}} v_{n,j}^m t_n u_m^{n,j}.$$

Proof. The statement is proved by induction on m . For $m=0$, we simply need to set $|J_{0,n}| = \delta_{0,n}$ and $u_0^0 = v_0^0 = f_0$. For $m > 0$, we decompose id_m as

$$\begin{array}{c} m \\ \hline \\ \\ \\ \\ m \end{array} = \sum_{n=0}^{2r-2} \sum_{j \in J_{m-1,n}} \begin{array}{c} m-1 \quad 1 \\ \hline \boxed{v_{n,j}^{m-1}} \\ \hline \boxed{t_n} \\ \hline \boxed{u_{m-1}^{n,j}} \\ \hline m-1 \quad 1 \end{array}.$$

Equations (20) and (37) allow us to rewrite $f_n \otimes f_1$ for every integer $0 \leq n \leq r-1$, equations (38) and (39) allow us to rewrite $g_n \otimes f_1$ for every integer $r \leq n \leq 2r-3$, and since equation (58) for $m=0$ implies in particular

$$\begin{array}{c} 2r-1 \\ \hline \boxed{f_{2r-1}} \\ \hline 2r-1 \end{array} = \begin{array}{c} 2r-1 \\ \hline \boxed{i_{2r-1}^+} \\ \hline \boxed{p_{2r-1}^+} \\ \hline 2r-1 \end{array} + \begin{array}{c} 2r-1 \\ \hline \boxed{i_{2r-1}^-} \\ \hline \boxed{p_{2r-1}^-} \\ \hline 2r-1 \end{array}$$

equation (40) allows us to rewrite $g_{2r-2} \otimes f_1$: something we would not be able to do in TL. For every integer $0 \leq n < 2r - 2$ and every $j \in J_{m-1,n}$, we set

$$\begin{array}{c} n+1 \\ | \\ \boxed{u_m^{n+1,j}} \\ | \\ m \end{array} := \begin{array}{c} n \quad 1 \\ | \quad | \\ \boxed{u_{m-1}^{n,j}} \\ | \\ m-1 \quad 1 \end{array} \quad \begin{array}{c} m \\ | \\ \boxed{v_{n+1,j}^m} \\ | \\ n+1 \end{array} := \begin{array}{c} m-1 \quad 1 \\ | \quad | \\ \boxed{v_{n,j}^{m-1}} \\ | \\ n \quad 1 \end{array}.$$

On the other hand, for $n = 2r - 2$ and for every $j \in J_{m-1,2r-2}$, we set

$$\begin{array}{c} r-1 \\ | \\ \boxed{u_m^{r-1,(j,\varepsilon)}} \\ | \\ m \end{array} := \begin{array}{c} r-1 \\ | \\ \boxed{p_{2r-1}^\varepsilon} \\ | \\ \boxed{u_{m-1}^{2r-2,j}} \\ | \\ m-1 \quad 1 \end{array} \quad \begin{array}{c} m \\ | \\ \boxed{v_{r-1,(j,\varepsilon)}^m} \\ | \\ r-1 \end{array} := \begin{array}{c} m-1 \quad 1 \\ | \quad | \\ \boxed{v_{2r-2,j}^{m-1}} \\ | \\ \boxed{i_{2r-1}^\varepsilon} \\ | \\ r-1 \end{array}.$$

When $0 < n < r - 1$, we set

$$\begin{array}{c} n-1 \\ | \\ \boxed{u_m^{n-1,j}} \\ | \\ m \end{array} := -\frac{[n]}{[n+1]} \cdot \begin{array}{c} n-1 \\ | \\ \boxed{f_n} \\ | \\ \boxed{u_{m-1}^{n,j}} \\ | \\ m-1 \quad 1 \end{array} \quad \begin{array}{c} m \\ | \\ \boxed{v_{n-1,j}^m} \\ | \\ n-1 \end{array} := \begin{array}{c} m-1 \quad 1 \\ | \quad | \\ \boxed{v_{n,j}^{m-1}} \\ | \\ \boxed{f_n} \\ | \\ n-1 \end{array}$$

for every $j \in J_{m-1,n}$. When $n = r$, we set

$$\begin{array}{c} r-1 \\ | \\ \boxed{u_m^{r-1,(j,+)}} \\ | \\ m \end{array} := \begin{array}{c} r-2 \quad 1 \\ | \quad | \\ \boxed{f_{r-1}} \\ | \\ \boxed{u_{m-1}^{r,j}} \\ | \\ m-1 \quad 1 \end{array} \quad \begin{array}{c} m \\ | \\ \boxed{v_{r-1,(j,+)}^m} \\ | \\ r-1 \end{array} := \begin{array}{c} m-1 \quad 1 \\ | \quad | \\ \boxed{v_{r,j}^{m-1}} \\ | \\ \boxed{f_{r-1}} \\ | \\ r-1 \end{array}$$

$$\begin{array}{c} r-1 \\ | \\ \boxed{u_m^{r-1,(j,-)}} \\ | \\ m \end{array} := \begin{array}{c} r-1 \\ | \\ \boxed{u_{m-1}^{r,j}} \\ | \\ m-1 \quad 1 \end{array} + [2] \cdot \begin{array}{c} r-2 \quad 1 \\ | \quad | \\ \boxed{f_{r-1}} \\ | \\ \boxed{u_{m-1}^{r,j}} \\ | \\ m-1 \quad 1 \end{array} \quad \begin{array}{c} m \\ | \\ \boxed{v_{r-1,(j,-)}^m} \\ | \\ r-1 \end{array} := \begin{array}{c} m-1 \quad 1 \\ | \quad | \\ \boxed{v_{r,j}^{m-1}} \\ | \\ \boxed{f_{r-1}} \\ | \\ r-2 \quad 1 \end{array}$$

for every $j \in J_{m-1,r}$. When $r < n \leq 2r - 2$, we set

$$\begin{array}{c} n-1 \\ | \\ \boxed{u_{m-1,j}^{n-1,j}} \\ | \\ m \end{array} := -\frac{[n]}{[n+1]} \cdot \begin{array}{c} n-1 \\ | \\ \boxed{g_n} \\ | \\ \boxed{u_{m-1}^{n,j}} \\ | \\ m-1 \quad 1 \end{array} + \frac{2}{[n+1]^2} \cdot \begin{array}{c} n-1 \\ | \\ \boxed{h_n} \\ | \\ \boxed{u_{m-1}^{n,j}} \\ | \\ m-1 \quad 1 \end{array}$$

$$\begin{array}{c} m \\ | \\ \boxed{v_{n-1,j}^m} \\ | \\ n-1 \end{array} := \begin{array}{c} m-1 \quad 1 \\ | \quad | \\ \boxed{v_{n,j}^{m-1}} \\ | \\ \boxed{g_n} \\ | \\ n-1 \end{array}$$

for every $j \in J_{m-1,n}$. Therefore we can set

$$J_{m,n} := \begin{cases} J_{m-1,1} & n = 0, \\ J_{m-1,n-1} \sqcup J_{m-1,n+1} & 0 < n < r-1, \\ J_{m-1,r-2} \sqcup ((J_{m-1,r} \sqcup J_{m-1,2r-2}) \times \{+, -\}) & n = r-1, \\ J_{m-1,r-1} & n = r, \\ J_{m-1,n-1} \sqcup J_{m-1,n+1} & r < n < 2r-2, \\ J_{m-1,2r-3} & n = 2r-2. \end{cases} \quad \square$$

The second step in the proof of Theorem 1 consists in exhibiting morphisms of $\tilde{\text{TL}}$ whose image under $F_{\tilde{\text{TL}}}$ recovers the missing morphisms between indecomposable projective $\bar{U}$ -modules. To do this, we define morphisms $c_m^\varepsilon \in \tilde{\text{TL}}(g_m, g_{3r-m-2})$ and $d_m^\varepsilon \in \tilde{\text{TL}}(g_{3r-m-2}, g_m)$ as

$$\begin{array}{c} 3r-m-2 \\ | \\ \boxed{c_m^\varepsilon} \\ | \\ m \end{array} := \begin{array}{c} 3r-m-2 \\ | \\ \boxed{i_{2r-1}^\varepsilon} \\ | \\ r-1 \quad m-r+1 \end{array}$$

$$\begin{array}{c} m \\ | \\ \boxed{d_m^\varepsilon} \\ | \\ 3r-m-2 \end{array} := \begin{array}{c} r-1 \quad m-r+1 \\ | \quad | \\ \boxed{p_{2r-1}^\varepsilon} \\ | \\ 3r-m-2 \end{array} \quad (62)$$

for every integer $r \leq m \leq 2r - 2$, which immediately gives

$$\begin{array}{c} 3r-m-2 \\ | \\ \boxed{c_m^\varepsilon} \\ | \\ m \end{array} = \begin{array}{c} 3r-m-2 \\ | \\ \boxed{c_{m-1}^\varepsilon} \\ | \\ m-1 \quad 1 \end{array}$$

$$\begin{array}{c} m \\ | \\ \boxed{d_m^\varepsilon} \\ | \\ 3r-m-2 \end{array} = \begin{array}{c} m-1 \quad 1 \\ | \quad | \\ \boxed{d_{m-1}^\varepsilon} \\ | \\ 3r-m-2 \end{array}. \quad (63)$$

The fact that $c_m^\varepsilon g_m = c_m^\varepsilon$ and that $g_m d_m^\varepsilon = d_m^\varepsilon$ can be proved by induction using equations (37) to (39), (61), and (63).

Lemma 8. *For every integer $r \leq m \leq 2r-2$ and every $\varepsilon \in \{+, -\}$ the morphisms c_m^ε and d_m^ε satisfy*

$$F_{\tilde{\text{TL}}}(c_m^+) = (-1)^m \frac{[m-r]!}{[m+1]} \gamma_m^+, \quad (64)$$

$$F_{\tilde{\text{TL}}}(c_m^-) = -(-1)^m [m-r]! \gamma_m^-, \quad (65)$$

$$F_{\tilde{\text{TL}}}(d_m^+) = \frac{1}{[m-r+1]!} \gamma_{3r-m-2}^-, \quad (66)$$

$$F_{\tilde{\text{TL}}}(d_m^-) = -\frac{1}{[m-r+1]![m+1]} \gamma_{3r-m-2}^+, \quad (67)$$

where γ_m^+ and γ_m^- are defined by equations (12) and (13).

Proof. The statement is proved by induction on m . For $m = r$ it follows from computations in the proof of Lemma 6 establishing the image under $F_{\tilde{\text{TL}}}$ of equation (59). For $r < m \leq 2r-2$, equation (3) gives

$$\begin{aligned} b_0^m &= \frac{q^{-m}}{[m+1]} a_0^{m-1} \otimes a_1^1 + [m] q b_0^{m-1} \otimes a_1^1 + \frac{[m]}{[m+1]} b_1^{m-1} \otimes a_0^1, \\ x_0^{3r-m-1} &= x_0^{3r-m-2} \otimes a_0^1, \\ x_1^{3r-m-1} &= q^{m+2} x_0^{3r-m-2} \otimes a_1^1 + x_1^{3r-m-2} \otimes a_0^1, \\ y_0^{3r-m-1} &= -\frac{q^{m+1}}{[m]} a_{m-r}^{3r-m-2} \otimes a_1^1 + \frac{[m+1]}{[m]} y_0^{3r-m-2} \otimes a_0^1, \\ y_1^{3r-m-1} &= \frac{[m+1] q^{m+2}}{[m]} y_0^{3r-m-2} \otimes a_1^1 + \frac{[m+1]}{[m]} y_1^{3r-m-2} \otimes a_0^1, \end{aligned}$$

and equation (63) gives

$$\begin{aligned} F_{\tilde{\text{TL}}}(c_m^+)(b_0^m) &= (-1)^m \frac{[m-r]!}{[m+1]} x_0^{3r-m-2}, \\ F_{\tilde{\text{TL}}}(c_m^-)(b_0^m) &= -(-1)^m [m-r]! y_0^{3r-m-2}. \end{aligned}$$

This proves equations (64) and (65). Furthermore, equations (3) and (8) give

$$\begin{aligned} b_0^{3r-m-2} \otimes a_0^1 &= \frac{q^{-m-1}}{[m+1]^2} x_{2r-m-1}^{3r-m-1} + \frac{q^{-m-1}}{[m+1]^2} a_0'^{3r-m-3} - \frac{[m+2]}{[m+1]} b_0'^{3r-m-3}, \\ b_0^{3r-m-2} \otimes a_1^1 &= \frac{\{m+1\}'}{[m+1]^3} a_0^{3r-m-1} - \frac{[m]}{[m+1]^2} b_0^{3r-m-1} + \frac{\{m+1\}'}{[m+1]^3} \tilde{a}_1^{3r-m-3} \\ &\quad - \frac{[m+2]}{[m+1]^2} \tilde{b}_1^{3r-m-3}, \end{aligned}$$

and equation (3) gives

$$x_0^m := x_0^{m-1} \otimes a_0^1, \quad y_0^m := \frac{q^{-m}}{[m+1]} a_{2r-m-1}^{m-1} \otimes a_1^1 + \frac{[m]}{[m+1]} y_0^{m-1} \otimes a_0^1,$$

so equation (63) gives

$$\begin{aligned} F_{\tilde{\text{TL}}}(d_m^+)(b_0^{3r-m-2}) &= \frac{1}{[m-r+1]!} y_0^m, \\ F_{\tilde{\text{TL}}}(d_m^-)(b_0^{3r-m-2}) &= -\frac{1}{[m-r+1]![m+1]} x_0^m. \quad \square \end{aligned}$$

We can now prove Theorem 1.

Proof of Theorem 1. For convenience of notation, let us set

$$T_m := \begin{cases} X_m & 0 \leq m \leq r-1 \\ P_m & r \leq m \leq 2r-2 \end{cases} \in \bar{U}\text{-mod}$$

for every integer $0 \leq m \leq 2r-2$. Let f be a morphism in $\text{Hom}_{\bar{U}}(X^{\otimes m}, X^{\otimes m'})$ for some $m, m' \in \tilde{\text{TL}}$. Thanks to Proposition 7, we have

$$f = \sum_{n, n'=0}^{2r-2} \sum_{j \in J_{m,n}} \sum_{j' \in J_{m',n'}} F_{\tilde{\text{TL}}}(v_{n',j'}^{m'}, t_{n'} u_{m'}^{n',j'}) \circ f \circ F_{\tilde{\text{TL}}}(v_{n,j}^m t_n u_m^{n,j})$$

where $J_{m,n}$ and $J_{m',n'}$ are finite sets, and where

$$\begin{aligned} u_m^{n,j} &\in \tilde{\text{TL}}(m, n), & v_{n,j}^m &\in \tilde{\text{TL}}(n, m), \\ u_{m'}^{n',j'} &\in \tilde{\text{TL}}(m', n'), & v_{n',j'}^{m'} &\in \tilde{\text{TL}}(n', m') \end{aligned}$$

for all integers $0 \leq n, n' \leq 2r-2$ and for all $j \in J_{m,n}$ and $j' \in J_{m',n'}$. Now, thanks to Lemmas 5 and 8, the linear map $F_{\tilde{\text{TL}}} : \tilde{\text{TL}}(t_n, t_{n'}) \rightarrow \text{Hom}_{\bar{U}}(T_n, T_{n'})$ is surjective for all integers $0 \leq n, n' \leq 2r-2$, and therefore

$$F_{\tilde{\text{TL}}}(t_{n'} u_{m'}^{n',j'}) \circ f \circ F_{\tilde{\text{TL}}}(v_{n,j}^m t_n) \in \text{Hom}_{\bar{U}}(T_n, T_{n'})$$

belongs to its image. This means that f belongs to the image of the linear map $F_{\tilde{\text{TL}}} : \tilde{\text{TL}}(m, m') \rightarrow \text{Hom}_{\bar{U}}(X^{\otimes m}, X^{\otimes m'})$. $\square$

7. Additional relations

In this section, we list additional relations satisfied by images of generating morphisms $p_{2r-1}^+, p_{2r-1}^-, i_{2r-1}^+, i_{2r-1}^-$ in $\bar{U}\text{-mod}$. In order to do this, we define the *small Temperley-Lieb category* TL as the quotient linear category

$$\bar{\text{TL}} := \tilde{\text{TL}} / \ker F_{\tilde{\text{TL}}}.$$

In other words, objects of $\bar{\text{TL}}$ are the same as objects of $\tilde{\text{TL}}$, while vector spaces of morphisms of $\bar{\text{TL}}$ are quotients of vector spaces of morphisms of $\tilde{\text{TL}}$ with respect to kernels of linear maps determined by the linear functor $F_{\tilde{\text{TL}}}$.

Lemma 9. In $\bar{\text{TL}}$ generating morphisms p_{2r-1}^ε and i_{2r-1}^ε satisfy

$$\begin{array}{c} 2r-m-1 \\ | \\ \boxed{i_{2r-1}^\varepsilon} \\ | \\ r-1 \end{array} \begin{array}{c} \text{---} \\ | \\ m \end{array} = \bar{\varepsilon}[r-1]! \cdot \begin{array}{c} r-1 \quad r-m \\ | \quad | \\ \boxed{p_{2r-1}^\varepsilon} \\ | \\ m+r-1 \end{array} \quad (68)$$

for every integer $0 \leq m \leq r$ and every $\varepsilon \in \{+, -\}$, where $\bar{\varepsilon} \in \{+, -\}$ denotes the opposite of ε , and they satisfy

$$\begin{array}{c} m \\ | \\ \boxed{i_{2r-1}^{\varepsilon'}} \\ | \\ \boxed{p_{2r-1}^\varepsilon} \\ | \\ m \end{array} = (-1)^{m+1} \frac{\delta_{\varepsilon, \varepsilon'}}{[m+1]} \cdot \begin{array}{c} m \\ | \\ \boxed{h_m} \\ | \\ m \end{array} \quad (69)$$

$$\begin{array}{c} r-1 \quad m-r+1 \\ | \quad | \\ \boxed{p_{2r-1}^{\varepsilon'}} \\ | \\ \boxed{i_{2r-1}^\varepsilon} \\ | \\ r-1 \end{array} \begin{array}{c} \text{---} \\ | \\ m-r+1 \end{array} = (-1)^{m+1} \frac{\delta_{\varepsilon, \varepsilon'}}{[m+1]} \cdot \begin{array}{c} m \\ | \\ \boxed{h_m} \\ | \\ m \end{array} \quad (70)$$

for every integer $r \leq m \leq 2r-2$ and all $\varepsilon, \varepsilon' \in \{+, -\}$.

Proof. The result follows directly from equations (64) to (67). $\square$

Lemma 10. In $\bar{\text{TL}}$ generating morphisms p_{2r-1}^+ , p_{2r-1}^- , i_{2r-1}^+ , i_{2r-1}^- satisfy

$$\sum_{\varepsilon \in \{+, -\}} \begin{array}{c} m+r \quad r-1 \\ | \quad | \\ \boxed{p_{2r-1}^\varepsilon} \\ | \\ \boxed{i_{2r-1}^\varepsilon} \\ | \\ r-1 \end{array} \begin{array}{c} \text{---} \\ | \\ m+r \end{array} = \begin{array}{c} r \quad m \quad r-1 \\ | \quad | \quad | \\ \boxed{f_{2r-1}} \\ | \\ \boxed{f_{2r-1}} \\ | \\ r-1 \end{array} \begin{array}{c} \text{---} \\ | \\ m \end{array} \quad \sum_{\varepsilon \in \{+, -\}} \begin{array}{c} r-1 \quad m+r \\ | \quad | \\ \boxed{p_{2r-1}^\varepsilon} \\ | \\ \boxed{i_{2r-1}^\varepsilon} \\ | \\ m+r \end{array} \begin{array}{c} \text{---} \\ | \\ r-1 \end{array} = \begin{array}{c} r-1 \quad m \quad r \\ | \quad | \quad | \\ \boxed{f_{2r-1}} \\ | \\ \boxed{f_{2r-1}} \\ | \\ r \end{array} \begin{array}{c} \text{---} \\ | \\ m-r-1 \end{array} \quad (71)$$

for every integer $0 \leq m \leq r$.

Proof. The result follows from a straightforward computation using equations (58), (59), and (68). $\square$

A. Computations

In this appendix, we provide computations to be used in Sections 5 and 7. We start with the identity

$$[r-j-1]! = (-1)^j \frac{[r-1]!}{[j]!}. \quad (72)$$

Lemma 11. *For all integers $0 \leq j \leq r-1$ we have*

$$\begin{aligned} \Delta(E^j) &= \sum_{k=0}^j \frac{[j]!}{[k]![j-k]!} q^{k(j-k)} E^{j-k} \otimes E^k K^{j-k}, \\ \Delta(F^j) &= \sum_{k=0}^j \frac{[j]!}{[k]![j-k]!} q^{k(j-k)} F^k K^{-j+k} \otimes F^{j-k}. \end{aligned} \quad (73)$$

Proof. The statement follows from a direct computation. $\square$

Next, let us fix some notation. For all integers $0 \leq j \leq m$, let us set

$$V_j^m := \text{span}_{\mathbb{C}}\{a_{j_1}^1 \otimes \dots \otimes a_{j_m}^1 \in X^{\otimes m} \mid j_1 + \dots + j_m = j\},$$

and let us give special names to highest and lowest weight vectors

$$v_0^m := a_0^1 \otimes \dots \otimes a_0^1 \in V_0^m, \quad v_m^m := a_1^1 \otimes \dots \otimes a_1^1 \in V_m^m.$$

Lemma 12. *For every integer $0 \leq m \leq r-1$ we have*

$$a_0^m = v_0^m, \quad a_m^m = [m]! v_m^m, \quad (74)$$

for every integer $r \leq m \leq 2r-2$ we have

$$x_0^m = v_0^m, \quad y_{m-r}^m = \frac{[m-r]![r-1]!}{[m+1]} v_m^m, \quad (75)$$

and for $m = 2r-1$ we have

$$a_0^{r-1,+} = v_0^{2r-1}, \quad a_{r-1}^{r-1,-} = [r-1]!^2 v_{2r-1}^{2r-1}. \quad (76)$$

Proof. The statement can be proved by induction on m . $\square$

Lemma 13. *For all integers $0 \leq m \leq r-1$ and $0 \leq j \leq m$ we have*

$$\begin{aligned} E^j \cdot a_j^m &= \frac{[m]![j]!}{[m-j]!} a_0^m, \\ F^{m-j} \cdot a_j^m &= a_m^m, \end{aligned} \quad (77)$$

for all integers $r \leq m \leq 2r-2$, $0 \leq k \leq m-r$, and $0 \leq \ell \leq 2r-m-2$ we have

$$\begin{aligned} E^k \cdot x_k^m &= \frac{[m-r]![k]!}{[m-k-r]!} x_0^m, \\ E^{m+\ell-r+1} \cdot b_\ell^m &= (-1)^\ell \frac{[m-r]![m+\ell-r+1]![\ell]!}{[m+1]} x_0^m, \end{aligned} \quad (78)$$

$$F^{m-k-r} \cdot y_k^m = y_{m-r}^m,$$

$$F^{r-\ell-1} \cdot b_\ell^m = y_{m-r}^m,$$

and for every integer $0 \leq j \leq r-1$ we have

$$\begin{aligned} E^j \cdot a_j^{r-1,+} &= (-1)^j [j]!^2 a_0^{r-1,+}, \\ F^{r-j-1} \cdot a_j^{r-1,-} &= a_{r-1}^{r-1,-}. \end{aligned} \quad (79)$$

Proof. The statement follows from equations (72) to (76). $\square$

Next, we recall the definition for every integer $0 \leq m \leq r-1$ of the idempotent endomorphism $\varphi_m \in \text{End}_{\bar{U}}(X^{\otimes m})$ associated with the decomposition specified in equation (50), and similarly for $m = 2r-1$ of the idempotent endomorphisms $\varphi_{2r-1}^+, \varphi_{2r-1}^- \in \text{End}_{\bar{U}}(X^{\otimes 2r-1})$ associated with the decomposition specified in equation (52). For every integer $0 \leq j \leq m \leq r-1$, we have

$$\ker \varphi_m \cap V_j^m = \ker E^j \cap V_j^m, \quad \ker \varphi_m \cap V_{m-j}^m = \ker F^j \cap V_{m-j}^m,$$

and similarly, for every integer $0 \leq j \leq m = 2r-1$, we have

$$\ker \varphi_{2r-1}^+ \cap V_j^{2r-1} = \ker E^j \cap V_j^{2r-1}, \quad \ker \varphi_{2r-1}^- \cap V_{2r-j-1}^{2r-1} = \ker F^j \cap V_{2r-j-1}^{2r-1}.$$

Lemma 14. *For all integers $0 \leq j \leq m \leq r-1$ and $0 \leq k \leq r-m-1$ we have*

$$\begin{aligned} \varphi_{r-1}(a_j^m \otimes a_k^{r-m-1}) &= (-1)^j \frac{[m+k]!}{[m-j]![j+k]!} q^{-j(m+k+1)} a_{j+k}^{r-1}, \\ \varphi_{r-1}(a_k^{r-m-1} \otimes a_j^m) &= (-1)^j \frac{[m+k]!}{[m-j]![j+k]!} q^{(m-j)k} a_{j+k}^{r-1}, \end{aligned} \quad (80)$$

for every integer $0 \leq \ell \leq m-j-1$ we have

$$\begin{aligned} \varphi_{2r-1}^+(a_j^m \otimes a_\ell^{2r-m-1}) &= 0, \\ \varphi_{2r-1}^+(a_j^m \otimes x_k^{2r-m-1}) &= (-1)^j \frac{[m+k]!}{[m-j]![j+k]!} q^{-j(m+k+1)} a_{j+k}^{r-1,+}, \\ \varphi_{2r-1}^+(a_j^m \otimes y_k^{2r-m-1}) &= 0, \\ \varphi_{2r-1}^+(a_j^m \otimes b_\ell^{2r-m-1}) &= (-1)^{m+\ell+1} \frac{[m-j-\ell-1]![\ell]!}{[m-j]![m]} q^{-j(\ell+1)} a_{j+\ell-m+r}^{r-1,+}, \\ \varphi_{2r-1}^+(a_\ell^{2r-m-1} \otimes a_j^m) &= 0, \\ \varphi_{2r-1}^+(x_k^{2r-m-1} \otimes a_j^m) &= (-1)^j \frac{[m+k]!}{[m-j]![j+k]!} q^{(m-j)k} a_{j+k}^{r-1,+}, \\ \varphi_{2r-1}^+(y_k^{2r-m-1} \otimes a_j^m) &= 0, \\ \varphi_{2r-1}^+(b_\ell^{2r-m-1} \otimes a_j^m) &= (-1)^{m+\ell+1} \frac{[m-j-\ell-1]![\ell]!}{[m-j]![m]} q^{-(m-j)(m-\ell)} a_{j+\ell-m+r}^{r-1,+}, \end{aligned} \quad (81)$$

and for every integer $m-j \leq \ell \leq m-1$ we have

$$\begin{aligned}
\varphi_{2r-1}^-(a_j^m \otimes a_\ell^{2r-m-1}) &= 0, \\
\varphi_{2r-1}^-(a_j^m \otimes x_k^{2r-m-1}) &= 0, \\
\varphi_{2r-1}^-(a_j^m \otimes y_k^{2r-m-1}) &= (-1)^{j+1} \frac{[m+k]!}{[m-j]![j+k]![m]} q^{-j(m+k+1)} a_{j+k}^{r-1,-}, \\
\varphi_{2r-1}^-(a_j^m \otimes b_\ell^{2r-m-1}) &= (-1)^{j+1} \frac{[\ell]!}{[m-j]![j+\ell-m]![m]} q^{-j(\ell+1)} a_{j+\ell-m}^{r-1,-}, \\
\varphi_{2r-1}^-(a_\ell^{2r-m-1} \otimes a_j^m) &= 0, \\
\varphi_{2r-1}^-(x_k^{2r-m-1} \otimes a_j^m) &= 0, \\
\varphi_{2r-1}^-(y_k^{2r-m-1} \otimes a_j^m) &= (-1)^{j+1} \frac{[m+k]!}{[m-j]![j+k]![m]} q^{(m-j)k} a_{j+k}^{r-1,-}, \\
\varphi_{2r-1}^-(b_\ell^{2r-m-1} \otimes a_j^m) &= (-1)^{j+1} \frac{[\ell]!}{[m-j]![j+\ell-m]![m]} q^{-(m-j)(m-\ell)} a_{j+\ell-m}^{r-1,+}.
\end{aligned} \tag{82}$$

Proof. For all integers $0 \leq m+n \leq r-1$, $0 \leq j \leq m$ and $0 \leq k \leq n$, thanks to equation (77) we have

$$\begin{aligned}
E^{j+k} \cdot a_{j+k}^{m+n} &= \frac{[m+n]![j+k]!}{[m+n-j-k]!} a_0^{m+n}, \\
F^{m+n-j-k} \cdot a_{j+k}^{m+n} &= a_{m+n}^{m+n},
\end{aligned}$$

and thanks to equations (73) and (74) we have

$$\begin{aligned}
E^{j+k} \cdot a_j^m \otimes a_k^n &= \frac{[m]![n]![j+k]!}{[m-j]![n-k]!} q^{j(n-k)} a_0^{m+n}, \\
F^{m+n-j-k} \cdot a_j^m \otimes a_k^n &= \frac{[m]![n]![m+n-j-k]!}{[m+n]![m-j]![n-k]!} q^{j(n-k)} a_{m+n}^{m+n}.
\end{aligned}$$

This means

$$\varphi_{m+n}(a_j^m \otimes a_k^n) = \frac{[m]![n]![m+n-j-k]!}{[m+n]![m-j]![n-k]!} q^{j(n-k)} a_{j+k}^{m+n},$$

which implies in particular equation (80) thanks to equation (72). For all integers $0 \leq j \leq m \leq r-1$, $0 \leq k \leq r-m-1$, and $0 \leq \ell \leq m-j-1$, thanks to equation (79) we have

$$\begin{aligned}
E^{j+k} \cdot a_{j+k}^{r-1,+} &= (-1)^{j+k} [j+k]!^2 a_0^{r-1,+}, \\
E^{j+\ell-m+r} \cdot a_{j+\ell-m+r}^{r-1,+} &= (-1)^{m+j+\ell+1} \frac{[r-1]!^2}{[m-j-\ell-1]!^2} a_0^{r-1,+},
\end{aligned}$$

and thanks to equations (73) to (76) we have

$$\begin{aligned}
 E^{j+\ell-m+r} \cdot a_j^m \otimes a_\ell^{2r-m-1} &= 0, \\
 E^{j+k} \cdot a_j^m \otimes x_k^{2r-m-1} &= (-1)^k \frac{[m+k]![j+k]!}{[m-j]!} q^{-j(m+k+1)} a_0^{r-1,+}, \\
 E^{j+k+r} \cdot a_j^m \otimes y_k^{2r-m-1} &= 0, \\
 E^{j+\ell-m+r} \cdot a_j^m \otimes b_\ell^{2r-m-1} &= (-1)^j \frac{[\ell]![r-1]!^2}{[m-j-\ell-1]![m-j]![m]} q^{-j(\ell+1)} a_0^{r-1,+}, \\
 E^{j+\ell-m+r} \cdot a_\ell^{2r-m-1} \otimes a_j^m &= 0, \\
 E^{j+k} \cdot x_k^{2r-m-1} \otimes a_j^m &= (-1)^k \frac{[m+k]![j+k]!}{[m-j]!} q^{(m-j)k} a_0^{r-1,+}, \\
 E^{j+k+r} \cdot y_k^{2r-m-1} \otimes a_j^m &= 0, \\
 E^{j+\ell-m+r} \cdot b_\ell^{2r-m-1} \otimes a_j^m &= (-1)^j \frac{[\ell]![r-1]!^2}{[m-j-\ell-1]![m-j]![m]} q^{-(m-j)(m-\ell)} a_0^{r-1,+}.
 \end{aligned}$$

For all integers $0 \leq j \leq m \leq r-1$, $0 \leq k \leq r-m-1$, and $m-j \leq \ell \leq m-1$, thanks to equation (79) we have

$$\begin{aligned}
 F^{r-j-k-1} \cdot a_{j+k}^{r-1,-} &= a_{r-1}^{r-1,-}, \\
 F^{r+m-j-\ell-1} \cdot a_{j+\ell-m}^{r-1,-} &= a_{r-1}^{r-1,-},
 \end{aligned}$$

and, thanks to equations (73) to (76), we have

$$\begin{aligned}
 F^{r-j-\ell+m-1} \cdot a_j^m \otimes a_\ell^{2r-m-1} &= 0, \\
 F^{2r-j-k-1} \cdot a_j^m \otimes x_k^{2r-m-1} &= 0, \\
 F^{r-j-k-1} \cdot a_j^m \otimes y_k^{2r-m-1} &= (-1)^{j+1} \frac{[m+k]!}{[m-j]![j+k]![m]} q^{-j(m+k+1)} a_{r-1}^{r-1,-}, \\
 F^{r-j-\ell+m-1} \cdot a_j^m \otimes b_\ell^{2r-m-1} &= (-1)^{j+1} \frac{[\ell]!}{[m-j]![j+\ell-m]![m]} q^{-j(\ell+1)} a_{r-1}^{r-1,-}, \\
 F^{r-j-\ell+m-1} \cdot a_\ell^{2r-m-1} \otimes a_j^m &= 0, \\
 F^{2r-j-k-1} \cdot x_k^{2r-m-1} \otimes a_j^m &= 0, \\
 F^{r-j-k-1} \cdot y_k^{2r-m-1} \otimes a_j^m &= (-1)^{j+1} \frac{[m+k]!}{[m-j]![j+k]![m]} q^{(m-j)k} a_{r-1}^{r-1,-}, \\
 F^{r-j-\ell+m-1} \cdot b_\ell^{2r-m-1} \otimes a_j^m &= (-1)^{j+1} \frac{[\ell]!}{[m-j]![j+\ell-m]![m]} q^{-(m-j)(m-\ell)} a_{r-1}^{r-1,-}. \quad \square
 \end{aligned}$$

Lemma 15. For all integers $0 \leq m \leq r-1$ and $0 \leq j \leq m$ we have

$$\begin{aligned}
 F^j \cdot a_0^m &= a_j^m, \\
 E^{m-j} \cdot a_m^m &= \frac{[m]![m-j]!}{[j]!} a_j^m,
 \end{aligned} \tag{83}$$

for all integers $r \leq m \leq 2r-2$, $0 \leq k \leq m-r$, and $0 \leq \ell \leq 2r-m-2$ we have

$$\begin{aligned}
F^k \cdot x_0^m &= x_k^m, \\
F^{m+\ell-r+1} \cdot x_0^m &= a_\ell^m, \\
E^{r-\ell-1} \cdot y_{m-r}^m &= (-1)^\ell \frac{[m-r]![r-1]!^2}{[m+\ell-r+1]![\ell]![m+1]} a_\ell^m, \\
E^{m-k-r} \cdot y_{m-r}^m &= \frac{[m-r]![m-k-r]!}{[k]!} y_j^m,
\end{aligned} \tag{84}$$

and for every integer $0 \leq j \leq r-1$ we have

$$\begin{aligned}
F^j \cdot a_0^{r-1,+} &= a_j^{r-1,+}, \\
E^{r-j-1} \cdot a_{r-1}^{r-1,-} &= (-1)^j \frac{[r-1]!^2}{[j]!^2} a_j^{r-1,-}.
\end{aligned} \tag{85}$$

Proof. The statement follows from a direct computation. $\square$

Lemma 16. For all integers $0 \leq m \leq r-1$ and $0 \leq j \leq r-1$ we have

$$\begin{aligned}
a_j^{r-1} &= \sum_{k=\max\{m+j-r+1,0\}}^{\min\{m,j\}} \frac{[j]!}{[k]![j-k]!} q^{-(m-k)(j-k)} a_k^m \otimes a_{j-k}^{r-m-1}, \\
a_j^{r-1} &= \sum_{k=\max\{m+j-r+1,0\}}^{\min\{m,j\}} \frac{[j]!}{[k]![j-k]!} q^{(m+j-k+1)k} a_{j-k}^{r-m-1} \otimes a_k^m, \\
a_j^{r-1,+} &= \sum_{k=0}^{\max\{m+j-r,-1\}} \frac{[j]!}{[k]![j-k]!} q^{-(m-k)(j-k)} a_k^m \otimes a_{j-k+m-r}^{2r-m-1} \\
&\quad + \sum_{k=\max\{m+j-r+1,0\}}^{\min\{m,j\}} \frac{[j]!}{[k]![j-k]!} q^{-(m-k)(j-k)} a_k^m \otimes x_{j-k}^{2r-m-1}, \\
a_j^{r-1,+} &= \sum_{k=0}^{\max\{m+j-r,-1\}} \frac{[j]!}{[k]![j-k]!} q^{(m+j-k+1)k} a_{j-k+m-r}^{2r-m-1} \otimes a_k^m \\
&\quad + \sum_{k=\max\{m+j-r+1,0\}}^{\min\{m,j\}} \frac{[j]!}{[k]![j-k]!} q^{(m+j-k+1)k} x_{j-k}^{2r-m-1} \otimes a_k^m, \\
a_j^{r-1,-} &= \sum_{k=\max\{m+j-r+1,0\}}^{\min\{m,j\}} -\frac{[j]![m]}{[k]![j-k]!} q^{-(m-k)(j-k)} a_k^m \otimes y_{j-k}^{2r-m-1} \\
&\quad + \sum_{k=\min\{m+1,j+1\}}^m \frac{[j]![r-1]!}{[k]![j-k+r]!} q^{-(m-k)(j-k)} a_k^m \otimes a_{j-k+m}^{2r-m-1}, \\
a_j^{r-1,-} &= \sum_{k=\max\{m+j-r+1,0\}}^{\min\{m,j\}} -\frac{[j]![m]}{[k]![j-k]!} q^{(m+j-k+1)k} y_{j-k}^{2r-m-1} \otimes a_k^m \\
&\quad + \sum_{k=\min\{m+1,j+1\}}^m \frac{[j]![r-1]!}{[k]![j-k+r]!} q^{(m+j-k+1)k} a_{j-k+m}^{2r-m-1} \otimes a_k^m.
\end{aligned} \tag{86}$$

Proof. We have

$$\begin{aligned}
 a_j^{r-1} &= F^j \cdot a_0^{r-1} \\
 &= \sum_{k=\max\{m+j-r+1,0\}}^{\min\{m,j\}} \frac{[j]!}{[k]![j-k]!} q^{k(j-k)} F^k K^{-j+k} \cdot a_0^m \otimes F^{j-k} \cdot a_0^{r-m-1} \\
 &= \sum_{k=\max\{m+j-r+1,0\}}^{\min\{m,j\}} \frac{[j]!}{[k]![j-k]!} q^{-(m-k)(j-k)} a_k^m \otimes a_{j-k}^{r-m-1},
 \end{aligned}$$

and

$$\begin{aligned}
 a_j^{r-1} &= F^j \cdot a_0^{r-1} \\
 &= \sum_{k=\max\{m+j-r+1,0\}}^{\min\{m,j\}} \frac{[j]!}{[k]![j-k]!} q^{k(j-k)} F^{j-k} K^{-k} \cdot a_0^{r-m-1} \otimes F^k \cdot a_0^m \\
 &= \sum_{k=\max\{m+j-r+1,0\}}^{\min\{m,j\}} \frac{[j]!}{[k]![j-k]!} q^{(m+j-k+1)k} a_{j-k}^{r-m-1} \otimes a_k^m.
 \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 a_j^{r-1} &= (-1)^j \frac{[j]!^2}{[r-1]!^2} E^{r-j-1} \cdot a_{r-1}^{r-1} \\
 &= \sum_{k=\max\{m+j-r+1,0\}}^{\min\{m,j\}} (-1)^{j+k} \frac{[j]![m+j-k]!}{[m-k]![r-1]!^2} q^{-(m-k)(m+j-k+1)} \\
 &\quad E^{m-k} \cdot a_m^m \otimes E^{r-m-j+k-1} K^{m-k} \cdot a_{r-m-1}^{r-m-1} \\
 &= \sum_{k=\max\{m+j-r+1,0\}}^{\min\{m,j\}} \frac{[j]!}{[k]![j-k]!} q^{-(m-k)(j-k)} a_k^m \otimes a_{j-k}^{r-m-1}
 \end{aligned}$$

and

$$\begin{aligned}
 a_j^{r-1} &= (-1)^j \frac{[j]!^2}{[r-1]!^2} E^{r-j-1} \cdot a_{r-1}^{r-1} \\
 &= \sum_{k=\max\{m+j-r+1,0\}}^{\min\{m,j\}} (-1)^{j+k} \frac{[j]![m+j-k]!}{[m-k]![r-1]!^2} q^{-(m-k)(m+j-k+1)} \\
 &\quad E^{r-m-j+k-1} \cdot a_{r-m-1}^{r-m-1} \otimes E^{m-k} K^{-m-j+k-1} \cdot a_m^m \\
 &= \sum_{k=\max\{m+j-r+1,0\}}^{\min\{m,j\}} \frac{[j]!}{[k]![j-k]!} q^{(m+j-k+1)k} a_{j-k}^{r-m-1} \otimes a_k^m.
 \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 a_j^{r-1,+} &= F^j \cdot a_0^{r-1,+} \\
 &= \sum_{k=0}^{\min\{m,j\}} \frac{[j]!}{[k]![j-k]!} q^{k(j-k)} F^k K^{-j+k} \cdot a_0^m \otimes F^{j-k} \cdot x_0^{2r-m-1} \\
 &= \sum_{k=0}^{\max\{m+j-r,-1\}} \frac{[j]!}{[k]![j-k]!} q^{-(m-k)(j-k)} a_k^m \otimes a_{j-k+m-r}^{2r-m-1} \\
 &\quad + \sum_{k=\max\{m+j-r+1,0\}}^{\min\{m,j\}} \frac{[j]!}{[k]![j-k]!} q^{-(m-k)(j-k)} a_k^m \otimes x_{j-k}^{2r-m-1}
 \end{aligned}$$

and

$$\begin{aligned}
 a_j^{r-1,+} &= F^j \cdot a_0^{r-1,+} \\
 &= \sum_{k=0}^{\min\{m,j\}} \frac{[j]!}{[k]![j-k]!} q^{k(j-k)} F^{j-k} K^{-k} \cdot x_0^{2r-m-1} \otimes F^k \cdot a_0^m \\
 &= \sum_{k=0}^{\max\{m+j-r,-1\}} \frac{[j]!}{[k]![j-k]!} q^{(m+j-k+1)k} a_{j-k+m-r}^{2r-m-1} \otimes a_k^m \\
 &\quad + \sum_{k=\max\{m+j-r+1,0\}}^{\min\{m,j\}} \frac{[j]!}{[k]![j-k]!} q^{(m+j-k+1)k} x_{j-k}^{2r-m-1} \otimes a_k^m.
 \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 a_j^{r-1,-} &= (-1)^j \frac{[j]!^2}{[r-1]!^2} E^{r-j-1} \cdot a_{r-1}^{r-1,-} \\
 &= \sum_{k=\max\{m+j-r+1,0\}}^m (-1)^{j+k+1} \frac{[j]![m+j-k]![m]}{[m-k]![r-1]!^2} q^{-(m-k)(m+j-k+1)} \\
 &\quad E^{m-k} \cdot a_m^m \otimes E^{r-m-j+k-1} K^{m-k} \cdot y_{r-m-1}^{2r-m-1} \\
 &= \sum_{k=\max\{m+j-r+1,0\}}^{\min\{m,j\}} - \frac{[j]![m]}{[k]![j-k]!} q^{-(m-k)(j-k)} a_k^m \otimes y_{j-k}^{2r-m-1} \\
 &\quad + \sum_{k=\min\{m+1,j+1\}}^m \frac{[j]![r-1]!}{[k]![j-k+r]!} q^{-(m-k)(j-k)} a_k^m \otimes a_{j-k+m}^{2r-m-1}
 \end{aligned}$$

and

$$\begin{aligned}
 a_j^{r-1,-} &= (-1)^j \frac{[j]!^2}{[r-1]!^2} E^{r-j-1} \cdot a_{r-1}^{r-1,-} \\
 &= \sum_{k=\max\{m+j-r+1,0\}}^m (-1)^{j+k+1} \frac{[j]![m+j-k]![m]}{[m-k]![r-1]!^2} q^{-(m-k)(m+j-k+1)} \\
 &\quad E^{r-m-j+k-1} \cdot y_{r-m-1}^{2r-m-1} \otimes E^{m-k} K^{-m-j+k-1} \cdot a_m^m \\
 &= \sum_{k=\max\{m+j-r+1,0\}}^{\min\{m,j\}} - \frac{[j]![m]}{[k]![j-k]!} q^{(m+j-k+1)k} y_{j-k}^{2r-m-1} \otimes a_k^m \\
 &\quad + \sum_{k=\min\{m+1,j+1\}}^m \frac{[j]![r-1]!}{[k]![j-k+r]!} q^{(m+j-k+1)k} a_{j-k+m}^{2r-m-1} \otimes a_k^m. \quad \square
 \end{aligned}$$

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Funding Information Open Access funding provided by Universität Zürich.

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