HIGHER GENERATION BY ABELIAN SUBGROUPS IN LIE GROUPS

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Abstract. To a compact Lie group G one can associate a space $E(2, G)$ akin to the poset of cosets of abelian subgroups of a discrete group. The space $E(2, G)$ was introduced by Adem, F. Cohen and Torres-Giese, and subsequently studied by Adem and Gómez, and other authors. In this short note, we prove that G is abelian if and only if $\pi_i(E(2, G)) = 0$ for $i = 1, 2, 4$. This is a Lie group analogue of the fact that the poset of cosets of abelian subgroups of a discrete group is simply connected if and only if the group is abelian.

1. Introduction

Suppose that G is a discrete group and $\mathcal{F}$ is a family of subgroups of G . One can associate to $\mathcal{F}$ a simplicial complex $\mathcal{C}(\mathcal{F}, G)$ whose n -simplices are the chains of cosets $g_0 A_0 \subset g_1 A_1 \subset \cdots \subset g_n A_n$ where $g_i \in G$ and $A_i \in \mathcal{F}$ for all $0 \leq i \leq n$.

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It is the order complex of what is commonly called the *coset poset* associated to the pair $(\mathcal{F}, G)$. A natural question to ask is how the topological properties of $\mathcal{C}(\mathcal{F}, G)$ are related to the algebraic properties of $\mathcal{F}$ and G . This question was studied by Abels and Holz [1] in some generality, in particular with regard to the higher connectivity of $\mathcal{C}(\mathcal{F}, G)$. For example, $\mathcal{C}(\mathcal{F}, G)$ is connected if and only if $\mathcal{F}$ covers G , and $\mathcal{C}(\mathcal{F}, G)$ is simply connected if and only if G is isomorphic to the amalgamated product of all $A \in \mathcal{F}$ along their intersections. In their terminology, $\mathcal{F}$ is *n-generating* if $\pi_i(\mathcal{C}(\mathcal{F}, G)) = 0$ for $i \leq n - 1$.

A simple situation arises if one considers the family $\mathcal{A}$ of all abelian subgroups of G . Then $\mathcal{C}(\mathcal{A}, G)$ is connected, and it is easy to show that $\mathcal{C}(\mathcal{A}, G)$ is simply connected if and only if G is abelian, see [14, Prop. 4.1]. When this is the case, $\mathcal{C}(\mathcal{A}, G)$ is contractible. On the other hand, it may be surprising that this statement has a direct analogue in the world of Lie groups. It is the objective of this note to formulate and prove this analogue.

First, one has to clarify the meaning of $\mathcal{C}(\mathcal{A}, G)$ when G itself carries a topology. The role of the complex $\mathcal{C}(\mathcal{A}, G)$ will be played by the geometric realization of a simplicial space denoted by $E(2, G)$ or $E_{\text{com}}G$ in the literature. It was introduced by Adem, Cohen and Torres-Giese [2], who studied basic properties of $E(2, G)$ as part of a more general construction involving families of nilpotent subgroups of G . For compact connected Lie groups G , further homological and homotopical properties of $E(2, G)$ were described by Adem and Gómez [3]. In particular, $E(2, G)$ can be related to the coset spaces G/A for closed abelian subgroups $A \subseteq G$, but the relationship is much more intricate than in the discrete case. When G is discrete, then $E(2, G)$ is homotopy equivalent to $\mathcal{C}(\mathcal{A}, G)$.

Our goal is then to establish a precise relationship between the vanishing of the homotopy groups of $E(2, G)$ and commutativity of G . To do this we promote the commutator map for G to a simplicial map

$$\mathbf{c}: E(2, G) \rightarrow B[G, G],$$

which will play a key role in the proof of our main result.

Theorem 1. *For a compact Lie group G the following assertions are equivalent:*

- (1) G is abelian,
- (2) $E(2, G)$ is contractible,
- (3) $\mathbf{c}$ is null-homotopic,
- (4) $\pi_i(E(2, G)) = 0$ for $i = 1, 2, 4$.

There are two situations in which a stronger statement can be made than that of Theorem 1, both of which are treated implicitly in our proof. Firstly, if G is an arbitrary discrete group, Proposition 9 will show that the statement of Theorem 1 remains valid if (4) is replaced by $\pi_1(E(2, G)) = 0$. For discrete groups the results of [1, Sect. I] imply that $E(2, G)$ is homotopy equivalent to $\mathcal{C}(\mathcal{A}, G)$. In this situation we obtain a new proof of the fact that $\mathcal{C}(\mathcal{A}, G)$ is simply connected if and only if G is abelian, and Theorem 1 may be viewed as a Lie group analogue thereof. Secondly, if G is a compact Lie group with abelian identity component, then Theorem 1 remains valid if (4) is replaced by $E(2, G)$ is 2-connected. This is Proposition 18.

It should be mentioned that the equivalence $(1) \Leftrightarrow (2)$ has a precursor in the work of Adem and Gómez [3] which concerns a variant of $E(2, G)$ denoted $E(2, G)_{\mathbb{1}}$. In general, the space of n -simplices of $E(2, G)$ is not connected, and $E(2, G)_{\mathbb{1}}$ is obtained by restricting to the basepoint component in each simplicial degree. It is proved in [3, Cor. 7.5] that for connected G , $E(2, G)_{\mathbb{1}}$ is rationally acyclic if and only if $E(2, G)_{\mathbb{1}}$ is contractible, if and only if G is abelian. This statement fails to hold when G is disconnected (it fails for every non-abelian discrete group, for instance). In this case, one must consider $E(2, G)$ instead. While their proof relies on a well-known description of the rational cohomology of spaces of commuting elements in Lie groups, we obtain our result by a rather different approach – more homotopical than homological.

Finally, it is worth mentioning that if not contractible, the spaces $E(2, G)$ have an interesting yet difficult to understand homotopy type. For example, by [6] we have

$$E(2, SU(2)) \simeq S^4 \vee \Sigma^4 \mathbb{R}P^2 \quad \text{and} \quad E(2, O(2)) \simeq S^2 \vee S^2 \vee S^3,$$

while for $SU = \operatorname{colim}_{n \rightarrow \infty} SU(n)$ it was shown in [10, Thm. 3.4] that

$$E(2, SU) \simeq BSU \times BSU\langle 6 \rangle \times BSU\langle 8 \rangle \times \cdots,$$

where $BSU\langle 2n \rangle$ is the $(2n - 1)$ -connected cover of BSU . If G is an extraspecial p -group whose Frattini quotient has rank $2r \geq 4$, then the universal cover of $E(2, G)$ is homotopy equivalent to a bouquet of r -dimensional spheres [15]. If G is a transitively commutative group, then $E(2, G)$ is homotopy equivalent to a bouquet of circles by [2, Prop. 8.8]. Other interesting properties of $E(2, G)$ are proved in [4], [17], [18], [19].

2. The simplicial space of affinely commuting elements

Let G be a group. We begin by recalling the simplicial bar construction for G , since it will form the basis for our constructions in the current and the following sections. The simplicial bar construction for the classifying space of G is the simplicial space $B_{\bullet}G$ with n -simplices

$$B_n(G) := G^m,$$

face maps

$$\partial_i: B_n(G) \rightarrow B_{n-1}(G)$$

$$\partial_i(g_1, \dots, g_n) := \begin{cases} (g_2, \dots, g_n) & \text{if } i = 0, \\ (g_1, \dots, g_i g_{i+1}, \dots, g_n) & \text{if } 0 < i < n, \\ (g_1, \dots, g_{n-1}) & \text{if } i = n, \end{cases}$$

and degeneracy maps $s_i: B_n(G) \rightarrow B_{n+1}(G)$ given by inserting the identity element $1 \in G$ in the $(i + 1)$ -st position. Similarly, one defines a simplicial space $E_{\bullet}G$ with n -simplices

$$E_n(G) := G^{n+1},$$

face maps

$$\partial_i: E_n(G) \rightarrow E_{n-1}(G), \quad \partial_i(g_0, \dots, g_n) := (g_0, \dots, \hat{g}_i, \dots, g_n),$$

and degeneracy maps $s_i: E_n(G) \rightarrow E_{n+1}(G)$ given by duplicating the i -th coordinate. For every $n \geq 0$ the group G acts on $E_n(G)$ diagonally by left translation, and this extends to an action on the simplicial space $E_\bullet G$. The quotient map $p: E_\bullet G \rightarrow E_\bullet G/G \cong B_\bullet G$ can be identified with the simplicial map given on n -simplices by

$$E_n(G) \rightarrow B_n(G), \quad (g_0, \dots, g_n) \mapsto (g_0^{-1}g_1, \dots, g_{n-1}^{-1}g_n).$$

Now let us assume for a moment that G is a discrete group. Let $\mathcal{A}$ be the set of abelian subgroups of G partially ordered by inclusion. We may form the union $\bigcup_{A \in \mathcal{A}} B_\bullet A$ inside $B_\bullet G$ and consider the pullback of simplicial sets

$$\begin{array}{ccc} E_\bullet(2, G) & \longrightarrow & E_\bullet G \\ \downarrow & & \downarrow p \\ \bigcup_{A \in \mathcal{A}} B_\bullet A & \hookrightarrow & B_\bullet G \end{array}.$$

The pullback, which we denote by $E_\bullet(2, G)$, can be identified with the simplicial subset of $E_\bullet G$ consisting of those simplices $(g_0, \dots, g_n) \in E_n(G)$ for which the condition $(g_0^{-1}g_1, \dots, g_{n-1}^{-1}g_n) \in B_n(A)$ holds for some abelian subgroup $A \subseteq G$. As EG is contractible, the geometric realization $E(2, G)$ is the homotopy fiber of the inclusion $\bigcup_{A \in \mathcal{A}} BA \rightarrow BG$. It is therefore a measure of how well $\bigcup_{A \in \mathcal{A}} BA$ approximates BG . In other words, it is a measure of the group's failure to be commutative.

By the results of [1, Sect. I], $E(2, G)$ is homotopy equivalent to $\mathcal{C}(\mathcal{A}, G)$. The same simplicial construction, however, can be carried out for an arbitrary topological group. First, observe:

Lemma 1. *Let G be a group. The following conditions on a finite subset $S = \{s_0, \dots, s_n\}$ of G are equivalent:*

- (1) *The elements $s_0^{-1}s_1, s_1^{-1}s_2, \dots, s_{n-1}^{-1}s_n$ pairwise commute.*
- (2) *The group $\langle S^{-1}S \rangle := \langle s_i^{-1}s_j \mid s_i, s_j \in S \rangle$ is abelian.*
- (3) *The set S is contained in a single left coset of some abelian subgroup of G .*

Proof. Condition (2) follows from (1), because each generator of $\langle S^{-1}S \rangle$ is either a product of some elements in (1), or the inverse of such a product. Condition (2) implies (3), because $S \subset s_0 \langle S^{-1}S \rangle$. The proof is completed by showing (3) $\Rightarrow$ (1), which is immediate. $\square$

Definition 2. We say that a finite subset $\{g_0, \dots, g_n\} \subset G$ is *affinely commutative* if it satisfies any of the equivalent conditions listed in Lemma 1.

Let G be a topological group. For each $n \geq 0$ consider the space

$$E_n(2, G) := \{(g_0, \dots, g_n) \in G^{n+1} \mid \{g_0, \dots, g_n\} \text{ is affinely commutative}\},$$

with the topology induced from G^{n+1} . These spaces form a simplicial subspace of $E_\bullet G$ as it can be readily seen that if $\{g_0, \dots, g_n\}$ is affinely commutative, then so are $\{g_0, \dots, \hat{g}_i, \dots, g_n\}$ as well as $\{g_0, \dots, g_i, g_i, \dots, g_n\}$ for any $0 \leq i \leq n$. We denote its geometric realization by

$$E(2, G) := |E_\bullet(2, G)|.$$

Remark 3. The space $E(2, G)$ was studied by Adem, Cohen and Torres-Giese in [2], where the construction was based on a different but isomorphic model of $E_\bullet G$. Namely, let $\mathcal{E}_\bullet G$ denote the simplicial space with n -simplices G^{n+1} , face maps $\partial_i(g_0, \dots, g_n) := (g_0, \dots, g_i g_{i+1}, \dots, g_n)$ for $0 \leq i < n$ and $\partial_n(g_0, \dots, g_n) = (g_0, g_1, \dots, g_{n-1})$, and degeneracy maps s_i given by inserting the identity element $1 \in G$ in the $(i+1)$ -st position. Then the map $E_\bullet G \rightarrow \mathcal{E}_\bullet G$ given on n -simplices by

$$(g_0, \dots, g_n) \mapsto (g_0, g_0^{-1} g_1, \dots, g_{n-1}^{-1} g_n)$$

is an isomorphism. Under this isomorphism $E_\bullet(2, G)$ becomes the simplicial space considered in [2].

We will need below a description of the fundamental group of $E(2, G)$ when G is discrete. In this case, $E(2, G)$ is the realization of a simplicial set and a standard presentation of its fundamental group can be given by picking a spanning tree of the 1-skeleton, with the following generators and relations (see, for example, [8, Prop. 2.7, p. 126]):

- there is a generator x_σ for each 1-simplex σ , which corresponds to the loop in the geometric realization that starts at the basepoint, goes along the spanning tree to the initial vertex of σ , runs across the 1-simplex, and then goes back to the basepoint along the spanning tree again; and
- there are two types of relations: (a) the generators corresponding to tree edges are trivial, and (b) for every triangle τ , the edges on the boundary of τ satisfy $x_{\partial_1 \tau} = x_{\partial_2 \tau} x_{\partial_0 \tau}$.

Let us choose the 0-simplex $1 \in G$ as the basepoint for $E(2, G)$ and the star centered at 1 as the spanning tree. We will have a generator $x_{g,h}$ for each $(g, h) \in G^2$; set $X := \{x_{g,h} \mid (g, h) \in G^2\}$. The generator $x_{g,h}$ represents the loop in $E(2, G)$ obtained by following the 1-simplices $(1, g)$, (g, h) and $(h, 1)$.

Lemma 4. *Let G be discrete. Then, the fundamental group of $E(2, G)$ admits the presentation*

$$\pi_1(E(2, G), 1) = \langle X \mid \{x_{g,1}, x_{1,g} \mid g \in G\} \cup \{x_{g,h} x_{h,k} x_{g,k}^{-1} \mid \{g, h, k\} \subset G \text{ is affinely commutative}\} \rangle.$$

3. The commutator map

In this section we introduce our key tool, a natural map $\mathfrak{c}: E(2, G) \rightarrow B[G, G]$ whose homotopy class will say something about the contractibility of $E(2, G)$. The construction of $\mathfrak{c}$ will be possible because of the following simple but crucial observation.

Lemma 5. *Let $\{g, h, k\} \subset G$ be an affinely commutative set. Then $[g, h][h, k] = [g, k]$.*

Proof. By hypothesis, $[h^{-1}g, k^{-1}h] = 1$, and thus

$$[g, h][h, k] = g^{-1}(h^{-1}g)(k^{-1}h)k = g^{-1}(k^{-1}h)(h^{-1}g)k = g^{-1}k^{-1}gk.$$

□

Lemma 5 is precisely what is needed to verify the following.

Corollary 6. *The maps*

$$\begin{aligned} \mathbf{c}_n: \quad E_n(2, G) &\rightarrow [G, G]^n, \\ (g_0, \dots, g_n) &\mapsto ([g_0, g_1], \dots, [g_{n-1}, g_n]), \end{aligned}$$

defined for all $n \geq 0$, assemble into a map of simplicial spaces $\mathbf{c}_\bullet: E_\bullet(2, G) \rightarrow B_\bullet[G, G]$.

Definition 7. Upon geometric realization $\mathbf{c}_\bullet$ defines a map

$$\mathbf{c}: E(2, G) \rightarrow B[G, G],$$

which we refer to as the *commutator map*.

The rest of this section is devoted to establishing some basic properties of $\mathbf{c}$. Let

$$c: G \times G \rightarrow G, \quad (x, y) \mapsto [x, y],$$

be the *algebraic commutator map* for G . Note that c factors through a map $\tilde{c}: G \wedge G \rightarrow [G, G]$, since $[g, h] = 1$ if either $g = 1$ or $h = 1$.

Definition 8. A topological group G is called *homotopy abelian* if the algebraic commutator map c is null-homotopic.

The following proposition summarizes the main features of the commutator map $\mathbf{c}$ that the proof of Theorem 1 will rely on.

Proposition 9. *Let G be either a discrete group or a compact Lie group, let $\mathbf{c}: E(2, G) \rightarrow B[G, G]$ be the commutator map, and let*

$$\mathbf{c}_*: \pi_1(E(2, G)) \rightarrow \pi_1(B[G, G])$$

be the map induced by $\mathbf{c}$ on fundamental groups.

- (1) *If G is discrete, then $\mathbf{c}_*$ satisfies $\mathbf{c}_*(x_{g,h}) = [g, h]$ for all $x_{g,h} \in \pi_1(E(2, G))$.*
- (2) *If either $\mathbf{c}$ is null-homotopic or $E(2, G)$ is $(2 \dim G + 1)$ -connected, then G is homotopy abelian.*
- (3) *The map $\mathbf{c}_*$ is surjective, and it is trivial if and only if $[G, G]$ is a connected Lie group.*

Proof. First, assume that G is discrete. As pointed out at the end of Section 2, the generator $x_{g,h} \in \pi_1(E(2, G))$ is represented by the path obtained by concatenating the 1-simplices $(1, g)$, (g, h) and $(h, 1)$. Statement (1) follows, because $\mathfrak{c}$ takes these 1-simplices to the 1-simplices $[1, g] = 1$, $[g, h]$ and $[h, 1] = 1$ in $B[G, G]$, respectively.

Next we prove (2). Since G is either discrete or a Lie group, the simplicial space $E_\bullet(2, G)$ is proper (cf. [3, Appendix]), hence the fat and thin realizations are naturally homotopy equivalent: $\|E_\bullet(2, G)\| \simeq E(2, G)$. If X is the geometric realization of a semi-simplicial space, we denote by $F_k X$ the k -th term in the skeletal filtration of X . Then, $F_1 \|E_\bullet G\| = F_1 \|E_\bullet(2, G)\| \cong G * G$ is the topological join, and $F_1 \|B_\bullet G\| \cong SG$ is the unreduced suspension. There is a commutative diagram

$$\begin{array}{ccc} G * G & \xrightarrow{c'} & S[G, G] \\ \downarrow & & \downarrow \\ \|E_\bullet(2, G)\| & \xrightarrow{\mathfrak{c}} & \|B_\bullet[G, G]\| \end{array} \quad (1)$$

where $c'(tg + (1-t)h) = t \wedge [g, h]$ for $t \in [0, 1]$ and $g, h \in G$. Up to homotopy, c' can be identified with the map $\Sigma\tilde{c}: \Sigma(G \wedge G) \rightarrow \Sigma[G, G]$. If $\mathfrak{c}$ is null-homotopic, diagram (1) implies that the composite

$$\Sigma(G \wedge G) \xrightarrow{\Sigma\tilde{c}} \Sigma[G, G] \rightarrow B[G, G]$$

is null-homotopic as well. This composite is also null-homotopic if $\|E_\bullet(2, G)\|$ is $(2 \dim G + 1)$ -connected, since in that case the map $G * G \rightarrow \|E_\bullet(2, G)\|$ appearing in the diagram is null-homotopic, by the standard fact from obstruction theory which states that any map from a k -dimensional CW-complex to a k -connected space is null-homotopic.

Maps between path connected spaces are null-homotopic if and only if they are based null-homotopic, so the map $\Sigma(G \wedge G) \rightarrow B[G, G]$ is based null-homotopic. Therefore its adjoint map

$$G \wedge G \xrightarrow{\tilde{c}} [G, G] \xrightarrow{\simeq} \Omega B[G, G],$$

and hence $\tilde{c}$, are null-homotopic. Since the algebraic commutator map $c: G \times G \rightarrow G$ factors through $\tilde{c}$, it is null-homotopic as well. This finishes the proof of (2).

Now we prove (3). If G is discrete it follows directly from statement (1). Assume G is a compact Lie group and let G^δ denote G equipped with the discrete topology. Let $d: G^\delta \rightarrow G$ be the canonical map. The commutator map $\mathfrak{c}$ for G and the commutator map $\mathfrak{c}^\delta$ for G^δ are related by a commutative diagram

$$\begin{array}{ccc} E(2, G^\delta) & \xrightarrow{\mathfrak{c}^\delta} & B[G^\delta, G^\delta] \\ E(2, d) \downarrow & & \downarrow Bd \\ E(2, G) & \xrightarrow{\mathfrak{c}} & B[G, G]. \end{array} \quad .$$

Recall that for Lie groups, the commutator subgroup $[G, G]$ is defined to be the closure of the algebraic commutator subgroup. But the algebraic commutator subgroup of a compact Lie group is always closed (see [12, Thm. 6.11]), so $[G, G]^\delta = [G^\delta, G^\delta]$.

The diagram induces a commutative diagram on fundamental groups. Now consider the composite homomorphism

$$\pi_1(E(2, G^\delta)) \xrightarrow{\mathfrak{c}_*^\delta} [G^\delta, G^\delta] \xrightarrow{\pi_1(Bd)} \pi_1(B[G, G]) \quad (2)$$

obtained by going through the top right corner of the diagram. Under the isomorphism $\pi_1(B[G, G]) \cong \pi_0([G, G])$ and the identification of $[G^\delta, G^\delta]$ with $[G, G]^\delta$ the map $\pi_1(Bd)$ corresponds to the canonical surjection $[G, G]^\delta \rightarrow \pi_0([G, G])$. Moreover, by part (1) the map $\mathfrak{c}_*^\delta$ is surjective. Together this implies that (2) is surjective, and by commutativity of the diagram $\mathfrak{c}_*$ must be surjective, as well. In particular, if $\mathfrak{c}_*$ is trivial, then $\pi_0([G, G]) = 1$. Conversely, if $[G, G]$ is path connected, then $B[G, G]$ is simply connected, hence $\mathfrak{c}_*$ is trivial. $\square$

Remark 10. Perhaps surprisingly, there exist homotopy abelian compact Lie groups G for which $\mathfrak{c}$ is not null-homotopic. Hence, the converse of part (2) of Proposition 9 fails to hold. An example illustrating this is the central extension

$$1 \rightarrow S^1 \rightarrow (S^1 \times Q_8)/\mathbb{Z}/2 \rightarrow \mathbb{Z}/2 \times \mathbb{Z}/2 \rightarrow 1,$$

where Q_8 is the quaternion group of order eight. The quotient $G = (S^1 \times Q_8)/\mathbb{Z}/2$ is taken over the central subgroup $\mathbb{Z}/2 = \langle (-1, -1) \rangle \subset S^1 \times Q_8$. It is indeed homotopy abelian; the commutator subgroup is $[G, G] = \{[(1, 1)], [(-1, 1)]\} \cong \mathbb{Z}/2$, which is a discrete subgroup of the path-connected group S^1 , thus making the algebraic commutator map null-homotopic. But by part (3) of Proposition 9, $\mathfrak{c}$ cannot be trivial on fundamental groups, since $[G, G]$ is not connected.

Remark 11. Let $j: B[G, G] \rightarrow BG$ be the map induced by the inclusion $[G, G] \subseteq G$. There is another description, up to homotopy, of the composition $j\mathfrak{c}: E(2, G) \rightarrow BG$. We shall not need it to prove our main theorem, but it seems worth mentioning, because it is not obvious from the definition. Let $C_n(G) \subseteq G^n$ denote the subspace of n -tuples of commuting elements in G . Then $C_\bullet(G) \subseteq B_\bullet G$ is a simplicial subspace, whose realization we denote by $B(2, G)$. The composite map

$$E(2, G) \subseteq EG \xrightarrow{p} BG$$

factors through the inclusion $i: B(2, G) \rightarrow BG$. By abuse of notation, we write $p: E(2, G) \rightarrow B(2, G)$ for the projection. Note that there is an automorphism $\phi^{-1}: B(2, G) \rightarrow B(2, G)$ induced by the map $G \rightarrow G$, $g \mapsto g^{-1}$. We claim that the diagram

$$\begin{array}{ccccc} E(2, G) & \xrightarrow{p} & B(2, G) & \xrightarrow{\phi^{-1}} & B(2, G) \\ \mathfrak{c} \downarrow & & & & \downarrow i \\ B[G, G] & \xrightarrow{j} & & & BG \end{array}$$

commutes up to homotopy. Indeed, it is tedious but straightforward to verify that the collection of maps $\{h_i\}_{0 \leq i \leq n}$ defined by

$$\begin{aligned} h_i: E_n(2, G) &\rightarrow B_{n+1}(G) = G^{n+1}, \\ (g_0, \dots, g_n) &\mapsto ([g_0, g_1], \dots, [g_{i-1}, g_i], g_i^{-1}, g_{i+1}^{-1}g_i, \dots, g_n^{-1}g_{n-1}), \end{aligned}$$

is a simplicial homotopy between $j\mathbf{c}$ and $i\phi^{-1}p$ in the sense of [13, Def. 9.1].

4. The proof of Theorem 1

The proof of Theorem 1 will require a couple of propositions, the first of which is a characterization of homotopy abelian compact Lie groups.

Proposition 12. *Let G be a compact Lie group. Then G is homotopy abelian if and only if $\pi_0(G)$ is abelian and G is a central extension of $\pi_0(G)$ by a torus.*

Proof. Suppose that G is homotopy abelian. Let $G_0 \subseteq G$ be the component of the identity and let $T \subseteq G_0$ be a maximal torus. As the commutator map $c: G \times G \rightarrow G$ is null-homotopic, it factors through G_0 and its restriction to G_0 is null-homotopic, too. It follows that G_0 is homotopy abelian. A result of Araki, James and Thomas [7] asserts that a compact, connected, homotopy abelian Lie group is abelian. Hence, $G_0 = T$. Thus G fits into an extension

$$1 \rightarrow T \rightarrow G \xrightarrow{p} \pi_0(G) \rightarrow 1.$$

It is clear that $\pi_0(G)$ is abelian, and so it remains to show that T is central.

Note that $\text{Aut}(T) \cong \text{Aut}(H_1(T; \mathbb{Z}))$ is discrete. For $g \in G$ let $\text{conj}_g \in \text{Aut}(T)$ denote the inner automorphism $t \mapsto g^{-1}tg$. The map $g \mapsto \text{conj}_g$ must be constant on connected components and thus factors through a representation

$$\rho: \pi_0(G) \rightarrow \text{Aut}(H_1(T; \mathbb{Z})).$$

To show that T is central, it is enough to show that ρ is constant. Fix $g \in G$. Let $c(-, g): T \rightarrow T$ denote the composition

$$T \xrightarrow{t \mapsto (t, g)} G \times G \xrightarrow{c} T,$$

which has image in $T = G_0$ because $c(1, g) = 1$. The composite map is null-homotopic, because the commutator map is null-homotopic. On the other hand, $c(-, g)$ can be identified with the composition

$$T \xrightarrow{t \mapsto (t^{-1}, t)} T \times T \xrightarrow{\text{id} \times \text{conj}_g} T \times T \xrightarrow{\cdot} T,$$

where the last map is multiplication in T . As this map is null-homotopic, the induced map on $H_1(T; \mathbb{Z})$ is zero. This implies that, for any $x \in H_1(T; \mathbb{Z})$, we must have

$$0 = -x + \rho(p(g))(x),$$

hence $\rho(p(g)) = id$. This finishes the proof that T is central.

Conversely, suppose that G is a central extension of $\pi_0(G)$ by a torus T and assume that $\pi_0(G)$ is abelian. The central extension is classified by a 2-cocycle $\omega: \pi_0(G) \times \pi_0(G) \rightarrow T$. As an abstract group, G is isomorphic to $T \times \pi_0(G)$ with group law

$$(t, x)(s, y) := (ts\omega(x, y), xy),$$

see [11, Rem. 18.1.14]. A short computation shows that, when $\pi_0(G)$ is abelian, the commutator of any two elements (t, x) and (s, y) of $T \times \pi_0(G)$ reads

$$[(t, x), (s, y)] = (\omega(x, y)\omega(y, x)^{-1}, 1).$$

Thus, the commutator map $c: G \times G \rightarrow G$ factors through $\pi_0(G) \times \pi_0(G)$ and has image in T . Since T is connected, c is null-homotopic. $\square$

Remark 13. The central extension $(S^1 \times Q_8)/\mathbb{Z}/2$ described in Remark 10 is an example of a homotopy abelian compact Lie group which is not abelian. It also illustrates that the theorem of Araki, James and Thomas [7] used in the proof of Proposition 12 fails to hold for disconnected groups.

Another statement that will enter into the proof of our main result is the following.

Proposition 14. *Let G be a compact Lie group. If $\pi_4(E(2, G)) = 0$, then the component G_0 of the identity is abelian, hence G is an extension of $\pi_0(G)$ by a torus.*

The proof of the proposition requires some preparation. Recall $C_n(G) \subseteq G^n$ denotes the subspace of n -tuples of commuting elements in G .

Lemma 15. *The realization of the simplicial subspace $C_{\bullet+1}(G) \subseteq E_{\bullet}(2, G)$ is contractible.*

Proof. Implicit in the statement is the claim that $C_{\bullet+1}(G)$ is a simplicial subspace of $E_{\bullet}(2, G)$. Since in $E_{\bullet}(2, G)$ the faces and degeneracies delete and duplicate coordinates (as they do in the simplicial model of EG described in Section 2) it is easy to check that $C_{\bullet+1}(G)$ is indeed a simplicial subspace.

To prove it is contractible we can straightforwardly adapt one of the usual proofs that EG is contractible: the simplicial model of EG can be augmented by adding a unique (-1) -simplex and this augmented simplicial space has an extra degeneracy given by $s_{-1}(g_0, \dots, g_n) = (1, g_0, \dots, g_n)$ for any $n \geq -1$. This extra degeneracy preserves $C_{\bullet+1}(G)$ and thus also shows that its geometric realization is contractible. $\square$

We now define a homotopy equivalent model for $E(2, G)$ which will turn out convenient. Consider the simplicial space $\bar{E}_{\bullet}(2, G)$ with n -simplices

$$\bar{E}_n(2, G) := E_n(2, G)/C_{n+1}(G)$$

and simplicial structure the one induced by $E_{\bullet}(2, G)$.

As $C_{\bullet+1}(G) \rightarrow E_{\bullet}(2, G)$ is a levelwise cofibration of good simplicial spaces the map of realizations $|C_{\bullet+1}(G)| \rightarrow E(2, G)$ is a cofibration. By Lemma 15 $|C_{\bullet+1}(G)|$ is contractible, so the map $E(2, G) \rightarrow E(2, G)/|C_{\bullet+1}(G)|$ is a homotopy equivalence. Since geometric realization commutes with taking cofibers, the levelwise quotient maps induce a homotopy equivalence

$$E(2, G) \xrightarrow{\simeq} \bar{E}(2, G).$$

Just like $E(2, G)$, the assignment $G \mapsto \bar{E}(2, G)$ is natural in homomorphisms of groups, and so is the equivalence $E(2, G) \simeq \bar{E}(2, G)$.

Remark 16. In the introduction we mentioned the space $E(2, G)_{\mathbb{1}}$, which is the geometric realization of the simplicial subspace $E_{\bullet}(2, G)_{\mathbb{1}} \subseteq E_{\bullet}(2, G)$ consisting of the connected component of $(1, \dots, 1)$ in each degree. This space also has a homotopy equivalent model obtained as above by setting

$$\bar{E}_n(2, G)_{\mathbb{1}} := E_n(2, G)_{\mathbb{1}}/C_{n+1}(G)_{\mathbb{1}}.$$

Indeed, the extra degeneracy used in the proof of Lemma 15 preserves the simplicial subspace $C_{\bullet+1}(G)_{\mathbb{1}}$ consisting in degree n of the connected component of $C_{n+1}(G)$ containing $(1, \dots, 1)$.

The commutator map $\mathfrak{c}: E(2, G) \rightarrow B[G, G]$ factors through $\bar{E}(2, G)$. To keep the notation simple we denote the resulting map $\mathfrak{c}: \bar{E}(2, G) \rightarrow B[G, G]$ by the same letter. Observe that $\bar{E}(2, G)$ is a reduced simplicial space, and the space of 1-simplices is $G^2/C_2(G)$. Therefore, the simplicial 1-skeleton is $\Sigma(G^2/C_2(G))$ and the restriction to the 1-skeleton of the commutator map,

$$\mathfrak{c}|: \Sigma(G^2/C_2(G)) \rightarrow \Sigma[G, G],$$

is simply the suspension of the map induced by the algebraic commutator map $c: G^2 \rightarrow [G, G]$.

Lemma 17. *After looping the commutator map $\mathfrak{c}: \bar{E}(2, SU(2)) \rightarrow BSU(2)$ has a section up to homotopy. Furthermore, a section $s: SU(2) \rightarrow \Omega\bar{E}(2, SU(2))$ can be chosen which is natural with respect to homomorphisms $f: SU(2) \rightarrow G$ in the following sense: for every f , the diagram*

$$\begin{array}{ccc} \Omega\bar{E}(2, SU(2)) & \xrightarrow{\Omega\bar{E}(2, f)} & \Omega\bar{E}(2, G) \\ \downarrow \scriptstyle s \quad \downarrow \scriptstyle \Omega\mathfrak{c} & & \downarrow \scriptstyle \Omega\mathfrak{c} \\ SU(2) & \xrightarrow{f} & [G, G] \end{array}$$

with the dotted arrow filled in commutes up to homotopy.

Proof. In the diagram we have implicitly used the canonical homotopy equivalence $[G, G] \simeq \Omega B[G, G]$ adjoint to the inclusion $\Sigma[G, G] \rightarrow B[G, G]$. By adjunction it

is enough to construct a map $s': \Sigma SU(2) \rightarrow \bar{E}(2, SU(2))$ making the following diagram commute:

$$\begin{array}{ccccc}
 & & \bar{E}(2, SU(2)) & \xrightarrow{\bar{E}(2, f)} & \bar{E}(2, G) \\
 & \nearrow s' & \downarrow c & & \downarrow c \\
 \Sigma SU(2) & \hookrightarrow & BSU(2) & \xrightarrow{Bf} & B[G, G].
 \end{array}$$

The desired section s may then be defined as the adjunct of s' . As the simplicial 1-skeleton of $\bar{E}(2, SU(2))$ is $\Sigma(SU(2)^2/C_2(SU(2)))$ it suffices to construct a section of the map

$$c|: \Sigma(SU(2)^2/C_2(SU(2))) \rightarrow \Sigma SU(2),$$

and s' may be defined as the composite of this section with the inclusion into $\bar{E}(2, SU(2))$.

It is shown in [5, Sect. III, 2(a), Prop. 2.1] that the algebraic commutator map is surjective and that its restriction to the non-commuting pairs in $SU(2)$,

$$c|: SU(2)^2 - C_2(SU(2)) \rightarrow SU(2) - \{1\},$$

is a submersion. Since $SU(2) \times SU(2)$ is compact, c and hence $c|$ are proper maps. Ehresmann's theorem says that the proper surjective submersion $c|$ is a fiber bundle. Let F denote the fiber. Note that $c| = \Sigma(c|)^+$, where $(c|)^+$ is the map induced by $c|$ on one-point compactifications. As $SU(2) - \{1\}$ is contractible the total space of the fiber bundle is homeomorphic to $(SU(2) - \{1\}) \times F$ in such a way that $c|$ corresponds to the projection onto the first factor. Now F is compact, since it is a closed subset of the compact space $SU(2)^2$. Thus, there is a homeomorphism

$$[(SU(2) - \{1\}) \times F]^+ \cong SU(2) \wedge F_+,$$

where F_+ denotes F with a disjoint basepoint added. Under this homeomorphism $(c|)^+$ can be identified with the map $SU(2) \wedge F_+ \rightarrow SU(2)$ induced by the projection $F_+ \rightarrow S^0$. A choice of basepoint of F gives a section $SU(2) \wedge S^0 \rightarrow SU(2) \wedge F_+$, and its suspension yields a section for $c|$. $\square$

Proof of Proposition 14. Let G_0 denote the component of the identity of G . We must show that G_0 is abelian. Clearly, this follows if we can show that $[G, G]_0$ is abelian, for $[G_0, G_0]$ is a subgroup of $[G, G]_0$, and the commutator group of a connected compact Lie group is semisimple.

Thus, assume for contradiction that $[G, G]_0$ is non-abelian. It is well known that the universal cover of a compact connected Lie group K decomposes as a product of simply connected simple Lie groups $\{K_i\}_{i=1, \dots, k}$ and a copy of $\mathbb{R}^m$, giving $\pi_3(K) = \pi_3(K_1) \oplus \dots \oplus \pi_3(K_k)$. For $K = [G, G]_0$ we must have $k \geq 1$, since $[G, G]_0$ is assumed non-abelian. In [9, Chap. III, Prop. 10.2] it is shown that in a simply connected simple Lie group K_i one can find a subgroup isomorphic to $SU(2)$ such that the inclusion $SU(2) \rightarrow K_i$ induces an isomorphism in $\pi_3(-)$. Thus we find a homomorphism $f: SU(2) \rightarrow [G, G]_0$ such that $\pi_3(f): \pi_3(SU(2)) \rightarrow \pi_3([G, G]_0)$

is injective. To reach a contradiction it suffices to show that the map $\pi_3(SU(2)) \rightarrow \pi_3([G, G])$ obtained by composition with the inclusion $[G, G]_0 \subseteq [G, G]$ is zero.

Application of $\pi_3(-)$ to the homotopy commutative diagram in Lemma 17 yields a commutative diagram

$$\begin{array}{ccc} \pi_3(\Omega\bar{E}(2, SU(2))) & \xrightarrow{\pi_3(f')} & \pi_3(\Omega\bar{E}(2, G)) \\ \pi_3(s) \uparrow & & \downarrow \pi_3(\Omega\mathfrak{c}) \\ \pi_3(SU(2)) & \xrightarrow{\pi_3(f)} & \pi_3([G, G]) \end{array}$$

where $f' := \Omega\bar{E}(2, f)$. Since $E(2, G)$ is homotopy equivalent with $\bar{E}(2, G)$, we have that

$$\pi_3(\Omega\bar{E}(2, G)) \cong \pi_4(\bar{E}(2, G)) \cong \pi_4(E(2, G)).$$

By assumption this group is zero, hence $\pi_3(f) = 0$. $\square$

The final item needed to prove Theorem 1 is the following proposition.

Proposition 18. *Let G be a compact Lie group and assume that the component of the identity G_0 is abelian. If $E(2, G)$ is 2-connected, then $\mathfrak{c}$ is null-homotopic.*

Proof. Since $\pi_1(E(2, G)) = 0$ by assumption, we deduce from Proposition 9 part (3) that $[G, G]$ is connected. Then $[G, G] \subseteq G_0$, and since $[G, G]$ is closed it is also a torus. Therefore, $B[G, G]$ is an Eilenberg–Mac Lane space of type $K(\mathbb{Z}^r, 2)$ for some $r \geq 0$, and the homotopy class of the commutator map

$$\mathfrak{c}: E(2, G) \rightarrow B[G, G] \simeq K(\mathbb{Z}^r, 2)$$

corresponds to a cohomology class in $H^2(E(2, G); \mathbb{Z}^r)$. Since $E(2, G)$ is assumed 2-connected, we have that $H^2(E(2, G); \mathbb{Z}^r) = 0$. Hence $\mathfrak{c}$ is null-homotopic, as desired. $\square$

We can now prove the main result of this paper.

Theorem 1. *For a compact Lie group G the following assertions are equivalent:*

- (1) G is abelian,
- (2) $E(2, G)$ is contractible,
- (3) $\mathfrak{c}$ is null-homotopic,
- (4) $\pi_i(E(2, G)) = 0$ for $i = 1, 2, 4$.

Proof. Clearly, if G is abelian, then $E(2, G)$ is contractible, because in this case every subset $\{g_0, \dots, g_n\} \subseteq G$ is affinely commutative, so $E(2, G) = EG$ and EG is contractible. If $E(2, G)$ is contractible, then it is obvious that $\mathfrak{c}$ is null-homotopic, and that $\pi_i(E(2, G)) = 0$ for $i = 1, 2$ and 4. To establish the theorem we shall prove that (3) $\Rightarrow$ (1), and (4) $\Rightarrow$ (3).

Suppose that $\mathfrak{c}$ is null-homotopic. Then clause (2) of Proposition 9 and Proposition 12 imply that G is a central extension of $\pi_0(G)$ by a torus. In addition the map $\mathfrak{c}_*$ of fundamental groups is trivial, hence $[G, G]$ is a connected Lie group by clause (3) of Proposition 9. Then it is a subgroup of $G_0 = T$, hence a torus. It will

suffice to show that $[G, G]$ is finitely generated, since a torus is finitely generated only if it is the trivial group. But as pointed out in the proof of Proposition 12, the commutator map for G factors through $\pi_0(G)$, hence its image is finite because $\pi_0(G)$ is finite. This shows that (3) $\Rightarrow$ (1).

We will now show that (4) $\Rightarrow$ (3). By assumption $\pi_4(E(2, G)) = 0$, so Proposition 14 implies that the identity component $G_0 \subseteq G$ is abelian. Proposition 18 now finishes the proof. $\square$

There is an intriguing relationship of $E(2, G)$ with bundle theory. In [3] it is explained how the i -th homotopy group of $E(2, G)$ can be interpreted as the set of “transitionally commutative structures” on the trivial principal G -bundle over S^i . We refer to [3] for more background. If G is non-abelian, then $\pi_i(E(2, G)) \neq 0$ for some $i \in \{1, 2, 4\}$. Therefore, our theorem has the following corollary.

Corollary 19. *Let G be a non-abelian compact Lie group. Then the trivial principal G -bundle over at least one of S^1 , S^2 or S^4 admits two distinct transitionally commutative structures.*

Remark 20. The compactness condition in Theorem 1 is necessary. For example, $SL(2, \mathbb{R})$ is a homotopy abelian Lie group as it deformation retracts onto $SO(2)$, but it is not abelian. On the other hand, a result of Pettet and Souto [16, Cor. 1.2] implies that $E(2, SL(2, \mathbb{R})) \simeq E(2, SO(2)) = ESO(2)$, which is contractible.

5. A potential splitting

The results in this paper would be well explained by a splitting up to homotopy of the looped commutator map $\Omega\mathfrak{c}$, and hence a splitting of spaces

$$\Omega E(2, G) \simeq [G, G] \times \Omega X,$$

for some space X . Indeed, if such a splitting exists, then any of the equivalent conditions listed in Theorem 1 readily implies that $[G, G] = 1$. Note that a connected and simply connected compact Lie group with trivial π_3 is necessarily trivial.

The splitting exists for $G = SU(2)$ as proved in Lemma 17. It also exists for $G = O(2)$ and for $G = SU$. For example, we proved in [6, Thm. 1.5] that $E(2, O(2)) \simeq \Sigma(S^1 \times S^1)$. For any group G there is a homotopy fiber sequence $G * G \rightarrow \Sigma G \rightarrow BG$ by Ganea’s theorem. After looping, the unit map $G \rightarrow \Omega \Sigma G$ splits the homotopy fiber sequence, hence

$$\Omega \Sigma G \simeq G \times \Omega(G * G).$$

In particular, there is a homotopy equivalence

$$\Omega E(2, O(2)) \simeq S^1 \times S^1 \times \Omega((S^1 \times S^1) * (S^1 \times S^1)).$$

Note that $S^1 = SO(2) = [O(2), O(2)]$. To prove that $\Omega\mathfrak{c}: \Omega E(2, O(2)) \rightarrow SO(2)$ splits up to homotopy it is enough to show that $\Omega\mathfrak{c}$ is surjective on fundamental groups. Because the inclusion $SO(2) \hookrightarrow O(2)$ induces an isomorphism on π_2 of the classifying spaces, it is equivalent to showing, using Remark 11, that the map

$i\phi^{-1}p: E(2, O(2)) \rightarrow BO(2)$ is surjective on π_2 . Surjectivity follows from results in [17] as we will now explain. The authors construct a map $f_1: S^2 \rightarrow B(2, O(2))$ such that if_1 is null-homotopic but $i\phi^{-1}f_1$ is a generator of $\pi_2(BO(2)) \cong \mathbb{Z}$ ([17, Prop. 3.5]). Thus we can find a lift of f_1 up to homotopy $\tilde{f}_1: S^2 \rightarrow E(2, O(2))$ such that $i\phi^{-1}p\tilde{f}_1$ is a generator of $\pi_2(BO(2))$.

We leave it to the reader to show that $\Omega\mathfrak{c}: \Omega E(2, SU) \rightarrow SU$ has a splitting up to homotopy using [10, Thm. 3.4] and Remark 11.

There are too few examples known to build a firm opinion, but the results of this paper suggest that the following question warrants further study.

Question 21. *Let G be a compact Lie group. Does the commutator map*

$$\mathfrak{c}: E(2, G) \rightarrow B[G, G]$$

split up to homotopy after looping?

One way of establishing a splitting is by showing that the restriction $\mathfrak{c}|$ of the commutator map to the simplicial 1-skeleton of $\bar{E}(2, G)$ has a splitting up to homotopy. This was carried out for $G = SU(2)$ in Lemma 17. However, one can show that $\mathfrak{c}|$ splits neither for $G = O(2)$ nor for $G = SO(3)$. For example, for $G = SO(3)$ we have $H_1(SO(3); \mathbb{Z}) \cong \mathbb{Z}/2$ but one can compute that

$$H_1(SO(3)^2/C_2(SO(3)); \mathbb{Z}) \cong \mathbb{Z}$$

using [20, Thm. 1.2]. This motivates the following question.

Question 22. *For which groups G does the commutator map*

$$\mathfrak{c}|: \Sigma(G^2/C_2(G)) \rightarrow \Sigma[G, G]$$

split up to homotopy?

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