PERIODIC AUTOMORPHISMS, COMPATIBLE POISSON BRACKETS, AND GAUDIN SUBALGEBRAS

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To the memory of our teacher Ernest Borisovich Vinberg

Abstract. Let $\mathfrak{q}$ be a finite-dimensional Lie algebra. The symmetric algebra $\mathcal{S}(\mathfrak{q})$ is equipped with the standard Lie–Poisson bracket. In this paper, we elaborate on a surprising observation that one naturally associates the second compatible Poisson bracket on $\mathcal{S}(\mathfrak{q})$ to any finite order automorphism ϑ of $\mathfrak{q}$. We study related Poisson-commutative subalgebras $\mathcal{Z}(\mathfrak{q}, \vartheta)$ of $\mathcal{S}(\mathfrak{q})$ and associated Lie algebra contractions of $\mathfrak{q}$. To obtain substantial results, we have to assume that $\mathfrak{q} = \mathfrak{g}$ is semisimple. Then we can use Vinberg’s theory of ϑ -groups and the machinery of Invariant Theory.

If $\mathfrak{g} = \mathfrak{h} \oplus \cdots \oplus \mathfrak{h}$ (sum of k copies), where $\mathfrak{h}$ is simple, and ϑ is the cyclic permutation, then we prove that the corresponding Poisson-commutative subalgebra $\mathcal{Z}(\mathfrak{q}, \vartheta)$ is polynomial and maximal. Furthermore, we quantise this $\mathcal{Z}(\mathfrak{g}, \vartheta)$ using a Gaudin subalgebra in the enveloping algebra $\mathcal{U}(\mathfrak{g})$.

Introduction

The ground field $\mathbb{k}$ is algebraically closed and $\text{char}(\mathbb{k}) = 0$. Let $\mathfrak{q} = (\mathfrak{q}, [\ , \])$ be a finite-dimensional algebraic Lie algebra, i.e., $\mathfrak{q} = \text{Lie } Q$, where Q is a connected affine algebraic group. The dual space $\mathfrak{q}^*$ is a Poisson variety, i.e., the algebra of polynomial functions on $\mathfrak{q}^*$, $\mathbb{k}[\mathfrak{q}^*] \simeq \mathcal{S}(\mathfrak{q})$, is equipped with the Lie–Poisson bracket $\{ \ , \ }$. Here $\{x, y\} = [x, y]$ for $x, y \in \mathfrak{q}$. Poisson-commutative subalgebras of $\mathbb{k}[\mathfrak{q}^*]$ are important tools for the study of geometry of the coadjoint representation of Q and representation theory of $\mathfrak{q}$.

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0.1. Compatible Poisson brackets

There is a well-known method, *the Lenard–Magri scheme*, for constructing “large” Poisson-commutative subalgebras of $\mathbb{k}[\mathfrak{q}^*]$, which is related to *compatible Poisson brackets*, see, e.g., [GZ00]. Two Poisson brackets $\{ , \}'$ and $\{ , \}''$ are said to be *compatible*, if any linear combination $\{ , \}_{a,b} := a\{ , \}' + b\{ , \}''$ with $a, b \in \mathbb{k}$ is a Poisson bracket. Then one defines a certain dense open subset $\Omega_{\text{reg}} \subset \mathbb{k}^2$ that corresponds to the *regular* brackets in the pencil $\mathcal{P} = \{ \{ , \}_{a,b} \mid (a,b) \in \mathbb{k}^2 \}$. If $\mathcal{Z}_{a,b} \subset \mathcal{S}(\mathfrak{q})$ denotes the Poisson centre of $(\mathcal{S}(\mathfrak{q}), \{ , \}_{a,b})$, then the subalgebra $\mathcal{Z} \subset \mathcal{S}(\mathfrak{q})$ generated by $\mathcal{Z}_{a,b}$ with $(a,b) \in \Omega_{\text{reg}}$ is Poisson-commutative with respect to $\{ , \}'$ and $\{ , \}''$, see Section 2.2 for more details. An obvious first step is to take the initial Lie–Poisson bracket $\{ , \}$ as $\{ , \}'$. The rest depends on a clever choice of $\{ , \}''$. The goal of this article is to introduce a new class of compatible Poisson brackets, study the respective subalgebras $\mathcal{Z}$, and provide applications.

Let us recall some known pencils of compatible Poisson brackets.

- For any $\gamma \in \mathfrak{q}^*$, the Poisson bracket $(x, y) \mapsto \{x, y\}_\gamma := \gamma([x, y])$, $x, y \in \mathfrak{q}$, is compatible with $\{ , \}$, cf. [DZ05, Example 1.8.16]. This leads to the fabulous *Mishchenko–Fomenko* (= MF) *subalgebras* $\mathcal{Z} = \mathcal{A}_\gamma \subset \mathcal{S}(\mathfrak{q})$, introduced in [MF78] for semisimple $\mathfrak{q}$. To the best of our knowledge, the very term “Mishchenko–Fomenko” was coined by Vinberg [Vi90].

- In [PY], we presented compatible Poisson brackets related to a $\mathbb{Z}_2$ -grading $\mathfrak{q} = \mathfrak{q}_0 \oplus \mathfrak{q}_1$ and studied the respective subalgebra $\mathcal{Z} = \mathcal{Z}(\mathfrak{q}, \mathfrak{q}_0)$. Here the second bracket is defined by the relations $\{x, y\}'' = \{x, y\}$ if $x \in \mathfrak{q}_0$ and $\{x, y\}'' = 0$ if $x, y \in \mathfrak{q}_1$. As well as with MF subalgebras, one cannot get too far if $\mathfrak{q}$ is arbitrary. Assuming that $\mathfrak{q} = \mathfrak{g}$ is reductive, we obtained a number of interesting results on $\mathcal{Z}(\mathfrak{g}, \mathfrak{g}_0)$. We proved that:

- $\mathcal{Z}(\mathfrak{g}, \mathfrak{g}_0)$ is a Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}_0}$ having the maximal possible transcendence degree, which equals $(\dim \mathfrak{g}_1 + \text{rk } \mathfrak{g} + \text{rk } \mathfrak{g}_0)/2$;

- with only four exceptions related to exceptional Lie algebras, $\mathcal{Z}(\mathfrak{g}, \mathfrak{g}_0)$ is a polynomial algebra whose algebraically independent generators are explicitly described;

- if $\mathfrak{g}$ is a classical Lie algebra and $\mathfrak{g}_0$ contains a regular nilpotent element of $\mathfrak{g}$, then $\mathcal{Z}(\mathfrak{g}, \mathfrak{g}_0)$ is a maximal Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}_0}$.

The proofs exploit numerous invariant-theoretic properties of the adjoint representation of $\mathfrak{g}$ [K63] as well as their analogues for the isotropy representation $G_0 \rightarrow \text{GL}(\mathfrak{g}_1)$ [KR71].

0.2. ϑ -groups

Results of the present paper stem from a surprising observation that if $\mathfrak{q}$ is equipped with a $\mathbb{Z}_m$ -grading with any $m \geq 2$, then one can naturally construct a compatible Poisson bracket $\{ , \}''$ (Section 2). In this case, all Poisson brackets in $\mathcal{P}$ are linear and there are two lines $l_1, l_2 \subset \mathbb{k}^2$ such that $\Omega = \mathbb{k}^2 \setminus (l_1 \cup l_2) \subset \Omega_{\text{reg}}$ and the Lie algebras corresponding to $(a, b) \in \Omega$ are isomorphic to $\mathfrak{q}$. The lines l_1 and l_2 give rise to new Lie algebras, denoted $\mathfrak{q}_{(0)}$ and $\mathfrak{q}_{(\infty)}$. These new algebras are different contractions of $\mathfrak{q}$. Let $\text{ind } \mathfrak{q}$ denote the *index* of $\mathfrak{q}$ (see Section 1). Then $\text{ind } \mathfrak{q} \leq \text{ind } \mathfrak{q}_{(n)}$, $n \in \{0, \infty\}$, and our first task is to realise whether it is true that $\text{ind } \mathfrak{q} = \text{ind } \mathfrak{q}_{(n)}$. Although basic theory can be developed for arbitrary $\mathfrak{q}$, essential

applications require a better class of Lie algebras, and we eventually stick to the semisimple case. Let $\vartheta \in \text{Aut}(\mathfrak{g})$ be an automorphism of order m and $\mathfrak{g} = \bigoplus_{j=0}^{m-1} \mathfrak{g}_j$ the associated $\mathbb{Z}_m$ -grading, i.e., if $\zeta = \sqrt[m]{1}$ is primitive, then

$$\mathfrak{g}_j = \{x \in \mathfrak{g} \mid \vartheta(x) = \zeta^j x\}.$$

The invariant-theoretic base for our consideration is Vinberg's theory of " ϑ -groups" (i.e., theory of orbits and invariants for representations of reductive groups related to the periodic automorphisms of $\mathfrak{g}$), see [Vi76], [Vi79].

A bad news is that, for $m \geq 3$, the invariant-theoretic picture related to ϑ and properties of $\mathfrak{g}_{(0)}$ become more complicated. For instance, if $m = 2$, then $\mathfrak{g}_{(0)} \simeq \mathfrak{g}_0 \ltimes \mathfrak{g}_1^{\text{ab}}$ (semi-direct product) and it is known that here $\text{ind } \mathfrak{g}_{(0)} = \text{ind } \mathfrak{g} = \text{rk } \mathfrak{g}$ [P07]. For $m \geq 3$, the number $\text{ind } \mathfrak{g}_{(0)}$ remains mysterious. We suspect that it is equal to $\text{rk } \mathfrak{g}$ for any ϑ . Other technical difficulties are discussed in Section 3.3. Nevertheless, we succeeded in computing $\text{ind } \mathfrak{g}_{(\infty)}$, see Theorem 3.2, and can state that $\text{ind } \mathfrak{g}_{(\infty)} = \text{rk } \mathfrak{g}$ if and only if $\mathfrak{g}_0$ is abelian.

A good news is that there are still many interesting cases (automorphisms ϑ), where analogues of results of [PY] are valid and also some unexpected applications pop up. Write $\mathcal{Z}(\mathfrak{g}, \vartheta)$ for the Poisson-commutative subalgebra associated with the pencil of compatible Poisson brackets related to ϑ . Let $\mathcal{N}$ be the set of nilpotent elements of $\mathfrak{g}$. Suppose that ϑ has the following properties:

- (1) $\mathfrak{g}_1$ contains a regular semisimple element of $\mathfrak{g}$;
- (2) each irreducible component of $\mathfrak{g}_1 \cap \mathcal{N}$ contains a regular nilpotent element of $\mathfrak{g}$.

Under these assumptions we prove that $\mathcal{Z}(\mathfrak{g}, \vartheta)$ has the same properties as the above algebra $\mathcal{Z}(\mathfrak{g}, \mathfrak{g}_0)$, see Sections 4, 5.

0.3. Cyclic permutations and Gaudin algebras

More good news is that there is a special case of $(\mathfrak{g}, \vartheta)$, when $\mathcal{Z}(\mathfrak{g}, \vartheta)$ is as good as possible and it has a nice quantisation. Namely, let $\mathfrak{h}$ be a non-abelian simple Lie algebra and $\mathfrak{g} = \mathfrak{h}^m$ be the direct sum of m copies of $\mathfrak{h}$. If ϑ is the cyclic permutation of the summands, then $\mathfrak{g}_0 = \Delta_{\mathfrak{h}} \simeq \mathfrak{h}$ and we prove in Section 6 that $\mathcal{Z} = \mathcal{Z}(\mathfrak{h}^m, \vartheta)$ is a polynomial ring in $\frac{1}{2}((m-1) \dim \mathfrak{h} + (m+1) \text{rk } \mathfrak{h})$ generators. Furthermore, $\mathcal{Z}$ is a maximal Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}_0}$. The quantisation problem asks for a lift of $\mathcal{Z}$ to the enveloping algebra $\mathcal{U}(\mathfrak{g})$, i.e., for a commutative algebra $\tilde{\mathcal{Z}} \subset \mathcal{U}(\mathfrak{g})$ such that $\text{gr}(\tilde{\mathcal{Z}}) = \mathcal{Z}$. To describe $\tilde{\mathcal{Z}}$ in this context, we need some preparations.

The enveloping algebra $\mathcal{U}(\mathfrak{h}[t, t^{-1}])$ of the loop algebra $\mathfrak{h}[t, t^{-1}]$ contains a large commutative subalgebra $\mathfrak{z}(\widehat{\mathfrak{h}})$ of infinite transcendence degree, known as the *Feigin–Frenkel centre* [FF92]. Actually $\mathfrak{z}(\widehat{\mathfrak{h}}) \subset \mathcal{U}(\widehat{\mathfrak{h}}_-)$, where $\widehat{\mathfrak{h}}_- = t^{-1}\mathfrak{h}[t^{-1}]$. Having a vector $\vec{z} \in (\mathbb{k}^*)^m$, one defines a homomorphism $\rho_{\vec{z}}: \mathcal{U}(\widehat{\mathfrak{h}}_-) \rightarrow \mathcal{U}(\mathfrak{g})$. The image of $\mathfrak{z}(\widehat{\mathfrak{h}})$ under $\rho_{\vec{z}}$ is called the *Gaudin subalgebra* $\mathcal{G}(\vec{z})$ [FFR]. For the case where the entries of $\vec{z}$ are pairwise distinct m -th roots of unity, we provide a simpler construction of $\mathcal{G}(\vec{z})$ (Proposition 7.2) and show that $\text{gr}(\mathcal{G}(\vec{z})) = \mathcal{Z}(\mathfrak{h}^m, \vartheta)$, see Theorem 7.4.

It is worth noting that, for the MF subalgebras $\mathcal{A}_\gamma \subset \mathcal{S}(\mathfrak{h})$, the quantisation problem was posed by Vinberg [Vi90]. A solution given by Rybnikov [R06] states that the image of $\mathfrak{z}(\widehat{\mathfrak{h}})$ under a certain homomorphism $\varrho_\gamma: \mathcal{U}(\widehat{\mathfrak{h}}_-) \rightarrow \mathcal{U}(\mathfrak{h})$, depending on $\gamma \in \mathfrak{h}^*$, is the *quantum MF subalgebra* $\widehat{\mathcal{A}}_\gamma$ and one has $\text{gr}(\widehat{\mathcal{A}}_\gamma) = \mathcal{A}_\gamma$ in many cases.

Let us identify $\widehat{\mathfrak{h}}_-$ with the quotient space $\mathfrak{h}[t, t^{-1}]/\mathfrak{h}[t]$. Then $\mathfrak{h}[t]$ acts on $t^{-1}\mathfrak{h}[t^{-1}]$ and correspondingly on $\mathcal{S}(\widehat{\mathfrak{h}}_-)$. By [FF92], [F07], one has

$$\text{gr}(\mathfrak{z}(\widehat{\mathfrak{h}})) = \mathcal{S}(\widehat{\mathfrak{h}}_-)^{\mathfrak{h}[t]}. \quad (0.1)$$

Yet another property of the Poisson-commutative algebra $\text{gr}(\mathfrak{z}(\widehat{\mathfrak{h}}))$ is that it is a polynomial ring in infinitely many variables, by a direct generalisations of a Raïs–Tauvel theorem [RT92].

Suppose that $\vartheta \in \text{Aut}(\mathfrak{h})$. In Section 8, we consider the ϑ -twisted version of $\text{gr}(\mathfrak{z}(\widehat{\mathfrak{h}}))$, a certain subalgebra $\mathcal{Z}(\widehat{\mathfrak{h}}_-, \vartheta)$ of $\mathcal{S}((\widehat{\mathfrak{h}}_-)^\vartheta)$ of infinite transcendence degree. Assuming the equality $\text{ind } \mathfrak{h}_{(0)} = \text{rk } \mathfrak{h}$, we prove that $\mathcal{Z}(\widehat{\mathfrak{h}}_-, \vartheta)$ is Poisson-commutative, see Theorem 8.2. In many cases, $\mathcal{Z}(\widehat{\mathfrak{h}}_-, \vartheta)$ is a polynomial ring.

Our general reference for semisimple Lie groups and algebras is [Lie3].

1. Preliminaries on Poisson brackets and polynomial contractions

Let Q be a connected affine algebraic group with Lie algebra $\mathfrak{q}$. The symmetric algebra of $\mathfrak{q}$ over $\mathbb{k}$ is $\mathbb{N}$ -graded, i.e., $\mathcal{S}(\mathfrak{q}) = \bigoplus_{i \geq 0} \mathcal{S}^i(\mathfrak{q})$. It is identified with the algebra of polynomial functions on the dual space $\mathfrak{q}^*$, and we also write $\mathbb{k}[\mathfrak{q}^*] = \bigoplus_{i \geq 0} \mathbb{k}[\mathfrak{q}^*]_i$ for it.

1.1. The coadjoint representation

The group Q acts on $\mathfrak{q}^*$ via the coadjoint representation and $\text{ad}^*: \mathfrak{q} \rightarrow \text{GL}(\mathfrak{q}^*)$ is the *coadjoint representation* of $\mathfrak{q}$. The algebra of Q -invariant polynomial functions on $\mathfrak{q}^*$ is denoted by $\mathcal{S}(\mathfrak{q})^Q$ or $\mathbb{k}[\mathfrak{q}^*]^Q$. Write $\mathbb{k}(\mathfrak{q}^*)^Q$ for the field of Q -invariant rational functions on $\mathfrak{q}^*$.

Let $\mathfrak{q}^\xi = \{x \in \mathfrak{q} \mid \text{ad}^*(x) \cdot \xi = 0\}$ be the *stabiliser* in $\mathfrak{q}$ of $\xi \in \mathfrak{q}^*$. The *index of $\mathfrak{q}$* , $\text{ind } \mathfrak{q}$, is the minimal codimension of Q -orbits in $\mathfrak{q}^*$. Equivalently,

$$\text{ind } \mathfrak{q} = \min_{\xi \in \mathfrak{q}^*} \dim \mathfrak{q}^\xi.$$

By the Rosenlicht theorem (see [Sp89, IV.2]), one also has $\text{ind } \mathfrak{q} = \text{tr.deg } \mathbb{k}(\mathfrak{q}^*)^Q$. Set $\mathbf{b}(\mathfrak{q}) = (\dim \mathfrak{q} + \text{ind } \mathfrak{q})/2$. Since the Q -orbits in $\mathfrak{q}^*$ are even-dimensional, $\mathbf{b}(\mathfrak{q})$ is an integer. If $\mathfrak{q}$ is reductive, then $\text{ind } \mathfrak{q} = \text{rk } \mathfrak{q}$ and $\mathbf{b}(\mathfrak{q})$ equals the dimension of a Borel subalgebra.

The Lie–Poisson bracket in $\mathcal{S}(\mathfrak{q})$ is defined on $\mathcal{S}^1(\mathfrak{q}) = \mathfrak{q}$ by $\{x, y\} := [x, y]$. It is then extended to higher degrees via the Leibniz rule. Hence $\mathcal{S}(\mathfrak{q})$ has the usual associative-commutative structure and additional Poisson structure. Whenever we refer to *subalgebras* of $\mathcal{S}(\mathfrak{q})$, we always mean the associative-commutative structure. Then a subalgebra $\mathcal{A} \subset \mathcal{S}(\mathfrak{q})$ is said to be *Poisson-commutative*, if $\{H, F\} = 0$ for all $H, F \in \mathcal{A}$. It is well known that if $\mathcal{A}$ is Poisson-commutative,

then $\text{tr.deg } \mathcal{A} \leq \mathbf{b}(\mathfrak{q})$, see, e.g., [Vi90, 0.2]. More generally, suppose that $\mathfrak{h} \subset \mathfrak{q}$ is a Lie subalgebra and $\mathcal{A} \subset \mathcal{S}(\mathfrak{q})^{\mathfrak{h}}$ is Poisson-commutative. Then

$$\text{tr.deg } \mathcal{A} \leq \mathbf{b}(\mathfrak{q}) - \mathbf{b}(\mathfrak{h}) + \text{ind } \mathfrak{h},$$

see [MY19, Prop. 1.1].

The *centre* of the Poisson algebra $(\mathcal{S}(\mathfrak{q}), \{, \})$ is

$$\mathcal{Z}(\mathfrak{q}) := \{H \in \mathcal{S}(\mathfrak{q}) \mid \{H, F\} = 0 \ \forall F \in \mathcal{S}(\mathfrak{q})\}.$$

Using the Leibniz rule, we obtain that $\mathcal{Z}(\mathfrak{q})$ is a graded Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{q})$, which coincides with the algebra of symmetric invariants of $\mathfrak{q}$, i.e.,

$$\mathcal{Z}(\mathfrak{q}) = \{H \in \mathcal{S}(\mathfrak{q}) \mid \{H, x\} = 0 \ \forall x \in \mathfrak{q}\} =: \mathcal{S}(\mathfrak{q})^{\mathfrak{q}} = \mathbb{k}[\mathfrak{q}^*]^{\mathfrak{q}}.$$

As Q is connected, we have $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}} = \mathcal{S}(\mathfrak{q})^Q = \mathbb{k}[\mathfrak{q}^*]^Q$. Since the quotient field of $\mathbb{k}[\mathfrak{q}^*]^Q$ is contained in $\mathbb{k}(\mathfrak{q}^*)^Q$, we deduce from the Rosenlicht theorem that

$$\text{tr.deg } (\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}) \leq \text{ind } \mathfrak{q}. \quad (1.1)$$

The set of Q -regular elements of $\mathfrak{q}^*$ is

$$\mathfrak{q}_{\text{reg}}^* = \{\eta \in \mathfrak{q}^* \mid \dim \mathfrak{q}^{\eta} = \text{ind } \mathfrak{q}\} = \{\eta \in \mathfrak{q}^* \mid \dim Q \cdot \eta \text{ is maximal}\}. \quad (1.2)$$

It is a dense open subset of $\mathfrak{q}^*$. Set $\mathfrak{q}_{\text{sing}}^* = \mathfrak{q}^* \setminus \mathfrak{q}_{\text{reg}}^*$. We say that $\mathfrak{q}$ has the *codim- n* property if $\text{codim } \mathfrak{q}_{\text{sing}}^* \geq n$. The *codim-2* property is going to be most important for us.

For $\gamma \in \mathfrak{q}^*$, let $\hat{\gamma}$ be the skew-symmetric bilinear form on $\mathfrak{q}$ defined by $\hat{\gamma}(\xi, \eta) = \gamma([\xi, \eta])$ for $\xi, \eta \in \mathfrak{q}$. It follows that $\ker \hat{\gamma} = \mathfrak{q}^{\gamma}$. The 2-form $\hat{\gamma}$ is related to the *Poisson tensor (bivector)* π of the Lie–Poisson bracket $\{, \}$ as follows.

Let dH denote the differential of $H \in \mathcal{S}(\mathfrak{q}) = \mathbb{k}[\mathfrak{q}^*]$. Then π is defined by the formula

$$\pi(dH \wedge dF) = \{H, F\} \text{ for } H, F \in \mathcal{S}(\mathfrak{q}).$$

Here $\pi(\gamma)(d_{\gamma}H \wedge d_{\gamma}F) = \{H, F\}(\gamma)$ and therefore $\hat{\gamma} = \pi(\gamma)$. In this term, $\text{ind } \mathfrak{q} = \dim \mathfrak{q} - \text{rk } \pi$, where $\text{rk } \pi = \max_{\gamma \in \mathfrak{q}^*} \text{rk } \pi(\gamma)$.

1.2. Contractions and invariants

We refer to [Lie3, Chap. 7, § 2] for basic facts on contractions of Lie algebras. In this article, we consider contractions of the following form. Let $\mathbb{k}^* = \mathbb{k} \setminus \{0\}$ be the multiplicative group of $\mathbb{k}$ and $\varphi : \mathbb{k}^* \rightarrow \text{GL}(\mathfrak{q})$, $s \mapsto \varphi_s$, a polynomial representation. That is, the matrix entries of $\varphi_s : \mathfrak{q} \rightarrow \mathfrak{q}$ are polynomials in s with respect to some (any) basis of $\mathfrak{q}$. Define a new Lie algebra structure on the vector space $\mathfrak{q}$ and associated Lie–Poisson bracket by

$$[x, y]_{(s)} = \{x, y\}_{(s)} := \varphi_s^{-1}[\varphi_s(x), \varphi_s(y)], \ x, y \in \mathfrak{q}, \ s \in \mathbb{k}^*. \quad (1.3)$$

The corresponding Lie algebra is denoted by $\mathfrak{q}_{(s)}$. Then $\mathfrak{q}_{(1)} = \mathfrak{q}$ and all these algebras are isomorphic. The induced $\mathbb{k}^*$ -action in the variety of structure constants

in not necessarily polynomial, i.e., $\lim_{s \rightarrow 0} [x, y]_{(s)}$ may not exist for all $x, y \in \mathfrak{q}$. Whenever such a limit exists, we obtain a new linear Poisson bracket, denoted $\{ , \}_0$, and thereby a new Lie algebra $\mathfrak{q}_{(0)}$, which is said to be a *contraction* of $\mathfrak{q}$. If we wish to stress that this construction is determined by φ , then we write $\{x, y\}_{(\varphi, s)}$ for the bracket in (1.3) and say that $\mathfrak{q}_{(0)}$ is the φ -contraction of $\mathfrak{q}$ or is the *zero limit of $\mathfrak{q}$ with respect to φ* . A criterion for the existence of $\mathfrak{q}_{(0)}$ can be given in terms of Lie brackets of the φ -eigenspaces in $\mathfrak{q}$, see [Y17, Sect. 4]. We identify all algebras $\mathfrak{q}_{(s)}$ and $\mathfrak{q}_{(0)}$ as vector spaces. The semi-continuity of index implies that $\text{ind } \mathfrak{q}_{(0)} \geq \text{ind } \mathfrak{q}$.

The map φ_s , $s \in \mathbb{k}^*$, is naturally extended to an invertible transformation of $\mathcal{S}^j(\mathfrak{q})$, which we also denote by φ_s . The resulting graded map $\varphi_s : \mathcal{S}(\mathfrak{q}) \rightarrow \mathcal{S}(\mathfrak{q})$ is nothing but the comorphism associated with $s \in \mathbb{k}^*$ and the dual representation $\varphi^* : \mathbb{k}^* \rightarrow \text{GL}(\mathfrak{q}^*)$. Since $\mathcal{S}^j(\mathfrak{q})$ has a basis that consists of $\varphi(\mathbb{k}^*)$ -eigenvectors, any $F \in \mathcal{S}^j(\mathfrak{q})$ can be written as $F = \sum_{i \geq 0} F_i$, where the sum is finite and $\varphi_s(F_i) = s^i F_i \in \mathcal{S}^j(\mathfrak{q})$. Let $F^\bullet$ denote the nonzero component F_i with maximal i .

Proposition 1.1 ([Y14, Lem. 3.3]). *If $F \in \mathcal{Z}(\mathfrak{q})$ and $\mathfrak{q}_{(0)}$ exists, then $F^\bullet \in \mathcal{Z}(\mathfrak{q}_{(0)})$.*

2. Automorphisms of finite order and compatible Poisson brackets

In this section, we associate a pencil of compatible Poisson brackets to any automorphism of finite order of a Lie algebra $\mathfrak{q}$, describe the limit algebras $\mathfrak{q}_{(0)}$ and $\mathfrak{q}_{(\infty)}$, and construct the related Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{q})$.

2.1. Periodic gradings of Lie algebras

Let $\vartheta \in \text{Aut}(\mathfrak{q})$ be a Lie algebra automorphism of finite order $m \geq 2$ and $\zeta = \sqrt[m]{1}$ a primitive root of unity. Write also $\text{ord}(\vartheta)$ for the order of ϑ . If $\mathfrak{q}_i$ is the ζ^i -eigenspace of ϑ , $i \in \mathbb{Z}_m$, then the direct sum $\mathfrak{q} = \bigoplus_{i \in \mathbb{Z}_m} \mathfrak{q}_i$ is a *periodic grading* or $\mathbb{Z}_m$ -grading of $\mathfrak{q}$. The latter means that $[\mathfrak{q}_i, \mathfrak{q}_j] \subset \mathfrak{q}_{i+j}$ for all $i, j \in \mathbb{Z}_m$. Here $\mathfrak{q}_0 = \mathfrak{q}^\vartheta$ is the fixed-point subalgebra for ϑ and each $\mathfrak{q}_i$ is a $\mathfrak{q}_0$ -module. We will be primarily interested in periodic gradings of **semisimple** Lie algebras, but such a general setting is going to be useful, too.

We choose $\{0, 1, \dots, m-1\} \subset \mathbb{Z}$ as a fixed set of representatives for $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$. Under this convention, we have $\mathfrak{q} = \mathfrak{q}_0 \oplus \mathfrak{q}_1 \oplus \dots \oplus \mathfrak{q}_{m-1}$ and

$$[\mathfrak{q}_i, \mathfrak{q}_j] \subset \begin{cases} \mathfrak{q}_{i+j}, & \text{if } i+j \leq m-1, \\ \mathfrak{q}_{i+j-m}, & \text{if } i+j \geq m. \end{cases} \quad (2.1)$$

This is needed below, when we consider $\mathbb{Z}$ -graded contractions of $\mathfrak{q}$ associated with ϑ .

The presence of ϑ allows us to split the Lie–Poisson bracket on $\mathfrak{q}^*$ into a sum of two compatible linear Poisson brackets, as follows. Consider the polynomial representation $\varphi : \mathbb{k}^* \rightarrow \text{GL}(\mathfrak{q})$ such that $\varphi_s(x) = s^j x$ for $x \in \mathfrak{q}_j$. As in Section 1.2, this defines a family of linear Poisson brackets in $\mathcal{S}(\mathfrak{q})$ parametrised by $s \in \mathbb{k}^*$, see (1.3).

Proposition 2.1. *For any $\vartheta \in \text{Aut}(\mathfrak{q})$ of finite order m and φ as above, we have*

- (i) *The map $\varphi_s^{-1} : \mathfrak{q} \rightarrow \mathfrak{q}$ provides an isomorphism between $(\mathfrak{q}, [\ , \])$ and $(\mathfrak{q}, [\ , \]_{(s)})$. In particular, the Poisson brackets $\{ \ , \ \}_{(s)}$, $s \in \mathbb{k}^*$, are isomorphic;*
- (ii) *there is a limit $\lim_{s \rightarrow 0} \{x, y\}_{(s)} =: \{x, y\}_0$, which is a linear Poisson bracket in $\mathcal{S}(\mathfrak{q})$;*
- (iii) *the difference $\{ \ , \ \} - \{ \ , \ \}_0 =: \{ \ , \ \}_\infty$ is a linear Poisson bracket on $\mathcal{S}(\mathfrak{q})$ and*

$$\{ \ , \ \}_{(s)} = \{ \ , \ \}_0 + s^m \{ \ , \ \}_\infty \text{ for any } s \in \mathbb{k}^*.$$

- (iv) *The Poisson bracket $\{ \ , \ \}_\infty$ is obtained as the zero limit with respect to the polynomial representation $\psi : \mathbb{k}^* \rightarrow \text{GL}(\mathfrak{q})$ such that $\psi_s = s^m \cdot \varphi_{s^{-1}} = s^m \cdot \varphi_s^{-1}$, $s \in \mathbb{k}^*$. In other words, we have $\{ \ , \ \}_\infty = \lim_{s \rightarrow 0} \{ \ , \ \}_{(\psi, s)}$.*

Proof. (i) This readily follows from Eq. (1.3).

- (ii) If $x \in \mathfrak{q}_i$ and $y \in \mathfrak{q}_j$, then

$$\{x, y\}_{(s)} = \begin{cases} [x, y], & \text{if } i + j < m; \\ s^m [x, y], & \text{if } i + j \geq m. \end{cases}$$

Therefore, the limit of $\{x, y\}_{(s)}$ as s tends to zero exists and is given by

$$\{x, y\}_0 := \begin{cases} [x, y], & \text{if } i + j < m; \\ 0, & \text{if } i + j \geq m. \end{cases}$$

The limit of Poisson brackets is again a Poisson bracket, hence the Jacobi identity is satisfied for $\{ \ , \ \}_0$, cf. Section 1.2. However, this is easily verified directly.

- (iii) By the above formula for $\{x, y\}_0$, we have

$$\{x, y\}_\infty = \begin{cases} 0, & \text{if } i + j < m; \\ [x, y], & \text{if } i + j \geq m. \end{cases}$$

Therefore $\{x, y\}_{(s)} = \{x, y\}_0 + s^m \{x, y\}_\infty$ for all $x, y \in \mathfrak{q}$. It is also easily verified that $\{ \ , \ \}_\infty$ satisfies the Jacobi identity.

- (iv) We have $\psi_s(x) = s^{m-i}x$ for $x \in \mathfrak{q}_i$. Then an easy calculation shows that the (ψ, s) -bracket is given by

$$\{x, y\}_{(\psi, s)} = \begin{cases} s^m [x, y], & \text{if } i + j < m; \\ [x, y], & \text{if } i + j \geq m. \end{cases} \quad \square$$

Two Poisson brackets on the algebra $\mathcal{S}(\mathfrak{q})$ are said to be *compatible*, if any linear combination of them is again a Poisson bracket. Actually, if $\{ \ , \ \}_1$ and $\{ \ , \ \}_2$ are Poisson brackets, then it suffices to check that just $\{ \ , \ \}_1 + \{ \ , \ \}_2$ is a Poisson bracket [PY, Lem. 1.1]. Anyhow, we have

Corollary 2.2. *For any $\vartheta \in \text{Aut}(\mathfrak{q})$ of finite order, the Poisson brackets $\{ , \}_0$ and $\{ , \}_\infty$ are compatible, and the corresponding pencil contains the initial Lie–Poisson bracket.*

All Poisson brackets involved in Proposition 2.1 are linear. Therefore, in place of Poisson brackets on the symmetric algebra $\mathcal{S}(\mathfrak{q})$, we can stick to the corresponding Lie algebra structures on the vector space $\mathfrak{q}$. Let $\mathfrak{q}_{(s)}$ be the Lie algebra corresponding to $\{ , \}_{(s)}$. Then all algebras with $s \in \mathbb{k}^\star$ are isomorphic, whereas the brackets $\{ , \}_0$ and $\{ , \}_\infty$ give rise to entirely different Lie algebras $\mathfrak{q}_{(0)}$ and $\mathfrak{q}_{(\infty)}$, respectively. Both $\mathfrak{q}_{(0)}$ and $\mathfrak{q}_{(\infty)}$ are Lie algebra contractions of $\mathfrak{q}$ in the sense of [Lie3, Chap. 7, § 2]. Therefore $\text{ind } \mathfrak{q}_{(0)} \geq \text{ind } \mathfrak{q}$ and $\text{ind } \mathfrak{q}_{(\infty)} \geq \text{ind } \mathfrak{q}$ (the semi-continuity of the index).

Proposition 2.3. *The Lie algebras $\mathfrak{q}_{(0)}$ and $\mathfrak{q}_{(\infty)}$ are $\mathbb{N}$ -graded. More precisely, if $\mathfrak{r}[i]$ stands for the component of grade $i \in \mathbb{N}$ in an $\mathbb{N}$ -graded Lie algebra $\mathfrak{r}$, then*

$$\mathfrak{q}_{(0)}[i] = \begin{cases} \mathfrak{q}_i & \text{for } i = 0, 1, \dots, m-1, \\ 0 & \text{otherwise,} \end{cases} \quad \mathfrak{q}_{(\infty)}[i] = \begin{cases} \mathfrak{q}_{m-i} & \text{for } i = 1, 2, \dots, m, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\mathfrak{q}_{(\infty)}$ is nilpotent and the subspace $\mathfrak{q}_0$, which is the highest grade component of $\mathfrak{q}_{(\infty)}$, belongs to the centre of $\mathfrak{q}_{(\infty)}$.

Proof. Use the formulae for $\{x, y\}_0$ and $\{x, y\}_\infty$ from the proof of Proposition 2.1. $\square$

2.2. Poisson-commutative subalgebras related to compatible Poisson brackets

There is a general method for constructing a Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{q})$ with “large” transcendence degree that exploits compatible Poisson brackets and the centre of $\mathcal{S}(\mathfrak{q})$, see, e.g., [DZ05, Sect. 1.8.3], [GZ00, Sect. 10], [PY, Sect. 2]. We recall this method in our present setting.

By Proposition 2.1, we have

$$\{ , \} = \{ , \}_0 + \{ , \}_\infty \quad \text{and} \quad \{ , \}_{(s)} = \{ , \}_0 + s^m \{ , \}_\infty.$$

Since $\{ , \}_{(s)} = \{ , \}_{(s')}$ if $s^m = (s')^m$, it is convenient to replace s^m with t and set

$$\{ , \}_t = \{ , \}_0 + t \{ , \}_\infty,$$

where $t \in \mathbb{P} := \mathbb{k} \cup \{\infty\}$ and the value $t = \infty$ corresponds to the bracket $\{ , \}_\infty$. We will use the parameter $s \in \mathbb{k}^\star$, when the multiplicative group $\varphi : \mathbb{k}^\star \rightarrow \text{GL}(\mathfrak{q})$ is needed.

Let $\mathfrak{q}_{(t)}$ stand for the Lie algebra corresponding to $\{ , \}_t$. All these Lie algebras have the same underlying vector space. Since the algebras $\mathfrak{q}_{(t)}$ with $t \in \mathbb{k}^\star$ are isomorphic, they have one and the same index. We say that $t \in \mathbb{P}$ is *regular* if $\text{ind } \mathfrak{q}_{(t)} = \text{ind } \mathfrak{q}$ and let $\mathbb{P}_{\text{reg}}$ stand for the set of regular values. Then $\mathbb{P}_{\text{sing}} := \mathbb{P} \setminus \mathbb{P}_{\text{reg}} \subset \{0, \infty\}$ is the set of singular values.

Let $\mathcal{Z}_t$ be the centre of the Poisson algebra $(\mathcal{S}(\mathfrak{q}), \{ , \}_t)$. In particular, $\mathcal{Z}_1 = \mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$. If $t \in \mathbb{k}^\star$, then $\mathcal{Z}_t = \varphi_s^{-1}(\mathcal{Z}_1)$, where $s^m = t$. By Eq. (1.1), we have

$\text{tr.deg } \mathcal{Z}_t \leq \text{ind } \mathfrak{q}_{(t)}$. Our main object is the subalgebra $\mathcal{Z} \subset \mathcal{S}(\mathfrak{q})$ generated by the centres $\mathcal{Z}_t$ with $t \in \mathbb{P}_{\text{reg}}$, i.e.,

$$\mathcal{Z} = \mathcal{Z}(\mathfrak{q}, \vartheta) = \text{alg}\langle \mathcal{Z}_t \mid t \in \mathbb{P}_{\text{reg}} \rangle.$$

By a general property of compatible brackets, the algebra $\mathcal{Z}$ is Poisson-commutative with respect to **all** brackets $\{, \}_t$ with $t \in \mathbb{P}$, cf. [PY, Sect. 2]. Note that the Lie subalgebra $\mathfrak{q}_0 \subset \mathfrak{q} = \mathfrak{q}_{(1)}$ is also the **same** Lie subalgebra in any $\mathfrak{q}_{(t)}$ with $t \neq \infty$ (cf. Proposition 2.3 for $\mathfrak{q}_{(0)}$). Therefore,

$$\mathcal{Z}_t \subset \mathcal{S}(\mathfrak{q})^{\mathfrak{q}_0} \quad \text{for } t \neq \infty. \quad (2.2)$$

In general, one cannot say much about $\mathcal{Z} \subset \mathcal{S}(\mathfrak{q})$. To arrive at more definite conclusions on $\mathcal{Z}$, a lot of extra information on $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}_0}$, $\mathfrak{q}_{(0)}$, and $\mathfrak{q}_{(\infty)}$ is required. In particular, one has to know whether 0 and/or ∞ belong to $\mathbb{P}_{\text{reg}}$. And this is the reason why we have to stick to semisimple Lie algebras.

3. Poisson-commutative subalgebras of $\mathcal{S}(\mathfrak{g})$: the semisimple case

From now on, G is a connected semisimple algebraic group and $\mathfrak{g} = \text{Lie } G$. We consider $\vartheta \in \text{Aut}(\mathfrak{g})$ of order $m \geq 2$ and freely use the previous notation and results, with $\mathfrak{q}$ being replaced by $\mathfrak{g}$. In particular,

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_{m-1},$$

where $\{0, 1, \dots, m-1\}$ is the fixed set of representatives for $\mathbb{Z}_m$, and G_0 is the connected subgroup of G with $\text{Lie}(G_0) = \mathfrak{g}_0$. Then $\mathfrak{g}_{(t)}$ is a family of Lie algebras parameterised by $t \in \mathbb{P} = \mathbb{k} \cup \{\infty\}$, where the algebras $\mathfrak{g}_{(t)}$ with $t \in \mathbb{k}^*$ are isomorphic to $\mathfrak{g} = \mathfrak{g}_{(1)}$, while $\mathfrak{g}_{(0)}$ and $\mathfrak{g}_{(\infty)}$ are different $\mathbb{N}$ -graded contractions of $\mathfrak{g}$. Next, $\mathcal{Z}_t$ is the Poisson centre of $(\mathcal{S}(\mathfrak{g}), \{, \}_t)$ and the construction of Section 2.2 provides a Poisson-commutative subalgebra $\mathcal{Z} = \text{alg}\langle \mathcal{Z}_t \mid t \in \mathbb{P}_{\text{reg}} \rangle \subset \mathcal{S}(\mathfrak{g})$. The connected algebraic group corresponding to $\mathfrak{g}_{(t)}$ is denoted by $G_{(t)}$.

Our goal is to demonstrate that there are many interesting cases, in which $\mathcal{Z}$ is a polynomial algebra having the maximal possible transcendence degree. Let us recall standard invariant-theoretic properties of semisimple Lie algebras.

The Poisson centre $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}} = \mathcal{S}(\mathfrak{g})^G$ is a polynomial algebra of Krull dimension $l = \text{rk } \mathfrak{g}$ and $\text{ind } \mathfrak{g} = l$. Hence one has now the equality in Eq. (1.1). Note also that $\mathfrak{g}_0$ is a reductive Lie algebra. Write $\mathcal{N}$ for the cone of nilpotent elements of $\mathfrak{g}$. Let $\varkappa$ be the Killing form on $\mathfrak{g}$. We identify $\mathfrak{g}$ and $\mathfrak{g}_0$ with their duals via $\varkappa$. Moreover, since $\varkappa(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ if $i + j \notin \{0, m\}$, the dual space of $\mathfrak{g}_j$, $\mathfrak{g}_j^*$, can be identified with $\mathfrak{g}_{m-j}$. Here $\mathfrak{g}_{\text{reg}}^* = \mathfrak{g}_{\text{reg}} = \{x \in \mathfrak{g} \mid \dim \mathfrak{g}^x = l\}$. By [K63], $\mathfrak{g}$ has the *codim*–3 property, i.e., $\text{codim}_{\mathfrak{g}}(\mathfrak{g} \setminus \mathfrak{g}_{\text{reg}}) = 3$. Recall also that $\mathcal{N} \cap \mathfrak{g}_{\text{reg}}$ is non-empty, and it is a sole G -orbit, the *regular nilpotent orbit*.

Convention. We think of $\mathfrak{g}^*$ as the dual space for any Lie algebra $\mathfrak{g}_{(t)}$ and sometimes omit the subscript ‘ (t) ’ in $\mathfrak{g}_{(t)}^*$. However, if $\xi \in \mathfrak{g}^*$, then the stabiliser of ξ with respect to the coadjoint representation of $\mathfrak{g}_{(t)}$ is denoted by $\mathfrak{g}_{(t)}^{\xi}$.

Each Lie algebra $\mathfrak{g}_{(t)}$ has its own singular set $\mathfrak{g}_{(t),\text{sing}}^* = \mathfrak{g}^* \setminus \mathfrak{g}_{(t),\text{reg}}^*$, which is regarded as a subset of $\mathfrak{g}^*$. If π_t is the Poisson tensor of the bracket $\{, \}_t$, then

$$\mathfrak{g}_{(t),\text{sing}}^* = \{\xi \in \mathfrak{g}^* \mid \text{rk } \pi_t(\xi) < \text{rk } \pi_t\},$$

which is the union of the coadjoint $G_{(t)}$ -orbits in $\mathfrak{g}^*$ having a non-maximal dimension. For simplicity, we write $\mathfrak{g}_{\infty,\text{sing}}^*$ or $\mathfrak{g}_{\infty,\text{reg}}^*$ in place of $\mathfrak{g}_{(\infty),\text{sing}}^*$ or $\mathfrak{g}_{(\infty),\text{reg}}^*$.

Proposition 3.1. *The closure of $\bigcup_{t \neq 0, \infty} \mathfrak{g}_{(t),\text{sing}}^*$ in $\mathfrak{g}^*$ is a subset of codimension at least 2.*

Proof. Let $\xi = \xi_0 + \xi_1 + \cdots + \xi_{m-1} \in \mathfrak{g}^*$, where $\xi_i \in \mathfrak{g}_i^*$. Using Proposition 2.1(i) and the dual representation $\varphi^* : \mathbb{k}^* \rightarrow \text{GL}(\mathfrak{g}^*)$, $s \mapsto \varphi_s^*$, one readily verifies that $\xi \in \mathfrak{g}_{\text{sing}}^* = \mathfrak{g}_{(1),\text{sing}}^*$ if and only if $\varphi_s^*(\xi) \in \mathfrak{g}_{(s),\text{sing}}^*$. Therefore,

$$\bigcup_{t \neq 0, \infty} \mathfrak{g}_{(t),\text{sing}}^* = \bigcup_{s \in \mathbb{k}^*} \varphi_s^*(\mathfrak{g}_{\text{sing}}^*) = \{\xi_0 + s^{-1}\xi_1 + \cdots + s^{1-m}\xi_{m-1} \mid \xi \in \mathfrak{g}_{\text{sing}}^*, s \in \mathbb{k}^*\}.$$

Since $\text{codim } \mathfrak{g}_{\text{sing}}^* = 3$, the closure of $\bigcup_{t \neq 0, \infty} \mathfrak{g}_{(t),\text{sing}}^*$ is a subset of $\mathfrak{g}^*$ of codimension at least 2. $\square$

In order to compute $\text{tr.deg } \mathcal{Z}$, we have to elaborate on some relevant properties of the limit Lie algebras $\mathfrak{g}_{(\infty)}$ and $\mathfrak{g}_{(0)}$.

3.1. Properties of $\mathfrak{g}_{(\infty)}$

By Proposition 2.3, $\mathfrak{g}_{(\infty)}$ is a nilpotent $\mathbb{N}$ -graded Lie algebra, hence $G_{(\infty)}$ is a unipotent algebraic group. Recall also that the subspace $\mathfrak{g}_0$ belongs to the centre of $\mathfrak{g}_{(\infty)}$.

Theorem 3.2. *For any $\vartheta \in \text{Aut}(\mathfrak{g})$ of finite order, one has the equality $\text{ind } \mathfrak{g}_{(\infty)} = \dim \mathfrak{g}_0 + \text{rk } \mathfrak{g} - \text{rk } \mathfrak{g}_0$.*

Proof. (1) For $\mu \in \mathfrak{g}_0^* \subset \mathfrak{g}_{(\infty)}^*$, the stabiliser $\mathfrak{g}_{(\infty)}^\mu$ is also $\mathbb{N}$ -graded. Furthermore, if $\eta \in \mathfrak{g}_j$ with $1 \leq j \leq m-1$, then $\varkappa(\mu, [\eta, \mathfrak{g}_{m-j}]_\infty) = 0$ if and only if $[\eta, \mu] = 0$. Hence

$$\mathfrak{g}_{(\infty)}^\mu = \mathfrak{g}^\mu + \mathfrak{g}_0, \quad (3.1)$$

where $\mathfrak{g}^\mu$ is the usual stabiliser of $\mu \in \mathfrak{g}^*$ (with respect to the initial Lie algebra structure on $\mathfrak{g}$). As is well known, the reductive subalgebra $\mathfrak{g}_0 = \mathfrak{g}^\vartheta$ contains regular semisimple elements of $\mathfrak{g}$, see, e.g., [Ka83, §8.8]. These elements form a dense open subset of $\mathfrak{g}_0$, which is denoted by Ω_0 . If $h \in \Omega_0$, then $\mathfrak{g}^h$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{g}^h \cap \mathfrak{g}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$. Using the identification of $\mathfrak{g}_0$ and $\mathfrak{g}_0^*$, we may think of the subset Ω_0^* of “regular semisimple” elements of $\mathfrak{g}_0^* \subset \mathfrak{g}_{(\infty)}^*$. For $\mu \in \Omega_0^*$, it follows from (3.1) that $\dim \mathfrak{g}_{(\infty)}^\mu = \dim \mathfrak{g}_0 + \text{rk } \mathfrak{g} - \text{rk } \mathfrak{g}_0$ and hence $\text{ind } \mathfrak{g}_{(\infty)} \leq \dim \mathfrak{g}_0 + \text{rk } \mathfrak{g} - \text{rk } \mathfrak{g}_0$.

(2) Let us prove the opposite inequality. We think of $\mathfrak{g}_{(\infty)}^*$ as a graded vector space of the form

$$\mathfrak{g}_{(\infty)}^* = \mathfrak{g}_0^* \oplus \mathfrak{g}_1^* \oplus \cdots \oplus \mathfrak{g}_{m-1}^*.$$

Write $\text{ad}_{(\infty)}^*$ for the coadjoint representation of $\mathfrak{g}_{(\infty)}$. The graded structure of $\mathfrak{g}_{(\infty)}$ described in Proposition 2.3 implies that $\text{ad}_{(\infty)}^*$ has the property that

$$\text{ad}_{(\infty)}^*(\mathfrak{g}_j) \cdot \mathfrak{g}_i^* \subset \mathfrak{g}_{i+m-j}^*. \quad (3.2)$$

Take any $\xi = \sum_{j=0}^{m-1} \xi_j \in \mathfrak{g}_{(\infty)}^*$ such that $\xi_0 \in \Omega_0^*$. Let $h \in \Omega_0$ be the regular semisimple element of $\mathfrak{g}$ corresponding to ξ_0 under our identifications. Set $\mathfrak{t} = \mathfrak{g}^h$. Then $[\mathfrak{g}, h] = \mathfrak{t}^\perp$ is the orthogonal complement of $\mathfrak{t}$ with respect to $\varkappa$ and $\mathfrak{g} = [\mathfrak{g}, h] \oplus \mathfrak{t}$. For $\mathfrak{t}_j = \mathfrak{t} \cap \mathfrak{g}_j$, this implies that $[\mathfrak{g}_j, h] = \mathfrak{g}_j \cap \mathfrak{t}_{m-j}^\perp$ and $[\mathfrak{g}_j, h] \oplus \mathfrak{t}_j = \mathfrak{g}_j$. Our goal is to prove that the orbit $G_{(\infty)} \cdot \xi$ contains an element γ such that

$$\dim \mathfrak{g}_{(\infty)}^\gamma \geq \dim \mathfrak{g}_0 + \text{rk } \mathfrak{g} - \text{rk } \mathfrak{g}_0.$$

We perform this step by step, as follows.

For $\eta \in \mathfrak{g}_{m-1}$, we have $\text{ad}_{(\infty)}^*(\eta) \cdot \xi_0 \in \mathfrak{g}_1^*$ and

$$\exp(\text{ad}_{(\infty)}^*(\eta)) \cdot \xi \in \xi_0 + (\xi_1 + \text{ad}_{(\infty)}^*(\eta) \cdot \xi_0) + \left(\bigoplus_{j \geq 2} \mathfrak{g}_j^* \right).$$

Since $\text{ad}_{(\infty)}^*(\mathfrak{g}_{m-1}) \cdot \xi_0 = \mathfrak{g}_1^* \cap \text{Ann}(\mathfrak{t}_1)$, there is $\eta_1 \in \mathfrak{g}_{m-1}$ such that

$$\gamma_1 := \xi_1 + \text{ad}_{(\infty)}^*(\eta_1) \cdot \xi_0 \in \mathfrak{t}_1^*.$$

Applying $\exp(\text{ad}_{(\infty)}^*(\eta))$ with $\eta \in \mathfrak{g}_{m-2}$ to $\exp(\text{ad}_{(\infty)}^*(\eta_1))\xi = \xi_0 + \gamma_1 + \xi'_2 + \dots$, we do not affect the summands ξ_0 and γ_1 . In doing so, we can replace ξ'_2 with $\gamma_2 \in \mathfrak{t}_2^*$. Continuing in this way, we get in $G_{(\infty)} \cdot \xi$ an element of the form $\gamma = \xi_0 + \sum_{j=1}^{m-1} \gamma_j$ with $\gamma_j \in \mathfrak{t}_j^*$. It is easily seen that $\mathfrak{t} \subset \mathfrak{g}_{(\infty)}^\gamma$, hence

$$\dim \mathfrak{g}_{(\infty)}^\gamma \geq \dim \mathfrak{g}_0 + \text{rk } \mathfrak{g} - \text{rk } \mathfrak{g}_0,$$

as required. $\square$

Corollary 3.3. *One has $\infty \in \mathbb{P}_{\text{reg}}$ if and only if $\dim \mathfrak{g}_0 = \text{rk } \mathfrak{g}_0$, i.e., $\mathfrak{g}_0$ is an abelian subalgebra of $\mathfrak{g}$.*

Remark 3.4. (1) There is a short proof of the corollary that does not use Theorem 3.2 in full strength. If $\infty \in \mathbb{P}_{\text{reg}}$, then $\mathcal{Z}_\infty \subset \mathcal{Z}$. Since $\mathfrak{g}_0$ belongs to the centre of $\mathfrak{g}_{(\infty)}$, we have $\mathfrak{g}_0 \subset \mathcal{Z}_\infty$. Hence $\mathfrak{g}_0$ has to be abelian in $\mathfrak{g}$. Conversely, if $\mathfrak{g}_0$ is abelian, then, for $\mu \in \Omega_0^*$ in the first part of the proof, we obtain $\dim \mathfrak{g}_{(\infty)}^\mu = \text{rk } \mathfrak{g}$. Hence $\text{ind } \mathfrak{g}_{(\infty)} = \text{rk } \mathfrak{g}$ and $\infty \in \mathbb{P}_{\text{reg}}$.

(2) By V. G. Kac's classification of elements of finite order in G [Ka83, Chap. 8], if $\mathfrak{g}$ is simple, ϑ is inner, and $\mathfrak{g}_0 = \mathfrak{g}^\vartheta$ is abelian, then $\text{ord}(\vartheta)$ is at least the Coxeter number of $\mathfrak{g}$. This means that, for many interesting examples with small $\text{ord}(\vartheta)$, we have $\infty \in \mathbb{P}_{\text{sing}}$.

Let us check another technical condition, which is required below. Recall from Section 1.1 that $\pi(\gamma) = \hat{\gamma}$ is a skew-symmetric bilinear form on $\mathfrak{g}$ associated with $\gamma \in \mathfrak{g}^*$. We then need the rank of the restriction $\pi(\xi)|_V$, if $\xi \in \mathfrak{g}^*$ is generic and $V = \ker \pi_\infty(\xi) = \mathfrak{g}_{(\infty)}^\xi$. Set $\mathfrak{g}_{>0} = \bigoplus_{j>0} \mathfrak{g}_j$ and $\mathfrak{g}_{>0}^* = \bigoplus_{j>0} \mathfrak{g}_j^*$.

Lemma 3.5. *For any ϑ and generic $\xi \in \mathfrak{g}^*$, we have $\mathrm{rk}(\pi(\xi)|_V) = \dim V - \mathrm{rk} \mathfrak{g}$.*

Proof. Take any $\xi \in \mathfrak{g}_{\mathrm{reg}}^* \cap \mathfrak{g}_{\infty, \mathrm{reg}}^*$. Then $\mathrm{rk}(\pi(\xi)|_V) \leq \dim V - \mathrm{rk} \mathfrak{g}$ by [PY, Appendix] and $\dim V = \dim \mathfrak{g}_0 + \mathrm{rk} \mathfrak{g} - \mathrm{rk} \mathfrak{g}_0$ by Theorem 3.2. Write $\xi = \xi_0 + \xi'$ with $\xi_0 \in \mathfrak{g}_0^*$ and $\xi' \in \mathfrak{g}_{>0}^*$. Assume also that $\xi_0 \in (\mathfrak{g}_0)_{\mathrm{reg}}^*$. Note that this is an open condition on ξ , too.

As $\mathfrak{g}_0 \subset V$ (cf. Eq. 3.1), one can write $V = \mathfrak{g}_0 \oplus V'$ for some $V' \subset \mathfrak{g}_{>0}$. Here we have $\dim V' = \mathrm{rk} \mathfrak{g} - \mathrm{rk} \mathfrak{g}_0$. Let $\tilde{V}$ be the kernel of $\pi(\xi)|_V$. Then $\tilde{V} \cap \mathfrak{g}_0 \subset \mathfrak{g}_0^{\xi_0}$ and

$$\mathrm{rk} \mathfrak{g}_0 \geq \dim(\tilde{V} \cap \mathfrak{g}_0) \geq \dim \tilde{V} - \dim V' = \dim \tilde{V} + \mathrm{rk} \mathfrak{g}_0 - \mathrm{rk} \mathfrak{g}.$$

Hence we have $\dim \tilde{V} \leq \mathrm{rk} \mathfrak{g}$ and thereby $\mathrm{rk}(\pi(\xi)|_V) \geq \dim V - \mathrm{rk} \mathfrak{g}$. This settles the claim. $\square$

For future use, we record yet another property of $\mathfrak{g}_{(\infty)}$.

Lemma 3.6. *If $\xi = \xi_0 + \xi'$, where $\xi' \in \mathfrak{g}_{>0}^*$, and $\xi_0 \in \mathfrak{g}_0^*$ is regular in $\mathfrak{g}$, then $\xi \in \mathfrak{g}_{\infty, \mathrm{reg}}^*$.*

Proof. As in Proposition 2.1(iv), consider the invertible linear map $\psi_s : \mathfrak{g} \rightarrow \mathfrak{g}$ such that $\psi(x_j) = s^{m-j}x_j$ for $x_j \in \mathfrak{g}_j$ and $s \in \mathbb{k}^*$. It follows from Proposition 2.3 that ψ_s is an automorphism of the graded Lie algebra $\mathfrak{g}_{(\infty)}$. Let ψ_s^* denote the induced action on $\mathfrak{g}_{(\infty)}^*$, i.e., $\psi_s^*(\xi_j) = s^{j-m}\xi_j$ if $\xi_j \in \mathfrak{g}_j^*$. Then $\dim \mathfrak{g}_{(\infty)}^\xi = \dim \mathfrak{g}_{(\infty)}^{\psi_s^*(\xi)}$ for any $\xi \in \mathfrak{g}^*$. Note that $\lim_{s \rightarrow 0} s^m \psi_s(\xi) = \xi_0$. If ξ_0 is regular in $\mathfrak{g}_{(\infty)}^*$, then so is ξ . It remains to deal with ξ_0 .

In view of (3.1), $\dim \mathfrak{g}_{(\infty)}^{\xi_0} = \dim \mathfrak{g}_0 + \dim \mathfrak{g}^{\xi_0} - \dim \mathfrak{g}_0^{\xi_0}$. Because ξ_0 is regular in $\mathfrak{g}$, it is also regular in $\mathfrak{g}_0$. Thus $\dim \mathfrak{g}_{(\infty)}^{\xi_0} = \mathrm{ind} \mathfrak{g}_{(\infty)}$, and we are done. $\square$

3.2. Pencils of skew-symmetric matrices and differentials

Recall that any $F \in \mathcal{S}(\mathfrak{g})$ is regarded as a polynomial function on $\mathfrak{g}^*$. Then dF is the differential of F , which is a polynomial mapping from $\mathfrak{g}^*$ to $\mathfrak{g}$. If $\gamma \in \mathfrak{g}^*$, then $d_\gamma F \in \mathfrak{g}$ stands for the value of dF at γ .

For a subalgebra $\mathcal{A} \subset \mathcal{S}(\mathfrak{g})$ and $\gamma \in \mathfrak{g}^*$, set $d_\gamma \mathcal{A} = \langle d_\gamma F \mid F \in \mathcal{A} \rangle_{\mathbb{k}}$. Then $\mathrm{tr.deg} \mathcal{A} = \max_{\gamma \in \mathfrak{q}^*} \dim d_\gamma \mathcal{A}$. By the definition of $\mathcal{Z}$, we have $d_\gamma \mathcal{Z} = \sum_{t \in \mathbb{P}_{\mathrm{reg}}} d_\gamma \mathcal{Z}_t$. Since $\mathcal{Z}_t$ is the centre of $(\mathcal{S}(\mathfrak{g}), \{ , \}_t)$, there is the inclusion $d_\gamma \mathcal{Z}_t \subset \ker \pi_t(\gamma)$ for each $t \in \mathbb{P}$ and each $\gamma \in \mathfrak{g}^*$.

Let $\{H_1, \dots, H_l\}$, $l = \mathrm{rk} \mathfrak{g}$, be a set of homogeneous algebraically independent generators of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$. For any $H \in \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$, we have $d_\xi H \in \mathfrak{g}^\xi$. The *Kostant regularity criterion* for $\mathfrak{g}$, see [K63, Thm. 9], asserts that

$$\langle d_\xi H_j \mid 1 \leq j \leq l \rangle_{\mathbb{k}} = \mathfrak{g}^\xi \quad \text{if and only if} \quad \xi \in \mathfrak{g}_{\mathrm{reg}}^*. \quad (3.3)$$

Clearly, this criterion applies to semisimple algebras $\mathfrak{g}_{(t)}$ when $t \neq 0, \infty$. That is,

$$\text{if } t \neq 0, \infty, \text{ then } \xi \in \mathfrak{g}_{(t), \mathrm{reg}}^* \Leftrightarrow d_\xi \mathcal{Z}_t = \ker \pi_t(\xi) \Leftrightarrow \dim \ker \pi_t(\xi) = \mathrm{rk} \mathfrak{g}. \quad (3.4)$$

As a step towards describing $\mathcal{Z}$, we first consider the smaller algebra

$$\mathcal{Z}_\times = \mathrm{alg} \langle \mathcal{Z}_t \mid t \in \mathbb{k}^* \rangle \subset \mathcal{Z}.$$

An open subset of an algebraic variety is said to be *big*, if its complement does not contain divisors. By Proposition 3.1, there is a big open subset $U_{\text{sr}} \subset \mathfrak{g}^*$ such that $\xi \in \mathfrak{g}_{(t),\text{reg}}^*$ for any $\xi \in U_{\text{sr}}$ and any $t \neq 0, \infty$. Suppose that $\xi \in U_{\text{sr}}$. Then

$$d_{\xi} \mathcal{Z}_{\times} = \sum_{t \in \mathbb{k}^*} \ker \pi_t(\xi) \subset \sum_{t: \text{rk } \pi_t(\xi) = \dim \mathfrak{g} - l} \ker \pi_t(\xi) =: L(\xi). \quad (3.5)$$

Let us recall a method that provides an upper bound on $\dim L(\xi)$ and thereby on $\text{tr.deg } \mathcal{Z}_{\times}$, see [PY20, Sect. 1.1].

Let $\mathcal{P}(\xi) = \{c\pi_t(\xi) \mid c \in \mathbb{k}, t \in \mathbb{P}\}$ be a pencil of skew-symmetric 2-forms on $\mathfrak{g}$, which is spanned by $\pi(\xi)$ and $\pi_0(\xi)$. A 2-form in this pencil is said to be *regular* if $\text{rk}(c\pi_t(\xi)) = \dim \mathfrak{g} - l$. Otherwise, it is *singular*. Set $U_{\text{srr}} = U_{\text{sr}} \cap \mathfrak{g}_{(0),\text{reg}}^* \cap \mathfrak{g}_{\infty,\text{reg}}^*$. It is a dense open subset of $\mathfrak{g}^*$, which may not be big.

Proposition 3.7. *If $\xi \in U_{\text{srr}}$, then $d_{\xi} \mathcal{Z}_{\times} = d_{\xi} \mathcal{Z} = L(\xi)$.*

Proof. Suppose that $\xi \in U_{\text{sr}}$. The space $L(\xi)$ is the sum of kernels over the regular lines in the pencil. Since $\mathbb{P}_{\text{reg}} \setminus \mathbb{k}^*$ is finite, we have $\sum_{t \in \mathbb{k}^*} \ker \pi_t(\xi) = L(\xi)$ by [PY08, Appendix]. Hence $d_{\xi} \mathcal{Z}_{\times} = L(\xi)$. If $\xi \in U_{\text{srr}}$, then $d_{\xi} \mathcal{Z} \subset L(\xi)$. Thereby $d_{\xi} \mathcal{Z}_{\times} \subset d_{\xi} \mathcal{Z} \subset L(\xi)$, which leads to the equality $d_{\xi} \mathcal{Z}_{\times} = d_{\xi} \mathcal{Z}$. $\square$

Corollary 3.8. *For a generic $\xi \in U_{\text{srr}}$, we have $\dim L(\xi) = \text{tr.deg } \mathcal{Z}_{\times} = \text{tr.deg } \mathcal{Z}$. In particular, $\mathcal{Z}_{\times} \subset \mathcal{Z}$ is an algebraic extension.*

Although three dimensions in Corollary 3.8 are equal, we do not know yet their exact value. For a given $\xi \in U_{\text{sr}}$, $\dim L(\xi)$ depends on the number of singular lines in $\mathcal{P}(\xi)$.

Proposition 3.9 ([PY20, Prop. 1.5]). *If $\xi \in U_{\text{sr}}$, then*

- (i) $\dim L(\xi) = \mathbf{b}(\mathfrak{g})$ if and only if all nonzero 2-forms in $\mathcal{P}(\xi)$ are regular;
- (ii) if $\mathcal{P}(\xi)$ contains a singular line, say $\mathbb{k}\pi_{t'}(\xi)$, then $\dim L(\xi) \leq l + \frac{1}{2} \text{rk } \pi_{t'}(\xi)$;
- (iii) furthermore, $\dim L(\xi) = l + \frac{1}{2} \text{rk } \pi_{t'}(\xi)$ if and only if
 - $\mathbb{k}\pi_{t'}(\xi)$ is the only singular line in $\mathcal{P}(\xi)$, and
 - $\text{rk}(\pi(\xi)|_V) = \dim V - l$ for $V = \ker \pi_{t'}(\xi)$.

By Corollary 3.8, $\mathcal{Z}$ is algebraic over $\mathcal{Z}_{\times}$, and it follows from (2.2) that $\mathcal{Z}_{\times} \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{g}_0}$. Since, being an algebra of invariants, $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}_0}$ is an algebraically closed subalgebra of $\mathcal{S}(\mathfrak{g})$, we conclude that

$$\mathcal{Z} \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{g}_0}. \quad (3.6)$$

By [MY19, Prop. 1.1], if $\mathcal{A}$ is a Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}_0}$, then

$$\text{tr.deg } \mathcal{A} \leq \mathbf{b}(\mathfrak{g}) - \mathbf{b}(\mathfrak{g}_0) + \text{rk } \mathfrak{g}_0 =: \mathbf{b}(\mathfrak{g}, \vartheta). \quad (3.7)$$

Note that $\mathbf{b}(\mathfrak{g}, \vartheta) \leq \mathbf{b}(\mathfrak{g})$ and the equality occurs if and only if $\mathfrak{g}_0$ is abelian. The most interesting case is that in which the upper bound in (3.7) is attained for $\mathcal{A} = \mathcal{Z}$.

Theorem 3.10. *If ϑ has the property that $\text{ind } \mathfrak{g}_{(0)} = \text{ind } \mathfrak{g}$, then $\text{tr.deg } \mathcal{Z} = \mathbf{b}(\mathfrak{g}, \vartheta)$.*

Proof. Suppose that $\xi \in U_{\text{srr}}$. Then $\text{rk } \pi_t(\xi) = \dim \mathfrak{g} - l$ if and only if $t \in \mathbb{P}_{\text{reg}}$. Since $\text{tr.deg } \mathcal{Z} \leq \mathbf{b}(\mathfrak{g}, \vartheta)$, it suffices to show that $\dim L(\xi) = \mathbf{b}(\mathfrak{g}, \vartheta)$ whenever $\xi \in U_{\text{srr}}$ is generic.

Suppose first that $\infty \in \mathbb{P}_{\text{reg}}$. Then $\text{rk } \pi_t(\xi) = \dim \mathfrak{g} - l$ for any $t \in \mathbb{P}$ and, by Proposition 3.9(i), we obtain $\dim L(\xi) = \mathbf{b}(\mathfrak{g})$. And the equality $\text{ind } \mathfrak{g} = \text{ind } \mathfrak{g}_{(\infty)}$ means that $\mathfrak{g}_0$ is abelian, hence $\mathbf{b}(\mathfrak{g}) = \mathbf{b}(\mathfrak{g}, \vartheta)$.

Suppose that $\infty \notin \mathbb{P}_{\text{reg}}$. Then $\pi_\infty(\xi)$ spans the unique singular line in the pencil $c\pi_t(\xi)$. By Lemma 3.5, there is a dense open subset $U \subset \mathfrak{g}^*$ such that, for any $\eta \in U$ and $V = \ker \pi_\infty(\eta)$, we have $\text{rk } (\pi(\eta)|_V) = \dim V - l$. If $\xi \in U_{\text{srr}} \cap U$, then conditions (ii) and (iii) of Proposition 3.9 are satisfied for $\pi_t(\xi)$. Thus, in this case,

$$\dim L(\xi) = l + \frac{1}{2} \text{rk } \pi_\infty = l + \frac{1}{2} (\dim \mathfrak{g} - \dim \mathfrak{g}_0 - l + \text{rk } \mathfrak{g}_0) = \mathbf{b}(\mathfrak{g}) - \mathbf{b}(\mathfrak{g}_0) + \text{rk } \mathfrak{g}_0 = \mathbf{b}(\mathfrak{g}, \vartheta),$$

since $\text{rk } \pi_\infty(\xi) = \text{rk } \pi_\infty = \dim \mathfrak{g} - \text{ind } \mathfrak{g}_{(\infty)}$ and $\text{ind } \mathfrak{g}_{(\infty)}$ is computed in Theorem 3.2. $\square$

Remark. One can readily prove that the inverse implication in Theorem 3.10 is also true. That is, if $\text{tr.deg } \mathcal{Z} = \mathbf{b}(\mathfrak{g}, \vartheta)$, then $\text{ind } \mathfrak{g}_{(0)} = \text{ind } \mathfrak{g}$.

3.3. Properties of $\mathfrak{g}_{(0)}$

Study of Lie algebras $\mathfrak{g}_{(0)}$ associated with arbitrary periodic automorphisms of $\mathfrak{g}$ have been initiated in [P09], where they are called *cyclic contractions* or $\mathbb{Z}_k$ -*contractions*. In [P09], these algebras are denoted by $\mathfrak{g}\langle k \rangle_0$, where $k = \text{ord}(\vartheta)$, because of their interpretation as the fixed-point subalgebras of an extension of ϑ to an automorphism of k -th Takiff algebra modelled on $\mathfrak{g}$. (We discuss Takiff algebras in Section 6.)

The case in which $\text{ord}(\vartheta) \geq 3$ appears to be more difficult than that of involutions. If $\text{ord}(\vartheta) = 2$, then $\text{ind } \mathfrak{g}_{(0)} = \text{ind } \mathfrak{g}$ and $\mathfrak{g}_{(0)}$ has the *codim-2* property [P07]. Whereas, for $\text{ord}(\vartheta) \geq 3$, it can happen that $\mathfrak{g}_{(0)}$ does not have the *codim-2* property, and the equality $\text{ind } \mathfrak{g}_{(0)} = \text{ind } \mathfrak{g}$ is only known under certain constraints.

In order to state some sufficient conditions, we first recall several results of E. B. Vinberg [Vi76]. Associated with $\mathbb{Z}_m$ -grading $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}_m} \mathfrak{g}_i$, one has the linear action of G_0 on $\mathfrak{g}_1$. Then

- the algebra of invariants $\mathbb{k}[\mathfrak{g}_1]^{G_0}$ is a polynomial (free) algebra;
- the morphism $\pi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_1 // G_0 = \text{Spec}(\mathbb{k}[\mathfrak{g}_1]^{G_0})$ is flat and surjective;
- $\pi^{-1}\pi(0) = \mathcal{N} \cap \mathfrak{g}_1$ and each fibre of π contains finitely many G_0 -orbits.

(It is worth mentioning that this is only a tiny fraction of fundamental results obtained in that great paper.) The fibre $\pi^{-1}\pi(0)$ is customarily called the *null-cone* (with respect to the G_0 -action on $\mathfrak{g}_1$) and we denote it by $\mathcal{N}_1$.

Definition 3.11. Following [P09], we say that

- (1) ϑ is *S-regular*, if $\mathfrak{g}_1$ contains a regular semisimple element of $\mathfrak{g}$;
- (2) ϑ is *N-regular*, if $\mathfrak{g}_1$ (i.e., $\mathcal{N}_1$) contains a regular nilpotent element of $\mathfrak{g}$;
- (3) ϑ is *very N-regular*, if each irreducible component of $\mathcal{N}_1$ contains a regular nilpotent element of $\mathfrak{g}$.

If $\text{ord}(\vartheta) = 2$, then it follows from [KR71] that properties (1)–(3) are equivalent. But this is not always the case if $\text{ord}(\vartheta) \geq 3$. It can happen that ϑ is S-regular, but

not $\mathcal{N}$ -regular; and vice versa. It can also happen that $\mathcal{N}_1$ is reducible and some irreducible components of $\mathcal{N}_1$ are not reduced (in the scheme-theoretic sense). Examples of automorphisms of order ≥ 3 such that good properties of Definition 3.11 hold are given in Examples 5.9 and 5.10 in [P09]. The following assertion provides sufficient conditions for some good properties of $\mathfrak{g}_{(0)}$ to hold.

Theorem 3.12 ([P09, Sect. 5]). *Let $\vartheta \in \text{Aut}(\mathfrak{g})$ be of finite order.*

- *If $\mathfrak{g}_1 \cap \mathfrak{g}_{\text{reg}} \neq \emptyset$, then $\text{ind } \mathfrak{g}_{(0)} = \text{ind } \mathfrak{g}$;*
- *If ϑ is both $\mathcal{S}$ -regular and very $\mathcal{N}$ -regular, then*
 - *$\mathfrak{g}_{(0)}$ has the codim-2-property;*
 - *$\mathcal{Z}_0$ is freely generated by $H_1^\bullet, \dots, H_l^\bullet$, $l = \text{rk } \mathfrak{g}$, where $\{H_1, \dots, H_l\}$ is any set of homogeneous generators of $\mathcal{S}(\mathfrak{g})^\vartheta = \mathcal{Z}_1$ that consists of ϑ -eigenvectors.*

The assumptions of Theorem 3.12 are not always satisfied, and the property of being “very $\mathcal{N}$ -regular” is difficult to check directly. Some methods for handling these properties and related examples can be found in Section 5 in [P09].

There is a nice special case, where all properties of Definition 3.11 hold and Theorem 3.12 applies. Namely, let $\mathfrak{g}$ be the direct sum of m copies of a semisimple Lie algebra $\mathfrak{h}$ and ϑ a cyclic permutation of the summands; hence $\text{ord}(\vartheta) = m$. Here we obtain a complete description of the Poisson-commutative subalgebra $\mathcal{Z}$, see Section 6.

4. On algebraically independent generators

If it is known that $\text{tr.deg } \mathcal{Z} = \mathbf{b}(\mathfrak{g}, \vartheta)$, then it becomes a meaningful task to compute the minimal number of generators of $\mathcal{Z}$ or $\mathcal{Z}_\times$.

Let $\{H_1, \dots, H_l\}$ be a set of homogeneous algebraically independent generators of $\mathcal{S}(\mathfrak{g})^\vartheta$ and $d_i = \deg H_i$. Then $\sum_{i=1}^l d_i = \mathbf{b}(\mathfrak{g})$. Recall from Sections 1.2 and 2.1 that associated with ϑ , we have the polynomial homomorphism $\varphi : \mathbb{k}^\star \rightarrow \text{GL}(\mathfrak{g})$ and its extension to invertible linear transformations of $\mathcal{S}^j(\mathfrak{g})$ for all j . Therefore, each H_j decomposes as $H_j = \sum_{i \geq 0} H_{j,i}$, where $\varphi_s(H_j) = \sum_{i \geq 0} s^i H_{j,i}$. The polynomials $H_{j,i}$ are called bi-homogeneous components of H_j . By definition, the φ -degree of $H_{j,i}$ is i , also denoted by $\deg_\varphi H_{j,i}$.

Let $H_j^\bullet$ be the nonzero bi-homogeneous component of H_j with maximal φ -degree. Then $\deg_\varphi H_j = \deg_\varphi H_j^\bullet$ and we set $d_j^\bullet = \deg_\varphi H_j^\bullet$. By Proposition 2.1(i), $\mathcal{Z}_t$ is generated by $\varphi_s^{-1}(H_1), \dots, \varphi_s^{-1}(H_l)$, where $t = s^m \in \mathbb{k}^\star$. By the standard argument with the Vandermonde determinant, we then conclude that $\mathcal{Z}_\times$ is generated by all bi-homogeneous components of $H_1, \dots, H_l$, i.e.,

$$\mathcal{Z}_\times = \text{alg}\langle H_{j,i} \mid 1 \leq j \leq l, 0 \leq i \leq d_j^\bullet \rangle. \quad (4.1)$$

Definition 4.1. Let us say that $H_1, \dots, H_l$ is a *good generating system* in $\mathcal{S}(\mathfrak{g})^\vartheta$ (g.g.s. for short) for ϑ , if $H_1^\bullet, \dots, H_l^\bullet$ are algebraically independent. Then we also say that ϑ *admits a g.g.s.*

The property of being ‘good’ really depends on a generating system. For instance, for $\mathfrak{g} = \mathfrak{sl}_n$ and $m = 2$, the coefficients of the characteristic polynomial of $A \in \mathfrak{sl}_n$

yield a g.g.s., while the polynomials $\text{tr}(A^i)$, $i = 2, \dots, n$ do not provide a g.g.s., see [P07].

The importance of g.g.s. is manifestly seen in the following fundamental result.

Theorem 4.2 ([Y14, Thm. 3.8]). *Let $H_1, \dots, H_l$ be an arbitrary set of homogeneous algebraically independent generators of $S(\mathfrak{g})^{\mathfrak{g}}$. Then*

- (i) $\sum_{j=1}^l \deg_{\varphi} H_j \geq \sum_{j=1}^{m-1} j \dim \mathfrak{g}_j =: D_{\vartheta}$;
- (ii) $H_1, \dots, H_l$ is a g.g.s. if and only if $\sum_{j=1}^l \deg_{\varphi} H_j = D_{\vartheta}$;
- (iii) if $\mathfrak{g}_{(0)}$ has the codim-2 property, $\text{ind } \mathfrak{g}_{(0)} = l$, and $H_1, \dots, H_l$ is a g.g.s., then $\mathcal{Z}_0 = S(\mathfrak{g}_{(0)})^{\mathfrak{g}_{(0)}}$ is a polynomial algebra freely generated by $H_1^{\bullet}, \dots, H_l^{\bullet}$.

Recall that $\dim \mathfrak{g}_j = \dim \mathfrak{g}_{m-j}$ for any $0 < j \leq m-1$. Hence

$$D_{\vartheta} = \sum_{j=1}^{m-1} j \dim \mathfrak{g}_j = \frac{1}{2} \sum_{j=1}^{m-1} m \dim \mathfrak{g}_j = \frac{m}{2} (\dim \mathfrak{g} - \dim \mathfrak{g}_0).$$

Not every $i \in \{0, 1, \dots, d_j^{\bullet}\}$ provides a **nonzero** bi-homogeneous component $H_{j,i}$. To make this precise, we first consider the case of inner automorphisms, which is technically easier.

Theorem 4.3. *Suppose that $\vartheta \in \text{Aut}(\mathfrak{g})$ is inner and admits a g.g.s. Let $H_1, \dots, H_l$ be such a g.g.s. in $S(\mathfrak{g})^{\mathfrak{g}}$ and $d_j^{\bullet} = \deg_{\varphi} H_j$. Then*

- (i) each $d_j^{\bullet} \in m\mathbb{Z}$ and $H_{j,i} \neq 0$ only if $i \in m\mathbb{Z}$ and $0 \leq i \leq d_j^{\bullet}$;
- (ii) if $\text{ind } \mathfrak{g}_{(0)} = l$, then $H_{j,i} \neq 0$ for every $i \in m\mathbb{Z}$ with $0 \leq i \leq d_j^{\bullet}$ and the polynomials $\{H_{j,i} \mid j = 1, \dots, l; \text{ \& } i = 0, m, \dots, d_j^{\bullet}\}$ are algebraically independent (and generate $\mathcal{Z}_{\times}$).

Proof. (i) Since ϑ is inner, $\text{rk } \mathfrak{g} = \text{rk } \mathfrak{g}_0$ and $\vartheta(H_j) = H_j$ for all j . On the other hand, $\vartheta(H_{j,i}) = \zeta^i H_{j,i}$. This proves everything.

(ii) It follows from (i) that the number of nonzero bi-homogeneous components of H_j is at most $(d_j^{\bullet}/m) + 1$. Hence the total number of nonzero bi-homogeneous components of all H_j is at most

$$\sum_{j=1}^l \left(\frac{d_j^{\bullet}}{m} + 1 \right) = \frac{D_{\vartheta}}{m} + l = \frac{1}{2} (\dim \mathfrak{g} - \dim \mathfrak{g}_0) + l = \mathbf{b}(\mathfrak{g}) - \mathbf{b}(\mathfrak{g}_0) + \text{rk } \mathfrak{g}_0 = \mathbf{b}(\mathfrak{g}, \vartheta),$$

where the equality $\text{rk } \mathfrak{g} = \text{rk } \mathfrak{g}_0$ is used. On the other hand, the hypothesis $\text{ind } \mathfrak{g}_{(0)} = l$ guarantees us that $\text{tr.deg } \mathcal{Z}_{\times} = \text{tr.deg } \mathcal{Z}_{\times} = \mathbf{b}(\mathfrak{g}, \vartheta)$, cf. Theorem 3.10 and Corollary 3.8. As the bi-homogeneous components $\{H_{j,i}\}$ generate $\mathcal{Z}_{\times}$ (4.1), all $H_{j,i}$ with $i/m \in \mathbb{Z}$ must be nonzero and algebraically independent. $\square$

With extra technicalities, the same idea works for the arbitrary automorphisms as well. Let $\vartheta \in \text{Aut}(\mathfrak{g})$ be an arbitrary automorphism of order m . Since ϑ acts on $S(\mathfrak{g})^{\mathfrak{g}}$, there is a set of homogeneous generators $\{H_j\} \subset S(\mathfrak{g})^{\mathfrak{g}}$ such that each H_j is an eigenvector of ϑ , i.e., $\vartheta(H_j) = \zeta^{r_j} H_j$ for some $r_j \in \mathbb{Z}$. However, we need a set of free generators that simultaneously is a g.g.s. and consists of ϑ -eigenvectors. For $m = 2$, this is proved in [PY, Lem. 3.4]. The following is an adaptation of that argument to any $m \geq 2$.

Lemma 4.4. *If ϑ admits a g.g.s., then there is also a g.g.s. that consists of ϑ -eigenvectors.*

Proof. Let $H_1, \dots, H_l$ be a g.g.s., hence $\sum_{j=1}^l \deg_{\varphi} H_j = D_{\vartheta}$ in view of Theorem 4.2.

Let $\mathcal{A}_+$ be the ideal in $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ generated by all homogeneous invariants of positive degree. Then $\mathcal{A} := \mathcal{A}_+ / \mathcal{A}_+^2$ is a finite-dimensional $\mathbb{k}$ -vector space. If $H \in \mathcal{A}_+$, then $\bar{H} := H + \mathcal{A}_+^2 \in \mathcal{A}$. As is well known, $F_1, \dots, F_m$ is a generating system for $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ if and only if the $\mathbb{k}$ -linear span of $\bar{F}_1, \dots, \bar{F}_m$ is the whole of $\mathcal{A}$. In our situation, $\dim_{\mathbb{k}} \mathcal{A} = l$ and $\mathcal{A} = \langle \bar{H}_1, \dots, \bar{H}_l \rangle_{\mathbb{k}}$.

If H_i is not a ϑ -eigenvector, i.e., $\vartheta(H_i) \notin \mathbb{k}H_i$, then we consider the (non-minimal) generating set

$$\{H_1, \dots, H_{i-1}, H_{i+1}, \dots, H_l\} \cup \left\{ H_i^{[k]} = \sum_{j=0}^{m-1} \zeta^{jk} \vartheta^j(H_i) \mid 0 \leq k < m \right\}$$

for $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ that includes $l + m - 1$ polynomials. Since $\bar{H}_1, \dots, \bar{H}_{i-1}, \bar{H}_{i+1}, \dots, \bar{H}_l$ are linearly independent in $\mathcal{A}$, one can pick out a suitable $H_i^{[k]}$ such that

$$\bar{H}_1, \dots, \bar{H}_{i-1}, H_i^{[k]}, \bar{H}_{i+1}, \dots, \bar{H}_l$$

is again a minimal generating set. Let us demonstrate that there is a unique suitable choice of $H_i^{[k]}$, and this yields a g.g.s. as well. Recall that $d_j^{\bullet} = \deg_{\varphi} H_j^{\bullet} = \deg_{\varphi} H_j$.

Clearly $\deg_{\varphi} H_i^{[k]} \leq d_i^{\bullet}$ for each k . Since $\sum_{j=0}^{m-1} \zeta^{jk'} = 0$ if $1 \leq k' \leq m-1$, we see that $\deg_{\varphi} H_i^{[k]} = d_i^{\bullet}$ if $(k + d_i^{\bullet}) \in m\mathbb{Z}$ and $\deg_{\varphi} H_i^{[k]} < d_i^{\bullet}$ for all other k . If $\deg_{\varphi} H_i^{[k]} < d_i^{\bullet}$, then the sum of φ -degrees for $H_1, \dots, H_{i-1}, H_i^{[k]}, H_{i+1}, \dots, H_l$ is less than D_{ϑ} . By Theorem 4.2, this means that such choice of $H_i^{[k]}$ in place of H_i provides a set of algebraically dependent polynomials. Thus, the only right choice is to take k such that $k + d_i^{\bullet} \equiv 0 \pmod{m}$, when Theorem 4.2 also guarantees us that we obtain a g.g.s.

The procedure reduces the number of generators that are not ϑ -eigenvectors, and we eventually obtain a g.g.s. that consists of ϑ -eigenvectors. $\square$

Let $\{H_1, \dots, H_l\} \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ be a generating set consisting of ϑ -eigenvectors. Then $\vartheta(H_i) = \zeta^{r_i} H_i$ with $0 \leq r_i < m$. The integers $\{r_i\}$ depend only on the connected component of $\text{Aut}(\mathfrak{g})$ that contains ϑ , and if a is the order of ϑ in $\text{Aut}(\mathfrak{g})/\text{Int}(\mathfrak{g})$, then $\zeta^{ar_i} = 1$. Therefore, if $\mathfrak{g}$ is simple, then $\zeta^{r_i} = \pm 1$ for all types but D_4 .

Lemma 4.5. *For any $\vartheta \in \text{Aut}(\mathfrak{g})$ of order m , we have*

- (1) $\vartheta(H_j) = H_j$ if and only if $d_j^{\bullet} \in m\mathbb{Z}$;
- (2) $\sum_{j=1}^l r_j = \frac{1}{2}m(\text{rk } \mathfrak{g} - \text{rk } \mathfrak{g}_0)$;
- (3) $\text{rk } \mathfrak{g}_0 = \#\{j \mid \vartheta(H_j) = H_j\}$.

Proof. (1) The proof is similar to that of Theorem 4.3(i).

(2) Recall that $\mathfrak{g}_0$ always contains regular semisimple elements of $\mathfrak{g}$ [Ka83, §8.8]. Therefore, if $\mathfrak{t}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$, then $\mathfrak{t} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{t}_0)$ is a ϑ -stable Cartan subalgebra of $\mathfrak{g}$. Since $\varkappa|_{\mathfrak{t}}$ is non-degenerate, $\dim(\mathfrak{t} \cap \mathfrak{g}_j) = \dim(\mathfrak{t} \cap \mathfrak{g}_{m-j})$ for all $j < m$.

Let us apply Kostant's regularity criterion (cf. (3.3)) to some $x \in \mathfrak{t}_0 \cap \mathfrak{g}_{\text{reg}}$. According to this criterion, $\langle d_x H_j \mid 1 \leq j \leq l \rangle_{\mathbb{k}} = \mathfrak{t}$. Here $d_x H_j \in \mathfrak{t} \cap \mathfrak{g}_i$ if and only if $r_j = i$. Therefore

$$\sum_{j=1}^l r_j = \sum_{i=1}^{m-1} i \dim(\mathfrak{t} \cap \mathfrak{g}_i) = \frac{1}{2} m (\dim \mathfrak{t} - \dim \mathfrak{t}_0).$$

(3) To prove this, it suffices to notice that $d_x H_j \in \mathfrak{g}_0$ if and only if we have $\vartheta(H_j) = H_j$. $\square$

Now, we can prove our main result on $\mathcal{Z}_{\times}$.

Theorem 4.6. *Suppose that $\vartheta \in \text{Aut}(\mathfrak{g})$ admits a g.g.s. and $\text{ind } \mathfrak{g}_{(0)} = \text{rk } \mathfrak{g}$. Then*

- (i) $\mathcal{Z}_{\times}$ is a polynomial Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}_0}$ having the maximal transcendence degree.
- (ii) More precisely, if $H_1, \dots, H_l$ is a g.g.s. that consists of ϑ -eigenvectors, then $\mathcal{Z}_{\times}$ is freely generated by the nonzero bi-homogeneous components of all $\{H_j\}$.

Proof. We already know that $\mathcal{Z}_{\times}$ is Poisson-commutative and $\text{tr.deg } \mathcal{Z}_{\times} = \mathbf{b}(\mathfrak{g}, \vartheta)$ is the maximal possible (Theorem 3.10). Hence we have to prove the polynomiality and (ii).

Recall that $\{H_j\}$ is a g.g.s. if and only if $\sum_{j=1}^l d_j^{\bullet} = D_{\vartheta} = m(\dim \mathfrak{g} - \dim \mathfrak{g}_0)/2$. If $\vartheta(H_j) = H_j$, i.e., $d_j^{\bullet} \in m\mathbb{Z}$, then H_j has at most $(d_j^{\bullet}/m) + 1$ nonzero bi-homogeneous components, as in the case of Theorem 4.3. In general, if $\vartheta(H_j) = \zeta^{r_j} H_j$, then $H_{j,i}$ can be nonzero only if $i \equiv r_j \pmod{m}$. Therefore, $d_j^{\bullet} \equiv r_j \pmod{m}$ and H_j has at most $1 + (d_j^{\bullet} - r_j)/m$ nonzero bi-homogeneous components. Using Lemma 4.5, we see that the total number of all nonzero bi-homogeneous components is at most

$$\begin{aligned} \sum_{j=1}^l \left(\frac{d_j^{\bullet} - r_j}{m} + 1 \right) &= \left(\sum_{j=1}^l \frac{d_j^{\bullet}}{m} \right) - \left(\sum_{j=1}^l \frac{r_j}{m} \right) + l \\ &= \frac{D_{\vartheta}}{m} - \frac{l - \text{rk } \mathfrak{g}_0}{2} + l = \mathbf{b}(\mathfrak{g}) - \mathbf{b}(\mathfrak{g}_0) + \text{rk } \mathfrak{g}_0 = \mathbf{b}(\mathfrak{g}, \vartheta). \end{aligned}$$

On the other hand, it follows from Eq. (4.1) that the total number of bi-homogeneous components of $H_1, \dots, H_l$ is at least $\mathbf{b}(\mathfrak{g}, \vartheta)$. Therefore, all admissible bi-homogeneous components must be nonzero and algebraically independent. $\square$

A precise relationship between $\mathcal{Z}$ and $\mathcal{Z}_{\times}$ depends on further properties of ϑ . Two complementary assertions are given below.

Corollary 4.7. *In addition to the hypotheses of Theorem 4.6, suppose that $\mathfrak{g}_{(0)}$ has the codim-2 property and $\mathfrak{g}_0 = \mathfrak{g}^{\vartheta}$ is not abelian. Then $\mathcal{Z} = \mathcal{Z}_{\times}$ is the polynomial algebra freely generated by all nonzero bi-homogeneous components $H_{j,i}$.*

Proof. In this case, $0 \in \mathbb{P}_{\text{reg}}$ and it follows from Theorem 4.2(iii) that $\mathcal{Z}_0$ is equal to $\mathbb{k}[H_1^{\bullet}, \dots, H_l^{\bullet}]$, which is contained in $\mathcal{Z}_{\times}$. On the other hand, $\infty \in \mathbb{P}_{\text{sing}}$ (Corollary 3.3), hence $\mathcal{Z}_{\infty}$ is not required for $\mathcal{Z}$. Thus, $\mathcal{Z} = \mathcal{Z}_{\times}$. $\square$

Corollary 4.8. *In addition to the hypotheses of Theorem 4.6, suppose that $\mathfrak{g}_{(0)}$ has the codim-2 property, ϑ is inner, and $\mathfrak{g}_0 = \mathfrak{g}^\vartheta$ is abelian. Then $\mathcal{Z}_\infty = \mathcal{S}(\mathfrak{g}_0)$ and $\mathcal{Z} = \text{alg}\langle \mathcal{Z}_\times, \mathfrak{g}_0 \rangle$ is a polynomial algebra.*

Proof. Recall that the subspace $\mathfrak{g}_0$ lies in the centre of $\mathfrak{g}_{(\infty)}$, hence $\mathcal{S}(\mathfrak{g}_0) \subset \mathcal{Z}_\infty$. If ϑ is inner, then $\text{ind } \mathfrak{g}_{(\infty)} = \dim \mathfrak{g}_0 = \text{rk } \mathfrak{g}$ (Theorem 3.2). Hence $\mathcal{S}(\mathfrak{g}_0) \subset \mathcal{Z}_\infty$ is an algebraic extension. Since $\mathcal{S}(\mathfrak{g}_0)$ is an algebraically closed subalgebra of $\mathcal{S}(\mathfrak{g})$, it coincides with $\mathcal{Z}_\infty$. As in the previous corollary, we have $\mathcal{Z}_0 \subset \mathcal{Z}_\times$. Therefore,

$$\mathcal{Z} = \text{alg}\langle \mathcal{Z}_\times, \mathcal{Z}_\infty \rangle = \text{alg}\langle \mathcal{Z}_\times, \mathfrak{g}_0 \rangle.$$

Among the algebraically independent generators of $\mathcal{Z}_\times$, one has l nonzero functions $H_{j,0} \in \mathcal{S}(\mathfrak{g}_0)$, $j = 1, \dots, l$. (Note that $d_j^\bullet \in m\mathbb{Z}$ for each j , since ϑ is inner.) Hence the passage from $\mathcal{Z}_\times$ to $\mathcal{Z}$ merely means that we have to replace $\{H_{j,0}\}$ with a basis for $\mathfrak{g}_0$. $\square$

Remark 4.9. If ϑ does not admit a g.g.s., then $H_1^\bullet, \dots, H_l^\bullet$ are algebraically dependent. Since $\{H_i^\bullet\}$ are certain bi-homogeneous components, this means that the number of all nonzero bi-homogeneous components of $\{H_j\}$ is larger than $\text{tr.deg } \mathcal{Z}$. Moreover, the case of involutions ($m = 2$) shows that then the algebra $\mathcal{Z}_0 = \mathcal{S}(\mathfrak{g}_{(0)})^{\mathfrak{g}_{(0)}}$, which is contained in $\mathcal{Z}$, is not polynomial, see [Y17, Sect. 5]. Therefore, it would be unwise to expect really good properties of $\mathcal{Z}_\times$ or $\mathcal{Z}$ without the presence of a g.g.s.

5. On the maximality problem

Since $\mathcal{Z} \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{g}_0}$ by (3-6), the algebra $\tilde{\mathcal{Z}} := \text{alg}\langle \mathcal{Z}, \mathcal{S}(\mathfrak{g}_0)^{\mathfrak{g}_0} \rangle$ is still Poisson-commutative. It was proved in [PY] that if ϑ is an involution admitting a g.g.s., then $\tilde{\mathcal{Z}}$ is a **maximal** Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}_0}$. For $m \geq 3$, the same problem becomes more difficult, and we obtain only partial results in this section. Our line of argument employs properties of graded polynomial algebras.

Let $F_1, \dots, F_N \in \mathbb{k}[\{x_i\}] = \mathbb{k}[\mathbb{A}^n]$ be algebraically independent homogeneous polynomials. Each differential dF_i is a regular 1-form on $\mathbb{A}^n$ with polynomial coefficients. Then

$$dF_1 \wedge \dots \wedge dF_N = \mathbf{f}R, \quad (5.1)$$

where $\mathbf{f} \in \mathbb{k}[\mathbb{A}^n]$ and R is a regular differential N -form that is nonzero on a big open subset. Note that $\mathbf{f}$ is defined uniquely up to multiplication with scalars. Let $\mathcal{F} = \mathbb{k}[\{F_j\}]$ be the subalgebra of $\mathbb{k}[\mathbb{A}^n]$ generated by the polynomials F_j .

Theorem 5.1 ([PPY, Thm. 1.1]). *If $\mathbf{f} = 1$, then $\mathcal{F}$ is an algebraically closed subalgebra of $\mathbb{k}[\mathbb{A}^n]$, i.e., if $H \in \mathbb{k}[\mathbb{A}^n]$ is algebraic over the field $\mathbb{k}(F_1, \dots, F_N)$, then $H \in \mathcal{F}$.*

Consider the following conditions on the Lie algebra $\mathfrak{g}_{(0)}$:

- ($\diamond_1$) $\text{ind } \mathfrak{g}_{(0)} = \text{rk } \mathfrak{g} = l$, i.e., $0 \in \mathbb{P}_{\text{reg}}$;
- ($\diamond_2$) $\mathcal{Z}_0$ is a polynomial ring generated by $H_i^\bullet$ with $1 \leq i \leq l$;
- ($\diamond_3$) $\dim \mathfrak{g}_{(0), \text{sing}}^* \leq \dim \mathfrak{g} - 2$, i.e., $\mathfrak{g}_{(0)}$ has the codim-2 property;
- ($\diamond_4$) either $\mathfrak{g}_0$ is non-abelian or $\mathfrak{g}_0$ is abelian and ϑ is inner.

These conditions imply that $\mathcal{Z}$ is a polynomial algebra, see Corollary 4.7 and 4.8. Moreover, the following is true.

Proposition 5.2. *If conditions $(\diamond_1)$ – $(\diamond_4)$ are satisfied, then $\tilde{\mathcal{Z}}$ is a polynomial algebra, too.*

Proof. By Lemma 4.5, we have $H_{j,0} \neq 0$ if and only if $\vartheta(H_j) = H_j$, and the number of such j 's equals $\text{rk } \mathfrak{g}_0$. Then we have $H_{j,0} \in \mathcal{S}(\mathfrak{g}_0)^{\mathfrak{g}_0}$. In the passage from $\mathcal{Z}$ to $\tilde{\mathcal{Z}}$, these nonzero generators $H_{j,0}$ are replaced with the basic symmetric invariants of $\mathfrak{g}_0$. $\square$

Lemma 5.3. *Assume that $(\diamond_1)$ and $(\diamond_3)$ hold. Suppose that there is a divisor $D \subset \mathfrak{g}^*$ such that $\dim d_\eta \mathcal{Z}_\times < \mathbf{b}(\mathfrak{g}, \vartheta)$ for any $\eta \in D$. Then $D = Y \times \mathfrak{g}_{>0}^*$, where $Y \subset \mathfrak{g}_0^* \cap \mathfrak{g}_{\text{sing}}^*$ is a G_0 -stable conical divisor in $\mathfrak{g}_0^*$.*

Proof. By Proposition 3.1, there is a big open subset $U_{\text{sr}} \subset \mathfrak{g}^*$ such that

$$U_{\text{sr}} \subset \bigcap_{t \neq 0, \infty} \mathfrak{g}_{(t), \text{reg}}^*.$$

By $(\diamond_3)$, $\mathfrak{g}_{(0), \text{reg}}^*$ is also a big open subset of $\mathfrak{g}^*$. Hence $\tilde{U} := D \cap U_{\text{sr}} \cap \mathfrak{g}_{(0), \text{reg}}^*$ is open and dense in D . Take any $\eta \in \tilde{U}$. If we write $\eta = \eta_0 + \eta'$ with $\eta \in \mathfrak{g}_0^*$ and $\eta' \in \mathfrak{g}_{>0}^*$, then we may also assume that $\eta_0 \in (\mathfrak{g}_0)_{\text{reg}}^*$. Assume that $\eta_0 \in \mathfrak{g}_{\text{reg}}^*$. Then $\eta \in \mathfrak{g}_{\infty, \text{reg}}^*$ by Lemma 3.6. Thus, in that case, $\eta \in U_{\text{sr}}$. Moreover, such η is generic in the sense of Lemma 3.5 and the conclusion of that lemma holds for it. Arguing as in the proof of Theorem 3.10 and using (3.5), we obtain that $\dim L(\eta) = \mathbf{b}(\mathfrak{g}, \vartheta)$, a contradiction!

Therefore we must have $\eta_0 \in \mathfrak{g}_{\text{sing}}^*$ for a generic $\eta \in D$ and hence for any $\eta \in D$. Since $\mathfrak{g}_0^* \cap \mathfrak{g}_{\text{sing}}^*$ is a proper subset of $\mathfrak{g}_0^*$, the divisor D is indeed of the form $Y \times \mathfrak{g}_{>0}^*$, where $Y \subset \mathfrak{g}_0^* \cap \mathfrak{g}_{\text{sing}}^*$.

The algebra $\mathcal{Z}_\times$ consists of G_0 -invariants and the group G_0 is connected. Therefore each irreducible component of the subset $\{\gamma \in \mathfrak{g}^* \mid \dim d_\gamma \mathcal{Z}_\times < \mathbf{b}(\mathfrak{g}, \vartheta)\}$ is G_0 -stable. In particular, D , and hence Y as well, is G_0 -stable. Since $\mathcal{Z}_\times$ is a homogeneous subalgebra, the divisor $Y \subset \mathfrak{g}_0^*$ is conical. $\square$

Theorem 5.4. *Suppose that $(\diamond_1)$, $(\diamond_2)$, and $(\diamond_3)$ hold. If $\mathfrak{g}_0 \cap \mathfrak{g}_{\text{sing}}^*$ does not contain divisors, then $\mathcal{S}(\mathfrak{g}_0)^{\mathfrak{g}_0} \subset \mathcal{Z}_\times$ and $\mathcal{Z}_\times = \mathcal{Z} = \tilde{\mathcal{Z}}$ is a maximal Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}_0}$.*

Proof. We know that $\mathcal{Z}_\times$ is a polynomial algebra (Theorem 4.6) and $\text{tr.deg } \mathcal{Z}_\times = \mathbf{b}(\mathfrak{g}, \vartheta)$ (Theorem 3.10). Suppose that $\mathcal{Z}_\times \subset \mathcal{A} \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{g}_0}$ and $\mathcal{A}$ is Poisson-commutative. Then we have $\text{tr.deg } \mathcal{A} \leq \mathbf{b}(\mathfrak{g}, \vartheta)$ by (3.7). In view of Lemma 5.3, the differentials dF_i of the algebraically independent generators $F_i \in \mathcal{Z}_\times$ are linearly independent on a big open subset. Then by Theorem 5.1, $\mathcal{Z}_\times$ is an algebraically closed subalgebra of $\mathcal{S}(\mathfrak{g})$. In particular, we must have $\mathcal{A} = \mathcal{Z}_\times$. This applies to $\mathcal{Z}$ and $\tilde{\mathcal{Z}}$ as well. $\square$

Remark 5.5. Conditions $(\diamond_1)$ – $(\diamond_4)$ are satisfied for involutions ϑ that have g.g.s..

Example 5.6. Theorem 5.4 applies to several outer automorphisms of semisimple Lie algebras, for instance, to $\vartheta \in \text{Aut}(\mathfrak{g})$ of order $2m$ in case $\mathfrak{g} = (\mathfrak{sl}_{2n})^m$ and $\mathfrak{g}_0$ is a diagonally embedded $\mathfrak{sp}_{2n}$, cf. [P09, (3.0)]. Automorphisms of this form are considered in Section 8.

Theorem 3.12 provides a bunch of automorphisms ϑ with $\text{ord}(\vartheta) \geq 3$ such that $\tilde{\mathcal{Z}}$ is a polynomial algebra, see Examples 5.9 and 5.10 in [P09]. In all these cases, $\mathcal{Z} \neq \tilde{\mathcal{Z}}$ and we conjecture that $\tilde{\mathcal{Z}}$ is a maximal Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}_0}$.

Summarising our previous considerations, we can say that in order to guarantee some good properties of the Poisson-commutative subalgebras $\mathcal{Z}_\times, \mathcal{Z}$, and $\tilde{\mathcal{Z}}$, the following properties of $\vartheta \in \text{Aut}(\mathfrak{g})$ and thereby of $\mathfrak{g}_{(0)}$ are needed:

- (a) $\text{ind } \mathfrak{g}_{(0)} = \text{ind } \mathfrak{g}$;
- (b) $\mathfrak{g}_{(0)}$ has the *codim-2* property;
- (c) ϑ admits a *g.g.s.*

If $\text{ord}(\vartheta) = 2$, then (a) and (b) are always satisfied [P07], and a complete description of involutions admitting a *g.g.s.* is available [Y17]. In a forthcoming paper, we are going to undertake a thorough substantial investigation of these properties for arbitrary ϑ .

6. The case of a cyclic permutation

Let $\mathfrak{h}$ be a simple non-abelian Lie algebra and $\mathfrak{g} = \mathfrak{h}^m$ the direct sum of $m \geq 2$ copies of $\mathfrak{h}$. Then $l = \text{rk } \mathfrak{g} = m \cdot \text{rk } \mathfrak{h}$. Let $\vartheta \in \text{Aut}(\mathfrak{g})$ be a cyclic permutation of the summands of $\mathfrak{g}$. That is,

$$\vartheta(x_1, \dots, x_m) = (x_m, x_1, \dots, x_{m-1}), \quad x_i \in \mathfrak{h}.$$

Then, for $i = 0, 1, \dots, m-1$ and $\zeta = \sqrt[m]{1}$, we have

$$\mathfrak{g}_i = \{(x, \zeta^{-i}x, \dots, \zeta^{(1-m)i}x) \mid x \in \mathfrak{h}\}. \quad (6.1)$$

In particular, $\mathfrak{g}_0 = \Delta_{\mathfrak{h}} \simeq \mathfrak{h}$ is the diagonal and each $\mathfrak{g}_i$ is isomorphic to $\mathfrak{h}$ as vector space and as $\mathfrak{g}_0$ -module. Here the Lie algebra $\mathfrak{g}_{(0)}$ is isomorphic to the truncated current algebra

$$\mathfrak{h}\langle m \rangle := \mathfrak{h} \otimes \mathbb{k}[t]/(t^m) = \mathfrak{h}[t]/(t^m),$$

see [P09, Corollary 3.6]. The isomorphism $\mathfrak{g}_{(0)} \simeq \mathfrak{h}\langle m \rangle$ takes $\mathfrak{g}_i$ to the image of $\mathfrak{h}t^i \subset \mathfrak{h} \otimes \mathbb{k}[t]$ in $\mathfrak{h}\langle m \rangle$. The Lie algebra $\mathfrak{h}\langle m \rangle$ is also known as a (generalised) *Takiff algebra* modelled on $\mathfrak{h}$. By [RT92, Thm. 2.8], we have $\text{ind } \mathfrak{g}_{(0)} = m \cdot \text{rk } \mathfrak{h} = \text{rk } \mathfrak{g}$, i.e., $0 \in \mathbb{P}_{\text{reg}}$. It then follows from Theorem 3.10 that in this case

$$\text{tr.deg } \mathcal{Z} = \mathbf{b}(\mathfrak{h}^m, \vartheta) = \frac{1}{2}((m-1)\dim \mathfrak{h} + (m+1)\text{rk } \mathfrak{h}) = (m-1)\mathbf{b}(\mathfrak{h}) + \text{rk } \mathfrak{h}.$$

On the other hand, $\mathfrak{g}_0$ is not abelian, hence $\mathbb{P}_{\text{sing}} = \{\infty\}$ and $\mathcal{Z}_{\infty}$ is not required for $\mathcal{Z}$. Since $\mathfrak{h}\langle m \rangle$ is $\mathbb{N}$ -graded and the zero part is semisimple, the nilpotent radical $\mathfrak{h}\langle m \rangle^u$ is equal to $\bigoplus_{i=1}^{m-1} \mathfrak{h}t^i$. Comparing this with Proposition 2.3 on the

graded structure of $\mathfrak{g}_{(\infty)}$, we conclude that here $\mathfrak{g}_{(\infty)} \simeq \mathfrak{h}\langle m+1 \rangle^u$. Since $\text{ind } \mathfrak{g}_{(\infty)}$ is computed for any ϑ in Theorem 3.2, we obtain a new observation that

$$\text{ind } \mathfrak{h}\langle m \rangle^u = (m-2)\text{rk } \mathfrak{h} + \dim \mathfrak{h}. \quad (6.2)$$

Upon the identification $\mathfrak{g}_1 \simeq \mathfrak{h}$, see Eq. (6.1), an element $x \in \mathfrak{h}$ is nilpotent (resp. semisimple, regular) in $\mathfrak{h}$ if and only if $(x, \zeta^{-i}x, \dots, \zeta^{(1-m)i}x) \in \mathfrak{g}_1$ is nilpotent (resp. semisimple, regular) in $\mathfrak{g}$. This also implies that the null-cone $\mathcal{N}_1$ is isomorphic to the null-cone of $\mathfrak{h}$. Hence $\mathcal{N}_1$ is irreducible. Thus, ϑ is both $\mathcal{S}$ -regular and very $\mathcal{N}$ -regular.

Let $\{H_1, \dots, H_l\}$ be a set of homogeneous generators of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ consisting of ϑ -eigenvectors. Since ϑ is $\mathcal{S}$ -regular and very $\mathcal{N}$ -regular, it follows from Theorem 3.12 that $\mathfrak{g}_{(0)}$ has the *codim-2*-property and $\mathcal{Z}_0 = \mathbb{k}[H_1^{\bullet}, \dots, H_l^{\bullet}]$, see also [RT92]. The last relation also means that $\{H_j\}$ is a *g.g.s.* for ϑ .

Theorem 6.1. *If $\mathfrak{g} = \mathfrak{h}^m$ and ϑ is a cyclic permutation, then the algebra $\mathcal{Z} = \mathcal{Z}(\mathfrak{h}^m, \vartheta)$ is freely generated by the nonzero bi-homogeneous components $H_{j,i}$ with $1 \leq j \leq l = m \cdot \text{rk } \mathfrak{h}$. Moreover, $\mathcal{Z}$ is a maximal Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}_0}$.*

Proof. The above discussion shows that conditions $(\diamond_1)$ – $(\diamond_4)$ hold for ϑ . Hence $\mathcal{Z}$ is freely generated by the nonzero bi-homogeneous components $H_{j,i}$ by Corollary 4.7.

A point $\xi \in \mathfrak{g}_0^*$ is regular in $\mathfrak{g}^*$ if and only if ξ is regular in $\mathfrak{g}_0^* \simeq \mathfrak{h}^*$. Thereby

$$\dim(\mathfrak{g}_0^* \cap \mathfrak{g}_{\text{sing}}^*) = \dim \mathfrak{h} - 3$$

and this intersection does not contain divisors of $\mathfrak{g}_0^*$. Therefore $\mathcal{Z}$ is a maximal Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}_0}$ by Theorem 5.4. $\square$

It is not hard to produce a generating set $\{H_j\} \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ consisting of ϑ -eigenvectors. Suppose that $\text{rk } \mathfrak{h} = r$ and that $\{F_1, \dots, F_r\}$ is a generating set of $\mathcal{S}(\mathfrak{h})^{\mathfrak{h}} \subset \mathcal{S}(\mathfrak{g})$ for the first copy of $\mathfrak{h}$ in $\mathfrak{g}$. For $0 \leq k < m$, set

$$F_i^{[k]} = \frac{1}{m} \left(F_i + \zeta^k \vartheta(F_i) + \zeta^{2k} \vartheta^2(F_i) + \dots + \zeta^{k(m-1)} \vartheta^{m-1}(F_i) \right). \quad (6.3)$$

Then $\vartheta(F_i^{[k]}) = \zeta^{-k} F_i^{[k]}$. Thus, we can take

$$\{H_j \mid 1 \leq j \leq l\} = \{F_i^{[k]} \mid 1 \leq i \leq r, 0 \leq k < m\}.$$

In Section 7, we will need an explicit description of the bi-homogeneous components $H_{j,i}$, which exploits the polarisation construction of [RT92]. Suppose that each F_i is homogeneous and $b_i = \deg F_i$.

For $\xi \in \mathfrak{h}^*$, let $\xi^{(k)} \in \mathfrak{g}^*$ be the corresponding element sitting in the k -th copy of $\mathfrak{h}^*$ in $\mathfrak{g}^*$. Then the induced action of ϑ in $\mathfrak{g}^*$ is given by the formula:

$$\vartheta(\xi_1^{(1)} + \dots + \xi_{m-1}^{(m-1)} + \xi_m^{(m)}) = \xi_2^{(1)} + \dots + \xi_m^{(m-1)} + \xi_1^{(m)}.$$

Accordingly, $\mathfrak{g}_j^* = \{\xi^{(1)} + \zeta^j \xi^{(2)} + \dots + \zeta^{(m-1)j} \xi^{(m)} \mid \xi \in \mathfrak{h}^*\}$ and we fix the corresponding isomorphisms $\nu_j : \mathfrak{h}^* \xrightarrow{\sim} \mathfrak{g}_j^*$, where

$$\xi \mapsto \nu_j(\xi) = (\xi^{(1)} + \zeta^j \xi^{(2)} + \dots + \zeta^{(m-1)j} \xi^{(m)}) \in \mathfrak{g}_j^*. \quad (6.4)$$

For $\xi_0, \dots, \xi_{m-1}$ with $\xi_j \in \mathfrak{g}_j^*$, let $\bar{\xi} = (\xi_0, \dots, \xi_{m-1})$ be the corresponding element of $\mathfrak{g}^*$ and $\xi_0 + \dots + \xi_{m-1}$ be an element of $\mathfrak{h}^*$. For $s \in \mathbb{k}$, set

$$\phi_s(\bar{\xi}) = (s^{m-1} \xi_0, s^{m-2} \xi_1, \dots, \xi_{m-1})$$

and define the ϕ -polarisations $F_{[k]}$ of $F \in \mathcal{S}^d(\mathfrak{h})$ in the following way:

$$\begin{aligned} F(\phi_s(\bar{\xi})) &:= F(s^{m-1} \xi_0 + s^{m-2} \xi_1 + \dots + \xi_{m-1}) \\ &=: F_{[0]}(\bar{\xi}) + s F_{[1]}(\bar{\xi}) + \dots + s^{d(m-1)} F_{[d(m-1)]}(\bar{\xi}). \end{aligned} \quad (6.5)$$

It is convenient to assume that $F_{[k]} = 0$ for $k > d(m-1)$.

Example 6.2. If $x \in \mathfrak{h}$ and $0 \leq k < m$, then $x_{[k]} = \frac{1}{m}(x, \zeta^{-i}x, \dots, \zeta^{(1-m)i}x) \in \mathfrak{g}_i$ with $i = m-1-k$.

According to [RT92], $\mathcal{Z}_0$ is freely generated by $(F_i)_{[k]}$ with $1 \leq i \leq r$ and $0 \leq k < m$, where $\mathcal{S}(\mathfrak{h})^{\mathfrak{h}} = \mathbb{k}[F_1, \dots, F_r]$.

Proposition 6.3. *We have*

$$\{H_{j,i} \neq 0 \mid 1 \leq j \leq l, 0 \leq i\} = \{(F_i)_{[k]} \mid 1 \leq i \leq r, 0 \leq k \leq b_i(m-1)\},$$

where $(F_i)_{[k]}$ are the ϕ -polarisations of F_i defined by (6.5).

Proof. Now we need to distinguish the first copy $\mathfrak{h}^{(1)} \subset \mathfrak{g}$ of $\mathfrak{h}$ from an abstract $\mathfrak{h}$. Suppose that $F^{(1)} \in \mathcal{S}^d(\mathfrak{h}^{(1)})$ is the image of $F \in \mathcal{S}^d(\mathfrak{h})$ under the tautological isomorphism $\mathcal{S}(\mathfrak{h}) \simeq \mathcal{S}(\mathfrak{h}^{(1)})$. The combination of (6.4) and the definition of φ_s , see Section 1.2, leads to

$$\varphi_s(F^{(1)}) = F_{[(m-1)d]} + s F_{[(m-1)d-1]} + \dots + s^{(m-1)d-1} F_{[1]} + s^{(m-1)d} F_{[0]}$$

for $s \in \mathbb{k}^*$. In this notation, $\vartheta(F_{[k]}) = \zeta^{-d-k} F_{[k]}$. Next we plug the formula for $\varphi_s(F^{(1)})$ into (6.3) and conclude that each bi-homogeneous component of each $F_i^{[k]}$ is a ϕ -polarisation of $F_i \in \mathcal{S}^{b_i}(\mathfrak{h})$. By Theorem 6.1, the total number of the nonzero bi-homogeneous components $(F_i^{[k]})_{i'}$ is

$$\text{tr.deg } \mathcal{Z} = \mathbf{b}(\mathfrak{g}, \vartheta) = (m-1) \cdot \mathbf{b}(\mathfrak{h}) + r.$$

Since $\mathfrak{h}$ is semisimple, we have $\mathbf{b}(\mathfrak{h}) = \sum_{i=1}^r b_i$. Finally, the total number of ϕ -polarisations $(F_i)_{[k]}$ equals

$$r + (m-1) \sum_{i=1}^r b_i = r + (m-1) \cdot \mathbf{b}(\mathfrak{h}) = \mathbf{b}(\mathfrak{g}, \vartheta).$$

Hence the two sets in question coincide. $\square$

7. Gaudin subalgebras

Let $\mathfrak{h}$ be the same as in the previous section. The enveloping algebra $\mathcal{U}(t^{-1}\mathfrak{h}[t^{-1}])$ contains a remarkable commutative subalgebra $\mathfrak{z}(\widehat{\mathfrak{h}})$, which is known as the *Feigin–Frenkel centre* [FF92] (see also [F07]). This subalgebra is defined as the centre of the universal affine vertex algebra associated with the affine Kac–Moody algebra $\widehat{\mathfrak{h}}$ at the critical level. In particular, each element of $\mathfrak{z}(\widehat{\mathfrak{h}})$ is annihilated by the adjoint action of $\mathfrak{h}$. The elements of $\mathfrak{z}(\widehat{\mathfrak{h}})$ give rise to higher Hamiltonians of the Gaudin model, which describes a completely integrable quantum spin chain [FFR].

A Gaudin model consists of n copies of $\mathfrak{h}$ and the Hamiltonians

$$\mathcal{H}_k = \sum_{j \neq k} \frac{\sum_{i=1}^{\dim \mathfrak{h}} x_i^{(k)} x_i^{(j)}}{z_k - z_j}, \quad 1 \leq k \leq n,$$

where $z_1, \dots, z_n \in \mathbb{K}$ are pairwise distinct and $\{x_i^{(k)} \mid 1 \leq i \leq \dim \mathfrak{h}\}$ is a basis for the k -th copy of $\mathfrak{h}$ that is orthonormal with respect to $\varkappa$. Letting $\mathfrak{g} = \mathfrak{h}^n$, these Gaudin Hamiltonians $\{\mathcal{H}_k\}$ can be regarded as elements of either $\mathcal{U}(\mathfrak{h})^{\otimes n} \simeq \mathcal{U}(\mathfrak{g})$ or $\mathcal{S}(\mathfrak{g})$. They commute (and hence Poisson-commute) with each other. Then higher Gaudin Hamiltonians are elements of $\mathcal{U}(\mathfrak{h})^{\otimes n}$ that commute with all $\mathcal{H}_k$.

Recall the construction of [FFR] that provides a *Gaudin subalgebra* $\mathcal{G}$ of $\mathcal{U}(\mathfrak{h})^{\otimes n}$. Set $\widehat{\mathfrak{h}}_- = t^{-1}\mathfrak{h}[t^{-1}]$ and let $\Delta\mathcal{U}(\widehat{\mathfrak{h}}_-) \simeq \mathcal{U}(\widehat{\mathfrak{h}}_-)$ be the diagonal of $\mathcal{U}(\widehat{\mathfrak{h}}_-)^{\otimes n}$. Then any vector $\vec{z} = (z_1, \dots, z_n) \in (\mathbb{K}^\star)^n$ defines a natural homomorphism

$$\rho_{\vec{z}}: \Delta\mathcal{U}(\widehat{\mathfrak{h}}_-) \rightarrow \mathcal{U}(\mathfrak{h})^{\otimes n} \quad \text{with} \quad \rho_{\vec{z}}(xt^k) = z_1^k x^{(1)} + z_2^k x^{(2)} + \dots + z_n^k x^{(n)} \quad \text{for } x \in \mathfrak{h}.$$

By definition, $\mathcal{G} = \mathcal{G}(\vec{z})$ is the image of $\mathfrak{z}(\widehat{\mathfrak{h}})$ under $\rho_{\vec{z}}$, and one can prove that $\mathcal{G}$ contains $\mathcal{H}_k$ for all k . Hence $[\mathcal{G}(\vec{z}), \mathcal{H}_k] = 0$ for each k . One also has $\mathcal{G} \subset \mathcal{U}(\mathfrak{g})^{\Delta \mathfrak{h}}$ by the construction.

Remark 7.1. Gaudin subalgebras have attracted a great deal of attention, see, e.g., [CFR] and references therein. It is standard to work with complex Lie algebras in this framework. Gaudin algebras are closely related to *quantum MF subalgebras* [R06] and share some of their properties. In particular, for a generic $\vec{z} \in (\mathbb{R})^n$, the action of $\mathcal{G}(\vec{z})$ on an irreducible finite-dimensional $\mathfrak{g}$ -module $V(\lambda_1) \otimes \dots \otimes V(\lambda_n)$ is diagonalisable and has a simple spectrum on the subspace of highest weight vectors with respect to the diagonal $\mathfrak{h} \simeq \Delta_{\mathfrak{h}} \subset \mathfrak{h}^{\oplus n}$ [R18].

The cyclic permutation ϑ is an automorphism of $\mathfrak{g}$ of order n . Let now ζ be a primitive n -th root of unity. Then the ϑ -eigenspace $\mathfrak{g}_j$ corresponding to ζ^j is

$$\mathfrak{g}_j = \{x^{(1)} + \zeta^{-j}x^{(2)} + \zeta^{-2j}x^{(3)} + \dots + \zeta^{j(1-n)}x^{(n)} \mid x \in \mathfrak{h}\} \simeq \mathfrak{h}. \quad (7.1)$$

Let $\bar{t}$ denote the image of t^{1-n} in $\mathbb{K}[t^{-1}]/(t^{-n} - 1)$. Then $\mathfrak{h}[t^{-1}]/(t^{-n} - 1)$ has the canonical $\mathbb{Z}_n$ -grading $\mathfrak{h} \oplus \mathfrak{h}\bar{t} \oplus \mathfrak{h}\bar{t}^2 \oplus \dots \oplus \mathfrak{h}\bar{t}^{n-1}$. Thus, $\mathfrak{h}[t^{-1}]/(t^{-n} - 1) \simeq \mathfrak{g}$ as a $\mathbb{Z}_n$ -graded Lie algebra. Fix an isomorphism $\mathfrak{g}_j \rightarrow \bar{t}^j \mathfrak{h}$ as the projection $\text{pr}_1: \mathfrak{g} \rightarrow \mathfrak{h}$ on the first summand of $\mathfrak{g}$ combined with the multiplication by $\bar{t}^j$. Now we regard $\mathfrak{g}$ as the quotient of $\mathfrak{h}[t^{-1}]$.

The above identifications related to ϑ provide a simpler approach to constructing certain Gaudin subalgebras.

Proposition 7.2. Take $z_k = \zeta^{1-k}$ and consider the corresponding Gaudin subalgebra $\mathcal{G} = \mathcal{G}(\vec{z})$ in $\mathcal{U}(\mathfrak{h})^{\otimes n}$. Then $\mathcal{G}$ coincides with the image of $\mathfrak{z}(\widehat{\mathfrak{h}})$ in the quotient $\mathcal{U}(\widehat{\mathfrak{h}}_-)/(t^{-n}-1) = \mathcal{U}(\mathfrak{g})$.

Proof. Take $a \in \mathbb{Z}_{>0}$ and write $a = nq + j$ with $1 \leq j \leq n$. Then for $xt^{-a} \in \mathfrak{h}[t^{-1}]$, we have

$$\rho_{\vec{z}}(xt^{-a}) = x^{(1)} + \zeta^j x^{(2)} + \zeta^{2j} x^{(3)} + \cdots + \zeta^{j(n-1)} x^{(n)} \in \mathfrak{g}_{n-j}.$$

Therefore $\rho_{\vec{z}}(xt^{-a})$ identifies with the image $x\bar{t}^{n-j}$ of xt^{-a} in $\mathfrak{h}[t^{-1}]/(t^{-n}-1)$. $\square$

By a theorem of Feigin and Frenkel [FF92], $\mathfrak{z}(\widehat{\mathfrak{h}})$ is a polynomial ring in infinitely many variables, with a distinguished set of generators. Set $\tau = -\partial_t$. There are algebraically independent elements $S_1, \dots, S_r \in \mathfrak{z}(\widehat{\mathfrak{h}})$ such that

$$\mathfrak{z}(\widehat{\mathfrak{h}}) = \mathbb{k}[\tau^k(S_i) \mid 1 \leq i \leq r, k \geq 0].$$

The symbols $\text{gr}(S_i)$ are homogeneous elements of $\mathcal{S}(\mathfrak{h}t^{-1})$ and if $F_i = \text{gr}(S_i)|_{t=1}$, then $\mathbb{k}[F_1, \dots, F_r] = \mathcal{S}(\mathfrak{h})^{\mathfrak{h}}$. The set $\{S_i\}$ is called a *complete set of Segal–Sugawara vectors*. Keep the notation $b_i = \deg F_i$.

Lemma 7.3. For $k \geq 0$, let $\mathbf{F}_{i,k}$ be the symbol of $\rho_{\vec{z}}(\tau^k(S_i))$. In the case $z_k = \zeta^{1-k}$ for $k = 1, \dots, n$, we have then

$$\langle \mathbf{F}_{i,k} \mid 0 \leq k \leq (n-1)b_i \rangle_{\mathbb{k}} = \langle (F_i)_{[k]} \mid 0 \leq k \leq (n-1)b_i \rangle_{\mathbb{k}}$$

for the ϕ -polarisations $(F_i)_{[k]}$ related to $\mathfrak{g}^* = \bigoplus_{j=0}^{n-1} \mathfrak{g}_j^*$, cf. (6.5).

Proof. Since $\text{gr}(S_i) \in \mathcal{S}(t^{-1}\mathfrak{h})$, we have $\mathbf{F}_{i,0} \in \mathcal{S}(\mathfrak{g}_{n-1})$ and $\mathbf{F}_{i,0} = n^{b_i}(F_i)_{[0]}$. Assuming $n \geq 2$, we can state that $\mathbf{F}_{i,1} = n^{b_i}(F_i)_{[1]}$. If $k < n$, then clearly $\frac{1}{k!}\mathbf{F}_{i,k} = n^{b_i}(F_i)_{[k]}$. More generally, as long as $k \leq (n-1)b_i$, we have

$$\frac{1}{k!n^{b_i}}\mathbf{F}_{i,k} = (F_i)_{[k]} + \sum_{0 < j \leq k/n} \binom{b_i - j - 1}{j} (F_i)_{[k-jn]},$$

where the leading term of $\frac{1}{k!n^{b_i}}\mathbf{F}_{i,k}$ corresponds to those summands of $\text{gr}(\tau^k(S_i))$ that belong to $\mathcal{S}(\mathfrak{h}t^{-n} \oplus \cdots \oplus \mathfrak{h}t^{-1})$. $\square$

Let $\mathcal{Z} \subset \mathcal{S}(\mathfrak{g})$ be the Poisson-commutative subalgebra associated with ϑ .

Theorem 7.4. If $z_k = \zeta^{1-k}$ for $k = 1, \dots, n$, then the Gaudin algebra $\mathcal{G}(\vec{z}) \subset \mathcal{U}(\mathfrak{g})$ is a quantisation of $\mathcal{Z}$.

Proof. By Theorem 6.1, $\mathcal{Z}$ is generated by the bi-homogeneous components $H_{j,i}$ with $1 \leq j \leq l$. Proposition 6.3 provides a description of these components. Then by Lemma 7.3, we have $\mathcal{Z} \subset \text{gr}(\mathcal{G}(\vec{z}))$. Clearly, $\text{gr}(\mathcal{G}(\vec{z})) \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{g}^0}$ is a Poisson-commutative subalgebra. Recall that $\mathcal{Z}$ is a maximal Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}^0}$ by Theorem 6.1. Thus $\mathcal{Z} = \text{gr}(\mathcal{G}(\vec{z}))$ as required. $\square$

Remark 7.5. If $z_k \neq z_j$ for $k \neq j$, then $\text{tr.deg } \mathcal{G}(\vec{z}) = (n-1)\mathbf{b}(\mathfrak{h}) + \text{rk } \mathfrak{h}$ by [CFR, Prop. 1]. That is, $\text{tr.deg } \mathcal{G}(\vec{z}) = \mathbf{b}(\mathfrak{g}, \vartheta)$ is the maximal possible for commutative subalgebras of $\mathcal{U}(\mathfrak{g})^{\mathfrak{g}^0}$. Theorem 7.4 shows that one particular Gaudin algebra is actually a maximal commutative subalgebra of $\mathcal{U}(\mathfrak{g})^{\mathfrak{g}^0}$. Using a connection with the MF subalgebras established in [R06], we can also prove that any such $\mathcal{G}(\vec{z})$ is a maximal commutative subalgebra of $\mathcal{U}(\mathfrak{g})^{\mathfrak{g}^0}$.

8. Fixed-point subalgebras in the infinite-dimensional case

Let $\mathfrak{h}$ be the same as before. Let now $\vartheta \in \text{Aut}(\mathfrak{h})$ be an automorphism of order m and $\zeta = \sqrt[m]{1}$ be primitive. To any such ϑ , one associates a ϑ -twisted loop algebra $\mathfrak{h}[t, t^{-1}]^\vartheta = (\mathfrak{h}[t, t^{-1}])^\vartheta$, where $\vartheta(t) = \zeta^{-1}t$ and $\vartheta(t^{-1}) = \zeta t^{-1}$. If ϑ is an outer automorphism of $\mathfrak{h}$, then $\mathfrak{h}[t, t^{-1}]^\vartheta$ is a subalgebra of a twisted Kac–Moody algebra, see [Ka83, Chap. 8] for details. Similarly to the case of $\widehat{\mathfrak{h}}_-$, we identify $\widehat{\mathfrak{h}}_-^\vartheta := (\widehat{\mathfrak{h}}_-)^\vartheta$ with $(\mathfrak{h}[t, t^{-1}]/\mathfrak{h}[t])^\vartheta$. This leads to an action of $(\mathfrak{h}[t])^\vartheta$ on $\widehat{\mathfrak{h}}_-^\vartheta$ and correspondingly on $\mathcal{S}(\widehat{\mathfrak{h}}_-^\vartheta)$. In this section, we consider the invariants of $(\mathfrak{h}[t])^\vartheta$ in $\mathcal{S}(\widehat{\mathfrak{h}}_-^\vartheta)$, i.e., the subalgebra

$$\mathcal{Z}(\widehat{\mathfrak{h}}_-, \vartheta) = \mathcal{S}(\widehat{\mathfrak{h}}_-^\vartheta)^{(\mathfrak{h}[t])^\vartheta},$$

which can be thought of as a ϑ -twisted analogue of $\text{gr}(\mathfrak{z}(\widehat{\mathfrak{h}}))$, cf. (0.1).

Set $\mathfrak{g} = \mathfrak{h}^n$. Let $\tilde{\vartheta} \in \text{Aut}(\mathfrak{g})$ be the composition of ϑ applied to one copy of $\mathfrak{h}$ only and a cyclic permutation of the summands. Formally speaking,

$$\tilde{\vartheta}(y_1^{(1)} + y_2^{(2)} + \cdots + y_n^{(n)}) = y_n^{(1)} + \vartheta(y_1)^{(2)} + y_2^{(3)} + \cdots + y_{n-1}^{(n)}$$

for any $y_1, \dots, y_n \in \mathfrak{h}$. The order of $\tilde{\vartheta}$ is $N = nm$. Similarly to the case of $\widehat{\mathfrak{h}}_-$, there are isomorphisms $\mathfrak{g} \simeq \mathfrak{h}[t]^\vartheta/(t^N - 1)$ and $\mathfrak{g}_{(0)} \simeq \mathfrak{h}[t]^\vartheta/(t^N)$.

Set $\mathfrak{q} = \mathfrak{h}_{(0)}$ transferring the notation of Section 3 to $\mathfrak{h}$. Next we want to understand the connection between the Takiff algebra $\mathfrak{q}\langle n \rangle = \mathfrak{q}[t]/(t^n)$, modelled on $\mathfrak{q}$, and $\mathfrak{g}_{(0)}$. Let $\tilde{\zeta} = \sqrt[n]{1}$ be such that $\zeta = \tilde{\zeta}^n$. Set also $\omega = \tilde{\zeta}^m$.

Lemma 8.1 (cf. [P09, Sect. 3]).

- (i) *The Takiff algebra $\mathfrak{q}\langle n \rangle$ is a contraction of $\mathfrak{g}_{(0)}$.*
- (ii) *If $\text{ind } \mathfrak{q} = \text{rk } \mathfrak{h}$, then $\text{ind } \mathfrak{g}_{(0)} = \text{rk } \mathfrak{g}$.*
- (iii) *If there is a g.g.s. for ϑ , then there is a g.g.s. for $\tilde{\vartheta}$.*
- (iv) *If $\mathfrak{q}$ has the codim-2 property, then so does $\mathfrak{g}_{(0)}$.*

Proof. (i) Let $\mathfrak{g} = \bigoplus_{0 \leq j < N} \mathbb{Z} \mathfrak{g}_j$ be the $\mathbb{Z}_N$ -grading of $\mathfrak{g}$ defined by $\tilde{\vartheta}$. Recall that $\mathfrak{g}_{(0)}$ is $\mathbb{N}$ -graded and the Lie bracket in $\mathfrak{g}_{(0)}$ is defined by $[\mathfrak{g}_j, \mathfrak{g}_{j'}]_0 = [\mathfrak{g}_j, \mathfrak{g}_{j'}] \subset \mathfrak{g}_{j+j'}$ if $j+j' < N$ and $[\mathfrak{g}_j, \mathfrak{g}_{j'}]_0 = 0$ otherwise. The Lie algebra $\mathfrak{q}\langle n \rangle$ is also $\mathbb{N}$ -graded and has the same components, however, the Lie bracket is different. Write $j = qm + i$ with $0 \leq i < m$ and similarly $j' = q'm + i'$. Then in $\mathfrak{q}\langle n \rangle$, the bracket between $\mathfrak{g}_j$ and $\mathfrak{g}_{j'}$ is the same as in $\mathfrak{g}_{(0)}$ if $i+i' < m$. If $i+i' \geq m$, then $\mathfrak{g}_j$ and $\mathfrak{g}_{j'}$ commute in $\mathfrak{q}\langle n \rangle$. This shows that $\mathfrak{q}\langle n \rangle$ is a contraction of $\mathfrak{g}_{(0)}$ corresponding to the following linear map:

$$\tilde{\psi}_s: \mathfrak{g}_{(0)} \rightarrow \mathfrak{g}_{(0)} \quad \text{with } s \in \mathbb{k}^\star, \tilde{\psi}_s|_{\mathfrak{g}_j} = s^i \text{id}_{\mathfrak{g}_j}.$$

(ii) By [RT92, Thm. 2.8(i)], one has $\text{ind } \mathfrak{q}\langle n \rangle = n \cdot \text{ind } \mathfrak{q}$. Therefore, if $\text{ind } \mathfrak{q} = \text{rk } \mathfrak{h}$, then $\text{ind } \mathfrak{q}\langle n \rangle = n \cdot \text{ind } \mathfrak{q} = \text{rk } \mathfrak{g}$. Because index cannot decrease under contractions, $\text{rk } \mathfrak{g} = \text{ind } \mathfrak{g} \leq \text{ind } \mathfrak{g}_{(0)} \leq \text{ind } \mathfrak{q}\langle n \rangle = \text{rk } \mathfrak{g}$. This settles (ii).

(iii) Let $F_1, \dots, F_r \in \mathcal{S}(\mathfrak{h})^\mathfrak{h}$ be a g.g.s. for ϑ such that $\vartheta(F_i) = \zeta^{\ell_i} F_i$ with $\ell_i \in \mathbb{Z}$ for each i . Regarding F_i as an element of $\mathcal{S}^{b_i}(\mathfrak{h}^{(1)})$, set

$$\begin{aligned} F_i^{[k]} = \frac{1}{n} & \left(F_i + \omega^k \tilde{\zeta}^{-\ell_i} \tilde{\vartheta}(F_i) + \omega^{2k} \tilde{\zeta}^{-2\ell_i} \tilde{\vartheta}^2(F_i) + \dots \right. \\ & \left. + \omega^{(n-1)k} \tilde{\zeta}^{(1-n)\ell_i} \tilde{\vartheta}^{n-1}(F_i) \right), \end{aligned} \quad (8.1)$$

cf. (6.3). Here $\tilde{\vartheta}^n(F_i) = \vartheta(F_i) = \zeta^{\ell_i} F_i \in \mathcal{S}(\mathfrak{h}^{(1)})$. Furthermore,

$$\tilde{\vartheta}(F_i^{[k]}) = \omega^{(n-1)k} \tilde{\zeta}^{\ell_i} F_i^{[k]}, \quad \text{where} \quad \omega^{(n-1)k} \tilde{\zeta}^{\ell_i} = \tilde{\zeta}^{\ell_i - mk}.$$

We claim that

$$\{H_j \mid 1 \leq j \leq l\} = \{F_i^{[k]} \mid 1 \leq i \leq r, 0 \leq k < n\}$$

is a g.g.s. for $\tilde{\vartheta}$.

Let $\tilde{\varphi}: \mathbb{k}^* \rightarrow \mathrm{GL}(\mathfrak{g})$ be the polynomial representation such that $\tilde{\varphi}_s(x) = s^j x$ for $x \in \mathfrak{g}_j$. Set $B_i^\bullet = b_i^\bullet + b_i(n-1)m$. Here $B_i^\bullet \equiv b_i^\bullet \equiv \ell_i \pmod{m}$. Furthermore, $\deg_{\tilde{\varphi}} F_i^{[k]} \leq B_i^\bullet$, because $\tilde{\vartheta}^v(\mathfrak{h}_j^{(1)}) \subset \bigoplus_{q=0}^{n-1} \mathfrak{g}_{qm+j}$ for $0 \leq j < m$ and $0 \leq v$. However, the equality $\deg_{\tilde{\varphi}} F_i^{[k]} = B_i^\bullet$ is possible only if $B_i^\bullet - \ell_i + mk \in N\mathbb{Z}$. If this is not the case for k , then $\deg_{\tilde{\varphi}} F_i^{[k]} \leq B_i^\bullet - m$. And again, the equality is possible only if $B_i^\bullet - \ell_i + m(k-1) \in N\mathbb{Z}$. Iterating these arguments, we obtain that $\sum_{k=0}^{n-1} \deg_{\tilde{\varphi}} F_i^{[k]} \leq nb_i^\bullet + m(2b_i - 1) \frac{n(n-1)}{2}$ for each $1 \leq i \leq r$. Hence

$$\sum_{1 \leq j \leq l} \deg_{\tilde{\varphi}} H_j \leq n \left(\sum_{i=1}^r b_i^\bullet \right) + m \frac{n(n-1)}{2} \sum_{i=1}^r (2b_i - 1) = nD_{\vartheta} + N \frac{n-1}{2} \dim \mathfrak{h} = D_{\tilde{\vartheta}}.$$

Now the claim follows from Theorem 4.2.

(iv) Suppose that $\mathfrak{q}$ has the *codim-2* property, then so does the Takiff algebra $\mathfrak{q}\langle n \rangle$ [RT92, PY20']. Furthermore, if the index does not change under a polynomial contraction, then the dimension of the singular subset can only increase [Y17, (4.1)]. Applying this to $\mathfrak{g}_{(0)}$ and its contraction $\mathfrak{q}\langle n \rangle$, we conclude that $\mathfrak{g}_{(0)}$ has the *codim-2* property. $\square$

Theorem 8.2.

(i) If $\mathrm{ind} \mathfrak{h}_{(0)} = \mathrm{rk} \mathfrak{h}$, then $\mathcal{Z}(\widehat{\mathfrak{h}}_-, \vartheta)$ is a Poisson-commutative subalgebra.

(ii) The algebra $\mathcal{Z}(\widehat{\mathfrak{h}}_-, \vartheta)$ is a polynomial ring (in infinitely many variables) if and only if $\mathcal{S}(\mathfrak{g}_{(0)})^{\mathfrak{g}_{(0)}}$ is a polynomial ring for each $n \geq 1$ and $\mathfrak{g} = \mathfrak{h}^n$ with $\tilde{\vartheta} \in \mathrm{Aut}(\mathfrak{g})$ as above.

Proof. For $N = nm$, set

$$\mathbb{W}_N := (\mathfrak{h}t^{-N} \oplus \dots \oplus \mathfrak{h}t^{-1})^{\vartheta} \subset \widehat{\mathfrak{h}}^{\vartheta}.$$

Then $\mathfrak{h}[t]^{\vartheta}$ acts on $\mathbb{W}_N$ via its quotient $\mathfrak{g}_{(0)}$ and $\mathbb{W}_N \simeq \mathfrak{g}_{(0)}$ as a $\mathfrak{g}_{(0)}$ -module. Therefore there is an isomorphism of commutative algebras $\mathcal{S}(\mathbb{W}_N)^{\mathfrak{h}[t]^{\vartheta}} \simeq \mathcal{S}(\mathfrak{g}_{(0)})^{\mathfrak{g}_{(0)}}$.

If $f_1, f_2 \in \mathcal{S}(\widehat{\mathfrak{h}}_-^{\vartheta})$, then there is $N' = n'm$ such that $f_1, f_2 \in \mathbb{W}_{N'}$. Set $n = 2n'$. Then $\{f_1, f_2\} = 0$ if and only if the images $\bar{f}_1, \bar{f}_2$ of f_1, f_2 in $\mathcal{S}(\mathfrak{g}) \simeq \mathcal{S}(\widehat{\mathfrak{h}}^{\vartheta})/(t^{-N} - 1)$ Poisson-commute. The $\mathbb{Z}_N$ -grading of $\mathfrak{g}$ inherited from $\widehat{\mathfrak{h}}_-$ coincides with the $\mathbb{Z}_N$ -grading defined by $\tilde{\vartheta}$. If $f_1, f_2 \in \mathcal{Z}(\widehat{\mathfrak{h}}_-, \vartheta)$, then $\bar{f}_1, \bar{f}_2 \in \mathcal{S}(\mathfrak{g}_{(0)})^{\mathfrak{g}_{(0)}}$ by the construction. The assumption $\mathrm{ind} \mathfrak{h}_{(0)} = \mathrm{rk} \mathfrak{h}$ implies that $\mathrm{ind} \mathfrak{g}_{(0)} = \mathrm{rk} \mathfrak{g}$, see Lemma 8.1. Thus $\mathcal{S}(\mathfrak{g}_{(0)})^{\mathfrak{g}_{(0)}}$ is contained in the Poisson-commutative subalgebra $\mathcal{Z}(\mathfrak{g}, \tilde{\vartheta}) \subset \mathcal{S}(\mathfrak{g})$ and hence $\{\bar{f}_1, \bar{f}_2\} = 0$.

(ii) The algebra $\mathcal{Z}(\widehat{\mathfrak{h}}_-, \vartheta) = \varinjlim \mathcal{S}(\mathbb{W}_{nm})^{\mathfrak{h}[t]^\vartheta}$ has a direct limit structure. If $\mathcal{Z}(\widehat{\mathfrak{h}}_-, \vartheta)$ is a polynomial ring, then $\mathcal{S}(\mathbb{W}_N)^{\mathfrak{h}[t]^\vartheta} \simeq \mathcal{S}(\mathfrak{g}_{(0)})^{\mathfrak{a}_{(0)}}$ has to be a polynomial ring for each n .

In order to prove the opposite implication, suppose that $\mathcal{S}(\mathbb{W}_N)^{\mathfrak{h}[t]^\vartheta}$ is a polynomial ring for each n . By a standard argument on graded algebras, see, e.g., the proof of Lemma 4.4, any algebraically independent set of generators of the ring $\mathcal{S}(\mathbb{W}_N)^{\mathfrak{h}[t]^\vartheta}$ extends to an algebraically independent set of generators of the ring $\mathcal{S}(\mathbb{W}_{N+m})^{\mathfrak{h}[t]^\vartheta}$. In the direct limit, one obtains a set of algebraically independent generators of $\mathcal{Z}(\widehat{\mathfrak{h}}_-, \vartheta)$. $\square$

If $\text{ind } \mathfrak{h}_{(0)} = \text{rk } \mathfrak{h}$, there is a g.g.s. for ϑ , and $\mathfrak{h}_{(0)}$ has the *codim-2* property, then $\mathcal{S}(\mathfrak{g}_{(0)})^{\mathfrak{a}_{(0)}}$ is a polynomial ring, see Lemma 8.1 and Theorem 4.2. Set $b_{i,\bullet} = b_i(m-1) - b_i^\bullet$.

Proposition 8.3. *If $\text{ind } \mathfrak{h}_{(0)} = \text{rk } \mathfrak{h}$, there is a g.g.s. $F_1, \dots, F_r \in \mathcal{S}(\mathfrak{h})^{\mathfrak{h}}$ for ϑ , and $\mathfrak{h}_{(0)}$ has the *codim-2* property, then $\mathcal{Z}(\widehat{\mathfrak{h}}_-, \vartheta)$ is freely generated by the (t^{-1}) -polarisations $(F_i)_{[b_{i,\bullet}+km]}$ with $k \geq 0$ related to the subspaces $((\mathfrak{h}t^{-j})^\vartheta)^*$.*

Proof. First we enlarge on t^{-1} -polarisations. For $k \in \mathbb{Z}$, let $\bar{k} \in \{0, \dots, m-1\}$ be the residue of k modulo m . Then $\widehat{\mathfrak{h}}_-^\vartheta = \bigoplus_{k \leq -1} \mathfrak{h}_{\bar{k}} t^k$. Let $\phi_s: (\widehat{\mathfrak{h}}_-^\vartheta)^* \rightarrow (\widehat{\mathfrak{h}}_-^\vartheta)^*$ with $s \in \mathbb{k}^*$ be the linear map multiplying elements of $(\mathfrak{h}_{\bar{k}} t^k)^*$ with s^{-k-1} . Next we canonically identify $(\mathfrak{h}_{\bar{k}} t^k)^*$ with $\mathfrak{h}_{\bar{k}}^*$. Then an element of $(\widehat{\mathfrak{h}}_-^\vartheta)^*$ is a finite sequence $\xi = (\xi_{-1}, \xi_{-2}, \dots, \xi_{-L})$ with $\xi_k \in \mathfrak{h}_{\bar{k}}^*$. For such a ξ , set $|\xi| = \sum_{i=1}^L \xi_{-i} \in \mathfrak{h}^*$. If $F \in \mathcal{S}(\mathfrak{h})$, then $F(|\phi_s(\xi)|) = \sum_{k \geq 0} s^k F_{[k]}(\xi)$, where the sum is actually finite. If $n = 1$, then $\mathbb{W}_m \simeq \mathfrak{h}_0$ and $(F_i)_{[b_{i,\bullet}]}$ identifies with $F_i^\bullet \in \mathcal{S}(\mathfrak{h}_{(0)})^{\mathfrak{h}_{(0)}}$.

Recall that $\mathcal{Z}(\widehat{\mathfrak{h}}_-, \vartheta) = \varinjlim \mathcal{S}(\mathbb{W}_{nm})^{\mathfrak{h}[t]^\vartheta}$ and that $\mathcal{S}(\mathbb{W}_N)^{\mathfrak{h}[t]^\vartheta} \simeq \mathcal{S}(\mathfrak{g}_{(0)})^{\mathfrak{a}_{(0)}}$ for $N = nm$. Now we identify the vector spaces $\mathbb{W}_N$ and $\mathfrak{g}$. For $0 \leq k < n$, let $F_i^{[k]} \in \mathcal{S}(\mathfrak{g})$ be defined by (8.1). Then $\mathcal{S}(\mathfrak{g}_{(0)})^{\mathfrak{a}_{(0)}}$ is freely generated by the highest components $(F_i^{[k]})^\bullet$ with $1 \leq i \leq r$, $0 \leq k < n$. The discussion in the proof of Lemma 8.1(iii) shows that if $B_i^\bullet - \ell_i + mk \in N\mathbb{Z}$, then

$$(F_i^{[k]})^\bullet = \frac{1}{n^{b_i}} (F_i)_{[b_{i,\bullet}]} \in \mathcal{S}(\mathbb{W}_m).$$

Analogously to the case of $\vartheta = \text{id}_{\mathfrak{h}}$, cf. Proposition 6.3, one can see that

$$\{n^{b_i} (F_i^{[k]})^\bullet \mid 0 \leq k < n\} = \{(F_i)_{[b_{i,\bullet}+km]} \mid 0 \leq k < n\}.$$

This completes the proof. $\square$

Remark 8.4. Theorem 8.2 emphasises the importance of the equality $\text{ind } \mathfrak{h}_{(0)} = \text{rk } \mathfrak{h}$. Let us recall that it holds for the involutions by [P07]. Furthermore, if $\mathfrak{h}$ is a classical Lie algebra and ϑ is an involution, then there is a g.g.s. for ϑ and $\mathfrak{h}_{(0)}$ has the *codim-2* property [P07], [Y14], i.e., Proposition 8.3 applies. This means that the ϑ -twisted Poisson analogue of the Feigin–Frenkel centre exists for many automorphisms and indicates that probably a ϑ -twisted version of $\mathfrak{z}(\widehat{\mathfrak{h}})$ can be constructed at least for some ϑ . Often both ϑ -twisted objects are expected to be polynomial rings in infinite number of variables. Finally, we mention that results of [PY20] can be used for the description of $\mathcal{S}(\mathfrak{g}_{(0)})^{\mathfrak{a}_{(0)}}$.

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