Transformation Groups
Vol. 27, No. 3, 2022, pp. 867–895


# COMBINATORICS OF CANONICAL BASES REVISITED: STRING DATA IN TYPE A

V. GENZ*

Faculty of Mathematics
Ruhr-University Bochum
Universitätsstraße 150
44780 Bochum, Germany
volker.genz@gmail.com

G. KOSHEVOY**

The Institute for
Information Transmission Problems
Bolshoy Karetny per. 19, Building 1
127051 Moscow, Russia
koshevoyga@gmail.com

B. SCHUMANN***

Mathematical Institute
University of Cologne
Weyertal 86-90
50931 Cologne, Germany
bschuman@math.uni-koeln.de

**Abstract.** We give a formula for the crystal structure on the integer points of the string polytopes and the $*$-crystal structure on the integer points of the string cones of type $\mathsf{A}$ for arbitrary reduced words. As a byproduct, we obtain defining inequalities for Nakashima–Zelevinsky string polytopes. Furthermore, we give an explicit description of the Kashiwara $*$-involution on string data for a special choice of reduced word.

## Introduction

Let $\mathfrak{g}$ be a simple complex Lie algebra of rank $n-1$ and $V$ a finite dimensional representation of $\mathfrak{g}$. Much information of $V$ is encoded in a directed graph with arrows colored by $\{1, 2, \ldots, n-1\}$, called the crystal graph of $V$ [K91]. For instance, this crystal graph is connected if and only if $V$ is irreducible, the character of $V$ is encoded in the vertices of the crystals graph and there exists a simple notion of the tensor product of two crystal graphs yielding the crystal graph of the tensor product of two representations.

For $V$ irreducible, its crystal graph has a unique source corresponding to a highest weight vector of $V$. Making use of this fact, Littelmann [Lit98] and Beren-

DOI: 10.1007/s00031-021-09668-7
*Supported by SFB/TRR 191.
**Supported by RSF 16-11-10075 and SFB/TRR 191.
***Supported by SFB/TRR 191.
Received November 10, 2019. Accepted March 26, 2021.
Published online September 7, 2021.
Corresponding Author: V. Genz, e-mail: volker.genz@gmail.com

stein–Zelevinsky [BZ93], [BZ01] gave a bijection between the vertices of this graph as integer points of a rational convex polytope called the Littelmann–Berenstein–Zelevinsky string polytope.

The rule for assigning an integer point in the Littelmann–Berenstein–Zelevinsky string polytope to a vertex $v$ is as follows. Let $x_1$ be the largest integer such that there are $x_1$ consecutive arrows of color $i_1$ ending in $v$. Let $v_1$ be the source of this sequence of arrows. Let $x_2$ be the length of the longest sequence of arrows of a color $i_2$ ending in $v_1$ and so on. If we pick the colors $i_1, i_2, \ldots, i_N$ according to the appearance in a reduced decomposition of the longest Weyl group element of $\mathfrak{g}$, this procedure ends at the source of the graph. Then the vertex $v$ maps to the integer point $(x_1, x_2, \ldots, x_N) \in \mathbb{N}^N$, called the string datum of $v$.

Littelmann–Berenstein–Zelevinsky string polytopes have a vast amount of applications. They are generalizations of Gelfand–Tsetlin polytopes [Lit98], and appear as Newton-Okounkov bodies for flag varieties [FFL17], [K15] and in Gross–Hacking –Keel–Kontsevich's construction of canonical bases for cluster varieties [BF16], [GKS17].

We consider the following problem for the string polytope of an irreducible representation $V$ associated to the reduced word $\mathbf{i} = (i_1, i_2, \ldots, i_N)$ of the longest Weyl group element of $\mathfrak{g}$.

**Problem 1.** *Give a formula for the operator $f_a$ on the integer points of the string polytope $P$ defined as follows. For two integer points $x$ and $x'$ in $P$ we have $f_a x = x'$, if the corresponding vertices $v$ and $v'$ in the crystal graph are connected by an arrow of color $a$.*

Problem 1 is easy to solve for $a = i_1$. In this case we have

$$f_a(x_1, x_2, \ldots, x_N) = (x_1 + 1, x_2, \ldots, x_N).$$

There is, however, no obvious solution for arbitrary $a$. For $sl_3(\mathbb{C})$ and the reduced word $s_1 s_2 s_1$, one can deduce from an explicit construction of the crystal graph (e.g., [DKKA07]) that $f_2(x_1, x_2, x_3)$ is equal to $(x_1, x_2 + 1, x_3)$ if $x_1 \leq x_2 - x_3$ and $(x_1 - 1, x_2 + 1, x_3 + 1)$ otherwise. In this work, we solve Problem 1 by establishing a formula for the operator $f_a$ for any $a$ in the case that $\mathfrak{g} = sl_n(\mathbb{C})$.

For $a \in \{1, 2, \ldots, n-1\}$ and a reduced word $\mathbf{i} = (i_1, i_2, \ldots, i_N)$ of the longest element of the Weyl group of $sl_n(\mathbb{C})$, we define in Section 4 finitely many sequences $\gamma = (\gamma_j)$ of positive roots of $sl_n(\mathbb{C})$ with certain properties that we call *a-crossings*. These sequences come with an order relation $\preceq$. We further introduce maps $r$, $s$ associating to $\gamma$ the vectors $r(\gamma)$, $s(\gamma) \in \mathbb{Z}^N$.

Our main result reads as follows, where $\langle \cdot , \cdot \rangle$ is the standard scalar product on $\mathbb{Z}^N$:

**Theorem 5.1.** *Let $\gamma$ be minimal such that $\langle x, r(\gamma) \rangle$ is maximal. Then*

$$f_a x = x + s(\gamma).$$

Theorem 5.1 is in analogy to the Crossing Formula established in [GKS21, Thm. 2.13, Prop. 2.20], which computes the operator $f_a$ on the polytopes arising from

Lusztig's parametrizations of the crystal graph. Indeed, the two formulae may be viewed as dual since the roles of maximum and minimum and the vectors $r(\gamma)$, $s(\gamma)$ interchange. We elaborate on this duality in [GKS19].

Theorem 5.1 gives rise to two applications. The Verma module of $\mathfrak{g}$ of weight 0 has a crystal graph $B(\infty)$ with a unique source. Kashiwara [K93] defined an involution $*$ on the vertices of $B(\infty)$, leading to a second crystal graph $B(\infty)^*$ with the same set of vertices. Namely, there is an arrow from $v_1$ to $v_2$ of color $a$ in $B(\infty)^*$ if and only if there is an arrow from $v_1^*$ to $v_2^*$ of color $a$ in $B(\infty)$.

Associating integer vectors to the vertices of $B(\infty)^*$ by taking their string data, we obtain a rational polyhedral cone called the string cone [Lit98], [BZ93], [BZ01] which contains the Littelmann–Berenstein–Zelevinsky string polytope.

A variation of Problem 1 now arises, replacing the Littelmann–Berenstein–Zelevinsky string polytope by the string cone and the crystal graph of an irreducible representation by $B(\infty)^*$. In Theorem 5.2, we provide a solution to this problem in the case $\mathfrak{g} = sl_n$. Indeed the crystal graph of each irreducible representation $V$ is a full subgraph of $B(\infty)^*$. Making use of this fact, we deduce Theorem 5.2 from Theorem 5.1.

Alternatively, the crystal graph for the irreducible representation $V$ can be realized as a full subgraph of $B(\infty)$. The set of corresponding string parameters is again the set of integer points in a rational polytope, called the Nakashima–Zelevinsky string polytope, which was shown by Fujita–Naito [FN17] based on work of Kashiwara [K93], Littelmann [Lit98] and Nakashima–Zelevinsky [NZ97], [N99]. These polytopes have been found to coincide with Newton-Okounkov bodies for flag varieties [FN17], [FO17]. They also appear in [CFL] among Newton-Okounkov bodies inducing semitoric degenerations of Schubert varieties associated to maximal chains in the corresponding Bruhat graphs.

For Nakashima–Zelevinsky polytopes, Problem 1 has been solved in the work of Kashiwara [K93] and Nakashima–Zelevinsky [NZ97], [N99]. It is, however, a difficult problem to compute the inequalities that cut the Nakashima–Zelevinsky polytopes out of the string cone. A few special cases are treated in [N99], [H05]. Using Theorem 5.2, we obtain these inequalities for all reduced words of the longest Weyl group element of $sl_n$ in Theorem 6.1. Previously, Joseph independently gave a description of these inequalities valid for all reduced words $\mathbf{i}$ in [J18, Thm. 3.1] using the notion of $\mathbf{i}$-trails introduced by Berenstein–Zelevinsky in [BZ01]. It would be interesting to further investigate the relation between $\mathbf{i}$-trails and $a$-crossings.

The paper is organized as follows. In Section 1, we recall the background on crystals. In Section 2, we recall facts about reduced words for elements of the symmetric group. In Section 3 ,string cones and Littelmann–Berenstein–Zelevinsky string polytopes, as well as their crystal structures, are discussed.

In Section 4, we introduce the main combinatorial tools of this paper, namely the notion of wiring diagrams and Reineke crossings. The main result (Theorem 5.1), providing a formula for the crystal structure on Littelmann–Berenstein–Zelevinsky string polytopes, is stated in Section 5. We further prove the Dual Crossing Formula for the $*$-crystal structure on the string cone in this section.

In Section 6, Nakashima–Zelevinsky string polytopes are introduced and their defining inequalities are computed.

Section 7 deals with Lusztig's parametrization of the canonical basis and recalls facts from [GKS21] which are used in the proof of Theorem 5.1, which is presented in Section 8.

In Section 9, we give a description of the piecewise linear Kashiwara $*$-involution on string data. In particular, we obtain a linear isomorphism between the Littelmann–Berenstein–Zelevinsky polytope and the Nakashima–Zelevinsky polytope for a specific reduced word.

*Acknowledgements.* V. Genz thanks the Independent University of Moscow and the Labaratoire J.-V. Poncelet for their hospitality. G. Koshevoy thanks the University of Cologne and the Ruhr-Univerity Bochum for their hospitality. B. Schumann thanks Xin Fang, Peter Littelmann, Valentin Rappel, Christian Steinert, and Shmuel Zelikson for helpful discussions.

## 1. Crystals

### 1.1. Notation

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the natural numbers and $\mathfrak{g} = sl_n(\mathbb{C})$, $\mathfrak{h} \subset \mathfrak{g}$ its Cartan subalgebra consisting of the diagonal matrices in $\mathfrak{g}$. We abbreviate

$$[n] := \{1, 2, \ldots, n\}$$

and define for $k \in [n]$ the function $\epsilon_k \in \mathfrak{h}^*$ by $\epsilon_k(\mathrm{diag}(h_1, h_2, \ldots, h_n)) = h_k$. We denote by $\Phi^+$ the set of positive roots of $\mathfrak{g}$ given by

$$\Phi^+ = \{\alpha_{k,\ell} = \epsilon_k - \epsilon_\ell \mid 1 \le k < \ell \le n\}.$$

For $a \in [n-1]$, the simple root $\alpha_a$ of $\mathfrak{g}$ is given by $\alpha_a = \alpha_{a,a+1} = \epsilon_a - \epsilon_{a+1}$. We denote by $N = n(n-1)/2$ the cardinality of $\Phi^+$.

To $a \in [n-1]$ we associate the fundamental weight $\omega_a = \sum_{s \in [a]} \epsilon_s$ of $\mathfrak{g}$. Let $P \subset \mathfrak{h}^*$ (resp. $P^+ \subset \mathfrak{h}^*$) be the $\mathbb{Z}$-span (resp. $\mathbb{Z}_{\ge 0}$-span) of the set of fundamental weights $\{\omega_a\}_{a \in [n-1]}$ of $sl_n(\mathbb{C})$. We call $P$ the weight lattice and $P^+$ the set of dominant integral weights.

Let $U_q(sl_n)$ be the $\mathbb{Q}(q)$-algebra with generators $E_a, F_a, K_a^{\pm 1}$, $a \in [n-1]$ and the following relations for $b \in [n-1] \setminus \{a\}$

$$K_a K_a^{-1} = K_a^{-1} K_a = 1, \quad K_a K_b = K_b K_a, \quad K_a E_a K_a^{-1} = q^2 E_a,$$
$$K_a F_a K_a^{-1} = q^{-2} F_a, \quad E_a F_b - F_b E_a = 0, \quad E_a F_a - F_a E_a = \frac{K_a - K_a^{-1}}{q - q^{-1}},$$

$$\begin{aligned}
\text{if } b = a \pm 1 : \; & E_a^2 E_b + E_b E_a^2 = (q + q^{-1}) E_a E_b E_a, \\
& F_a^2 F_b + F_b F_a^2 = (q + q^{-1}) F_a F_b F_a, \\
& K_a E_b K_a^{-1} = q^{-1} E_b, \quad K_a F_b K_a^{-1} = q F_b, \\
\text{if } b \neq a \pm 1 : \; & E_a E_b = E_b E_a, \quad F_a F_b = F_b F_a, \\
& K_a E_b K_a^{-1} = E_b, \quad K_a F_b K_a^{-1} = F_b.
\end{aligned}$$

For $m \in \mathbb{N}$, let $[m]_q := q^{m-1} + q^{m-3} + \cdots + q^{-m+1}$. For $x \in U_q(sl_n)$ we set

$$x^{(m)} := \frac{x^m}{([m]_q[m-1]_q \cdots [2]_q)}. \tag{1}$$

For $\lambda \in P^+$ we denote by $V(\lambda)$ the irreducible $U_q(sl_n)$-module of highest weight $\lambda$.

We finally denote by $U_q^- \subset U_q(\mathrm{sl}_n)$ the subalgebra generated by $\{F_a\}_{a \in [n-1]}$.

### 1.2. Crystals

We recall the definition of crystals from [K94, Sect. 7].

**Definition 1.1.** A *crystal* $B$ is a set endowed with the following maps:

$$\begin{aligned} &\mathrm{wt} : B \to P, && \varepsilon_a : B \to \mathbb{Z} \sqcup \{-\infty\}, \quad \varphi_a : B \to \mathbb{Z} \sqcup \{-\infty\}, \\ &e_a : B \to B \sqcup \{0\}, && f_a : B \to B \sqcup \{0\} \quad \text{for } a \in [n-1]. \end{aligned}$$

Here 0 is an element not included in $B$. The above maps satisfy the following axioms for $a \in [n-1]$ and $b, b' \in B$

(C1) $\varphi_a(b) = \varepsilon_a(b) + \mathrm{wt}(b)(h_a)$,
(C2) if $b \in B$ satisfies $e_a b \neq 0$ then

$$\mathrm{wt}(e_a b) = \mathrm{wt}(b) + \alpha_a, \quad \varphi_a(e_a b) = \varphi_a(b) + 1, \quad \varepsilon_a(e_a b) = \varepsilon_a(b) - 1,$$

(C3) if $b \in B$ satisfies $f_a b \neq 0$ then

$$\mathrm{wt}(f_a b) = \mathrm{wt}(b) - \alpha_a, \quad \varphi_a(f_a b) = \varphi_a(b) - 1, \quad \varepsilon_a(f_a b) = \varepsilon_a(b) + 1,$$

(C4) $e_a b = b'$ if and only if $f_a b' = b$,
(C5) if $\varepsilon_a b = -\infty$, then $e_a b = f_a b = 0$.

Here we put $-\infty + k = -\infty$ for $k \in \mathbb{Z}$.

Let $B_1$ and $B_2$ be crystals. A map $\Lambda : B_1 \sqcup \{0\} \to B_2 \sqcup \{0\}$ satisfying $\Lambda(0) = 0$ is called a *strict morphism of crystals* if $\Lambda$ commutes with all $f_a$, $e_a$ $(a \in [n-1])$ and if for $b \in B_1$, $\Lambda(b) \in B_2$ we have

$$\mathrm{wt}(\Lambda(b)) = \mathrm{wt}(b), \quad \varepsilon_a(\Lambda(b)) = \varepsilon_a(b), \quad \varphi_a(\Lambda(b)) = \varphi_a(b)$$

for all $a \in [n-1]$. An injective strict morphism is called a *strict embedding of crystals* and a bijective strict morphism is called an *isomorphism of crystals.*

**Definition 1.2.** Let $B_1$ and $B_2$ be crystals. The set

$$B_1 \otimes B_2 := \{b_1 \otimes b_2 \mid b_1 \in B_2,\ b_2 \in B_2\}$$

equipped with the following crystal structure is called the *tensor product* of $B_1$ and $B_2$. For $a \in [n-1]$,

$$\begin{aligned} \mathrm{wt}(b_1 \otimes b_2) &= \mathrm{wt}(b_1) + \mathrm{wt}(b_2), \\ \varepsilon_a(b_1 \otimes b_2) &= \max\{\varepsilon_a(b_1), \varepsilon_a(b_2) - \mathrm{wt}(b_1)(h_a)\}, \\ \varphi_a(b_1 \otimes b_2) &= \max\{\varphi_a(b_2), \varphi_a(b_1) + \mathrm{wt}(b_2)(h_a)\}, \\ e_a(b_1 \otimes b_2) &= \begin{cases} e_a b_1 \otimes b_2 & \text{if } \varphi_a(b_1) \geq \varepsilon_a(b_2), \\ b_1 \otimes e_a b_2 & \text{else,} \end{cases} \\ f_a(b_1 \otimes b_2) &= \begin{cases} f_a b_1 \otimes b_2 & \text{if } \varphi_a(b_1) > \varepsilon_a(b_2), \\ b_1 \otimes f_a b_2 & \text{else.} \end{cases} \end{aligned}$$

### 1.3. Crystals of representations

We recall the *crystal bases* $B(\infty)$ and $B(\lambda)$ of $U_q^-$ and $V(\lambda)$, respectively, from [K91, Sects. 2 and 3].

Let $a \in [n-1]$. For $P \in U_q^-$ there exist unique $Q, R \in U_q^-$ such that

$$E_a P - P E_a = Q K_a + R K_a^{-1}.$$

We define $e_a'(P) = R$. As vector spaces, we have

$$U_q^- = \bigoplus_{m \geq 0} F_a^{(m)} \ker(e_a').$$

We define the *Kashiwara operators* $e_a$, $f_a$ on $U_q^-$ for $u \in \ker(e_a')$ by

$$f_a(F_a^{(m)} u) = F_a^{(m+1)} u, \quad e_a(F_a^{(m)} u) = F_a^{(m-1)} u. \tag{2}$$

Let $A$ be the subring of $\mathbb{Q}(q)$ consisting of rational functions $g(q)$ without a pole at $q = 0$. Let $\mathcal{L}(\infty)$ be the $A$-lattice generated by all elements of the form

$$f_{i_1} f_{i_2} \cdots f_{i_\ell}(1) \tag{3}$$

and let $B(\infty) \subset \mathcal{L}(\infty)/q\mathcal{L}(\infty)$ be the subset of all residues of elements of the form (3).

For $b \in B(\infty)$, let $\mathrm{wt}(b)$ be the weight of the corresponding element in $U_q^-$. For $a \in [n-1]$ we furthermore set $\varepsilon_a(b) = \max\{e_a^k \neq 0 \mid k \in \mathbb{N}\}$. This endows $B(\infty)$ with the structure of an crystal (see Definition 1.1).

We let $* : U_q^- \to U_q^-$ be the $\mathbb{Q}(q)$-anti-automorphism of $U_q^-$ such that $E_a^* = E_a$ for all $a \in [n-1]$. By [K93, Thm. 2.1.1], we have $B(\infty)^* = B(\infty)$. Clearly $*$ preserves the function wt. We denote by $f_a^*(x) = (f_a x^*)^*$, $e_a^*(x) = (e_a x^*)^*$ and $\varepsilon_a^*(x) = \varepsilon_a(x^*)$ the $*$-twisted maps. This endows $B(\infty)$ with a second structure of a crystal. We denote the crystal given by the set $B(\infty)$ and the twisted maps by $B(\infty)^*$. By construction, $*$ induces a crystal isomorphism between $B(\infty)$ and $B(\infty)^*$.

For $\lambda \in P^+$ let $\pi_\lambda : U_q^- \to V(\lambda)$ be the surjection $u \mapsto u v_\lambda$, where $v_\lambda$ is a highest weight vector of $V(\lambda)$. The operators $e_a$ and $f_a$ defined in (2) descend to $V(\lambda)$ and we denote by $\mathcal{L}(\lambda)$ the $A$-lattice generated by all elements of the form

$$f_{i_1} f_{i_2} \cdots f_{i_\ell}(v_\lambda) \tag{4}$$

and by $B(\lambda) \subset \mathcal{L}(\lambda)/q\mathcal{L}(\lambda)$ the subsets of all residues of elements of the form (4).

For $b \in B(\lambda)$, let $\mathrm{wt}(b)$ be the weight of the corresponding element in $V(\lambda)$. For $a \in [n-1]$, we furthermore set

$$\varepsilon_a(b) = \max\left\{e_a^k b \neq 0 \,\middle|\, k \in \mathbb{N}\right\},$$
$$\varphi_a(b) = \max\left\{f_a^k b \neq 0 \,\middle|\, k \in \mathbb{N}\right\}.$$

This endows $B(\lambda)$ with the structure of a crystal (see Definition 1.1).

We embed $B(\lambda)$ into $B(\infty)$ with accordingly shifted weight as follows. By [K91, Thm. 4] we have $\pi_\lambda(\mathcal{L}(\infty)) = \mathcal{L}(\lambda)$ inducing a map

$$\overline{\pi}_\lambda : \mathcal{L}(\infty)/q\mathcal{L}(\infty) \to \mathcal{L}(\lambda)/q\mathcal{L}(\lambda)$$

with the following properties:

(1) $f_a \circ \overline{\pi}_\lambda = \overline{\pi}_\lambda \circ f_a$ for all $a \in [n-1]$,
(2) If $\overline{\pi}_\lambda(b) \neq 0$ we have $e_a\overline{\pi}_\lambda(b) = \overline{\pi}_\lambda(e_a b)$ for all $a \in [n-1]$,
(3) $\overline{\pi}_\lambda : B(\infty) \setminus \overline{\pi}_\lambda^{-1}(0) \to B(\lambda)$ is bijective.

For $\lambda \in P$ an integral weight, let $R_\lambda = \{r_\lambda\}$ be the crystal consisting of one element satisfying $\mathrm{wt}(r_\lambda) = \lambda$, $\varepsilon_a(r_\lambda) = -\lambda(h_a)$, $\varphi_a(r_\lambda) = 0$ and $e_a r_\lambda = f_a r_\lambda = 0$ for all $a \in [n-1]$.

By [J95, Cor. 5.3.13], [N99, Thm. 3.1]

$$\widetilde{B}(\lambda) := \{b \otimes r_\lambda \in B(\infty) \otimes R_\lambda \mid \overline{\pi}_\lambda(b) \neq 0\}$$

is a subcrystal of $B(\infty) \otimes R_\lambda$ and $\overline{\pi}_\lambda$ induces an isomorphism of crystals $\widetilde{B}(\lambda) \cong B(\lambda)$. Furthermore,

$$\widetilde{B}(\lambda) = \{b \otimes r_\lambda \in B(\infty) \otimes R_\lambda \mid \varepsilon_a^*(b) \leq \lambda(h_a)\ \forall a \in [n-1]\} \cong B(\lambda). \tag{5}$$

## 2. Symmetric groups, reduced words and wiring diagrams

### 2.1. Symmetric groups and reduced words

Let $\mathfrak{S}_n$ be the symmetric group in $n$ letters. The group $\mathfrak{S}_n$ is generated by the simple transpositions $\sigma_a$ $(a \in [n-1])$ interchanging $a$ and $a+1$.

A reduced expression of $w \in \mathfrak{S}_n$ is a decomposition of $w$

$$w = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}$$

into a product of simple transpositions with a minimal possible number of factors. We call $k$ the *length* $\ell(w)$ *of* $w$. For a reduced expression of $w \in \mathfrak{S}_n$ we write $\mathbf{i} := (i_1, i_2, \ldots, i_k)$ and call $\mathbf{i}$ a *reduced word* (for $w$). The set of reduced words for $w$ is denoted by $\mathcal{W}(w)$.

The group $\mathfrak{S}_n$ has a unique longest element $w_0$ of length $N := n(n-1)/2$. We have two operations on the set of reduced words $\mathcal{W}(w_0)$.

**Definition 2.1.** A reduced word $\mathbf{j} = (j_1, \ldots, j_N) \in \mathcal{W}(w_0)$ is said to be obtained from $\mathbf{i} = (\mathbf{i}_1, i_k, i_{k+1}, \mathbf{i}_2) \in \mathcal{W}(w_0)$ by a 2-*move at position* $k \in [N-1]$ if $\mathbf{j} = (\mathbf{i}_1, i_{k+1}, i_k, \mathbf{i}_2)$ and $|i_k - i_{k+1}| > 1$.

A reduced word $\mathbf{j} = (j_1, \ldots, j_N)$ is said to be obtained from

$$\mathbf{i} = (\mathbf{i}_1, i_k, i_{k+1}, i_{k+2}, \mathbf{i}_2) \in \mathcal{W}(w_0)$$

by a 3-*move at position* $k \in [N-1]$ if $i_k = i_{k+2}$, $\mathbf{j} = (\mathbf{i}_1, i_{k+1}, i_k, i_{k+1}, \mathbf{i}_2)$ and $|i_k - i_{k+1}| = 1$.

A pair $(p, q) \in [n]^2$ with $p < q$ is called an *inversion for* $w \in \mathfrak{S}_n$ if $w(p) > w(q)$. Let $I(w)$ be the set of inversions for $w \in \mathfrak{S}_n$. A total ordering $<$ on $I(w)$ is called a *reflection ordering* or *convex ordering* if for any triple $(p,q),(p,r),(q,r) \in I(w)$ of pairwise distinct inversions we either have $(p,q) < (p,r) < (q,r)$ or $(q,r) < (p,r) < (p,q)$.

It is well known that the set of reflection orders on $I(w_0)$ is in natural bijection to $\mathcal{W}(w_0)$ (see, e.g., [D93, Prop. 2.13]). Under this bijection, the reflection order corresponding to $\mathbf{i} = (i_1, \dots, i_k) \in \mathcal{W}(w_0)$ is given by

$$(p_1, q_1) < \cdots < (p_k, q_k),$$

where $p_j = \sigma_{i_1} \cdots \sigma_{i_{j-1}}(i_j)$, $q_j = \sigma_{i_1} \cdots \sigma_{i_{j-1}}(i_j + 1)$.

*Remark 2.2.* Let $\mathbf{i} \in \mathcal{W}(w_0)$. The set $I(w_0)$ is in bijection with $\Phi^+$ via the map

$$(p,q) \mapsto \alpha_{p,q}, \tag{6}$$

where $\alpha_{p,q}$ is defined in Section 1.1. The reflection order corresponding to $\mathbf{i}$ induces a total ordering on $\Phi^+$ in this case.

## 3. String parametrizations

### 3.1. String parametrization

*Kashiwara embedding and string parameters.* Let $\mathbf{i} \in \mathcal{W}(w_0)$ and $b \in B(\infty)$. For $1 \le k \le N$ we recursively define

$$x_k = \varepsilon_{i_k}\left(e_{i_{k-1}}^{x_{k-1}} \cdots e_{i_1}^{x_1} b\right)$$

and call $\mathrm{str}_{\mathbf{i}}(b) := (x_1, \dots, x_N)$ the *string datum of $b$ in direction* $\mathbf{i}$.

By [Lit94, Lem. 5.3] we have

$$e_{i_N}^{x_N} \cdots e_{i_1}^{x_1} b = b_\infty, \tag{7}$$

where $b_\infty$ is the element in $B(\infty)$ of highest weight.

By (7) the map $\mathrm{str}_{\mathbf{i}}$ is injective. We denote by $\mathcal{S}_{\mathbf{i}} = \mathrm{str}_{\mathbf{i}}(B(\infty))$ the image of $\mathrm{str}_{\mathbf{i}}$. Let $\mathcal{S}_{\mathbf{i}}^{\mathbb{R}} \subset \mathbb{R}^N$ be the cone spanned by $\mathcal{S}_{\mathbf{i}}$. By [Lit98, Prop. 1.5], [BZ01, Prop. 3.5] $\mathcal{S}_{\mathbf{i}}^{\mathbb{R}}$ is a rational polyhedral cone, called the *string cone*, and $\mathcal{S}_{\mathbf{i}}$ are the integral points of $\mathcal{S}_{\mathbf{i}}^{\mathbb{R}}$.

Recall the definition of $\varepsilon_a^*$ and $e_a^*$ from Section 1.3. Now let

$$x_k = \varepsilon_{i_k}^*\left((e_{i_{k-1}}^*)^{x_{k-1}} \cdots (e_{i_1}^*)^{x_1} b\right).$$

We call $\mathrm{str}_{\mathbf{i}}^*(b) := (x_1, \dots, x_N)$ the $*$-*string datum of $b$ in direction* $\mathbf{i}$. The following is well known:

**Lemma 3.1.** *For $b \in B(\infty)$ we have*

$$\mathrm{str}_{\mathbf{i}}(b^*) = \mathrm{str}_{\mathbf{i}}^*(b), \tag{8}$$

$$b_\infty = (e_{i_N}^*)^{x_N} \cdots (e_{i_1}^*)^{x_1} b, \tag{9}$$

$$\mathcal{S}_{\mathbf{i}} = \mathrm{str}_{\mathbf{i}}^*(B(\infty)). \tag{10}$$

### 3.2. Crystal structures on string data

In this section, we equip $S_{\mathbf{i}}$ with two crystal structures isomorphic to $B(\infty)$.

For $a \in [n-1]$ and $k \in \mathbb{Z}$, let $b_a(k)$ be a formal symbol. We denote by $B_a := \{b_a(k) \mid k \in \mathbb{Z}\}$ the crystal, such that for $a' \in [n-1]$

$$\begin{aligned}
&\varepsilon_{a'}(b_a(k)) = \varphi_{a'}(b_a(-k)) = \begin{cases} -k, & \text{if } a = a', \\ -\infty, & \text{else,} \end{cases} \\
&\mathrm{wt}(b_a(k)) = k\alpha_a, \\
&f_{a'}(b_a(k)) = \begin{cases} b_a(k-1) & \text{if } a' = a, \\ 0 & \text{else,} \end{cases} \\
&e_{a'}(b_a(k)) = \begin{cases} b_a(k+1) & \text{if } a' = a, \\ 0 & \text{else.} \end{cases}
\end{aligned} \tag{11}$$

By [K93, Thm. 2.2.1] there exists for any $a \in [n-1]$ a unique strict embedding of crystals given by

$$\begin{aligned} \Lambda_a : B(\infty) &\hookrightarrow B(\infty) \otimes B_a \\ b_\infty &\mapsto b_\infty \otimes b_a(0). \end{aligned} \tag{12}$$

In [K93, Thm. 2.2.1 and its proof] (see also [NZ97, Sect. 2.4]) the following statement is proved.

**Lemma 3.2.** *Let $b \in B(\infty)$ and $m = \varepsilon_a^*(b)$. We have*

$$\Lambda_a(b) = (e_a^*)^m \, b \otimes b_a(-m).$$

Lemma 3.2 naturally provides two crystal structures on $\mathcal{S}_{\mathbf{i}}$ as follows. Let $\mathbf{i} = (i_1, \ldots, i_N) \in \mathcal{W}(w_0)$. We iterate the map (12) along $\mathbf{i}$ by setting

$$\Lambda_{\mathbf{i}} = \Lambda_{i_N} \circ \Lambda_{i_{N-1}} \circ \cdots \circ \Lambda_{i_1}.$$

Combining Lemma 3.1 with Lemma 3.2 we obtain the strict embedding

$$\Lambda_{\mathbf{i}}(b) = b_\infty \otimes b_{i_1}(-x_N) \otimes b_{i_2}(-x_{N-1}) \cdots \otimes b_{i_N}(-x_1),$$

where $(x_1, x_2, \ldots, x_N) = \mathrm{str}_{\mathbf{i}}^*(b) = \mathrm{str}_{\mathbf{i}}(b^*)$. Identifying $\mathcal{S}_{\mathbf{i}}$ with $\Lambda_{\mathbf{i}}(B(\infty))$ via

$$(x_1, \ldots, x_N) \mapsto b_\infty \otimes b_{i_N}(-x_N) \otimes \cdots \otimes b_{i_1}(-x_1).$$

yields two crystal structures $B(\infty)$ and $B(\infty)^*$ on $\mathcal{S}_{\mathbf{i}}$.

From $\Lambda_{\mathbf{i}}(B(\infty)) \subset \{b_\infty\} \otimes B_{i_N} \otimes \cdots \otimes B_{i_1}$, we obtain the following explicit description of the crystal structure on $\mathcal{S}_{\mathbf{i}}$ resulting from $B(\infty)$. Let $(c_{i,j})$ be the Cartan matrix of $\mathrm{sl}_n(\mathbb{C})$. For $k \in [N]$ and $x \in \mathcal{S}_{\mathbf{i}}$ we set

$$\eta_k(x) := x_k + \sum_{k < \ell \leq N} c_{i_k, i_\ell} x_\ell. \tag{13}$$

**Lemma 3.3** ([K02]). *The crystal structure on $\mathcal{S}_{\mathbf{i}}$ obtained from $B(\infty)$ via the bijection $b \mapsto \operatorname{str}^*_{\mathbf{i}}(b)$ is given as follows. For $x \in \mathcal{S}_{\mathbf{i}}$ and $a \in [n-1]$*

$$\begin{aligned}
&\varepsilon_a(x) = \max\left\{\eta_k(x) \mid k \in [N],\, i_k = a\right\}, \qquad \operatorname{wt}(x) = -\sum_{k=1}^{N} x_k \alpha_{i_k}, \\
&f_a(x) = x + (\delta_{k,\ell^x})_{k\in[N]}, \\
&e_a(x) = \begin{cases} x - (\delta_{k,\ell_x})_{k\in[N]} & \text{if } \varepsilon_a(x) > 0, \\ 0 & \text{else,} \end{cases}
\end{aligned} \tag{14}$$

*where $\ell^x \in [N]$ is minimal such that $i_{\ell^x} = a$ and $\eta_{\ell^x}(x) = \varepsilon_a(x)$ and where $\ell_x \in [N]$ is maximal such that $i_{\ell_x} = a$ and $\eta_{\ell_x}(x) = \varepsilon_a(x)$.*

The crystal structure on $\mathcal{S}_{\mathbf{i}}$ obtained from $B(\infty)^*$ via the bijection $b \mapsto \operatorname{str}^*_{\mathbf{i}}(b)$ is given as follows.

By [Lit98, Prop. 2.3] (see also [BZ93, Thm. 2.7]) we introduce piecewise linear bijections $\Psi^{\mathbf{i}}_{\mathbf{j}} : \mathcal{S}^{\mathbb{R}}_{\mathbf{i}} \to \mathcal{S}^{\mathbb{R}}_{\mathbf{j}}$ between the string cones associated to reduced words $\mathbf{i}, \mathbf{j} \in \mathcal{W}(w_0)$ satisfying for $b \in B(\infty)$

$$\Psi^{\mathbf{i}}_{\mathbf{j}} \circ \operatorname{str}_{\mathbf{i}}(b) = \operatorname{str}_{\mathbf{j}}(b) \tag{15}$$

as follows. If $\mathbf{j} \in \mathcal{W}(w_0)$ is obtained from $\mathbf{i} \in \mathcal{W}(w_0)$ by a 3-move at position $k$ we set $y = \Psi^{\mathbf{i}}_{\mathbf{j}}(x)$ with

$$\begin{gathered}
y = (x_1, \dots, x_{k-2}, x'_{k-1}, x'_k, x'_{k+1}, x_{k+2}, \dots, x_N), \\
x'_{k-1} = \max(x_{k+1}, x_k - x_{k-1}), \quad x'_k = x_{k+1} + x_{k-1} \text{ and} \\
x'_{k+1} = \min(x_k - x_{k-1}, x_{k+1}).
\end{gathered}$$

If $\mathbf{j} \in \mathcal{W}(w_0)$ is obtained from $\mathbf{i} \in \mathcal{W}(w_0)$ by a 2-move at position $k$ we set

$$\Psi^{\mathbf{i}}_{\mathbf{j}}(x_1, \dots, x_N) = (x_1, \dots, x_{k-1}, x_{k+1}, x_k, x_{k+2}, \dots, x_N).$$

For arbitrary $\mathbf{i}, \mathbf{j} \in \mathcal{W}(w_0)$ we define $\Psi^{\mathbf{i}}_{\mathbf{j}} : \mathcal{S}_{\mathbf{i}} \to \mathcal{S}_{\mathbf{j}}$ as the composition of the transition maps corresponding to a sequence of 2- and 3-moves transforming $\mathbf{i}$ into $\mathbf{j}$.

The following is well known:

**Lemma 3.4.** *Let $x \in \mathcal{S}_{\mathbf{i}}$, $a \in [n-1]$ and $\mathbf{j} \in \mathcal{W}(w_0)$ with $j_1 = a$. Setting $y := \Psi^{\mathbf{i}}_{\mathbf{j}}(x) \in \mathcal{S}_{\mathbf{j}}$ we have*

$$\begin{aligned}
&\varepsilon^*_a(x) = y_1, \qquad \operatorname{wt}(x) = -\sum_{k=1}^{N} x_k \alpha_{i_k}, \\
&f^*_a(x) = \Psi^{\mathbf{j}}_{\mathbf{i}}\left(y + (1,0,0,\dots)\right), \\
&e^*_a(x) = \begin{cases} \Psi^{\mathbf{j}}_{\mathbf{i}}\left(y - (1,0,0,\dots)\right) & \text{if } \varepsilon^*_a(x) > 0, \\ 0 & \text{else.} \end{cases}
\end{aligned} \tag{16}$$

In Theorem 5.2 we give a formula for the crystal structure of Lemma 3.4.

### 3.3. String polytopes and their crystals structures

Let $\lambda \in P^+$ and $\mathbf{i} \in \mathcal{W}(w_0)$. Recall from (5) that the crystal $B(\lambda)$ is isomorphic to the subcrystal $\widetilde{B}(\lambda)$ of $B(\infty) \otimes R_\lambda$. Hence, using (7) we get a bijection between $B(\lambda)$ and

$$\mathcal{S}_{\mathbf{i}}^*(\lambda) := \{\mathrm{str}_{\mathbf{i}}(b) \mid b \otimes r_\lambda \in B(\infty) \otimes R_\lambda, \varepsilon_a^*(b) \leq \lambda(h_a)\ \forall a \in [n-1]\}. \tag{17}$$

In [Lit98, Prop. 1.5] it is shown that $\mathcal{S}_{\mathbf{i}}^*(\lambda)$ is the set of integer points of the rational polytope

$$\mathcal{S}_{\mathbf{i}}^*(\lambda)^{\mathbb{R}} = \left\{ x \in \mathcal{S}_{\mathbf{i}}^{\mathbb{R}} \,\middle|\, x_k + \sum_{k<\ell\leq N} c_{i_k,i_\ell} x_k \leq \lambda_{i_k}\ \forall k \in [N] \right\} \subset \mathbb{R}^N. \tag{18}$$

We call $\mathcal{S}_{\mathbf{i}}^*(\lambda)^{\mathbb{R}}$ the *Littelmann–Berenstein–Zelevinsky string polytope.*

By (17) we obtain the following crystal structure isomorphic to $B(\lambda)$ on $\mathcal{S}_{\mathbf{i}}^*(\lambda) \subset \mathcal{S}_{\mathbf{i}}$. Denoting by $\iota_\lambda : \mathcal{S}_{\mathbf{i}}^*(\lambda) \hookrightarrow \mathcal{S}_{\mathbf{i}}$ the natural embedding we obtain

**Lemma 3.5.** *For $x \in \mathcal{S}_{\mathbf{i}}^*(\lambda)$ and $a \in [n-1]$ we have*

$$\varepsilon_a(x) = \varepsilon_a^*(\iota_\lambda(x)), \qquad \mathrm{wt}(x) = \lambda + \mathrm{wt}(\iota_\lambda(x)), \qquad \iota_\lambda e_a(x) = e_a^* \iota_\lambda(x),$$
$$\iota_\lambda f_a(x) = \begin{cases} f_a^* \iota_\lambda(x) & \text{if } \varphi_a(x) > 0, \\ 0 & \text{else.} \end{cases}$$

In Theorem 5.1 we give a formula for the crystal structure of Lemma 3.5.

## 4. Wiring diagrams and Reineke crossings

Following [BFZ96], we recall the notion of a wiring diagram, which is a graphical presentation of the reduced word $\mathbf{i} \in \mathcal{W}(w_0)$.

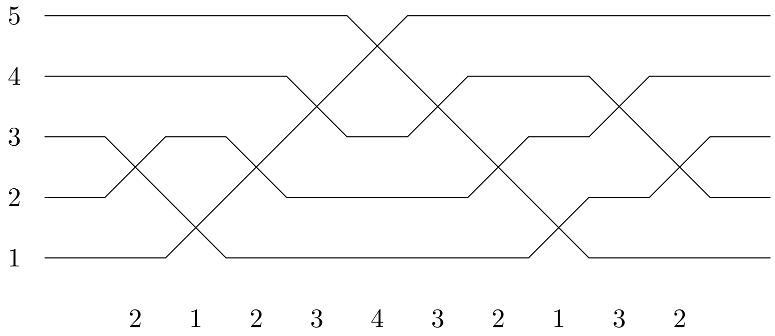
**Definition 4.1** (wiring diagram)**.** Let $\mathbf{i} = (i_1, i_2, \ldots i_N) \in \mathcal{W}(w_0)$. The *wiring diagram* $\mathcal{D}_{\mathbf{i}}$ consists of a family of $n$ piecewise straight lines, called *wires*, which can be viewed as graphs of $n$ continuous piecewise linear functions defined on the same interval. The wires have labels in the set $[n]$. Each vertex of $\mathcal{D}_{\mathbf{i}}$ (i.e., an intersection of two wires) represents a letter $j$ in $\mathbf{i}$. If the vertex corresponds to the letter $j \in [n-1]$, then $j-1$ is equal to the number of wires running below this intersection. We call

$$\mathrm{level}(v) := j$$

the *level of the vertex $v$.*

The word $\mathbf{i}$ can be read off from $\mathcal{D}_{\mathbf{i}}$ by reading the levels of the vertices from left to right.

**Example 4.2.** Let $n = 5$ and $\mathbf{i} = (2, 1, 2, 3, 4, 3, 2, 1, 3, 2)$. The corresponding wiring diagram $\mathcal{D}_{\mathbf{i}}$ is depicted below:




The condition $\mathbf{i} \in \mathcal{W}(w_0)$ implies that two lines $p, q$ with $p \neq q$ in $\mathcal{D}_{\mathbf{i}}$ intersect exactly once.

Each vertex of the wiring diagram $\mathcal{D}_{\mathbf{i}}$, $\mathbf{i} \in \mathcal{W}(w_0)$ corresponds to an inversion $(p, q) \in I(w_0)$, where $p$ and $q$ are the labels of the wires intersecting in that vertex. Thus the vertices of $\mathcal{D}_{\mathbf{i}}$ are in bijection with the positive roots by (6). The reflection order on $I(w_0)$ and the induced total order on $\Phi^+$ can be read off of $\mathcal{D}_{\mathbf{i}}$ by reading the vertices from left to right. We identify

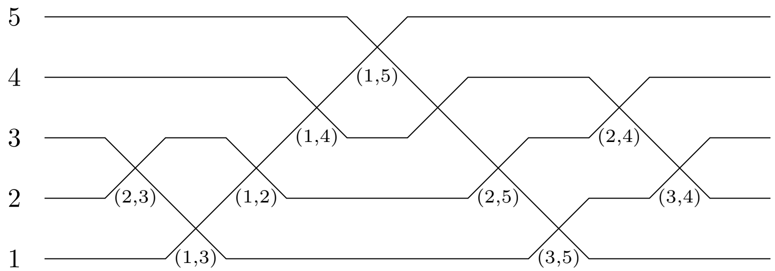
$$[N] \leftrightarrow I(w_0) = \left\{(p, q) \in [n]^2 \,\middle|\, p < q\right\} \tag{19}$$

such that $k \in [N]$ corresponds to the $k$-th vertex $(p, q) \in I(w_0)$ in $\mathcal{D}_{\mathbf{i}}$ from left.

**Example 4.3.** We continue with Example 4.2. The reflection ordering

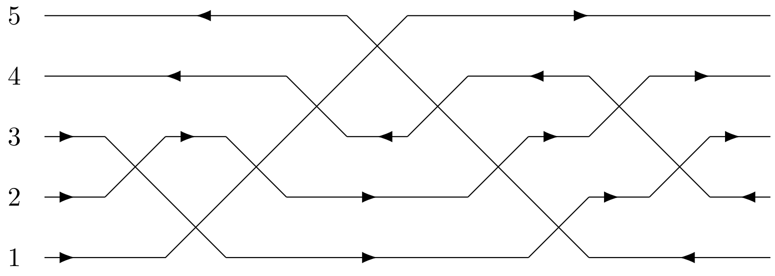
$$(2,3) < (1,3) < (1,2) < (1,4) < (1,5) < (4,5) < (2,5) < (3,5) < (2,4) < (3,4)$$

corresponding to $\mathbf{i}$ is depicted in the wiring diagram $\mathcal{D}_{\mathbf{i}}$ below:




**Definition 4.4.** Let $\mathbf{i} \in \mathcal{W}(w_0)$ and $\mathcal{D}_{\mathbf{i}}$ be the corresponding wiring diagram. For $a \in [n-1]$ we denote by $\mathcal{D}_{\mathbf{i}}(a)$ the oriented graph obtained from $\mathcal{D}_{\mathbf{i}}$ by orienting its wires $p$ from left to right if $p \leq a$, and from right to left if $p > a$.

**Example 4.5.** Let $a = 3$ and $\mathcal{D}_{\mathbf{i}}$ as in Example 4.2. The oriented graph $\mathcal{D}_{\mathbf{i}}(3)$ looks as follows:

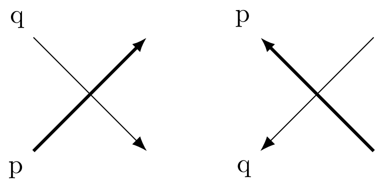



An oriented path in $\mathcal{D}_{\mathbf{i}}(a)$ is a sequence $(v_1, \ldots, v_k)$ of vertices of $\mathcal{D}_{\mathbf{i}}$, which are connected by oriented edges $v_1 \to v_2 \to \ldots \to v_k$ in $\mathcal{D}_{\mathbf{i}}(a)$.

**Definition 4.6 (Reineke crossings).** For $a \in [n-1]$ an *a-crossing* is an oriented path $\gamma = (v_1, \ldots, v_k)$ in $\mathcal{D}_{\mathbf{i}}(a)$ that starts with the leftmost vertex of the wire $a$ and ends with the leftmost vertex of the wire $a+1$. We say an $a$-crossing $\gamma$ is a Reineke crossing if $\gamma$ additionally satisfies the following condition: Whenever $v_j, v_{j+1}, v_{j+2}$ lie on the same wire $p$ in $\mathcal{D}_{\mathbf{i}}$ and the vertex $v_{j+1}$ lies on the intersection the wires $p$ and $q$, then

$$\begin{aligned} p > q \quad & \text{if } q \leq a, \\ p < q \quad & \text{if } a < q. \end{aligned}$$

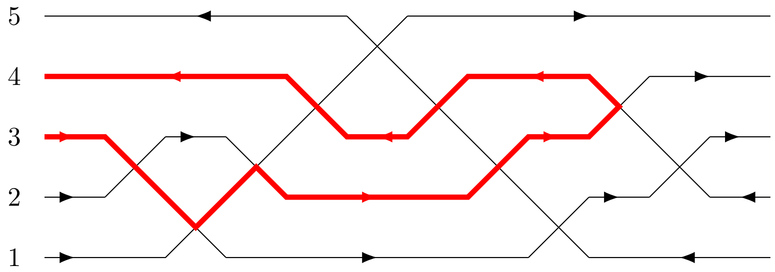
In other words, the path $\gamma$ avoids the following two fragments:




We denote the set of all $a$-Reineke crossings by $\Gamma_a$.

*Remark 4.7.* Reineke crossings appear as rigorous paths in [GP00].

**Example 4.8.** Let $n = 5$. The vertices lying on the path as highlighted below form the 3-Reineke crossing $\gamma = (v_{3,2}, v_{3,1}, v_{1,2}, v_{2,5}, v_{2,4}, v_{4,5}, v_{4,1})$:




In the remainder of this section, we adopt the following convention: we label each vertex $v = v_{p,q} \in \gamma$ by the wires $p$ and $q$ that intersect in this vertex where $p$ is the wire of the oriented edge in $\gamma$ whose source is $v_{p,q}$.

**Definition 4.9.** Let $a \in [n-1]$ and $\gamma = (v_{p_1,q_1}, v_{p_2,q_2}, \dots, v_{p_m,q_m}) \in \Gamma_a$. We call the set of vertices $v_{p_s,q_s}$ such that $p_{s+1} = q_s$ the turning points $T_\gamma$ of $\gamma$.

**Example 4.10.** For $\gamma = (v_{3,2}, v_{3,1}, v_{1,2}, v_{2,5}, v_{2,4}, v_{4,5}, v_{4,1})$ as in Example 4.8 we have $T_\gamma = \{v_{3,1}, v_{1,2}, v_{2,4}\}$.

Using the identification (19) we introduce:

**Definition 4.11.** The maps $r : \Gamma_a \to \mathbb{Z}^N$ and $s : \Gamma_a \to \mathbb{Z}^N$ are given by

$$(r(\gamma))_{p,q} := \begin{cases} \operatorname{sgn}(q-p) & \text{if } v_{p,q} \in T_\gamma, \\ 0 & \text{else,} \end{cases}$$

$$(s(\gamma))_{p,q} := \begin{cases} 1 & \text{if } v_{p,q} \in \gamma,\ p \le a < q \text{ or } q \le a < p, \\ -1 & \text{if } v_{p,q} \in \gamma \setminus T_\gamma,\ \ a < p,q \text{ or } p,q \le a, \\ 0 & \text{else.} \end{cases}$$

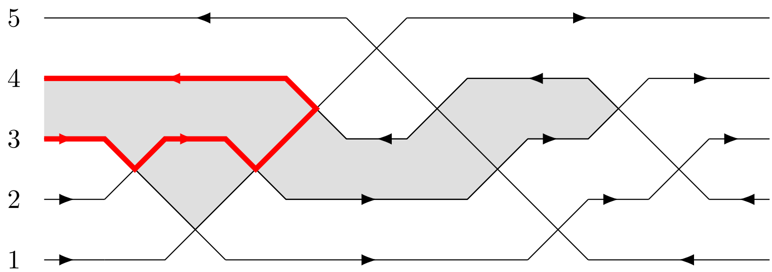
**Example 4.12.** Let $\gamma = (v_{3,2}, v_{3,1}, v_{1,2}, v_{2,5}, v_{2,4}, v_{4,5}, v_{4,1})$ be as in Example 4.8. We have

$$r(\gamma) = (0,-1,1,0,0,0,0,0,1,0), \quad s(\gamma) = (-1,0,0,1,0,-1,1,0,1,0).$$

By [GKS21, Prop. 2.2] we have the following order relation $\preceq$ on $\Gamma_a$:

**Definition 4.13.** Let $\gamma_1, \gamma_2 \in \Gamma_a$. We say $\gamma_1 \preceq \gamma_2$ if all vertices of $\gamma_1$ lie in the region of $\mathcal{D}_{\mathbf{i}}$ cut out by $\gamma_2$.

**Example 4.14.** Let $\gamma$ be as in Example 4.8 and $\gamma' = (v_{3,2}, v_{2,1}, v_{1,4})$. In the picture below, the region cut out by $\gamma$ is shaded grey while $\gamma'$ consists of all vertices lying on the highlighted path. Thus, $\gamma' \preceq \gamma$.




## 5. Dual Crossing Formula for string parametrizations

Let $\lambda \in P^+$ and $\mathbf{i} \in \mathcal{W}(w_0)$. In this section, we state our main result, which is a formula for the crystal structure on the integer points of the Littelmann–Berenstein–Zelevinsky string polytope $\mathcal{S}_{\mathbf{i}}^*(\lambda)^{\mathbb{R}}$ defined in (18).

Recall the notion of the set of $a$-Reineke crossings $\Gamma_a$ from Definition 4.6 and their associated vectors from Definition 4.11. We denote by $\langle \cdot, \cdot \rangle$ the standard scalar product on $\mathbb{Z}^N$. The crystal structure on $\mathcal{S}_{\mathbf{i}}^*(\lambda)$ from Lemma 3.5 is explicitly computed by:

**Theorem 5.1.** *For $\lambda \in P^+$, $a \in [n-1]$ and $x \in \mathcal{S}_{\mathbf{i}}^*(\lambda)$ we have*

$$\varepsilon_a(x) = \max\left\{\langle x, r(\gamma)\rangle \,|\, \gamma \in \Gamma_a\right\}, \tag{20}$$

$$\mathrm{wt}(x) = \lambda - \sum_{k\in[N]} x_k \alpha_{i_k}, \tag{21}$$

$$f_a(x) = \begin{cases} x + s(\gamma^x) & \text{if } \varphi_a(x) > 0,\\ 0 & \text{else,}\end{cases} \tag{22}$$

$$e_a(x) = \begin{cases} x - s(\gamma_x) & \text{if } \varepsilon_a(x) > 0,\\ 0 & \text{else,}\end{cases} \tag{23}$$

*where $\gamma^x \in \Gamma_a$ is minimal such that $\langle x, r(\gamma^x)\rangle = \varepsilon_a(x)$ and $\gamma_x \in \Gamma_a$ is maximal such that $\langle x, r(\gamma_x)\rangle = \varepsilon_a(x)$.*

Theorem 5.1 is proved in Section 8. A formula for the $*$-crystal structure on $\mathcal{S}_{\mathbf{i}}$ given in Lemma 3.4 can directly deduced from Theorem 5.1:

**Theorem 5.2** (Dual Crossing Formula)**.** *For $a \in [n-1]$ and $x \in \mathcal{S}_{\mathbf{i}}$ we have*

$$\varepsilon_a^*(x) = \max\left\{\langle x, r(\gamma)\rangle \,|\, \gamma \in \Gamma_a\right\},$$
$$f_a^*(x) = x + s(\gamma^x),$$
$$e_a^*(x) = \begin{cases} x - s(\gamma_x) & \text{if } \varepsilon_a(x) > 0,\\ 0 & \text{else,}\end{cases}$$

*where $\gamma^x \in \Gamma_a$ is minimal such that $\langle x, r(\gamma^x)\rangle = \varepsilon_a^*(x)$ and $\gamma_x \in \Gamma_a$ is maximal such that $\langle x, r(\gamma_x)\rangle = \varepsilon_a^*(x)$.*

*Proof.* Since $\mathcal{S}_{\mathbf{i}} = \cup_{\lambda\in P^+}\mathcal{S}_{\mathbf{i}}^*(\lambda)$, we can find for each $x \in \mathcal{S}_{\mathbf{i}}$ a $\lambda \in P^+$ such that $f_a^* x \in \mathcal{S}_{\mathbf{i}}^*(\lambda) = \{x \in \mathcal{S}_{\mathbf{i}} \mid \varepsilon_a(x) \le \lambda_a \ \forall a \in [n-1]\}$. Thus the claim follows from Lemma 3.5 and Theorem 5.1. □

*Remark 5.3.* The $*$-crystal structure on the string cone $\mathcal{S}_{\mathbf{i}}$ is dual to the crystal structure on Lusztig data, which is governed by the Crossing Formula 7.3 recalled below. By duality we understand the following: Maximum and minimum swap place as do the maps $r : \Gamma_a \to \mathbb{Z}^N$ and $s : \Gamma_a \to \mathbb{Z}^N$.

The $*$-crystal structure on Lusztig data $x \in \mathbb{N}^N$ is described by the $*$-Crossing Formula [GKS21, Thm. 2.20], which is completely analogous to the Crossing Formula for Lusztig data. In [GKS21, Thm. 4.4] we show that $\mathcal{S}_{\mathbf{i}}$ is polar to the set

$$\mathbf{R}^* = \left\{f_a^* x - x \,\middle|\, a \in [n-1],\, x \in \mathbb{N}^N\right\}, \tag{24}$$

i.e., the vectors $f_a^* x - x$ of the $*$-crystal structure on Lusztig data provide defining inequalities for $\mathcal{S}_{\mathbf{i}}$. For the special case of reduced words adapted to quivers, (24) was obtained in [Z13].

Similarly, the set of Lusztig data $\mathbb{N}^N$ is polar to

$$\{f_a x - x \,|\, x \in \mathcal{S}_{\mathbf{i}}\} = \left\{(\delta_{k,\ell})_{k\in[N]} \,\middle|\, \ell \in [N]\right\},$$

i.e., the vectors $f_a x - x$ of the crystal structure (14) on $\mathcal{S}_{\mathbf{i}}$ provide defining inequalities for the cone of Lusztig data $\mathbb{N}^N$. We refer to [GKS19] for more details.

## 6. Defining inequalities of Nakashima–Zelevinsky string polytopes

Theorem 5.1 provides a formula for the crystal structure on the Littelmann–Berenstein–Zelevinsky string polytope $\mathcal{S}^*_{\mathbf{i}}(\lambda)$. Switching the roles of $B(\infty)$ and $B(\infty)^*$ in the definition of $\mathcal{S}^*_{\mathbf{i}}(\lambda)$ one arrives at

$$\mathcal{S}_{\mathbf{i}}(\lambda) := \{x \in \mathcal{S}_{\mathbf{i}} \mid \varepsilon^*_a(x) \leq \lambda_a \ \forall a \in [n-1]\}.$$

Building up on [NZ97], $\mathcal{S}_{\mathbf{i}}(\lambda)$ and its crystal structure is defined in [N99].

By Lemma 3.4, the set $\mathcal{S}_{\mathbf{i}}(\lambda)$ consists of the integer points of the *Nakashima–Zelevinsky string polytope*

$$\mathcal{S}_{\mathbf{i}}(\lambda)^{\mathbb{R}} := \{x \in \mathcal{S}^{\mathbb{R}}_{\mathbf{i}} \mid \varepsilon^*_a(x) \leq \lambda_a \ \forall a \in [n-1]\},$$

where $\varepsilon^*_a$ on $\mathcal{S}_{\mathbf{i}}(\lambda)^{\mathbb{R}}$ is defined as in (16). By [FN17], the convex polytope $\mathcal{S}_{\mathbf{i}}(\lambda)^{\mathbb{R}}$ is rational. In this section, we solve the problem of deriving defining inequalities for $\mathcal{S}_{\mathbf{i}}(\lambda)^{\mathbb{R}} \subset \mathbb{R}^N$.

The Dual Crossing Formula (Theorem 5.2) immediately implies

**Theorem 6.1.** *The set $\mathcal{S}_{\mathbf{i}}(\lambda)^{\mathbb{R}} \subset \mathcal{S}^{\mathbb{R}}_{\mathbf{i}}$ is explicitly described by*

$$\mathcal{S}_{\mathbf{i}}(\lambda)^{\mathbb{R}} = \left\{x \in \mathcal{S}^{\mathbb{R}}_{\mathbf{i}} \,\middle|\, \langle x, r(\gamma)\rangle \leq \lambda_a \textit{ for all } a \in [n-1] \textit{ and for all } \gamma \in \Gamma_a\right\}.$$

*Remark 6.2.* Previously, Joseph independently gave a description of a set of defining inequalities for $\mathcal{S}_{\mathbf{i}}(\lambda)^{\mathbb{R}}$ in [J18, Thm. 3.1] using the notion of $\mathbf{i}$-trails introduced by Berenstein–Zelevinsky in [BZ01]. It would be interesting to further investigate the relation between $\mathbf{i}$-trails and $a$-crossings.

Using the explicit description of defining inequalities of $\mathcal{S}^{\mathbb{R}}_{\mathbf{i}}$ obtained in [GP00] we obtain defining inequalities of $\mathcal{S}_{\mathbf{i}}(\lambda)^{\mathbb{R}} \subset \mathbb{R}^N$. We recall the result of [GP00] for the convenience of the reader.

Using the notation of Section 4, let $\mathcal{D}_{\mathbf{i}}$ be the wiring diagram associated to $\mathbf{i} \in \mathcal{W}(w_0)$. For $a \in [n-1]$, let $\mathcal{D}_{\mathbf{i}}(a)^{\vee}$ be the graph obtained from $\mathcal{D}_{\mathbf{i}}(a)$ by reversing all arrows. For $a \in [n-1]$, an *$a$-rigorous path* is an oriented path $\gamma = (v_1, \dots, v_k)$ in $\mathcal{D}_{\mathbf{i}}(a)^{\vee}$ that starts with the rightmost vertex of the wire $a$ and ends with the rightmost vertex of the wire $a+1$. Additionally $\gamma$ satisfies the following condition: whenever $v_j, v_{j+1}, v_{j+2}$ lie on the same wire $p$ in $\mathcal{D}_{\mathbf{i}}$ and the vertex $v_{j+1}$ lies on the intersection the wires $p$ and $q$, we have

$$\begin{aligned} p > q \quad &\text{if } q \leq a, \\ p < q \quad &\text{if } a+1 \leq q. \end{aligned}$$

We denote the set of all $a$-rigorous paths by $\Gamma^*_a$.

For $\gamma \in \Gamma^*_a$, we define the set of turning points and the vector $r(\gamma)$ as in Definitions 4.9 and 4.11, respectively.

As a direct consequence of [GP00, Cor. 5.8] and Theorem 6.1, we obtain

**Corollary 6.3.** *The Nakashima–Zelevinsky string polytope $\mathcal{S}_{\mathbf{i}}(\lambda)^{\mathbb{R}}$ is explicitly described by*

$$\mathcal{S}_{\mathbf{i}}(\lambda)^{\mathbb{R}} = \left\{x \in \mathbb{R}^N \,\middle|\, \langle x, r(\gamma)\rangle \geq 0,\ \langle x, r(\gamma')\rangle \leq \lambda_a \ \forall a \in [n-1], \gamma \in \Gamma_a^*, \gamma' \in \Gamma_a \right\}.$$

For the sake of completeness we recall the crystal structure on $\mathcal{S}_{\mathbf{i}}(\lambda)$. For $k \in [N]$ we consider the function $\eta_k$ on $\mathcal{S}_{\mathbf{i}}(\lambda)$ defined in (13). Analogously to Lemma 3.5 we have:

**Lemma 6.4** ([N99])**.** *The following defines a crystal structure on $\mathcal{S}_{\mathbf{i}}(\lambda)$ isomorphic to $B(\lambda)$. For $x \in \mathcal{S}_{\mathbf{i}}(\lambda)$ and $a \in [n-1]$*

$$\varepsilon_a(x) = \max\{\eta_k(x) \mid k \in [N],\, i_k = a\}, \qquad \mathrm{wt}(x) = \lambda - \sum_{k\in[N]} x_k \alpha_{i_k},$$

$$f_a(x) = \begin{cases} x + (\delta_{k,\ell_x})_{k\in[N]} & \textit{if } \varphi_a(x) > 0, \\ 0 & \textit{else}, \end{cases}$$

$$e_a(x) = \begin{cases} x - (\delta_{k,\ell^x})_{k\in[N]} & \textit{if } \varepsilon_a(x) > 0, \\ 0 & \textit{else}, \end{cases}$$

*where $\ell^x \in [N]$ is minimal such that $i_{\ell^x} = a$ and $\eta_{\ell^x}(x) = \varepsilon_a(x)$ and where $\ell_x \in [N]$ is maximal such that $i_{\ell_x} = a$ and $\eta_{\ell_x}(x) = \varepsilon_a(x)$.*

## 7. The Crossing Formula on Lusztig data

The main ingredient in the proof of Theorem 5.1 is the Crossing Formula proved in [GKS21], which we recall in this section.

### 7.1. Lusztig's parametrization of the canonical basis

Lusztig [L90] associated to a reduced word $\mathbf{i} = (i_1, i_2, \ldots, i_N) \in \mathcal{W}(w_0)$ a PBW-type basis $B_{\mathbf{i}}$ of $U_q^-$ as follows. Let $\beta_1 < \beta_2 < \cdots < \beta_N$ be the total ordering of $\Phi^+$ corresponding to $\mathbf{i}$ via Remark 2.2. We set

$$F_{\mathbf{i},\beta_m} := T_{i_1} T_{i_2} \cdots T_{i_{m-1}} F_{i_m},$$

where $T_i$ acts via the braid group action defined in [Lu90, Sect. 1.3]. The divided powers $x^{(m)}$ for $x \in U_q^-$ are defined in (1). Then the PBW-type basis

$$\mathbf{B}_{\mathbf{i}} := \left\{ F_{\mathbf{i},\beta_1}^{(x_1)} F_{\mathbf{i},\beta_2}^{(x_2)} \cdots F_{\mathbf{i},\beta_N}^{(x_N)} \,\middle|\, (x_1, x_2, \ldots, x_N) \in \mathbb{N}^N \right\}$$

is in natural bijection with the canonical basis $\mathbf{B}$ of $U_q^-$ (see [L90, Prop. 2.3, Theorem 3.2]).

**Definition 7.1.** We call $x = (x_1, x_2, \ldots, x_N) \in \mathbb{N}^N$, the $\mathbf{i}$-*Lusztig datum* of the element $F_{\mathbf{i},\beta_1}^{(x_1)} F_{\mathbf{i},\beta_2}^{(x_2)} \cdots F_{\mathbf{i},\beta_N}^{(x_N)} \in \mathbf{B}_{\mathbf{i}}$.

### 7.2. Crystal structures on Lusztig's parametrizations

Let $\mathbf{i}$ and $\mathbf{j}$ be two reduced words for $w_0$. A piecewise linear bijection $\Phi_{\mathbf{j}}^{\mathbf{i}} : \mathbb{N}^N \to \mathbb{N}^N$ from the set of $\mathbf{i}$-Lusztig data to the set of $\mathbf{j}$-Lusztig data is defined in [L90, Sect. 2.1] using the fact that any reduced word $\mathbf{j}$ can be obtained from any other reduced word $\mathbf{i}$ by applying a sequence of 2- and 3-moves given in Definition 2.1.

Let $\mathbf{i} \in \mathcal{W}(w_0)$ with corresponding total ordering $\beta_1 < \beta_2 < \cdots < \beta_N$ of $\Phi^+$ as in Remark 2.2. The crystal structure on $\mathbf{i}$-Lusztig data $\mathbb{N}^N$ obtained from $B(\infty)$ via the bijection

$$(x_1, \ldots, x_N) \mapsto b_{\mathbf{i}}(x) := F_{\mathbf{i},\beta_1}^{(x_1)} F_{\mathbf{i},\beta_2}^{(x_2)} \cdots F_{\mathbf{i},\beta_N}^{(x_N)} \in \mathbf{B}_{\mathbf{i}} \simeq \mathbf{B} \tag{25}$$

is given as follows (see [L93], also [BZ01, Prop. 3.6]).

**Proposition 7.2.** *Let $a \in [n-1]$ and $\mathbf{j} \in \mathcal{W}(w_0)$ with $j_1 = a$. For an $\mathbf{i}$-Lusztig datum $x \in \mathbb{N}^N$ and $y := \Phi_{\mathbf{j}}^{\mathbf{i}}(x)$*

$$\begin{aligned}
&\varepsilon_a(x) = y_1, \qquad \mathrm{wt}(x) = -\sum_{k\in[N]} x_k\beta_k,\\
&f_a(x) = \Phi_{\mathbf{i}}^{\mathbf{j}}\left(y + (1,0,0,\ldots)\right),\\
&e_a(x) = \begin{cases} \Phi_{\mathbf{i}}^{\mathbf{j}}\left(y - (1,0,0,\ldots)\right) & \text{if } \varepsilon_a(x) > 0\\ 0, & \text{else.}\end{cases}
\end{aligned}$$

The main result of [GKS21] is the Crossing Formula for the crystal structure from Proposition 7.2. Using (5) this leads for $\lambda \in P^+$ to a formula for the crystal structure on $\mathcal{L}_{\mathbf{i}}(\lambda) := \{x \in \mathbb{N}^N \mid \varepsilon_a^*(x) \leq \lambda_a \ \forall a \in [n-1]\}$ isomorphic to $B(\lambda)$:

**Theorem 7.3** ([GKS21, Thm. 2.13, Prop. 2.20])**.** *Let $a \in [n-1]$, $\lambda \in P^+$ and $x \in \mathcal{L}_{\mathbf{i}}(\lambda)$. We have*

$$\begin{aligned}
&\varepsilon_a(x) = \max\left\{\langle x, s(\gamma)\rangle \mid \gamma \in \Gamma_a\right\}, \qquad \mathrm{wt}(x) = \lambda - \sum_{k\in[N]} x_k\beta_k,\\
&f_a(x) = \begin{cases} x + r(\gamma_x) & \text{if } \varphi_a(x) > 0,\\ 0 & \text{else,}\end{cases}\\
&e_a(x) = \begin{cases} x - r(\gamma^x) & \text{if } \varepsilon_a(x) > 0,\\ 0 & \text{else,}\end{cases}
\end{aligned}$$

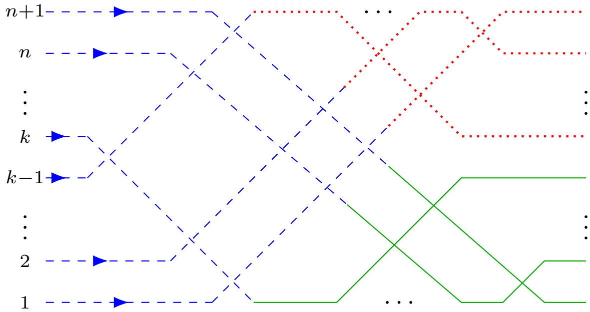
*where $\gamma^x \in \Gamma_a$ is minimal such that $\langle x, s(\gamma^x)\rangle = \varepsilon_a(x)$ and $\gamma_x \in \Gamma_a$ is maximal such that $\langle x, s(\gamma_x)\rangle = \varepsilon_a(x)$.*

*Remark 7.4.* An explicit form of the crystal structure on $\mathbf{i}$-Lusztig data was known in several cases before. Let $\mathfrak{g}$ be a simple, finite dimensional, complex Lie algebra and $w_0$ be the longest element of the Weyl group of $\mathfrak{g}$.

In [R97], a rule was given for all reduced words corresponding to a quiver satisfying a certain homological condition. In type $\mathsf{A}_n$, this condition is always satisfied and our Crossing Formula in [GKS21] is a generalization of Reineke's rule to all reduced words in type $\mathsf{A}_n$ (i.e., not necessarily adapted to a quiver).

In [SST18], a combinatorial "bracketing rule" describing the crystal structure for so-called "simply braided" reduced words for $w_0$ has been established. For $\mathfrak{g}$ of type $\mathsf{A}_n$, a word is simply braided if and only if for all $a \in I$ all paths in $\Gamma_a$ consist only of vertices on the $a$-wire, and $a+1$-wire which is the case if and only if $\Gamma_a$ is linearly oriented. As a consequence, restricting to simply-braided words $\mathbf{i}$ Theorem 5.2 becomes a bracketing rule for the computation of the $\star$-crystal structure on $\mathcal{S}_{\mathbf{i}}$ in type $\mathsf{A}$.

In [K18], Reineke's rule [R97] is applied for $\mathfrak{g}$ of type $\mathsf{A}_n$ and for reduced words adapted to quivers with a single sink to give a crystal isomorphism to Young tableaux. By [BFZ96, Prop. 4.4.1] and [E97, Lem. 2.1], the reduced words adapted to quivers with a single sink are simply braided and correspond to wiring diagrams of the following form, where the sink is at vertex $k$:




An essential ingredient of the crystal isomorphism in [K18] is the following tensor product decomposition given in [K18, Thm. 4.2]:

$$\mathbb{N}^N \simeq \mathbb{N}^J \otimes \mathbb{N}^{J_1} \otimes \mathbb{N}^{\mathbf{J_2}}, \tag{26}$$

$$J = \{\alpha_{p,q} \,|\, p \leq a < q\}, \quad J_1 = \{\alpha_{p,q} \,|\, p,q \leq a\}, \quad J_2 = \{\alpha_{p,q} \,|\, p,q > a\}.$$

Here $\mathbb{N}^I$ denotes for $I \subset \Phi^+$ the crystal obtained by applying the Crossing Formula with $\Gamma_a$ replaced by $\{\gamma \in \Gamma_a \,|\, r(\gamma_a) \subset \mathbb{N}^I\}$ and $s(\gamma)$ replaced by $s(\gamma)|_{\mathbb{N}^I}$. The sets $J$, $J_1$ and $J_2$ correspond to the dashed, dotted and solid parts in the above picture, respectively.

More generally, the decomposition (26) can be deduced for all simply braided reduced words from the bracketing rule or alternatively from the Crossing Formula as follows. We denote the restriction of the crystal $\mathbb{N}^I$ obtained by forgetting the root operators $f_b$ for $b \neq a$ by $\mathbb{N}^I|_a$. We abbreviate $I_{p,q} := \{\alpha_{p,q}, \alpha_{p+1,q}\}$ and $C_q := \mathbb{N}^{I_{a,q}}|_a$. Since for simply braided words the Reineke lattice $\Gamma_a$ is linearly ordered, we obtain alternatively from the Crossing Formula or the bracketing rule the tensor product decomposition

$$\mathbb{N}^N|_a \simeq \begin{cases} C_k \otimes \ldots \otimes C_{n+1} \otimes C_{k-1} \otimes \ldots \otimes C_{a+2} \otimes \mathbb{N}^{\{\alpha_{a,a+1}\}}|_a \otimes \mathbb{N}^{I_a}|_a, & \text{if } a<k, \\ \mathbb{N}^{\{\alpha_{a,a+1}\}}|_a \otimes \mathbb{N}^{I_a}|_a, & \text{if } a=k, \\ C_{k-1} \otimes \ldots \otimes C_1 \otimes C_k \otimes \ldots \otimes C_{a-1} \otimes \mathbb{N}^{\{\alpha_{a,a+1}\}}|_a \otimes \mathbb{N}^{I_a}|_a, & \text{if } a>k \end{cases} \tag{27}$$

for suitable $I_a \subset \Phi^+$. We remark that $\mathbb{N}^{I_a}|_a$ is the trivial crystal with weight function obtained by restricting the weight function on $\mathbb{N}^N$. Similarly, we obtain

$$\begin{aligned}
\mathbb{N}^J|_a &\simeq \begin{cases} C_k \otimes \cdots \otimes C_{n+1} \otimes \mathbb{N}^{I'_a}|_a, & \text{if } a < k, \\ \mathbb{N}^{\{\alpha_{a,a+1}\}}|_a \otimes \mathbb{N}^{I'_a}|_a, & \text{if } a = k, \\ C_{k-1} \otimes \cdots \otimes C_1 \otimes \mathbb{N}^{I'_a}|_a, & \text{if } a > k, \end{cases} \\
\mathbb{N}^{J_1}|_a &\simeq \begin{cases} C_{k-1} \otimes \cdots \otimes C_{a+2} \otimes \mathbb{N}^{\{\alpha_{a,a+1}\}}|_a \otimes \mathbb{N}^{I''_a}|_a, & \text{if } a < k, \\ \mathbb{N}^{I''_a}|_a, & \text{else}, \end{cases} \\
\mathbb{N}^{J_2}|_a &\simeq \begin{cases} C_k \otimes \cdots \otimes C_{a-1} \otimes \mathbb{N}^{\{\alpha_{a,a+1}\}}|_a \otimes \mathbb{N}^{I'''_a}|_a, & \text{if } a > k, \\ \mathbb{N}^{I'''_a}|_a, & \text{else}. \end{cases}
\end{aligned} \tag{28}$$

From (27) and (28) we conclude (26).

## 8. Proof of Theorem 5.1

We fix $\mathbf{i} = (i_1, \ldots, i_N) \in \mathcal{W}(w_0)$ as well as $\lambda = \sum_{b \in [n-1]} \lambda_b \omega_b \in P^+$ and set

$$\lambda^* := \sum_{b \in [n-1]} \lambda_{n-b} \omega_b \in P^+,$$
$$\underline{\lambda} := (\lambda_{i_1}, \lambda_{i_2}, \ldots \lambda_{i_N}) \in \mathbb{N}^N.$$

### 8.1. A bijection between string and Lusztig data

Let $(c_{i,j})$ be the Cartan matrix of $sl_n$. For $x \in \mathbb{Z}^N$ we define

$$F_{\mathbf{i}}(x) := \left( x_k + \sum_{k < \ell \le N} c_{i_k, i_\ell} x_\ell \right)_{k \in [N]} \in \mathbb{Z}^N,$$
$$G_{\mathbf{i}}^\lambda(x) := \underline{\lambda} - F_{\mathbf{i}}(x) \in \mathbb{Z}^N.$$

By [MG03, Cor. 3.5], [CMMG04, Lem. 6.3] (see also [GKS17, Lem. 6.4, Lem. 7.4, Prop. 8.2]) we have

**Proposition 8.1.** *The map $G_{\mathbf{i}}^\lambda$ restricts to a bijection*

$$G_{\mathbf{i}}^\lambda : \mathcal{S}_{\mathbf{i}}^*(\lambda) \xrightarrow{\sim} \mathcal{L}_{\mathbf{i}}(\lambda^*).$$

*Further, $G_{\mathbf{j}}^\lambda \circ \Psi_{\mathbf{j}}^{\mathbf{i}} = \Phi_{\mathbf{j}}^{\mathbf{i}} \circ G_{\mathbf{i}}^\lambda$ for any $\mathbf{j} \in \mathcal{W}(w_0)$.*

The bijection $G_{\mathbf{i}}^\lambda$ between $\mathcal{S}_{\mathbf{i}}^*(\lambda)$ and $\mathcal{L}_{\mathbf{i}}(\lambda^*)$ intertwines the crystal structures given in Lemma 3.5 and Proposition 7.2 as follows.

**Lemma 8.2.** *For $a \in [n-1]$ we have on $\mathcal{S}_{\mathbf{i}}^*(\lambda)$*

$$\varepsilon_a = \varphi_a \circ G_{\mathbf{i}}^\lambda, \tag{29}$$
$$G_{\mathbf{i}}^\lambda \circ e_a = f_a \circ G_{\mathbf{i}}^\lambda, \tag{30}$$
$$\mathrm{wt} = -\,\mathrm{wt} \circ G_{\mathbf{i}}^\lambda. \tag{31}$$

*Proof.* Clearly, (29) and (30) hold for $i_1 = a$ and thus by Proposition 8.1 for arbitrary $\mathbf{i} \in \mathcal{W}(w_0)$.

By (30) and the crystal axiom (C3) in Definition 1.1 it is enough to show (31) for the highest weight element $x_\lambda$ of $\mathcal{S}^*_{\mathbf{i}}(\lambda)$. By (29) we have for $a' \in [n-1]$

$$\varphi_{a'} \circ G^\lambda_{\mathbf{i}}(x_\lambda) = \varepsilon_{a'}(x_\lambda) = 0,$$

i.e., $G^\lambda_{\mathbf{i}}(x_\lambda)$ is the lowest weight element of $\mathcal{L}_{\mathbf{i}}(\lambda^*)$. Thus,

$$\mathrm{wt}(x_\lambda) = \lambda = -\,\mathrm{wt} \circ G^\lambda_{\mathbf{i}}(x_\lambda). \qquad \square$$

*Remark 8.3.* The map $G^\lambda_{\mathbf{i}}$ between Lusztig and string data also appears in [K07] by passing through Mirković–Vilonen polytopes. Namely Kamnitzer defines in op. cit. crystal isomorphisms between MV polytopes in $B(\lambda)$ and $\mathbf{i}$-Lusztig data of $B(\lambda)$ and $\mathbf{i}$-string data of $B(\lambda^*)^\vee$, respectively. The crystal $B(\lambda^*)^\vee$ is here defined as in [K94, Sect. 7.4]. In a different context, the composition of Kamnitzer's bijections inducing a map $\mathcal{S}^*_{\mathbf{i}}(\lambda) \to \mathcal{L}_{\mathbf{i}}(\lambda)$ was computed in [GKS17, Prop. 8.2] and turns out to coincide with the map $G^\lambda_{\mathbf{i}}$.

### 8.2. Reineke crossings and the bijection $G^\lambda_{\mathbf{i}}$

For $a \in [n-1]$, we attach in Definition 4.11 to $\gamma \in \Gamma_a$ the vectors $s(\gamma), r(\gamma) \in \mathbb{Z}^N$. In [G18, Thm. 3.11] it is shown that the map $F_{\mathbf{i}}$ relates $s(\gamma)$ and $r(\gamma) \in \mathbb{Z}^N$ as follows:

**Proposition 8.4** ([G18])**.** *For $a \in [n-1]$ we have $r = F_{\mathbf{i}} \circ s$ on $\Gamma_a$.*

In this section we use Proposition 8.4 to show

**Proposition 8.5.** *For $x \in \mathcal{S}^*_{\mathbf{i}}(\lambda)$, $a \in [n-1]$ and $\gamma \in \Gamma_a$ we have*

$$\left\langle G^\lambda_{\mathbf{i}}(x), s(\gamma)\right\rangle - \langle x, r(\gamma)\rangle = \mathrm{wt}(x)(h_a).$$

For this we define for $a \in [n-1]$ the function

$$\begin{aligned} \ell_a : \mathbb{Z}^N &\to \mathbb{Z}, \\ x = (x_k)_{k\in[N]} &\mapsto \sum_{k:\ i_k = a} x_k. \end{aligned}$$

To prove Proposition 8.5 we use

**Lemma 8.6.** *For $a, b \in [n-1]$ and $\gamma \in \Gamma_a$ we have $\ell_b(s(\gamma)) = \delta_{a,b}$.*

*Proof of Proposition* 8.5*.* From Proposition 8.4 we obtain

$$\begin{aligned} \left\langle G^\lambda_{\mathbf{i}}(x), s(\gamma)\right\rangle - \langle x, r(\gamma)\rangle &= \langle \underline{\lambda}, s(\gamma)\rangle - \langle F_{\mathbf{i}}(x), s(\gamma)\rangle - \langle x, r(\gamma)\rangle \\ &= \langle \underline{\lambda}, s(\gamma)\rangle - \langle F_{\mathbf{i}}(x), s(\gamma)\rangle - \langle x, F_{\mathbf{i}}(s(\gamma))\rangle . \end{aligned} \tag{32}$$

By Lemma 8.6 we have

$$\langle \underline{\lambda}, s(\gamma)\rangle = \sum_{k\in[N]} \lambda_{i_k}(s(\gamma))_k = \sum_{b\in[n-1]} \lambda_b \ell_b(s(\gamma)) = \lambda_a. \tag{33}$$

Furthermore, since $c_{b,b} = 2$,

$$\begin{aligned}
\big\langle F_{\mathbf{i}}(x), s(\gamma)\big\rangle + \langle x, F_{\mathbf{i}}(s(\gamma))\rangle
&= \sum_{k\in[N]} (F_{\mathbf{i}}(x))_k (s(\gamma))_k + \sum_{k\in[N]} x_k \big(F_{\mathbf{i}}(s(\gamma))\big)_k \\
&= \sum_{k\in[N]} \Big(x_k + \sum_{\ell>k} c_{i_k,i_\ell} x_\ell\Big)(s(\gamma))_k + \sum_{k\in[N]} x_k \Big((s(\gamma))_k + \sum_{\ell>k} c_{i_k,i_\ell}(s(\gamma))_\ell\Big) \\
&= \sum_{k,\ell\in[N]} c_{i_k,i_\ell} x_k (s(\gamma))_\ell = \sum_{i,j\in[n-1]} c_{i,j}\ell_i(x)\ell_j(s(\gamma)).
\end{aligned}$$

Thus, by Lemma 8.6

$$\langle F_{\mathbf{i}}(x), s(\gamma)\rangle + \langle x, F_{\mathbf{i}}(s(\gamma))\rangle = \sum_{i\in[n-1]} c_{a,i}\ell_i(x). \tag{34}$$

Combining (32), (33) and (34) yields

$$\big\langle G_{\mathbf{i}}^{\lambda}(x), s(\gamma)\big\rangle - \langle x, r(\gamma)\rangle = \lambda_a - \sum_{i\in[n-1]} \sum_{k:\ i_k=i} c_{a,i} x_k = \mathrm{wt}(x)(h_a). \qquad \square$$

It remains to prove Lemma 8.6. Recall the notion of the level of a vertex $v$ of $\mathcal{D}_{\mathbf{i}}$ from Definition 4.1. For each vertex $v$ of $\gamma$, we define

$$\mathrm{level}_\gamma^-(v) = \begin{cases} \mathrm{level}(v)+1 & \text{the oriented edge of } \mathcal{D}_{\mathbf{i}}(a) \text{ with target } v \text{ that } \gamma \\ & \text{follows is headed downwards,} \\ \mathrm{level}(v) & \text{the oriented edge of } \mathcal{D}_{\mathbf{i}}(a) \text{ with target } v \text{ that } \gamma \\ & \text{follows is headed upwards,} \end{cases}$$

and

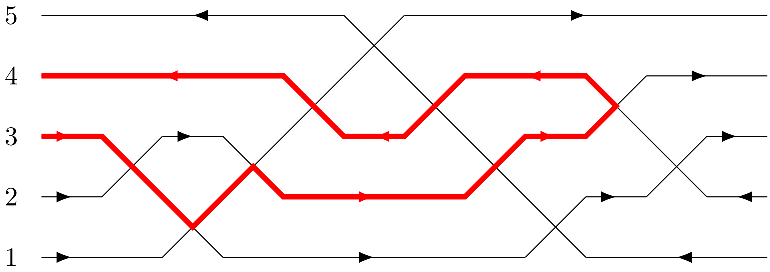
$$\mathrm{level}_\gamma^+(v) = \begin{cases} \mathrm{level}(v) & \text{the oriented edge of } \mathcal{D}_{\mathbf{i}}(a) \text{ with source } v \text{ that } \gamma \\ & \text{follows is headed downwards,} \\ \mathrm{level}(v)+1 & \text{the oriented edge of } \mathcal{D}_{\mathbf{i}}(a) \text{ with source } v \text{ that } \gamma \\ & \text{follows is headed upwards.} \end{cases}$$

Here we understand "headed upwards" and "headed downwards" with respect to a small neighborhood around the vertex $v$.

We give an example for this notion.

**Example 8.7.** Let $n = 5$. And $\gamma = (v_{3,2}, v_{3,1}, v_{1,2}, v_{2,5}, v_{2,4}, v_{4,5}, v_{4,1})$ the 3 - Reineke crossing from Example 4.8 highlighted below. We have

$$\begin{aligned}
&\mathrm{level}_\gamma^-(v_{3,2}) = 3,\ \mathrm{level}_\gamma^+(v_{3,2}) = 2,\ \mathrm{level}_\gamma^-(v_{3,1}) = 2,\ \mathrm{level}_\gamma^+(v_{3,1}) = 2,\\
&\mathrm{level}_\gamma^-(v_{1,2}) = 2,\ \mathrm{level}_\gamma^+(v_{1,2}) = 2,\ \mathrm{level}_\gamma^-(v_{2,5}) = 2,\ \mathrm{level}_\gamma^+(v_{2,5}) = 3,\\
&\mathrm{level}_\gamma^-(v_{2,4}) = 3,\ \mathrm{level}_\gamma^+(v_{2,4}) = 4,\ \mathrm{level}_\gamma^-(v_{4,5}) = 4,\ \mathrm{level}_\gamma^+(v_{4,5}) = 3,\\
&\mathrm{level}_\gamma^-(v_{4,1}) = 3,\ \mathrm{level}_\gamma^+(v_{4,1}) = 4.
\end{aligned}$$

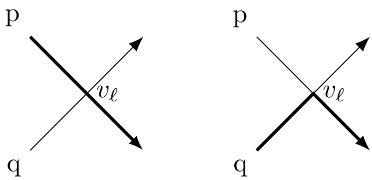



Note that, by definition, for $\gamma = (v_1, v_2, \ldots, v_m) \in \Gamma_a$, we have $\mathrm{level}^-_\gamma(v_1) = a$, $\mathrm{level}^+_\gamma(v_\ell) = \mathrm{level}^-_\gamma(v_{\ell+1})$ and $\mathrm{level}^+_\gamma(v_m) = a + 1$. Thus, Lemma 8.6 is a direct consequence of

**Lemma 8.8.** *For* $1 \leq \ell \leq m$ *we have* $\mathit{level}^+_\gamma(v_\ell) - \mathit{level}^-_\gamma(v_\ell) = (s(\gamma))_\ell$.

*Proof.* Assume that the vertex $v_\ell = v_{p,q}$ of $\gamma$ lies at the intersection of wires $p$ and $q$, where $p$ is the wire of the oriented edge in $\gamma$ whose source is $v_{p,q}$. We assume first $q \leq a$, hence the wire $q$ is oriented from left to right in $\mathcal{D}_{\mathbf{i}}(a)$. We proceed by a case-by-case analysis.

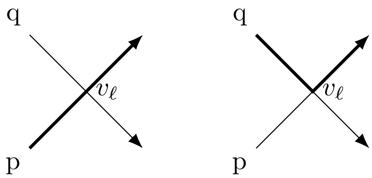
$\underline{q < p \leq a}$ : Locally around $v_\ell$ there are two possibilities for $\gamma$:




.

In the left case, we have $(s(\gamma))_\ell = -1$, $\mathrm{level}^-_\gamma(v_\ell) = \mathrm{level}(v_\ell) + 1$ and $\mathrm{level}^+_\gamma(v_\ell) = \mathrm{level}(v_\ell)$. In the right case, we have $(s(\gamma))_\ell = 0$, $\mathrm{level}^-_\gamma(v_\ell) = \mathrm{level}(v_\ell)$ and $\mathrm{level}^+_\gamma(v_\ell) = \mathrm{level}(v_\ell)$.

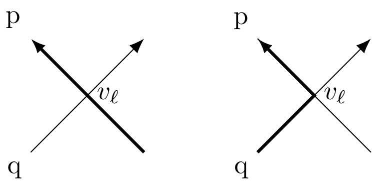
$\underline{p < q \leq a}$ : Locally around $v_\ell$ there are two possibilities for $\gamma$:




.

The left case cannot appear since $\gamma$ is an $a$-Reineke crossing. In the right case, we have $(s(\gamma))_\ell = 0$, $\mathrm{level}^-_\gamma(v_\ell) = \mathrm{level}(v_\ell) + 1$ and $\mathrm{level}^+_\gamma(v_\ell) = \mathrm{level}(v_\ell) + 1$.

$\underline{q \leq a < p}$ : Locally around $v_\ell$ there are two possibilities for $\gamma$:




.

In the both cases, we have $(s(\gamma))_\ell = 1$, $\operatorname{level}^-_\gamma(v_\ell) = \operatorname{level}(v_\ell)$ and $\operatorname{level}^+_\gamma(v_\ell) = \operatorname{level}(v_\ell) + 1$.

The argument for the assumption $a + 1 \le q$ is symmetrical. □

### 8.3. Proof of the Dual Crossing Formula

*Proof of Theorem 5.1.* Equation (21) was established in Lemma 3.5.

We prove (20). By Lemma 8.2 and the crystal axiom (C1) in Definition 1.1

$$\varepsilon_a(x) = \varphi_a(G^\lambda_{\mathbf{i}}(x)) = \operatorname{wt}(G^\lambda_{\mathbf{i}}(x))(h_a) + \varepsilon_a(G^\lambda_{\mathbf{i}}(x)). \tag{35}$$

By Proposition 8.1 we have $G^\lambda_{\mathbf{i}}(x) \in \mathcal{L}_{\mathbf{i}}(\lambda^*)$. Using Theorem 7.3 to compute the value of $\varepsilon_a$ on this Lusztig-datum we obtain

$$\begin{aligned} \varepsilon_a(G^\lambda_{\mathbf{i}}(x)) &= \max\{\left\langle G^\lambda_{\mathbf{i}}(x), s(\gamma)\right\rangle \mid \gamma \in \Gamma_a\} \\ &= \max\left\{\langle x, r(\gamma)\rangle \mid \gamma \in \Gamma_a\right\} + \operatorname{wt}(x)\,(h_a)\,, \end{aligned} \tag{36}$$

where (36) follows from Proposition 8.5. By Lemma 8.2,

$$\operatorname{wt}(x)(h_a) = -\operatorname{wt}(G^\lambda_{\mathbf{i}}(x))(h_a). \tag{37}$$

Plugging (36) and (37) into (35) yields (20).

We next prove (22). If $\varphi_a(x) = 0$, the claim follows from Lemma 8.2.

Assume now that $\varphi_a(x) > 0$. By Lemma 8.2 we have

$$f_a x = f_a (G^\lambda_{\mathbf{i}})^{-1} \circ G^\lambda_{\mathbf{i}}(x) = (G^\lambda_{\mathbf{i}})^{-1}\left(e_a G^\lambda_{\mathbf{i}}(x)\right). \tag{38}$$

By Proposition 8.1, we have that $G^\lambda_{\mathbf{i}}(x) \in \mathcal{L}_{\mathbf{i}}(\lambda^*)$ and by Lemma 8.2 that

$$\varepsilon_a\left(G^\lambda_{\mathbf{i}}(x)\right) > 0.$$

Thus by Theorem 7.3,

$$e_a G^\lambda_{\mathbf{i}}(x) = G^\lambda_{\mathbf{i}}(x) - r(\gamma^x), \tag{39}$$

where $\gamma^x \in \Gamma_a$ is minimal such that $\langle G^\lambda_{\mathbf{i}}(x), s(\gamma^x)\rangle = \max\{\langle G^\lambda_{\mathbf{i}} x, s(\gamma)\rangle \mid \gamma \in \Gamma_a\}$. By Proposition 8.5 $\left\langle G^\lambda_{\mathbf{i}}(x), s(\gamma)\right\rangle - \langle x, r(\gamma)\rangle = \operatorname{wt}(x)(h_a)$ is independent of $\gamma \in \Gamma_a$. Thus, $\gamma^x \in \Gamma_a$ is minimal such that

$$\langle x, r(\gamma^x)\rangle = \max\left\{\langle x, r(\gamma)\rangle \mid \gamma \in \Gamma_a\right\} = \varepsilon_a(x),$$

where we used (20) in the last equality. Furthermore, by (38) and (39)

$$f_a x = (G^\lambda_{\mathbf{i}})^{-1}\left(G^\lambda_{\mathbf{i}}(x) - r(\gamma^x)\right) = x + F^{-1}_{\mathbf{i}}(r(\gamma^x))$$

and (22) follows from Proposition 8.4.

The proof of (23) works analogously to the proof of (22). □

## 9. Kashiwara $*$-involution on String data

In this section we denote by $\mathcal{S}_{\mathbf{i}}$ and $\mathcal{S}_{\mathbf{i}}^*$ the set of $\mathbf{i}$-string data equipped with the crystal structure inherited from $B(\infty)$ and $B(\infty)^*$, respectively, via the bijection $\mathrm{str}_{\mathbf{i}}^*$ (see (14) and (16)). We denote by $\mathcal{L}_{\mathbf{i}} = \mathbb{N}^N$ and $\mathcal{L}_{\mathbf{i}}^* = \mathbb{N}^N$ the set of $\mathbf{i}$-Lusztig data with the crystal structure inherited from $B(\infty)$ and $B(\infty)^*$, respectively, via the bijection $b_{\mathbf{i}}$ defined in (25). We write $\mathcal{L}_{\mathbf{i}}^{\mathbb{R}} := \mathbb{R}_{\geq 0}^N$. Using $\varepsilon_a$ from the crystal $\mathcal{L}_{\mathbf{i}}$ and $\varepsilon_a^*$ from $\mathcal{L}_{\mathbf{i}}^*$ we define the polytopes

$$\mathcal{L}_{\mathbf{i}}(\lambda)^{\mathbb{R}} := \{x \in \mathcal{L}_{\mathbf{i}}^{\mathbb{R}} \mid \varepsilon_a^*(x) \leq \lambda_a \ \forall a \in [n-1]\},$$
$$\mathcal{L}_{\mathbf{i}}^*(\lambda)^{\mathbb{R}} := \{x \in \mathcal{L}_{\mathbf{i}}^{\mathbb{R}} \mid \varepsilon_a(x) \leq \lambda_a \ \forall a \in [n-1]\}.$$

The integral points of $\mathcal{L}_{\mathbf{i}}(\lambda)^{\mathbb{R}}$ and $\mathcal{L}_{\mathbf{i}}^*(\lambda)^{\mathbb{R}}$ are $\mathcal{L}_{\mathbf{i}}(\lambda)$ and $\mathcal{L}_{\mathbf{i}}^*(\lambda)$, respectively.

For a reduced word $\mathbf{i} = (i_1, \dots, i_N) \in \mathcal{W}(w_0)$ we define

$$\mathbf{i}^* := (n - i_1, \dots, n - i_N) \in \mathcal{W}(w_0),$$
$$\mathbf{i}^{\mathrm{op}} := (i_N, \dots, i_1) \in \mathcal{W}(w_0).$$

For $\mathbf{i}, \mathbf{j} \in \mathcal{W}(w_0)$ the Kashiwara $*$-involution $* : B(\infty) \to B(\infty)^*$ introduced in Section 1.3 on string data is given by the isomorphism of crystals

$$\mathrm{str}_{\mathbf{j}}^* \circ \mathrm{str}_{\mathbf{i}}^{-1} : \mathcal{S}_{\mathbf{i}}^* \xrightarrow{\sim} \mathcal{S}_{\mathbf{j}}. \tag{40}$$

In general, the map (40) is piecewise linear. We show that (40) is linear for $\mathbf{i} = \mathbf{i}_0 := (1, 2, 1, 3, 2, 1, \dots, n-1, n-2, \dots, 1)$ and $\mathbf{j} = \mathbf{i}_0^*$.

Using the Crossing Formula [GKS21, Thm. 2.13], we compute $\mathrm{str}_{\mathbf{i}_0} \circ b_{\mathbf{i}_0}$: If $(i_\ell, i_{\ell+1}, \dots, i_{\ell+m})$ is a maximal subword of $\mathbf{i}_0$ of the form $(k, k-1, \dots, 1)$ we have for $j \in \{0, 1, \dots, m\}$

$$(\mathrm{str}_{\mathbf{i}_0} \circ b_{\mathbf{i}_0}(x))_{\ell+j} = x_\ell + x_{\ell+1} + \cdots x_{\ell+m-j}.$$

From the $*$-Crossing Formula [GKS21, Thm. 2.20] we compute

$$\mathrm{str}_{\mathbf{i}_0^*}^* \circ b_{\mathbf{i}_0^{\mathrm{op}}}(x_1, \dots, x_N) = \mathrm{str}_{\mathbf{i}_0} \circ b_{\mathbf{i}_0}(x_N, \dots, x_1).$$

Since $\mathbf{i}_0$ and $\mathbf{i}_0^{\mathrm{op}}$ are related by a sequence of 2-moves the isomorphism of crystals $\Phi_{\mathbf{i}_0^{\mathrm{op}}}^{\mathbf{i}_0}$ sending $\mathbf{i}_0$-Lusztig data to $\mathbf{i}_0^{\mathrm{op}}$-Lusztig data is linear. We thus obtain the linear isomorphism of crystals

$$* = \mathrm{str}_{\mathbf{i}_0^*}^* \circ \mathrm{str}_{\mathbf{i}_0}^{-1} = \mathrm{str}_{\mathbf{i}_0^*}^* \circ b_{\mathbf{i}_0^{\mathrm{op}}} \circ \Phi_{\mathbf{i}_0^{\mathrm{op}}}^{\mathbf{i}_0} \circ b_{\mathbf{i}_0}^{-1} \circ \mathrm{str}_{\mathbf{i}_0}^{-1} : \mathcal{S}_{\mathbf{i}_0}^* \xrightarrow{\sim} \mathcal{S}_{\mathbf{i}_0^*}.$$

Since $* = \mathrm{str}_{\mathbf{i}_0^*}^* \circ \mathrm{str}_{\mathbf{i}_0}^{-1} : \mathcal{S}_{\mathbf{i}_0} \xrightarrow{\sim} \mathcal{S}_{\mathbf{i}_0^*}^*$ is an isomorphism of crystals as well, we obtain for $\lambda \in P^+$ the linear isomorphism of crystals

$$* = \mathrm{str}_{\mathbf{i}_0^*}^* \circ \mathrm{str}_{\mathbf{i}_0}^{-1} : \mathcal{S}_{\mathbf{i}_0}^*(\lambda) \xrightarrow{\sim} \mathcal{S}_{\mathbf{i}_0^*}(\lambda)$$

and the unimodular isomorphismus of polytopes

$$* = \operatorname{str}^*_{\mathbf{i}_0^*} \circ \operatorname{str}^{-1}_{\mathbf{i}_0} : \mathcal{S}^*_{\mathbf{i}_0}(\lambda)^{\mathbb{R}} \xrightarrow{\sim} \mathcal{S}_{\mathbf{i}_0^*}(\lambda)^{\mathbb{R}}.$$

For $\mathbf{i}, \mathbf{j} \in \mathcal{W}(w_0)$ arbitrary we obtain the piecewise linear isomorphisms

$$\begin{aligned}
\Psi_{\mathbf{j}}^{\mathbf{i}_0^*} \circ \operatorname{str}^*_{\mathbf{i}_0^*} \circ \operatorname{str}^{-1}_{\mathbf{i}_0} \circ \Psi^{\mathbf{i}}_{\mathbf{i}_0} &: \mathcal{S}^*_{\mathbf{i}} \xrightarrow{\sim} \mathcal{S}_{\mathbf{j}}, \\
\Psi_{\mathbf{j}}^{\mathbf{i}_0^*} \circ \operatorname{str}^*_{\mathbf{i}_0^*} \circ \operatorname{str}^{-1}_{\mathbf{i}_0} \circ \Psi^{\mathbf{i}}_{\mathbf{i}_0} &: \mathcal{S}^*_{\mathbf{i}}(\lambda) \xrightarrow{\sim} \mathcal{S}_{\mathbf{j}}(\lambda)
\end{aligned}$$

and the piecewise linear volume preserving bijections

$$\begin{aligned}
\Psi_{\mathbf{j}}^{\mathbf{i}_0^*} \circ \operatorname{str}^*_{\mathbf{i}_0^*} \circ \operatorname{str}^{-1}_{\mathbf{i}_0} \circ \Psi^{\mathbf{i}}_{\mathbf{i}_0} &: \mathcal{S}^{\mathbb{R}}_{\mathbf{i}} \xrightarrow{\sim} \mathcal{S}^{\mathbb{R}}_{\mathbf{j}}, \\
\Psi_{\mathbf{j}}^{\mathbf{i}_0^*} \circ \operatorname{str}^*_{\mathbf{i}_0^*} \circ \operatorname{str}^{-1}_{\mathbf{i}_0} \circ \Psi^{\mathbf{i}}_{\mathbf{i}_0} &: \mathcal{S}^*_{\mathbf{i}}(\lambda)^{\mathbb{R}} \xrightarrow{\sim} \mathcal{S}_{\mathbf{j}}(\lambda)^{\mathbb{R}}.
\end{aligned}$$

By [BZ01, Prop. 3.3 (iii)] the $*$-involution is given on Lusztig data by the linear map

$$\begin{aligned}
* : \mathcal{L}_{\mathbf{i}} &\xrightarrow{\sim} \mathcal{L}^*_{\mathbf{i}^{*,\mathrm{op}}}, \\
x = (x_1, \dots, x_N) &\mapsto x^{\mathrm{op}} = (x_N, \dots, x_1).
\end{aligned}$$

For $\lambda \in P^+$ we thus have the following commutative diagrams of isomorphisms of crystals that are linear for $\mathbf{i} = \mathbf{i}_0$:

$$\begin{tikzcd}
\mathcal{L}_{\mathbf{i}} \arrow[r, "\operatorname{str}_{\mathbf{i}} \circ b_{\mathbf{i}}"] \arrow[d, "*"] & \mathcal{S}^*_{\mathbf{i}} \arrow[d, "*"] \\
\mathcal{L}^*_{\mathbf{i}^{*\mathrm{op}}} \arrow[r, "\operatorname{str}_{\mathbf{i}^{*\mathrm{op}}} \circ b_{\mathbf{i}^{*\mathrm{op}}}"] & \mathcal{S}_{\mathbf{i}^{*\mathrm{op}}}
\end{tikzcd}, \qquad
\begin{tikzcd}
\mathcal{L}_{\mathbf{i}}(\lambda) \arrow[r, "\operatorname{str}_{\mathbf{i}} \circ b_{\mathbf{i}}"] \arrow[d, "*"] & \mathcal{S}^*_{\mathbf{i}}(\lambda) \arrow[d, "*"] \\
\mathcal{L}^*_{\mathbf{i}^{*\mathrm{op}}}(\lambda) \arrow[r, "\operatorname{str}_{\mathbf{i}^{*\mathrm{op}}} \circ b_{\mathbf{i}^{*\mathrm{op}}}"] & \mathcal{S}_{\mathbf{i}^{*\mathrm{op}}}(\lambda)
\end{tikzcd}.$$

Furthermore, the following are commutative diagrams of volume preserving piecewise linear bijections that are linear for $\mathbf{i} = \mathbf{i}_0$:

$$\begin{tikzcd}
\mathcal{L}^{\mathbb{R}}_{\mathbf{i}} \arrow[r, "\operatorname{str}_{\mathbf{i}} \circ b_{\mathbf{i}}"] \arrow[d, "*"] & \mathcal{S}^{\mathbb{R}}_{\mathbf{i}} \arrow[d, "*"] \\
\mathcal{L}^{\mathbb{R}}_{\mathbf{i}^{*\mathrm{op}}} \arrow[r, "\operatorname{str}_{\mathbf{i}^{*\mathrm{op}}} \circ b_{\mathbf{i}^{*\mathrm{op}}}"] & \mathcal{S}^{\mathbb{R}}_{\mathbf{i}^{*\mathrm{op}}}
\end{tikzcd}, \qquad
\begin{tikzcd}
\mathcal{L}_{\mathbf{i}}(\lambda)^{\mathbb{R}} \arrow[r, "\operatorname{str}_{\mathbf{i}} \circ b_{\mathbf{i}}"] \arrow[d, "*"] & \mathcal{S}^*_{\mathbf{i}}(\lambda)^{\mathbb{R}} \arrow[d, "*"] \\
\mathcal{L}^*_{\mathbf{i}^{*\mathrm{op}}}(\lambda)^{\mathbb{R}} \arrow[r, "\operatorname{str}_{\mathbf{i}^{*\mathrm{op}}} \circ b_{\mathbf{i}^{*\mathrm{op}}}"] & \mathcal{S}_{\mathbf{i}^{*\mathrm{op}}}(\lambda)^{\mathbb{R}}
\end{tikzcd}.$$

## References


[BFZ96] A. Berenstein, S. Fomin, A. Zelevinsky, *Parametrizations of canonical bases and totally positive matrices*, Adv. Math. **122** (1996), 49–149.

[BZ93] A. Berenstein, A. Zelevinsky, *String bases for quantum groups of type* $\mathsf{A}_r$, in: *I. M. Gelfand Seminar* (S. I. Gelfand and S. G. Gindikin, eds), Adv. Soviet Math. **16** (1993), Amer. Math. Soc., 51–89.

[BZ01] A. Berenstein, A. Zelevinsky, *Tensor product multiplicities, canonical bases and totally positive varieties*, Invent. Math. **143** (2001), no.1, 77–128.

[BF16] L. Bossinger, G. Fourier, *String cone and Superpotential combinatorics for flag and Schubert varieties in type* $\mathsf{A}$, J. Combin. Theory Ser. A. **167** (2019), 213–256.

[CMMG04] P. Caldero, R. Marsh, S. Morier-Genoud, *Realisation of Lusztig cones*, Represent. Theory **8** (2019), 458–478.

[CFL] R. Chirivì, X. Fang, P. Littelmann, *Semitoric degenerations via Newton–Okounkov bodies, LS-algebras and standard monomial theory*, preprint.

[DKKA07] V. I. Danilov, A. V. Karzanov, G. A. Koshevoy, *Combinatorics of regular* $\mathsf{A}_2$*-crystals*, J. Algebra **310** (2007), 218–234.

[FFL17] X. Fang, G. Fourier, P. Littelmann, *Essential bases and toric degenerations arising from birational sequences*, Adv. Math. **312** (2017), 107–149.

[D93] M. Dyer, *Hecke algebras and shellings of Bruhat intervals*, Comp. Math. **89** (1993), 91–115.

[E97] S. Elnitsky, *Rhombic tilings of polygons and classes of reduced words in Coxeter groups.*, J. Combin. Theory Ser. A **77** (1997), 193–221.

[FN17] N. Fujita, S. Naito, *Newton–Okounkov convex bodies of Schubert varieties and polyhedral realizations of crystal bases*, Math. Z. **285** (2017), 325–352.

[FO17] N. Fujita, H. Oya, *A comparison of Newton–Okounkov polytopes of Schubert varieties*, J. Lond. Math. Soc. **96** (2017), 201–227.

[G18] V. Genz, *Crystal combinatorics and mirror symmetry for cluster varieties*, Dissertation, University of Cologne (2018).

[GKS17] V. Genz, G. Koshevoy, B. Schumann, *Polyhedral parametrizations of canonical bases & cluster duality*, Adv. Math. **369** (2020), 107178.

[GKS19] V. Genz, G. Koshevoy, B. Schumann, *On the interplay of the parametrizations of canonical bases by Lusztig and string data*, `arXiv:1901.03500` (2019).

[GKS21] V. Genz, G. Koshevoy, B. Schumann, *Combinatorics of canonical bases revisited: type* $\mathsf{A}$, Sel. Math. New Ser. **27** (2021), article number 67, `https://doi.org/10.1007/s00029-021-00658-x`.

[GP00] O. Gleizer, A. Postnikov, *Littlewood–Richardson coefficients via Yang-Baxter equation*, Internat. Math. Res. Notices **14** (2000), 741–774.

[H05] A. Hoshino, *Polyhedral realizations of crystal bases for quantum algebras of finite types*, J. Math. Phys. **46** (2005), 113514.

[J95] A. Joseph, *Quantum Groups and their Primitive Ideals*, Ergebnisse der Mathematik, Vol. 29, Springer-Verlag, Berlin, 1995.

[J18] A. Joseph, *Trails for minuscule modules and dual Kashiwara functions for the* $B(\infty)$ *crystal*, in: *Quantum Theory and Symmetries with Lie Theory and its Applications in Physics*, Vol. 1, Springer Proc. Math. Stat., Vol. 263, Springer, Singapore, 2018, pp. 37–53.

[K07] J. Kamnitzer, *The crystal structure on the set of Mirković–Vilonen polytopes*, Adv. Math. **215** (2007), 66–93.

[K91] M. Kashiwara, *On crystal bases of the q-analogue of universal enveloping algebras*, Duke Math. J. **63** (1991), 465–516.

[K93] M. Kashiwara, *The crystal base and Littelmann's refined Demazure character formula*, Duke Math. J. **71** (1993), 839–858.

[K94] M. Kashiwara, *On crystal bases*, in: *Representations of Groups*, Proceedings of the 1994 Annual Seminar of the Canadian Math. Soc. (B.N. Allison and G.H. Cliff, eds), CMS Conference Proceedings, Vol. 16, Amer. Math. Soc., 1995, pp. 155–197.

[K02] M. Kashiwara, *Bases Cristallines des Groupes Quantiques* (Notes by Charles Cochet), Cours Spécialisés, Vol. 9, Société Mathématique de France, Paris, 2002.

[K15] K. Kaveh, *Crystal bases and Newton–Okounkov bodies*, Duke Math. J. **164** (2015), 2461–2506.

[K18] J.-H. Kwon, *A crystal embedding into Lusztig data of type* $\mathsf{A}$, J. Combin. Theory Ser. A **154** (2018), 422–443.

[Lit94] P. Littelmann, *A Littlewood–Richardson rule for symmetrizable Kac–Moody algebras*, Invent. Math. **116** (1994), 329–346.

[Lit98] P. Littelmann, *Cones, crystals, and patterns*, Transform. Groups **3** (1998), 145–179.

[L90] G. Lusztig, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. **3** (1990), 447–498.

[Lu90] G. Lusztig, *Finite-dimensional Hopf algebras arising from quantized universal enveloping algebra*, J. Amer. Math. Soc. **3** (1990), 257–296.

[MG03] S. Morier-Genoud, *Relèvement géométrique de la base canonique et involution de Schützenberger*, C. R. Acad. Sci. Paris. Ser. I **337** (2003), 371–374.

[L93] G. Lusztig, *Introduction to Quantum Groups*, Progress in Mathematics, Vol. 110, Birkhäuser Boston, Boston, MA, 1993.

[N99] T. Nakashima, *Polyhedral realizations of crystal bases for integrable highest weight modules*, J. Algebra **219** (1999), no. 2, 571–597.

[NZ97] T. Nakashima, A. Zelevinsky, *Polyhedral realizations of crystal bases for quantized Kac–Moody algebras*, Adv. Math. **131** (1997), 253–278.

[R97] M. Reineke, *On the coloured graph structure of Lusztig's canonical basis*, Math. Ann. **307** (1997), 705–723.

[SST18] B. Salisbury, A. Schultze, P. Tingley, *Combinatorial descriptions of the crystal structure on certain PBW bases*, Transform. Groups **23** (2018), 501–525.

[Z13] S. Zelikson, *On crystal operators in Lusztig's parametrizations and string cone defining inequalities*, Glasg. Math. J. **55** (2013), 177–200.

**Funding Information** Open Access funding enabled and ogranized by Projeckt DEAL.