TAME DISCRETE SETS IN ALGEBRAIC GROUPS

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Abstract. Rosay and Rudin introduced the notion of tame discrete subsets of the affine complex space and investigated their properties. We generalize this theory to the case of a complex linear algebraic group with trivial character group.

1. Introduction

For discrete subsets in $\mathbb{C}^n$ the notion of being “tame” was defined in the important paper of Rosay and Rudin [10]. A discrete subset $D \subset \mathbb{C}^n$ is called “tame” if and only if there exists an automorphism ϕ of $\mathbb{C}^n$ such that $\phi(D) = \mathbb{N} \times \{0\}^{n-1}$. (In this paper a subset D of a topological space X is called a “discrete subset” if every point p in X admits an open neighbourhood W such that $W \cap D$ is finite.)

In [14] we introduced a new definition of tameness which applies to arbitrary complex manifolds and agrees with the tameness notion of Rosay and Rudin for the case $X \simeq \mathbb{C}^n$.

Definition 1. Let X be a complex manifold. An infinite discrete subset D is called (weakly) *tame* if for every exhaustion function $\rho: X \rightarrow \mathbb{R}^+$ and every map $\zeta: D \rightarrow \mathbb{R}^+$ there exists an automorphism ϕ of X such that $\rho(\phi(x)) \geq \zeta(x)$ for all $x \in D$.

In terms of Nevanlinna theory a similar tameness notion may be formulated as follows: A discrete set is tame if its counting function $N(r, D)$ may be made as small as desired. (See §3 for details.)

First results about this notion obtained in [14] suggested that best results are to be expected in the case of complex manifolds whose automorphism group is very large in a certain sense. In particular, in [14] we proved some significant results for the case $X = \mathrm{SL}_n(\mathbb{C})$ and found some evidence suggesting that our tameness notion might not work very well for non-Stein manifolds like $\mathbb{C}^n \setminus \{(0, \dots, 0)\}$ or partially hyperbolic complex manifolds like $\Delta \times \mathbb{C}$.

DOI: 10.1007/s00031-020-09608-x

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Received March 25, 2019. Accepted January 2020.

Published online August 28, 2020.

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In this paper we show that for complex manifolds biholomorphic to complex linear algebraic groups without non-trivial morphisms to the multiplicative group $\mathbb{C}^*$ we obtain a theory of tame discrete sets essentially as strong as the theory which Rosay and Rudin developed for $\mathbb{C}^n$.

Andrist and Ugolini ([1]) have proposed a different notion, namely the following:

Definition 2. Let X be a complex manifold. An infinite discrete subset D is called (strongly) *tame* if for every injective map $f: D \rightarrow D$ there exists an automorphism ϕ of X such that $\phi(x) = f(x)$ for all $x \in D$.

They proved that every complex linear algebraic groups admits a *strongly tame* discrete subset ([1]).

It is easily verified that “strongly tame” implies “weakly tame” (see [14]). For $X \simeq \mathbb{C}^n$ and $X \simeq \mathrm{SL}_n(\mathbb{C})$ both tameness notions coincide ([14]). Furthermore, for $X = \mathbb{C}^n$ both notions agree with tameness as defined by Rosay and Rudin.

However, for arbitrary manifolds “strongly tame” and “weakly tame” are not equivalent ([14]).

In this article, unless explicitly stated otherwise, tame always means weakly tame, i.e., tame in the sense of Definition 1.

2. Main results

2.1. Equivalence of tameness notions

This article is focused on the study of tame discrete subsets in complex linear-algebraic groups whose character group is trivial, i.e., which do not admit any non-constant morphism of algebraic groups to the multiplicative group $\mathbb{C}^*$. For simplicity, we call these algebraic groups “character-free”.

First we show that for this class of complex manifolds many different possible notions of tameness agree. This indicates that character-free complex linear algebraic groups form a good class of complex manifolds for the purpose of investigating discrete subsets.

Theorem 1. Let G be a character-free complex linear algebraic group and $\dim(G) \geq 2$. Let D be a discrete subset of G . Then the following statements are equivalent:

- (i) D is (weakly) tame (in the sense of Definition 1), i.e., for every exhaustion function $\rho: G \rightarrow \mathbb{R}^+$ and every map $\zeta: D \rightarrow \mathbb{R}^+$ there exists a biholomorphic self-map ϕ of the complex manifold G such that $\rho(\phi(x)) \geq \zeta(x)$ for all $x \in D$.
- (ii) For every exhaustion function $\rho: G \rightarrow \mathbb{R}^+$ and every increasing positive sequence r_k with $\lim r_k = \infty$ there exists a biholomorphic self-map ϕ of G such that

$$\#\{x \in \phi(D) : \rho(x) \leq r_k\} \leq k \quad \forall k \in \mathbb{N}.$$

- (iii) D is strongly tame (in the sense of Definition 2), i.e., for every injective map $f: D \rightarrow D$ there exists a biholomorphic self-map ϕ of G such that $\phi(x) = f(x)$ for all $x \in D$.
- (iv) There is a biholomorphic self-map ϕ of G and an algebraic subgroup U of G such that $U \simeq (\mathbb{C}, +)$ (as a complex algebraic group) and $\phi(D) \subset U$.

- (v) *There exists a biholomorphic self-map ϕ of G and an algebraic subgroup H of G with $\dim(H) > 0$ such that the quotient $Z = G/H$ is affine and such that the natural projection map $\pi: G \rightarrow G/H = Z$ restricts to a proper map from $\phi(D)$ onto a discrete subset of Z .*
- (vi) *There exists a biholomorphic self-map ϕ of G and an algebraic subgroup H of G with $\dim(H) > 0$ such that the quotient $Z = G/H$ is affine and such that the natural projection map $\pi: G \rightarrow G/H = Z$ restricts to an injective map from $\phi(D)$ onto a discrete subset of Z .*
- (vii) *There exists a biholomorphic self-map ϕ of G , a unipotent subgroup U with $U \simeq (\mathbb{C}, +)$ and an open embedding $i: G/U \hookrightarrow M$ into an affine variety M such that $i(\pi(\phi(D)))$ is discrete in M (where π denotes the natural projection $\pi: G \rightarrow G/U$).*
- (viii) *For every monotone increasing continuous function $h: [1, \infty[\rightarrow \mathbb{R}^+$ with $\lim_{r \rightarrow \infty} h(r)/\log r = \infty$ there exists a biholomorphic self map ϕ of G such that $N(r, \phi(D)) < h(r) \ \forall r \geq 1$ where $N(r, D)$ is the counting function as defined in §3.*
- (ix) *For every infinite discrete subset D' of G there exists a biholomorphic self-map ϕ of G with $\phi(D) \subset D'$.*

Proof. (i) $\Rightarrow$ (ii) If $D = \{a_k : k \in \mathbb{N}\}$, we may choose ϕ such that $\rho(\phi(a_k)) > r_k$ for every k .

(i) $\Rightarrow$ (iii) follows from Proposition 10.

(ii) $\Rightarrow$ (i) Due to Proposition 8 the manifold G admits a *threshold sequence* R_k . This notion (defined in Definition 3) applied to the case $r_k = R_k$ in condition (ii) implies tameness.

(iii) $\Rightarrow$ (i) Obvious.

(i) $\Rightarrow$ (iv) Due to Corollary 7 there exists an algebraic subgroup $U \simeq \mathbb{C}$. By Corollary 9 every unbounded set (hence in particular U) contains a tame discrete subset D' and due to Proposition 10 there is a holomorphic self-map ϕ of G with $\phi(D) = D' \subset U$.

(iv) $\Rightarrow$ (i) For dimension reasons we have $U \neq G$. Hence this implication follows from Proposition 12.

(i) $\Rightarrow$ (vi) Let T be a maximal torus of G . If $\dim(T) > 0$, the implication follows from Theorem 14. If $\dim(T) = 0$, then G is unipotent and therefore G is biholomorphic to the complex affine space $\mathbb{C}^n$. In this case the assertion follows from the classical results of Rosay and Rudin ([10]).

(vi) $\Rightarrow$ (v) Obvious.

(v) $\Rightarrow$ (i) [14, Prop. 8.4].

(i) $\Rightarrow$ (vii) Lemma 7 combined with Fact 2(ii) and Proposition 9.

(vii) $\Rightarrow$ (i) Corollary 10.

(i) $\Leftrightarrow$ (viii) Proposition 1.

(i) $\Rightarrow$ (ix) As an infinite discrete set, D' is unbounded and therefore contains a tame discrete set $D'' \subset D'$ (Corollary 9). Now any two tame discrete subsets are equivalent (Corollary 1), therefore there is a biholomorphic self-map ϕ with $\phi(D) = D'' \subset D'$.

(ix) $\Rightarrow$ (ii) We fix an exhaustion function ρ on G . Given a sequence r_k we choose a discrete subset D' such that

$$\#\{x \in D' : \rho(x) \leq r_k\} \leq k \quad \forall k \in \mathbb{N}.$$

Now (ix) implies the existence of a biholomorphic self map ϕ such that $\phi(D) \subset D'$. By construction of D' this implies (ii). $\square$

Remarks.

— The implication (i) $\Rightarrow$ (iii) provides a positive answer to [1, Question 7.7] for character-free complex linear algebraic groups. (Andrist and Ugolini asked for this implication for the class of Stein manifold with density property — a class including every character-free complex linear algebraic group.)

— A discrete subset of a one-dimensional complex manifold is never tame ([14]); thus we need to require G to have at least dimension two.

— Condition (iii) implies in particular that every permutation of a tame discrete subset extends to a biholomorphic self-map.

The latter assertion may be strengthened to the following result.

Theorem 2 (=Proposition 11). *Let G be a character-free complex linear algebraic group. Let D and D' be tame discrete subsets. Then every injective map $\zeta : D \rightarrow D'$ may be extended to a biholomorphic self-map of G .*

Thus for example it follows from our results that every injective map from $\mathrm{SL}_n(\mathbb{Z})$ to $\mathrm{SL}_n(\mathbb{Z}[i])$ extends to a biholomorphic self-map of $\mathrm{SL}_n(\mathbb{C})$.

Corollary 1. *Under the assumptions of the theorem any two tame discrete subsets D, D' are “equivalent”, i.e., there exists a biholomorphic self-map of G with $\phi(D) = D'$.*

Corollary 2. *Let G be a character-free complex linear algebraic group. Let D, D' be infinite discrete subsets of G such that the “symmetric difference” $(D \setminus D') \cup (D' \setminus D)$ is finite. Then D is tame if and only if D' is tame.*

Proof. This follows immediately from characterization (v) of the theorem: Since $(D \setminus D') \cup (D' \setminus D)$ is finite, the restricted projection map $\pi|_D : D \rightarrow Z$ is proper if and only if $\pi|_{D'} : D' \rightarrow Z$ is proper. $\square$

Corollary 3. *Let G be a character-free complex linear algebraic group. Let D be a tame discrete subset. Let A denote the group of all biholomorphic self-maps of G fixing D point-wise. Then A acts transitively on $G \setminus D$.*

Proof. Let $p, q \in G \setminus D$. Due to Corollary 2 both $D \cup \{p\}$ and $D \cup \{q\}$ are tame. Define $\zeta : D \cup \{p\} \rightarrow D \cup \{q\}$ as $\zeta(x) = x$ for $x \in D$ and $\zeta(p) = q$. Now Proposition 11 guarantees us that there exists a biholomorphic self-map ϕ of G with $\phi|_D = \mathrm{id}_D$ (i.e., $\phi \in A$) and $\phi(p) = q$. $\square$

In the same way one deduces the following generalization:

Corollary 4. *Let G be a character-free complex linear algebraic group and let D be a tame discrete subset. Then the automorphism group of the complex manifold $G \setminus D$ acts m -transitively on $G \setminus D$ for every $m \in \mathbb{N}$, i.e., if S and S' are two subsets of $G \setminus D$, then there is a biholomorphic self-map of $G \setminus D$ mapping S to S' .¹*

It should be noted that from the definition 1 it is not obvious that there exist any tame discrete subsets and indeed there are complex manifolds without any tame discrete subset ([14]). However, every character-free complex linear algebraic group of dimension at least two does contain tame discrete subsets. In fact, each of the characterizations (iv)–(vii) of Theorem 1 provides an easy way to construct tame discrete subsets. In this way we obtain a confirmation of the following known fact: *Every complex linear algebraic group of dimension at least two admits both tame and non-tame discrete subsets.* (The existence of tame discrete subsets has been proved by Andrist and Ugolini ([1]) for an arbitrary complex linear algebraic group G , the existence of a non-tame discrete subset by the author for arbitrary Stein manifolds ([11]).

History. In [14] the equivalence (i) $\Leftrightarrow$ (ii) was proved for the special cases $G \simeq \mathrm{SL}_n$ and $G \simeq (\mathbb{C}^n, +)$ and (i) $\Leftrightarrow$ (v) for $G \simeq \mathrm{SL}_2$.

2.2. Threshold Sequences

The technical notion of a “Threshold sequence” (which was first considered in [14]) plays an important role:

Definition 3. Let G be a complex manifold with exhaustion function ρ . A “threshold sequence” for G with respect to ρ is an increasing sequence R_k of positive real numbers such that the following property holds: *If $D \subset G$ is a discrete subset with*

$$\#\{z \in D : \rho(z) \leq R_k\} \leq k \quad \forall k \in \mathbb{N}$$

then D is tame.

Theorem 3 (=Proposition 8). *Let G be a character-free complex linear algebraic group. Then G admits a threshold sequence (for any exhaustion function ρ).*

2.3. Oka manifolds

A complex manifold X is called Oka iff for every $N \in \mathbb{N}$ and every convex compact subset $C \subset \mathbb{C}^N$ every holomorphic map with values in X defined on a neighbourhood of C may be approximated by holomorphic maps defined on all of $\mathbb{C}^N$.

(See [4] for more information about Oka manifolds.)

Complex Lie groups are known to be Oka manifolds. Now let G be a complex linear algebraic group with $\mathrm{Hom}(G, \mathbb{C}^*) = \{1\}$. We show that certain manifolds obtained from G by certain modifications along tame discrete subsets are still Oka:

Theorem 4. *Let G be a character-free complex linear algebraic group. Let D be a tame discrete subset. Let $X = G \setminus D$ and let Y denote the complex manifold obtained from G by blowing up D . Then both X and Y are Oka manifolds.*

¹This property is frequently called “infinite transitivity” and has been studied intensely in the context of “flexible varieties” (see, e.g., [2]).

Proof. Theorem 20 and Proposition 16. $\square$

Remark. For the class of character-free complex linear algebraic groups this answers a question raised by Andrist and Ugolini ([1, Question 7.8]) for Stein manifolds with density property.

In contrast, every Stein manifold X (in particular, every complex linear algebraic group) admits a discrete subset D such that $X \setminus D$ is not Oka. In ([11]) it is proved: Every Stein manifold X admits a discrete subset D such that every holomorphic map from $\mathbb{C}^n$ (with $n = \dim(X)$) to $X \setminus D$ is degenerate, a property which implies that $X \setminus D$ cannot be an Oka manifold.

2.4. Discrete subgroups

We conjecture that every discrete subgroup of a complex linear algebraic group is a tame discrete set. In this direction we have some partial results.

Theorem 5. *Let G be a complex linear algebraic group and let Γ be a discrete subgroup which contains a unipotent element other than the neutral element.*

Assume that furthermore at least one of the following conditions is fulfilled:

- (i) $G \simeq \mathrm{SL}_2(\mathbb{C})$ or $G \simeq \mathrm{PSL}_2(\mathbb{C})$.
- (ii) G is defined over $\mathbb{Q}$ or a totally imaginary quadratic number field K and Γ is contained in $G(\mathcal{O})$ where $\mathcal{O}$ denotes the ring of algebraic integers of $\mathbb{Q}$ resp. K .

Then Γ is a tame discrete subset of the complex manifold G .

Proof. Case (i) follows from Proposition 19 and case (ii) from Theorem 23. $\square$

The theorem implies in particular that $\mathrm{SL}_n(\mathbb{Z})$ and $\mathrm{SL}_n(\mathbb{Z}[i])$ are tame discrete subsets of $\mathrm{SL}_n(\mathbb{C})$.

In the context of this theorem it is an interesting fact that, if Γ is a discrete subgroup in a complex linear algebraic group G such that G/Γ has finite volume with respect to the Haar measure of G , but is not compact, then $\Gamma \setminus \{e\}$ contains a unipotent element. (Theorem of Kazhdan–Margulis, [5].)

Thus, if $G = \mathrm{SL}_2(\mathbb{C})$ and Γ is a discrete subgroup, such that G/Γ has finite volume, but is not compact, then Γ is a tame discrete subset.

If Γ is the fundamental group of a hyperbolic punctured Riemann surface, then Γ embedded into $\mathrm{PSL}_2(\mathbb{R})$ by its action of deck transformations on the upper half plane becomes a tame discrete subset of $\mathrm{PSL}_2(\mathbb{C})$.

Other examples are provided by fundamental groups of hyperbolic knot complements. Except for certain special cases, the complement $M = S \setminus C$ of a knot (or link) C in the 3-sphere S admits a complete Riemannian metric of negative constant sectional curvature, i.e., M is a “real hyperbolic manifold”. This implies that the universal covering $\tilde{M}$ of M is isomorphic to $H = G/K$ with $G = \mathrm{SO}(3, \mathbb{C})$ and $K = \mathrm{SO}(3, \mathbb{R})$. Moreover, G is the group of isometries of the “hyperbolic 3-space” G/K . Let Γ be the fundamental group of M . Then Γ acts by “deck transformations” on its universal covering $\tilde{M} = G/K$. Thus Γ admits a natural

embedding into $G = \mathrm{SO}(3, \mathbb{C})$ which is the group of isometries of $\widetilde{M}$. In this way $\Gamma = \pi_1(M)$ is naturally embedded into the complex linear algebraic group $G = \mathrm{SO}(3, \mathbb{C})$ as a discrete subgroup. By our results it follows that this is a *tame* discrete subset. (see §13.2.)

2.5. Union of tame sets

Theorem 6 (=Theorem 17). *Let D be a discrete subset of a linear algebraic group G with $\dim(G) \geq 2$. Then D can be realized as the union of two tame subsets of the complex manifold G .*

3. Relation to Nevanlinna theory

Here we want to explain the connection between tameness and the usual notions of Nevanlinna theory. (See, e.g., [9] for more information about Nevanlinna theory.)

Given a divisor $D = \sum_k n_k \{p_k\}$ in $\mathbb{C}$, in Nevanlinna theory one introduces the “counting function” $N(r, D)$ which may be defined as

$$N(r, D) = \int_1^r n(t, D) \frac{dt}{t}$$

with

$$n(t, D) = \sum_{k: |p_k| \leq t} n_k.$$

In this spirit we define for a discrete subset D in a complex manifold X with exhaustion function ρ

$$N(r, D) = \int_1^r n(t, D) \frac{dt}{t}$$

with

$$n(t, D) = \#\{x \in D : \rho(x) \leq t\}.$$

We recall the notation $\log^+(x) \stackrel{\text{def}}{=} \max\{\log x, 0\}$.

Remark 1. We may replace a given exhaustion function ρ by $\tilde{\rho}(p) = \max\{1, \rho(p)\}$ and therefore without loss of generality assume that $\rho \geq 1$. Under this assumption we have

$$N(r, D) = \sum_{p \in D} \log^+ \frac{r}{\rho(p)}.$$

The goal of this section is to establish the following result.

Proposition 1. *Let X be a complex manifold. A discrete subset D in X is tame if and only if for every real continuous increasing function $h : [1, \infty[\rightarrow \mathbb{R}^+$ with $\lim_{r \rightarrow \infty} h(r)/\log r = +\infty$ there exists a biholomorphic self-map ϕ of X such that $N(r, \phi(D)) \leq h(r) \ \forall r \geq 1$.*

Proof. This is an immediate consequence of Corollary 2 and Proposition 6 below.

□

Lemma 7. *Let $h: [1, \infty) \rightarrow \mathbb{R}^+$ be a continuous increasing function having the property $\lim_{r \rightarrow \infty} h(r)/\log(r) = +\infty$. Then there exists a positive real number $R > 1$ such that*

$$\log^+ \frac{r}{R} < h(r) \quad \forall r \geq 1.$$

Proof. Since $\lim_{r \rightarrow \infty} h(r)/\log(r) = +\infty$, we have

$$\lim_{r \rightarrow \infty} (\log r - h(r)) = -\infty$$

and therefore there exists an upper bound B for $\log r - h(r)$ on $[1, +\infty)$. Now it suffices to choose R such that $\log R > B$. $\square$

Corollary 5. *Let $h: [1, \infty) \rightarrow \mathbb{R}^+$ be a continuous increasing function having the property $\lim_{r \rightarrow \infty} h(r)/\log(r) = +\infty$. Then there exists a sequence of positive real numbers R_k with $\lim_{k \rightarrow \infty} R_k = +\infty$ such that*

$$\sum_k \log^+ \frac{r}{R_k} < h(r) \quad \forall r \geq 1.$$

Proof. For each $k \in \mathbb{N}$ we apply the lemma to $h_k(r) = 2^{-k}h(r)$ in order to obtain a real number R_k such that

$$\log^+ \frac{r}{R_k} < h_k(r) = 2^{-k}h(r) \quad \forall r \geq 1.$$

Since this inequality is only strengthened by increasing R_k there is no problem in choosing the constants R_k such that $\lim_{k \rightarrow \infty} R_k = +\infty$.

Summing up over k yields the desired inequality. (We observe that due to $\lim_{k \rightarrow \infty} R_k = \infty$ the sum $\sum_k \log^+(r/R_k)$ is actually locally finite. Hence there are no convergence problems.) $\square$

Corollary 6. *Let $h: [1, \infty[\rightarrow \mathbb{R}^+$ be an increasing continuous function having the property $\lim_{r \rightarrow \infty} h(r) = +\infty$. Then there exists an increasing sequence of positive real numbers $R_k \geq 1$ with $\lim_{k \rightarrow \infty} R_k = \infty$ such that for every discrete subset $D \subset X$ the property*

$$\#\{x \in D : \rho(x) \leq R_k\} \leq k \quad \forall k \tag{1}$$

implies $N(r, D) \leq h(r) \quad \forall r \geq 1$.

Proof. Due to Corollary 5 there exists such a sequence R_k with

$$\sum_k \log^+ \frac{r}{R_k} < h(r) \quad \forall r \geq 1.$$

On the other hand, 1 implies

$$N(r, D) \leq \sum_k \log^+ \frac{r}{R_k}.$$

Hence the statement. $\square$

Proposition 2. *Let R_k be an increasing sequence of positive real numbers $R_k \geq 1$ with $\lim R_k = \infty$. Then there exists an increasing continuous function $h : [1, \infty[\rightarrow \mathbb{R}^+$ with $\lim_{r \rightarrow \infty} h(r)/\log r = \infty$ such that the following implication holds for every discrete set D : If $N(r, D) \leq h(r)$ for all $r \geq 1$, then*

$$\#\{x \in D : \rho(x) \leq R_k\} \leq k \quad \forall k.$$

Proof. Let R_k be a given real positive sequence with $\lim R_k = +\infty$. We choose $c_1 > 1$ such

$$\frac{2c_1}{c_1 + \log R_1} > \frac{1}{2}.$$

(This is possible, since $\lim_{c \rightarrow \infty} 2c/(c + \log R_1) = 2$.) Then we recursively choose c_k for every $k \in \mathbb{N} \setminus \{1\}$ such that

$$1 < c_k, \quad c_{k-1} < c_k, \quad e^{c_{k-1}} R_{k-1} < e^{c_k} R_k, \quad \frac{c_k(k+1)}{c_k + \log R_k} > \frac{k}{2}. \quad (2)$$

Next we observe that the sequence $e^{c_k} R_k$ is strictly increasing and define a continuous function $h : [1, \infty[\rightarrow \mathbb{R}^+$ by stipulating the following conditions:

- (1) $h(e^{c_k} R_k) = c_k(k+1)$ and
- (2) For every $k \in \mathbb{N}$ there are constants a_k, b_k such that $h(t) = a_k + b_k \log t$ on the interval $[e^{c_k} R_k, e^{c_{k+1}} R_{k+1}]$.

Define $p_k = e^{c_k} R_k$. By construction we have $h(p_k) = c_k(k+1)$, implying $h(p_k) < h(p_{k+1}) \quad \forall k$. It follows that $h(t) = a_k + b_k \log t$ with $b_k > 0$ on $[p_k, p_{k+1}]$. Thus h is monotonously increasing. We also observe that

$$\frac{h(t)}{\log t} = \frac{a_k + b_k \log t}{\log t} = \frac{a_k}{\log t} + b_k$$

is monotone on each interval $[p_k, p_{k+1}]$. Since

$$\frac{h(p_k)}{\log p_k} = \frac{c_k(k+1)}{\log p_k} = \frac{c_k(k+1)}{c_k + \log R_k} > \frac{k}{2},$$

it follows that $\lim_{t \rightarrow \infty} h(t)/\log t = +\infty$.

Now let us assume that there exists a number k with

$$\#\{x \in D : \rho(x) \leq R_k\} > k \quad (3)$$

This implies that $n(r, D) \geq k+1$ for all $r \geq R_k$. Hence

$$\begin{aligned} N(p_k, D) &= N(e^{c_k} R_k, D) \geq \int_{R_k}^{e^{c_k} R_k} n(t, D) \frac{dt}{t} \\ &\geq \int_{R_k}^{e^{c_k} R_k} (k+1) \frac{dt}{t} = c_k(k+1) = h(p_k). \end{aligned}$$

Thus the existence of a natural number k with 3 implies that there exists a number $r \geq 1$ with $N(r, D) \geq h(r)$. $\square$

4. Tools

Here we summarize some mostly well-known material which we will need.

Fact 1. Let G be an algebraic group with algebraic subgroups H and I .

Then the following conditions are equivalent:

- (i) The H -action (by left multiplication) on G/I is trivial.
- (ii) $H \subset gIg^{-1}$ for every $g \in G$.
- (iii) $gHg^{-1} \subset I$ for every $g \in G$.

Fact 2. Let G be a linear algebraic group and let H be an algebraic subgroup.

- (i) If H is normal or reductive, then G/H is an affine variety. ([8]).
- (ii) If every homomorphism of algebraic groups from H to $\mathbb{C}^*$ is trivial (e.g., if H is unipotent), then G/H is a quasi-affine variety.

Proof of (ii). The quotient G/H may be embedded equivariantly into some projective space $\mathbb{P}_n(\mathbb{C})$. The G -action on $\mathbb{P}_n(\mathbb{C})$ is linear and therefore lifts to a G -action on $V = \mathbb{C}^{n+1} \setminus \{(0, \dots, 0)\}$. Let $Z = G/I$ be a G -orbit in V which is mapped to G/H by the canonical projection from V to $\mathbb{P}_n(\mathbb{C})$. We obtain a fibration $G/I \rightarrow G/H$. Since the natural projection $V \rightarrow \mathbb{P}_n(\mathbb{C})$ is a $\mathbb{C}^*$ -principal bundle, $I = \ker \alpha$ for some homomorphism of algebraic groups $\alpha: H \rightarrow \mathbb{C}^*$. By assumption on H the homomorphism α must be trivial. Thus $I = H$, i.e., we have found an embedding of G/H in $\mathbb{C}^{n+1} \setminus \{(0, \dots, 0)\}$. Hence G/H is quasi-affine. $\square$

Fact 3. Let D be a discrete subset of a Stein manifold X , let G be a connected complex Lie group and let $\alpha: D \rightarrow G$ be a map.

Then there exists a holomorphic map $F: X \rightarrow G$ with $F(x) = \alpha(x)$ for all $x \in D$.

Proof. For a Stein manifold, the values of a holomorphic function on a discrete subset may be prescribed arbitrarily. Hence the assertion for the case $G = \mathbb{C}^n$. For an arbitrary connected complex Lie group G , there is always a surjective holomorphic map $f: \mathbb{C}^N \rightarrow G$ for some $N \in \mathbb{N}$ ([13, Prop. 2.1]). Thus we may choose a map $\beta: D \rightarrow \mathbb{C}^N$ such that $f \circ \beta = \alpha$, then choose a holomorphic map $\Phi: X \rightarrow \mathbb{C}^N$ with $\Phi|_D = \beta$ and finally define $F = f \circ \Phi$. $\square$

Lemma 8 (nil cone). *Let G be a character-free complex linear algebraic group. Let N denote the “nil cone” defined as the set of all elements v of the Lie algebra $\text{Lie}(G)$ for which $\text{ad}(v)$ is nilpotent, i.e.,*

$$N = \{v \in \text{Lie}(G) : \exists n : \text{ad}(v)^n = 0\}.$$

Then N is an irreducible analytic subset of $\text{Lie}(G)$ which generates $\text{Lie}(G)$ as a complex vector space.

Proof. We required G to be character-free. Hence an element $v \in \text{Lie}(G)$ is nilpotent if and only if the corresponding one-parameter group

$$\{\exp(vt) : t \in \mathbb{C}\}$$

is a unipotent subgroup of G . Let B be a Borel subgroup of G and let U denote the unipotent radical of B . Every unipotent subgroup of G is conjugate to a subgroup of U . Therefore N is the set of all elements of $\text{Lie}(G)$ which can be written as $c = \text{Ad}(g)(v_0)$ with $g \in G$, $v_0 \in \text{Lie}(U)$. This proves irreducibility of N .

The nil cone N is evidently invariant under the adjoint action. It follows that the vector space V spanned by N is likewise invariant under the adjoint action and therefore an ideal of the Lie algebra $\text{Lie}(G)$. However, structure theory of linear algebraic groups implies that the only normal Lie subgroup of G containing U is G itself. Hence $V = \text{Lie}(G)$. $\square$

Corollary 7. *Let G be a character-free complex linear algebraic group. Then G contains an algebraic subgroup $U \simeq (\mathbb{C}, +)$.*

Proof. Define $U = \{\exp(vt) : t \in \mathbb{C}\}$ for some element $v \in N \setminus \{0\}$. $\square$

Lemma 9. *Let G be a complex linear-algebraic group, and let I be an algebraic subgroup such that G/I is quasi-affine. Let $i: G/I \rightarrow Y$ be an open dense embedding into an affine variety Y . Let H be an algebraic subgroup of G and let $M_0 = Hp$ be a positive-dimensional orbit in G/I . Then $M = i(M_0)$ cannot be relatively compact in Y .*

Proof. Assume the contrary, i.e., assume that M is relatively compact in Y . Since Y is affine, points on Y may be separated by regular functions. Hence there must be a regular function f which is not constant on M . Since M is assumed to be relatively compact in Y , every continuous function on Y is bounded on M .

Now consider the exponential map $\exp: \text{Lie}(H) \rightarrow H$. Via $F: v \mapsto f(\exp(v) \cdot p)$ we obtain a holomorphic function on the complex vector space $\text{Lie}(H)$. Note that F is bounded, because f is bounded. Hence F must be constant due to the theorem of Liouville. Since $\exp$ is locally biholomorphic near the origin in $\text{Lie}(H)$, it follows that f must be constant on M which yields a contradiction. Thus M cannot be relatively compact in Y . $\square$

Lemma 10. *Let G be a character-free complex linear algebraic group, T a maximal torus, and U a T -invariant maximal unipotent subgroup (i.e., U is maximal unipotent and $gUg^{-1} = U$ for all $g \in T$). Then there exists an dense open embedding of G/U into an affine variety V , and a point $p \in G/U$ such that*

- (1) *The isotropy group $T_p = \{x \in T : x(p) = p\}$ is trivial and*
- (2) *the T -orbit $T(p)$ is closed in V .*

Proof. By taking the quotient by the unipotent radical we may assume that G is semisimple. We choose a faithful irreducible representation ρ of G . Thus we have an injective morphism of algebraic groups $\rho: G \rightarrow \text{SL}(n, \mathbb{C})$ for some n . We may assume that $\rho(T)$ is contained in the diagonal matrices, because $\rho(T)$ is contained in a maximal torus in $\text{SL}(n, \mathbb{C})$, hence up to conjugation in the set D of diagonal matrices of determinant 1. We may also assume that $\rho(U)$ is contained in the set U^+ of upper triangular matrices. We consider $\mathbb{C}^n \setminus \{(0, \dots, 0)\} = \text{SL}_n(\mathbb{C})/H$, with

$$H = \left\{ \left(\begin{array}{c|c} 1 & v \\ \hline 0 & A \end{array} \right) : A \in \text{SL}_{n-1}(\mathbb{C}), v \in \mathbb{C}^{n-1} \right\}.$$

We note that $U^+ \subset H$. Define

$$\Omega = \{z \in \mathbb{C}^n : z_i \neq 0 \ \forall i\}.$$

We consider

$$G/U \longrightarrow \mathrm{SL}_n(\mathbb{C})/H \simeq \mathbb{C}^n \setminus \{(0, \dots, 0)\}$$

with the base point eU being mapped to $e = (1, 0, \dots, 0)$. Since the representation ρ was assumed to be irreducible, the G -orbit through e intersects Ω . Hence there is an element $g_0 \in G$ such that g_0U is mapped to a point q in Ω . Observe that the D -orbit through q is closed in $\mathbb{C}^n$, because it equals

$$\{(z_1, \dots, z_n) : \Pi z_i = c\}$$

for some $c \in \mathbb{C}^*$. Recall that ρ embeds T as a Zariski closed subgroup into D . It follows that the T -orbit through q is closed as well. Moreover the T -isotropy group at q is trivial, because the isotropy group of D at q is trivial. Now let $\zeta : G/U \rightarrow W$ be an open embedding into an affine variety W . (Note that G/U is quasi-affine, because U is unipotent.) The coordinates z_i on $\mathbb{C}^n$ induce regular functions ξ_i on G/U via

$$\xi_i : gU \mapsto z_i(\rho(g)(q)).$$

We define

$$F : G/U \rightarrow W \times \mathbb{C}^n$$

as

$$F(gU) = (\zeta(gU), \xi_1(gU), \dots, \xi_n(gU))$$

and take V to be the closure of $F(G/U)$ in $W \times \mathbb{C}^n$. Define $p = g_0U$. Note that the T -orbit through p is closed and with trivial isotropy. Hence the statement. $\square$

Proposition 3. *Let S be a complex semisimple Lie group with exhaustion function $\rho : S \rightarrow \mathbb{R}^+$. Then there exists an increasing sequence R_n of positive real numbers such that the following holds: If D is a discrete subset of S with*

$$\#\{x \in D : \rho(x) \leq R_n\} \leq n \ \forall n \in \mathbb{N}$$

then there exists a maximal torus T and a biholomorphic self map ϕ of S such that the natural projection map $\pi : S \rightarrow S/T$ restricts to an injective proper map from $\phi(D)$ to S/T .

Proof. [14, Prop. 9.7]. $\square$

Proposition 4. *Let G be a complex Lie group, H a closed complex Lie subgroup, D a discrete subset of G , and let $\pi : G \rightarrow G/H$ be the projection map. Assume that G/H is Stein, and that the projection map π restricts to a finite map from D to a discrete subset of G/H . Then D is tame.*

Proof. See [14, Prop. 8.4]. $\square$

An easy modification of the proof given in [14] yields:

Proposition 5. *Let G be a linear-algebraic group with an algebraic subgroup H , such that the quotient variety G/H admits an open dense embedding $i: G/H \rightarrow M$ into an affine variety M . Let D be a discrete subset. If the natural projection morphism $\pi: G \rightarrow G/H$ composed with the embedding $i: G/H \rightarrow M$ restricts to a finite map from D to a discrete subset of M , then D is tame.*

Proposition 6. *Let G be a character-free complex linear algebraic group. Let T be a connected reductive commutative subgroup of G . Then every infinite discrete subset of T is a tame discrete subset of G .*

Proof. Since G is character-free, we have $T \neq G$. Let U be a T -invariant maximal unipotent subgroup of G . Due to Lemma 10 we obtain a dense open embedding of G/U into an affine variety V such that there is a free T -orbit in G/U which is closed in V . By applying a suitable translation, we may assume that the image of D in G/U is contained in this T -orbit. Since T acts freely on this orbit, it follows that D is injectively mapped onto a discrete subset of V . Now Proposition 5 implies that D is tame. $\square$

5. $\text{Aut}(X)$ is a Baire space

Lemma 11. *The automorphism group of a complex manifold X , endowed with the compact-open-topology, is a Baire space.*

Proof. Let d_0 be a complete metric on X . Then $d_1 = \min\{1, d_0\}$ is a bounded complete metric. Let Ω_k be a countable family of relatively compact open subsets of X which cover all of X . We define a metric on the automorphism group G of X as follows

$$d(f, g) = \sum_k \frac{1}{2^k} \sup_{x \in \Omega_k} (\max\{d_1((f(x), g(x))), d_1(f^{-1}(x), g^{-1}(x))\}).$$

This turns G into a complete metric space. It is well-known that complete metric spaces are Baire spaces. $\square$

Lemma 12. *$\text{Aut}(X)$ has a countable basis of topology because X does.*

Proof. Let $(W_k)_{k \in \mathbb{N}}$ be a countable basis of the topology of X . Assume that all the W_k are relatively compact in X . For every $k \in \mathbb{N}$ and every finite subset $I \subset \mathbb{N}$ let

$$\Omega_{k,I} = \{f \in \text{Aut}(X) : f(\overline{W_k}) \subset \bigcup_{i \in I} W_i\}.$$

We claim that the sets $\Omega_{k,I}$ form a subbasis of the compact-open topology of $\text{Aut}(X)$.

We have to show: Given an automorphism f of the complex manifold X , a compact subset K of X and an open subset U of X with $f(K) \subset U$ there are finitely many such sets $\Omega_{k_1, I_1}, \dots, \Omega_{k_s, I_s}$ with

$$f \in \bigcap_{m=1}^s \Omega_{k_m, I_m} \subset \{ \phi \in \text{Aut}(X) : \phi(K) \subset U \}.$$

First we note that there is a finite subset $J \subset \mathbb{N}$ with

$$K \subset \bigcup_{j \in J} \overline{W_j} \subset f^{-1}(U)$$

because K is compact and the sets W_k form a basis of the topology of X . Define $L = \bigcup_{j \in J} \overline{W_j}$. Second there is a finite subset $H \subset \mathbb{N}$ with

$$f(L) \subset \bigcup_{h \in H} W_h \subset U.$$

Since $K \subset L$ and $f(L) \subset U$, it follows that $f \in \Omega_{j,H}$ for all $j \in J$. On the other hand, we have

$$\phi(K) \subset \phi(L) \subset \bigcup_{h \in H} W_h \subset U \quad \forall \phi \in \Omega_{j,H} \quad \forall j \in J$$

which implies

$$\Omega_{j,H} \subset \{\phi \in \text{Aut}(X) : \phi(K) \subset U\} \quad \forall j \in J. \quad \square$$

Lemma 13. *Let X be a complex manifold on which its automorphism group $G = \text{Aut}(X)$ acts transitively. Let $p \in X$ and let $Z \subset X$ be a nowhere dense closed analytic subset. Then $B = \{g \in G : g(p) \notin Z\}$ is open and dense in G .*

Proof. Openness of B is obvious.

Let S be a countable dense subset of G . If B is not dense, then $A = G \setminus B$ has a non-empty interior, and there is an open subset $W \subset G$ with $Wp \subset Z$. But then $SW = G$, implying $Gp \subset \bigcup_{s \in S} s \cdot Z$ which contradicts the assumption of transitivity (because S is countable and Z is a nowhere dense analytic subset). $\square$

Corollary 8. *Let X be a complex manifold on which its automorphism group $\text{Aut}(X)$ acts transitively. Let Z be a nowhere dense closed analytic subset of X and let D be a countable subset of X . Then there exists an automorphism g of X with $g(p) \notin Z \quad \forall p \in D$.*

Proof. Due to Lemma 13, the set $U_p = \{g \in \text{Aut}(X) : g(p) \notin Z\}$ is open and dense in $\text{Aut}(X)$ for every $p \in D$. Now $\text{Aut}(X)$ is Baire (Lemma 11). Hence the intersection of countably many dense open subsets of $\text{Aut}(X)$ is not empty. This implies the claim. $\square$

6. Projections

A key method in this theory is to exploit the interplay between two different projection maps. For this method the proposition below is a key technical result.

Proposition 7. *Let G be a linear-algebraic group with discrete subsets D, D' and algebraic subgroups H and I . Assume that $X = G/H$ admits an open embedding into an affine variety X' and that $Y = G/I$ admits an open embedding into an affine variety Y' . Furthermore we assume that the natural projection $\pi : G \rightarrow X = G/H$ restricts to a proper map from D and D' to X' , i.e., the fibers are finite and the images $\pi(D)$ and $\pi(D')$ are discrete in X' . Let $\tau : G \rightarrow Y = G/I$ denote the natural projection. Let $\zeta : D \rightarrow D'$ be a bijective map.*

- (i) *If the H -left action on G/I is non-trivial, then there exists a holomorphic automorphism ϕ of the complex manifold G such that $\tau \circ \phi$ maps D to a discrete subset of Y' with finite fibers. Furthermore ϕ can be chosen such that $\tau \circ \phi$ is injective if H acts freely on G/I or $\pi|_D$ is injective.*
- (ii) *Assume that the H -action on G/I has a dense orbit and that the projection map to G/H is injective on $D \cup D'$. Then there exists a holomorphic automorphism ϕ of the complex manifold G such that $\tau \circ \phi$ is injective on D' , $\tau \circ \phi = \tau \circ \phi \circ \zeta$ on D and such that $\tau(\phi(D')) = \tau(\phi(\zeta(D)))$ is discrete in Y' .*

Proof. Let $D = \{a_k : k \in \mathbb{N}\}$ and $D' = \{b_k : k \in \mathbb{N}\}$ with $\zeta(a_k) = b_k$.

We assume that the H -action on G/I is non-trivial. (We may do this, because the statement is empty if H acts trivially on G/I .)

Let E_0 denote the set of all points in Y where the dimension of the H -orbit is less than the generic H -orbit dimension. Note that E_0 is a nowhere dense closed analytic subset of Y . Let $E = \tau^{-1}(E_0)$. E is a closed analytic set, $E \neq G$ and $D \cup D'$ is countable. Therefore there exists an element $g \in G$ such that $ga_k, gb_k \notin E \ \forall k$, i.e., we may wlog assume that $(D \cup D') \cap E = \{\}$.

(i) Since $D \cap E = \{\}$, the H -orbit through $\tau(a_k)$ is non-trivial for every k . Now every non-trivial H -orbit is unbounded (i.e., has non-compact closure) in Y' (Lemma 9). Hence we may choose a sequence $h_k \in H$ such that $h_k \cdot \tau(a_k) = \tau(h_k \cdot a_k)$ is a discrete sequence in Y' . Since π restricts to a map with finite fibers on D , we may choose this sequence in such a way that furthermore $h_k = h_m$ whenever $\pi(a_k) = \pi(a_m)$. If $\pi|_D : D \rightarrow X$ is injective, we chose h_k in such a way that $k \mapsto h_k \cdot \tau(a_k)$ is injective, too. Observe that $\pi(a_k) \mapsto h_k$ is a well-defined map from the discrete subset $\pi(D) \subset X'$ to H . Because H is a complex Lie group and X' is affine (hence Stein), we can find a holomorphic function $f : X' \rightarrow H$ such that $f(\pi(a_k)) = h_k$ for every $k \in \mathbb{N}$. We define a biholomorphic self-map of G by

$$\phi : x \mapsto x \cdot f(\pi(x))$$

This proves statement (i).

(ii) We assume that H has a dense orbit in G/I . Due to algebraicity this dense orbit is open. For every k both $\tau(a_k)$ and $\tau(b_k)$ are contained in this (unique) open H -orbit in Y . Hence we may choose $h_k, \tilde{h}_k \in H$ such that $\tau(h_k a_k) = \tau(\tilde{h}_k b_k)$ and such that this sequence $\tau(h_k a_k)$ is discrete in Y' .

Next we observe that $\alpha : g \mapsto g^{-1}$ is a biholomorphic self-map of G inducing an isomorphism between the left quotient $H \backslash G$ and the right quotient G/H . Therefore we may assume that the natural projection $\tilde{\pi} : G \rightarrow H \backslash G = Z$ maps D injectively onto a discrete subset of Z' where Z' is an affine variety containing $H \backslash G$ as a Zariski open subvariety.

In both cases it suffices now to choose a holomorphic map f from Z to H such that $f(\pi(a_k)) = h_k$ and $f(\pi(b_k)) = \tilde{h}_k$.

Then the automorphism defined by

$$\phi : g \mapsto f(Hg) \cdot g$$

does the job. $\square$

Theorem 14. *Let G be a character-free complex linear algebraic group. Let D be a tame discrete subset. Let T denote the maximal torus of G . Then there exists a biholomorphic self-map ϕ of G such that the projection map $\pi: G \rightarrow G/T$ restricts to an injective proper map on $\phi(D)$.*

Proof. Let R denote the unipotent radical of G and let S be a maximal semisimple algebraic subgroup of G . If $R = \{e\}$, then G is semisimple and we may invoke Proposition 3. If $S = \{e\}$, then $T = \{e\}$ and the statement becomes trivial. Thus we may assume that both R and S are positive-dimensional.

We observe that group multiplication yields a bijective map $R \times S \rightarrow G$ given by $(x, y) \mapsto x \cdot y$. Let a_k, b_k be sequences in R resp. S such that

$$D = \{a_k \cdot b_k : k \in \mathbb{N}\}.$$

Since all maximal tori and all maximal semisimple subgroups are conjugate, we may assume that $T \subset S$. We fix an exhaustion function ρ_S on S . As a variety, R is isomorphic to $\mathbb{C}^n$ and we may use the euclidean norm to define an exhaustion function ρ_R on R in such a way that there is a holomorphic action of a compact Lie group K (namely, $\mathrm{SU}(n, \mathbb{C})$) on the complex manifold R which leaves ρ_R invariant. Now we define an exhaustion function ρ on G by imposing

$$\rho(xy) = \rho_R(x) + \rho_S(y) \quad \forall x \in R, y \in S.$$

Let R_k be a threshold sequence for S with respect to the exhaustion function ρ_S (cf. [14, Prop. 9.7]). Since we assume D to be tame, we may assume that

$$\#\{x \in D : \rho(x) \leq 2R_k + 1\} \leq k \quad \forall \mathbb{N}.$$

Applying a small translation if necessary, we may assume that $D \cap S$ is empty while

$$\#\{x \in D : \rho(x) \leq 2R_k\} \leq k \quad \forall \mathbb{N}. \quad (4)$$

holds.

Now we define

$$\begin{aligned} D' &= \{a_k \cdot b_k : k \in \mathbb{N}, \rho_R(a_k) > \rho_S(b_k)\} \\ D'' &= \{a_k \cdot b_k : k \in \mathbb{N}, \rho_R(a_k) \leq \rho_S(b_k)\}. \end{aligned}$$

These definitions imply that

$$\{a_k : k \in \mathbb{N}, a_k \cdot b_k \in D'\}$$

is discrete in R and

$$\{b_k : k \in \mathbb{N}, a_k \cdot b_k \in D''\}$$

is discrete in S .

For $(a_k \cdot b_k) \in D''$ we have $\rho_R(a_k) \leq \rho_S(b_k)$ implying

$$\rho_S(b_k) \geq \frac{1}{2}\rho(a_k \cdot b_k) = \frac{1}{2}(\rho_R(a_k) + \rho_S(b_k))$$

which in combination with 4 implies

$$\#\{a_k : k \in \mathbb{N}, a_k \cdot b_k \in D'', \rho_S(b_k) \leq R_k\} \leq k \quad \forall k \in \mathbb{N}.$$

It follows that

$$\{a_k : k \in \mathbb{N}, a_k \cdot b_k \in D''\}$$

is a tame discrete subset of S . Hence, due to Proposition 3, the projection map $R \times S \rightarrow S/T$ restricts to an injective proper map from D'' to S/T (after applying a suitable biholomorphic self-map of S).

Now D'' maps properly to S/T , while D' maps properly to R .

Put together, these facts imply that $D = D' \cup D''$ maps properly to $G/T \simeq (R \times S)/T$.

We observe that the intersection of a reductive and a unipotent subgroup is necessarily trivial. Therefore T acts freely on G/R and R acts freely on G/T . Using Proposition 7 (with T as H and R as I) we obtain that — after applying a suitable biholomorphic self-map — the projection of D to G/R is proper and injective. Now another invocation of Proposition 7 (this time with R as H and T as I) yields the statement. $\square$

The above prove provides some additional information. In fact, we have shown:

Let G be a complex linear algebraic group with maximal semisimple subgroup S such that every morphism of algebraic groups from G to $\mathbb{C}^$ is trivial and such that $\dim(S) > 0$. If R_k is a threshold sequence for S , then $2R_k + 1$ is a threshold sequence for G .*

As a consequence we obtain:

Proposition 8. *Let G be a character-free complex linear algebraic group with $\dim(G) \geq 2$. Then there exists a “threshold sequence” R_k .*

Proof. If G is not solvable, then $\dim(S) > 0$ and the statement follows from the above considerations. If G is solvable, then the triviality of $\text{Hom}(G, \mathbb{C}^*)$ implies that G is unipotent. In particular, $G \simeq \mathbb{C}^n$ as an algebraic variety. Then the existence of a threshold sequence follows from [12]. $\square$

Corollary 9. *Let G be a character-free complex linear algebraic group with $\dim(G) \geq 2$. Then every unbounded subset A of G contains a tame discrete subset of G .*

Proof. Let ρ be an exhaustion function on G . By Proposition 8 there exists a threshold sequence R_k for G with respect to ρ . If a subset A of G is unbounded, we may choose a sequence of points p_k of A with $\rho(p_k) > R_k$. Now $D = \{p_k : k \in \mathbb{N}\}$ is a tame discrete subset of G which is contained in A . $\square$

Proposition 9. *Let G be a character-free complex linear algebraic group. Let D be a tame discrete subset. Let H be an algebraic subgroup of G such that $Y = G/H$ is a quasi-affine open subvariety of an affine variety Y' . Then there exists a biholomorphic self-map ϕ of G such that the projection map $\pi : G \rightarrow G/H$ restricts to an injective map from $\phi(D)$ to its image $\pi(\phi(D))$ such that $\pi(\phi(D))$ is discrete in Y' .*

Proof. Due to Proposition 14 we may assume that there is a maximal torus T of G such that the projection $\rho: G \rightarrow G/T$ restricts to an injective proper map on D . Let N denote the smallest normal subgroup of G containing T . Standard structure theory of linear algebraic groups implies that N contains one (and therefore all) maximal semisimple algebraic subgroups of G .

We first deal with the case where H does not contain a maximal semisimple algebraic subgroup of G . Then evidently N is not contained in H . By the definition of N it follows that there exists a conjugate gTg^{-1} of T which is not contained in H . Thus the T -action on G/H is non-trivial and we may use Proposition 7 to conclude the proof.

Now assume that H contains a maximal semisimple algebraic subgroup. Let U denote a maximal unipotent algebraic subgroup of G . Clearly U does not contain any semisimple algebraic subgroup. Therefore, by the preceding consideration, after applying a biholomorphic self-map of G we may assume that the projection $p: G \rightarrow G/U$ restricts to an injective proper map on D . From the Levi decomposition of G we deduce that $G = H \cdot U$. This implies that the U -action on G/H is non-trivial. Now a second invocation of Proposition 7 yields the desired result.

Finally we discuss the case where G is unipotent. In this case $G \simeq \mathbb{C}^n$ as a complex manifold and the projection $G \rightarrow G/H$ is simply the projection onto the first k coordinates (for some $0 < k < n$) after a suitable change of coordinates. Thus the statement follows from the fact that $\phi(D) = \mathbb{N} \times \{0\}^{n-1}$ for some biholomorphic self-map ϕ . $\square$

Lemma 15. *Let G be a linear-algebraic group with $\dim(G) > 1$ which is not locally isomorphic to $\mathrm{SL}_2(\mathbb{C})$. Then there exist algebraic subgroups H and I of G such that*

- (1) $0 < \dim(H), \dim(I) < \dim(G)$.
- (2) *The quotient varieties G/H and G/I are quasi-affine.*
- (3) *The H action on G/I (by left multiplication) has a dense orbit.*

Proof. *Case 1.* G is neither reductive nor unipotent. In this case let V denote the unipotent radical and let H denote a maximal reductive subgroup. Then G is a semi-direct product of H and V . Therefore $G/V \simeq H$ and $G/H \simeq V$. In particular, both quotient varieties are affine and H acts transitively on G/V .

Case 2. G is unipotent. Let G' denote the commutator group. We choose I as a codimension one subgroup of G with $G' \subset I$ and H as a one-dimensional unipotent subgroup with $H \not\subset I$. The quotients G/H and G/I are quasi-affine, because G (and therefore every algebraic subgroup of G) is unipotent. Since I is a codimension one normal subgroup and $H \not\subset I$, we have $H \cdot I = G$. In particular, the H -action on G/I has a dense orbit (because H acts transitively on G/I).

Case 3. G is reductive, but not simple. Then we can find normal reductive subgroups H, I of G such that G equals the quotient of $I \times H$ by a finite subgroup. The quotients G/H and G/I are affine, because H and I are reductive. Furthermore H acts transitively on G/I , because $G = H \cdot I$ by construction.

Case 4. G is simple. We use root system theory. Let T be a maximal torus of G , $\Delta \subset \mathrm{Hom}(\mathrm{Lie} T, \mathbb{C})$ the root system, Δ^+ a set of positive roots and S a set of simple positive roots. Due to our assumption that G is not locally isomorphic to $\mathrm{SL}_2(\mathbb{C})$ we know that the rank of G is at least two. Hence there are at least

two different simple positive roots. For a simple root $x \in S$ let Δ_x^+ denote the set of all positive roots which can be written as $\mathbb{Z}^+$ -linear combinations of simple roots in $S \setminus \{x\}$. Furthermore let T_x be the connected Lie subgroup of T with Lie algebra $\ker(x: T \rightarrow \mathbb{C})$. For each root $v \in \Delta$ there is a one-dimensional unipotent subgroup U_v of G with

$$\mathrm{Lie}(U_v) = \{x \in \mathrm{Lie} G : [z, x] = v(z) \cdot x \ \forall z \in \mathrm{Lie}(T)\}.$$

We choose two distinct simple roots α, β . Associated to α, β we define two different algebraic subgroups H, I as follows:

$$\mathrm{Lie}(H) = \mathrm{Lie}(T_\alpha) + \bigoplus_{v \in \Delta^+} U_v + \bigoplus_{v \in \Delta_\alpha^+} U_{-v}$$

and

$$\mathrm{Lie}(I) = \mathrm{Lie}(T_\beta) + \bigoplus_{v \in \Delta_\beta^+} U_v + \bigoplus_{v \in \Delta^+} U_{-v}.$$

The construction immediately gives us the Levi decompositions of H and I : The Lie algebras given by

$$\mathrm{Lie}(T_\alpha) + \bigoplus_{v \in \Delta_\alpha^+} U_v + \bigoplus_{v \in \Delta_\alpha^+} U_{-v}$$

and

$$\mathrm{Lie}(T_\beta) + \bigoplus_{v \in \Delta_\beta^+} U_v + \bigoplus_{v \in \Delta_\beta^+} U_{-v}$$

define the Lie algebras of the semisimple subgroups of G corresponding to the subsets of simple positive roots $S \setminus \{\alpha\}$ and $S \setminus \{\beta\}$ whereas

$$\bigoplus_{v \in \Delta^+ \setminus \Delta_\alpha^+} U_v$$

and

$$\bigoplus_{v \in \Delta^+ \setminus \Delta_\beta^+} U_v$$

clearly are ideals in the Lie algebra of H resp. I corresponding to unipotent subgroups. Thus we see that both H and I are semi-direct products of a semisimple Lie group with a unipotent group. In particular, neither H nor I admit any character (i.e., a homomorphism of algebraic groups to $\mathbb{C}^*$). It follows that the quotient varieties G/H and G/I are quasi-affine.

Also by construction $\mathrm{Lie} H + \mathrm{Lie} I = \mathrm{Lie} G$. Hence the H -action on G/I has a dense orbit. Thus H and I have the desired properties. $\square$

Proposition 10. *Let G be a character-free complex linear algebraic group. Then every tame discrete set is strongly tame and any two tame discrete sets are equivalent.*

Proof. This is an immediate consequence of the next proposition. $\square$

Proposition 11. *Let G be a character-free complex linear algebraic group. Given two tame discrete sets D, D' every injective map ζ from D to D' extends to a biholomorphic self-map F of G .*

Proof. Upon replacing D' by $\zeta(D)$ it suffices to consider the case where ζ is bijective.

If $G \simeq \mathrm{SL}_2(\mathbb{C})$ or $G \simeq \mathrm{PSL}_2(\mathbb{C})$, this is known due to [14]. Thus we may assume that G is not locally isomorphic to $\mathrm{SL}_2(\mathbb{C})$.

Fix an exhaustion function ρ on G . Let R_k be threshold sequence (in the sense of Definition 3) for G . (The existence is due to Theorem 8.) Since D and D' are tame, there are holomorphic self-maps ϕ and $\tilde{\phi}$ of G such that

$$\#\{x \in D : \rho(\phi(x)) \leq R_{2k+1}\} \leq k \quad \forall k \in \mathbb{N}_0$$

and

$$\#\{x \in D' : \rho(\tilde{\phi}(x)) \leq R_{2k+1}\} \leq k \quad \forall k \in \mathbb{N}_0.$$

Then

$$\#\{x \in \phi(D) \cup \tilde{\phi}(D') : \rho(x) \leq R_{2k+1}\} \leq 2k \quad \forall k \in \mathbb{N}_0.$$

It follows that

$$\#\{x \in \phi(D) \cup \tilde{\phi}(D') : \rho(x) \leq R_m\} \leq m \quad \forall m \in \mathbb{N}_0.$$

Since the (R_k) form a threshold sequence, $\phi(D) \cup \tilde{\phi}(D')$ is tame. Thus, upon replacing D by $\phi(D)$, D' by $\tilde{\phi}(D')$ and ζ by $\tilde{\phi} \circ \zeta \circ \phi^{-1}$ we may assume that $D \cup D'$ is tame.

Now we choose H and I as given by Lemma 15. Then G/H is quasi-affine and there exists an embedding $G/H \hookrightarrow V$ into an affine variety V . We may assume that after application of a suitable biholomorphic self-map α of G the natural projection $\pi: G \rightarrow G/H$ maps $D \cup D'$ injectively onto a discrete subset of V . (Proposition 9).

Let $\tau: G \rightarrow G/I$ be the natural projection map. Proposition 7(ii) implies that there exists a biholomorphic self-map ϕ of G such that

$$\tau \circ \phi(x) = \tau \circ \phi \circ \zeta(x) \quad \forall x \in D.$$

and such that $\tau \circ \phi$ is injective on D' .

It follows that there is a map $\beta: D \rightarrow I$ such that

$$\phi(x)\beta(x) = \phi(\zeta(x)) \quad \forall x \in D.$$

In addition, due to Proposition 7, $\tau(\phi(D'))$ is discrete in some affine completion Y' of $Y = G/I$. Since ζ is bijective, $\tau \circ \phi$ also maps D bijectively onto a discrete subset of the affine variety Y' .

Hence we may choose a holomorphic map $\gamma: G/I \rightarrow I$ with

$$\gamma(\tau(\phi(x))) = \beta(x) \quad \forall x \in D$$

and define a self-map Ψ of G as

$$\Psi: y \mapsto y \cdot \gamma(\tau(y)).$$

Then $\phi^{-1} \circ \Psi \circ \phi$ is a biholomorphic self-map of G whose restriction to D agrees with ζ . $\square$

Proposition 12. *Let G be a complex linear algebraic group and let $H \subsetneq G$ be a unipotent algebraic subgroup. Let $D \subset H$ be a discrete subset. Then D is a tame discrete subset of G .*

Proof. First, we assume that G is not unipotent. Let T be a “maximal torus”. Note that $\dim(T) > 0$, because G is not unipotent. For any $g \in G$ the group gHg^{-1} is unipotent and therefore has trivial intersection with T . It follows that the H -action on G/T is free. In particular, all orbits have the same dimension. Since the group action is algebraic, it follows that all orbits are closed. Thus H acts freely on G/T with closed orbits. Furthermore G/T is an affine variety, because T is reductive. Thus we may invoke Proposition 4 to conclude that D is tame.

Second, assume that G is unipotent. Now H is a non-trivial unipotent subgroup of a unipotent group. By standard theory of nilpotent Lie groups it follows that $H \cdot G' \neq G$ where G' denotes the commutator group of G . Therefore there exists a complex one-parameter Lie subgroup A of G with $A \not\subset HG'$. Because G is unipotent, its exponential map is algebraic and every connected complex Lie subgroup of G is algebraic. Thus A is an algebraic subgroup. Now gHg^{-1} is contained in HG' for every $g \in G$, because HG' is normal in G . It follows that $gHg^{-1} \cap A$ is trivial for every $g \in G$, i.e., H acts freely on G/A . By the same reasoning as in the first case above we know that every H -orbit in G/T is closed. Finally note that G/A is isomorphic to an affine space, because both G and A are unipotent. $\square$

7. Unions of tame sets

Lemma 16. *Let G be a complex linear algebraic group with maximal connected reductive commutative subgroup (“maximal torus”) T . If $\dim(T) > 1$, then T contains two subgroups isomorphic to $\mathbb{C}^*$ which are not conjugate in G .*

Proof. Let R denote the radical of G . If $T \cap R = T$, then $T = G$, i.e., G is solvable. Then T maps injectively into $G/[G : G]$, which implies that two different subgroups of T cannot be conjugate.

If $0 < \dim(R \cap T) < \dim T$, the assertion is obvious, because subgroups of $R \cap T$ cannot be conjugate to subgroups which are not contained in R .

The remaining case is the case where $\dim(R \cap T) = 0$. Then R is unipotent, and there is no loss in generality in replacing G by G/R . Hence we may assume that G is semisimple. Let $\Delta \subset \text{Hom}(T, \mathbb{C})$ be the root system, i.e.,

$$\Delta = \{ \lambda \in \text{Hom}(\text{Lie } T, \mathbb{C}) : \exists v \in \text{Lie } (G) \setminus \{0\} : \text{ad}(g)(v) = \lambda(g) \cdot v \ \forall g \in \text{Lie } T \}.$$

We define

$$R_\alpha = \{ v \in \text{Lie } (G) : \text{ad}(g)(v) = \lambda(g) \cdot v \ \forall g \in \text{Lie } T \}$$

for $\alpha \in \Delta$.

Now $\Sigma = \bigcup_{\alpha \in \Delta} \ker \alpha$ denotes the set of all $g \in T$ whose centralizer contains a unipotent element. Evidently a one-dimensional subgroup contained in Σ cannot be conjugate to a subgroup of T which is not contained in Σ . Therefore there do exist non-conjugate subgroups isomorphic to $\mathbb{C}^*$. $\square$

Proposition 13. *Let G be a connected complex Lie group with positive-dimensional connected Lie subgroups H and I such that G/H and G/I are Stein manifolds and such that the diagonal map $\delta: G \rightarrow (G/H) \times (G/I)$ is a proper map. Then every discrete subset of G is the union of two tame subsets.*

Proof. Let ρ_1 resp. ρ_2 be an exhaustion function on G/H and G/I . Via δ we obtain an induced exhaustion function ρ on G :

$$\rho(g) = \rho_1(gH) + \rho_2(gI).$$

Let D be a discrete subset of G . We define $D_1 = \{g \in D : \rho_1(gH) \geq \rho_2(gI)\}$ and $D_2 = \{g \in D : \rho_1(gH) < \rho_2(gI)\}$. We observe that $\rho(g) \leq 2\rho_1(gH)$ for every $g \in D_1$. Hence D_1 admits a proper map to G/H . This implies that D_1 is tame due to Proposition 4.

Similarly for D_2 . $\square$

Theorem 17. *Let G be a connected complex linear algebraic group, $\dim(G) \geq 2$. Then every discrete subset of G is the union of two tame subsets.*

Proof. Let T be a maximal torus.

Case 1: $\dim(T) > 1$. Then G admits two subgroups H, I which are isomorphic to $\mathbb{C}^*$ and which are not conjugate. The quotients G/H and G/I are Stein, because H and I are reductive. We consider the G -action on $Y = (G/H) \times (G/I)$. The isotropy group at (xH, yI) equals $xHx^{-1} \cap yIy^{-1}$. This is necessarily finite, since H and I are one-dimensional and not conjugate. Thus all isotropy groups are finite. It follows that each G -orbit in Y has the same dimension. By algebraicity this implies that every G -orbit in Y is closed. Thus every orbit map $G \rightarrow Y$ (with $g \mapsto (gxH, gyI)$) is a proper map. Hence we may apply Proposition 13 and deduce the desired assertion.

Case 2: $\dim(T) < 2$ and G is solvable. If G is solvable, G is biholomorphic to $(\mathbb{C})^k \times (\mathbb{C}^*)^m$. Thus G is isomorphic as a variety to a product $\tilde{G}$ of two linear algebraic groups H and I , which allows us to employ Proposition 13.

Case 3: $\dim(T) < 2$ and G is neither solvable nor semisimple. Then necessarily T is one-dimensional. It follows that T is a maximal torus in a maximal connected semisimple subgroup S of G . Therefore the intersection $T \cap R$ of T with the radical R of G is finite and we may argue by choosing $H = R$ and $I = T$.

Case 4: G is (up to a finite covering) isomorphic to $\mathrm{SL}_2(\mathbb{C})$. Here we may refer to our earlier paper [14]. $\square$

8. Oka theory

A complex manifold X is called an *Oka manifold* if for every compact convex subset C of $\mathbb{C}^n$ the space of holomorphic functions $\mathrm{Hol}(\mathbb{C}^n, X)$ is dense (with respect to the topology of locally uniform convergence) in $\mathrm{Hol}(C, X)$. This is known to imply the property that for every Stein complex manifold Y and every continuous map from Y to X there exists a homotopic holomorphic map.

One instrument to prove this Oka property is a “spray”:

Definition 4. Let X be a complex manifold. A spray is a pair (E, π, s) where $\pi: E \rightarrow X$ is a vector bundle and $s: E \rightarrow X$ is a holomorphic map such that $s = \pi$ on the zero-section S of E . A spray is called “dominating” if $s|_{E_x}: E_x \rightarrow X$ has maximal rank at x for every $x \in S$.

A complex manifold with a dominating spray in this sense is necessarily an Oka manifold.

This generalizes to the notion of “subellipticity” or the notion of a dominating family of sprays.

Theorem 18. *Let X be a complex manifold and let $\pi: E_i \rightarrow X$ be a finite family of holomorphic vector bundles with maps $s_i: E_i \rightarrow X$ such that s_i agrees with π_i on the zero-section of E_i and such that for each point $x \in X$ we have $T_x X = \bigoplus_k (ds_i)(T_{p_i} E_i)$ where p_i is the point in the zero section of E_i above x . Then X is an Oka manifold.*

See [4] for more information about Oka manifolds.

Proposition 14. *Let X be a complex manifold on which its automorphism group $\text{Aut}(X)$ acts transitively. Let $\pi: E \rightarrow X$ be a vector bundle, and let $\tau: E \rightarrow X$ be a holomorphic map such that τ equals the identity map on the zero-section and such that there exists a point where τ has maximal rank. Then X is an Oka manifold.*

Proof. Let S denote the zero-section of E . Let $Z_0 = Z$ denote the set of points of $S \simeq X$ where τ does not have maximal rank. For each $p \in Z$ the set $\{\phi \in \text{Aut}(X) : \phi(p) \in Z_0\}$ is nowhere dense. We choose a countable subset $D \subset Z_0$ such that D intersects every irreducible component of Z_0 . Due to Corollary 8, there is an element $g_0 \in \text{Aut}(X)$ with $g_0(D) \subset X \setminus Z$. Define $Z_1 = Z_0 \cap g_0(Z_0)$. Then $\dim(Z_1) < \dim(Z_0)$. Continuing recursively, we obtain finitely many elements $g_0, \dots, g_s$ of $\text{Aut}(X)$ such that $\bigcap_k g_k(Z_k)$ is empty.

Now the pull-back bundles $g_k^* E$ combined with the maps $v \mapsto g_k(\tau(v))$ constitute a subelliptic family of dominating sprays in the sense of Theorem 18.

Therefore X is an Oka manifold. $\square$

Lemma 19. *Let V be a normal quasi-affine variety, and let $f: \mathbb{C} \rightarrow V$ be an injective morphism. Then f is a proper map, and every holomorphic function on $f(\mathbb{C})$ extends to a holomorphic function on V .*

Proof. Let V be embedded into a normal affine variety V_1 which in turn is embedded into a normal projective variety $\bar{V}$. Then f extends to a morphism from $\mathbb{P}_1 = \mathbb{C} \cup \{\infty\}$ to $\bar{V}$. Since $f(\mathbb{P}_1)$ is compact and V_1 is affine, it is clear that $f(\mathbb{P}_1) \not\subset V_1$. Therefore $f(\infty) \notin V_1$. It follows that $f(\mathbb{C}) = V_1 \cap f(\mathbb{P}_1)$ which in turn implies that $f(\mathbb{C})$ is closed in V_1 . Combined with V_1 being normal and the injectivity of f , we deduce that $f: \mathbb{C} \rightarrow V_1$ is a closed embedding and in particular proper.

Now $f(\mathbb{C})$ is a closed analytic subset of the Stein space V_1 . Therefore every holomorphic function extends from $f(\mathbb{C})$ to V_1 . Now the statement of the lemma is obvious, since V is an open subset of V_1 . $\square$

Proposition 15. *Let G be a complex linear algebraic group, U a one-dimensional unipotent subgroup, and H a connected algebraic subgroup of G with $H \cap U = \{e\}$.*

If H is either reductive or unipotent, then every holomorphic function on U extends to an H -invariant holomorphic function on G .

Proof. Since H is reductive or unipotent, we know that G/H is quasi-affine. Due to $H \cap U = \{e\}$ the projection map $\pi: G \rightarrow G/H$ restricts to an injective map on U . Lemma 19 implies that the image $\pi(U)$ is a closed subvariety of G/H such that every holomorphic function on $\pi(U)$ extends to the complex manifold G/H . Since holomorphic functions on G/H correspond to H -invariant holomorphic functions on G , the assertion follows. $\square$

Theorem 20. *Let G be a complex linear algebraic group, and let D be a tame discrete subset. Then $X = G \setminus D$ is an Oka manifold. Furthermore there exists an infinite discrete subset $D' \subset G$ such that $G \setminus D'$ is not an Oka manifold.*

Proof. Let U be a one-dimensional unipotent subgroup of G , i.e., an algebraic subgroup isomorphic to the additive group $(\mathbb{C}, +)$. (Such a subgroup exists due to Lemma 7.) Thanks to the equivalence (i) $\iff$ (iv) of Theorem 1 we may assume that $D \subset U$.

Let N denote the “nil cone” of G and let $v = (v_1, \dots, v_n)$ be a vector space basis of $\text{Lie}(G)$ such that $v_i \in N$ for all i . (This is possible by Lemma 8.) Since $\text{ad}(v_i)$ is nilpotent, the algebraic groups U_i generated by $\{\exp(v_i t) : t \in \mathbb{C}\}$ are all one-dimensional. We choose v such that every U_i is different from U .

We define

$$\sum_v \stackrel{\text{def}}{=} \bigcup_{1 \leq i \leq n} \bigcup_{x \in D} U_i \cdot x.$$

This is a closed analytic subset of G , because D is discrete and the orbits of the U_i are closed in G .

We obtain flows ϕ_i :

$$\phi_i(t; g) \stackrel{\text{def}}{=} \exp(v_i t) \cdot g.$$

We choose a holomorphic function β on U such that $\{x \in U : \beta(x) = 0\} = D$.

For every $i \in \{1, \dots, n\}$ the function β on U extends to a U_i -invariant holomorphic function α_i on G (due to Proposition 15).

Now we can modify the flows ϕ_i :

$$\tilde{\phi}_i(t; g) \stackrel{\text{def}}{=} \exp(v_i t \alpha_i(g)) \cdot g.$$

By construction, $\tilde{\phi}_i$ acts trivially on

$$\{u \cdot x : u \in U_i, x \in D\}.$$

We define further a map $\Psi_v: \mathbb{C}^n \times G \rightarrow G$ as follows

$$\Psi_v(t_1, \dots, t_n; p) = \tilde{\phi}_1(t_1) \circ \dots \circ \tilde{\phi}_n(t_n)(p).$$

Since the $v_1, \dots, v_n$ form a basis of $\text{Lie}(G)$, it is clear that for a fixed element $p \in G$ the map $t \mapsto \Psi_v(t)$ has maximal rank at $(0; p)$ if $\alpha_i(p) \neq 0 \ \forall i$.

Let A denote the set of all holomorphic automorphisms of G which fix D pointwise. We recall that A acts transitively on $Y = G \setminus D$.

Therefore we may now invoke Proposition 14 (with the trivial vector bundle as E) and deduce that $G \setminus D$ is an Oka manifold.

Finally we recall that every Stein manifold (hence in particular every complex linear algebraic group) G admits a discrete subset D' such that every holomorphic map from $\mathbb{C}^n$ ($n = \dim_{\mathbb{C}}(X)$) to $G \setminus D'$ is degenerate ([11]). Evidently it follows that the complement $G \setminus D'$ is not an Oka manifold. $\square$

9. Oka of blow up

Lemma 21. *Let $D = \mathbb{Z} \times \{0\}^{n-1} \subset \mathbb{C}^n$. Then the manifold $\hat{X}$ obtained from $\mathbb{C}^n$ by blowing up D is Oka.²*

Proof. We observe that

$$D = \{(z, 0, \dots, 0) : f(z) = 0\}$$

for $f(z) = \exp(2\pi iz) - 1$. The blow up of $\mathbb{C}^n$ along D may be described as

$$\begin{aligned} \hat{X} = \{(z_1, \dots, z_n; [w_1 : \dots : w_n]) \in \mathbb{C}^n \times \mathbb{P}_{n-1} : \\ f(z_1)w_j = z_jw_1 \ \forall j > 1, z_jw_k = z_kw_j \ \forall j, k > 1\}. \end{aligned}$$

Now

$$\phi: (z_1, \dots, z_n) \mapsto (z_1, f(z_1)z_2, \dots, f(z_1)z_n; [1 : z_2 : \dots : z_n])$$

defines a biholomorphic map from $\mathbb{C}^n$ to $\hat{X} \setminus H$ where

$$H = \{(z_1, \dots, z_n; [w_1 : \dots : w_n]) : f(z_1) = 0 = w_1\}$$

is the strict transform of the hypersurface defined by $f(z_1) = 0$. For every linear map $L: \mathbb{C}^{n-1} \rightarrow \mathbb{C}$ we obtain an automorphism α_L of $\mathbb{C}^n$ given by

$$(z_1, z_2, \dots, z_n) \mapsto (z_1 + L(z_2, \dots, z_n), z_2, \dots, z_n).$$

This automorphism of $\mathbb{C}^n$ lifts to an automorphism β_L of $\hat{X}$, because it stabilizes the exceptional locus of the blow up defined by $f(z_1) = 0 = z_2 = \dots = z_n$. Now $\beta_L \circ \phi$ defines a biholomorphic from $\mathbb{C}^n$ to the complement of the strict transform of the hypersurface defined by $f(z_1 + L(z_2, \dots, z_n)) = 0$. By taking $(n-1)$ linear independent linear forms L_j and the corresponding maps $\beta_{L_j} \circ \phi$ we see that $\hat{X}$ may be covered by finitely many Zariski open subsets biholomorphic to $\mathbb{C}^n$. Since $\mathbb{C}^n$ is Oka, the Oka property for $\hat{X}$ now follows from [6, Thm. 1.4]. $\square$

Lemma 22. *Let $H = (\mathbb{C}^*)^k \times \mathbb{C}^n$ with $k \geq 0$, $n \geq 1$ and $k + n \geq 2$. Let D be a discrete subset of H which is contained in $(\mathbb{C}^*)^k \times \{\mathbb{C}\}^{n-1} \times \{0\}$ and let Z denote the complex manifold obtained from H by blowing up D . Then Z is an Oka manifold.*

²The anonymous referee informed us that this fact has already been proved (by different methods) in [4, Prop. 6.4.12].

Proof. Let $\tau: \tilde{H} = \mathbb{C}^{k+n} \rightarrow H$ denote the universal covering of H . Then the universal covering of Z is the complex manifold obtained from $\tilde{H}$ by blowing up the discrete subset $D' = \tau^{-1}(D)$. Observe that D' is tame in $\tilde{H}$, because $D' \subset \mathbb{C}^{k+n-1} \times \{0\}$. ([10], Corollary 3.6.a.) Hence $\tilde{Z}$ is biholomorphic to the complex manifold obtained from $\mathbb{C}^{k+n}$ by blowing up $\mathbb{Z} \times \{0\}^{n+k-1}$. Therefore $\tilde{Z}$ is Oka. (Lemma 21). Since $\tilde{Z} \rightarrow Z$ is a holomorphic covering, Z must be Oka, too ([4, Prop. 5.5.2]). $\square$

Proposition 16. *Let G be a character-free complex linear algebraic group. Let D be a tame discrete subset and let X denote the complex manifold obtained from G by blowing up D . Then X is an Oka manifold.*

Proof. Let S denote a maximal semisimple Lie subgroup of G and let R denote the unipotent radical. Let T be a maximal torus of S .

Case (1): Assume that $S = \{e\}$. Then $G = R$. Hence G is biholomorphic to some $\mathbb{C}^n$ with $n \geq 2$ and the assertion follows from Lemma 22.

Case (2): Assume that $\dim(S) > 0$. Then T is non-trivial and thus unbounded. Consequently T contains a tame discrete subset D' (Corollary 9). Since all tame discrete subsets of G are equivalent, we may assume that $D \subset T$ (Corollary 1).

Fix a root system for S and let U^+ resp. U^- denote the maximal unipotent subgroup of D corresponding to all positive resp. negative roots. Next we claim that the map $\alpha: W_0 = U^- \times T \times U^+ \rightarrow S$ defined as

$$(x, y, z) \mapsto x \cdot y \cdot z$$

is an open embedding. It suffices to check injectivity. However

$$x \cdot y \cdot z = x' \cdot y' \cdot z', \quad x, x' \in U^-, y, y' \in T, z, z' \in U^+$$

implies

$$(x')^{-1} \cdot x = y' \cdot z' \cdot (y \cdot z)^{-1}.$$

Now $(x')^{-1} \cdot x$ is an element of the unipotent group U^- while the right-hand side is an element in the Borel group TU^+ . It follows that both sides equal e_G which implies the claim.

Now G is a semi-direct product of S with the unipotent radical R , thus we obtain a Zariski open embedding of $W = W_0 \times R$ into G via $(w, r) \mapsto \alpha(w) \cdot r$.

The complex manifold $\widehat{W}$ obtained from W by blowing up D (remember that we assumed $D \subset T \subset W$) is Oka due to Lemma 22.

Define $V = (S \setminus T) \cdot R$. Now G is flexible (see [3]) and T is of codimension at least 2 in G . Therefore V is still flexible ([3, Thm. 1.1]) and as a consequence Oka ([4, Prop. 5.5.17]).

We thus have found two Zariski open submanifolds of X , namely, V and $\widehat{W}$ which are both Oka. Furthermore $X = V \cup \widehat{W}$.

Therefore X is Oka due to [6, Thm. 1.4]. $\square$

10. Generalized overshears

Proposition 17. *Let Y_1 be a Stein space, E a nowhere dense analytic subset of Y_1 , $Y = Y_1 \setminus E$, $\pi: X \rightarrow Y$ a $\mathbb{C}$ -principal bundle with a meromorphic section σ , and $D \subset Y_1$ a discrete subset such that $D \cap (E \cup \{\sigma = \infty\}) = \emptyset$. For each $p \in D$, let an automorphism α_p of the fiber $X_p = \pi^{-1}(p)$ be given. Then there exists an automorphism ϕ of X such that $\pi = \pi \circ \phi$ and such that $\phi|_{X_p} = \alpha_p$ for all $p \in D$.*

Proof. Let $(U_i)_i$ be a trivializing open cover of Y and let $\phi_{ij}: U_i \cap U_j \rightarrow \mathbb{C}$ be the corresponding transition functions. Then the meromorphic section σ is given by a collection of meromorphic functions $s_i \in \mathcal{M}(U_i)$ such that $s_i = s_j + \phi_{ij}$ on each $U_i \cap U_j$. A fiber-preserving automorphism ϕ of X is given by a holomorphic function $\lambda: Y_1 \rightarrow \mathbb{C}^*$ and a family of holomorphic functions $c_i \in \mathcal{O}(U_i)$ such that in local coordinates ϕ may be described as

$$\phi: (x, z_i) \mapsto (x, \lambda(x)z_i + c_i(x)).$$

The functions c_i must fulfill

$$\lambda z_j + c_j + \phi_{ij} = \lambda \cdot (z_j + \phi_{ij}) + c_i,$$

i.e., $c_i = c_j + \phi_{ij}(1 - \lambda)$.

Now we may choose $\lambda \in \mathcal{O}^*(Y_1)$ arbitrarily subject only to the condition that $1 - \lambda$ has to vanish (with the respective multiplicities) along the poles of σ . Once λ is fixed, we may choose $h \in \mathcal{O}(Y_1)$ freely, and define

$$c_i = s_i(1 - \lambda) + h.$$

This yields the assertion, because on a Stein space we may prescribe the values of holomorphic functions on a discrete subset. $\square$

Proposition 18. *Let G be a complex linear algebraic group, H a commutative reductive subgroup, $U \simeq (\mathbb{C}, +)$ a one-dimensional unipotent subgroup and let $D \subset G$ be a discrete subset. Let $j: G/U = Y \hookrightarrow Y_1$ be an open algebraic embedding into an affine variety Y_1 , let $\pi: G \rightarrow G/U$, $\tau: G \rightarrow G/H$ denote the natural projections and assume that $\pi(D)$ is discrete in Y_1 . Then there exists a holomorphic automorphism ϕ of G such that $\tau \circ \phi|_D: D \rightarrow G/H$ is a proper map.*

Proof. There is no non-constant algebraic morphism from $\mathbb{C}$ to H , therefore the restriction of τ to an U -orbit gU is always non-constant. Since G/H is an affine variety (because H is reductive), it follows that the morphism $\tau|_{gU}: gU \rightarrow G/H$ is actually a finite proper morphism for every $g \in G$.

Since $G \rightarrow G/U$ is an algebraic $\mathbb{C}$ -bundle, there exists a meromorphic section σ . Let $P \subset Y = G/U$ denote its polar divisor. Since D is countable, we may replace D by a translate and thereby without loss of generality assume that $\pi(D)$ does not intersect P .

Therefore we may, using the above Proposition 17, prescribe automorphisms along $\pi(D)$.

From this fact we deduce the assertion in the following way:

We fix an enumeration $\pi(D) = \{q_k : k \in \mathbb{N}\}$. We choose an exhaustion function ρ on G/H . We observe:

Given a discrete subset Λ of $\mathbb{C}$ and a compact subset C of $\mathbb{C}$, it is always possible to choose an automorphism $\zeta : z \mapsto \alpha z + \beta$ ($\alpha \in \mathbb{C}^$, $\beta \in \mathbb{C}$) such that $\zeta(\Lambda) \cap C = \{\}$.*

Therefore, for each $k \in \mathbb{N}$, we may choose an automorphism ζ_k of $L_k \stackrel{\text{def}}{=} \pi^{-1}(q_k)$ such that

$$\tau(\zeta_k(D \cap L_k)) \cap \{x \in G/H; \rho(x) \leq k\} = \{\}.$$

Now Proposition 17 yields an automorphism ϕ of the complex manifold G which for each $k \in \mathbb{N}$ agrees with ζ_k on L_k . This implies that

$$\{x \in D : \rho(\tau(\phi(x))) \leq n\} \subset \bigcup_{k \leq n} L_k.$$

In view of the fact that $\tau|_{L-k} : L_k \rightarrow G/H$ is a proper map for each k , we may deduce that

$$\{x \in D : \rho(\tau(\phi(x))) \leq n\}$$

is finite for every $n \in \mathbb{N}$. $\square$

Corollary 10. *Let G be a complex linear algebraic group, $U \simeq (\mathbb{C}, +)$ a one-dimensional unipotent subgroup and let $D \subset G$ be a discrete subset. Let $j : G/U = Y \hookrightarrow Y_1$ be an open algebraic embedding into an affine variety Y_1 , let $\pi : G \rightarrow G/U$, denote the natural projection and assume that $\pi(D)$ is discrete in Y_1 . Then D is tame.*

Proof. If G is unipotent, then $G \simeq \mathbb{C}^n$ as a complex manifold. Moreover in this case we may choose coordinates on $\mathbb{C}^n$ such that π is a linear projection. Then the assertion follows from standard facts on tame discrete sets in $\mathbb{C}^n$. ([10].)

Hence we may assume that G is not unipotent. Then there exists a positive-dimensional commutative reductive subgroup of G and we may invoke the preceding proposition. In combination with Proposition 4 we obtain tameness of D . $\square$

11. Arithmeticity

In this section we exploit arithmeticity conditions to show that certain discrete subgroups are tame as discrete subsets.

The fact that $\mathbb{Z}$ is discrete in $\mathbb{R}$ generalizes as follows. Let K be a number field of degree $d = 2s + r$ with r real embeddings $\tau_i : K \rightarrow \mathbb{R}$ and s pairs of conjugate complex embeddings $(\sigma_j, \bar{\sigma}_j)$, $\sigma_j : K \rightarrow \mathbb{C}$. Let $\mathcal{O}_K$ be its ring of algebraic integers, i.e., the integral closure of $\mathbb{Z}$ in K . Then the diagonal map $x \mapsto (\tau_1(x), \dots, \tau_r(x); \sigma_1(x), \dots, \sigma_s(x))$ embeds $\mathcal{O}_K$ as a discrete subring in $\mathbb{R}^r \times \mathbb{C}^s$. If G is an algebraic group defined over K , in the same way $G(\mathcal{O}_K)$ embeds as a discrete subgroup in $G(\mathbb{R})^r \times G(\mathbb{C})^s$.

Fact 4. Let K be a number field with ring of algebraic integers $\mathcal{O}_K$. Let $P : \mathbb{A}^n \rightarrow \mathbb{A}^m$ be a polynomial map which is defined over K , i.e., all the coefficients of the polynomial functions $P_1, \dots, P_m$ are in K . Then $P(\mathcal{O}_K^n)$ is contained in $(1/N)\mathcal{O}_K^m$ for some $N \in \mathbb{N}$.

In fact, we may simply chose N as the lowest common multiple of the denominators of the coefficients of P .

This implies the following fact which is crucial in the sequel.

Fact 5. Let X, Y be affine algebraic varieties defined over a number field K and let $f: X \rightarrow Y$ be a K -morphism. Fix a closed embedding of X into an affine space $\mathbb{A}^n$. Then there is a closed embedding of Y into an affine space $\mathbb{A}^m$ such that $f(X \cap \mathbb{A}^n(\mathcal{O}_K)) \subset Y \cap \mathbb{A}^m(\mathcal{O}_K)$.

Given a group G with two subgroups Γ_i ($i = 1, 2$) these two subgroups are called *commensurable* if the intersection $\Gamma_1 \cap \Gamma_2$ is of finite index in both Γ_1 and Γ_2 .

Theorem 23. Let K be a number field of degree $d = 2s + r$ with r real embeddings $\tau_i: K \rightarrow \mathbb{R}$ and s pairs of conjugate complex embeddings $(\sigma_j, \bar{\sigma}_j)$, $\sigma_j: K \rightarrow \mathbb{C}$. Let $\mathcal{O}_K$ be its ring of algebraic integers. Let G be a character-free complex linear algebraic K -group such that there is a unipotent element $u \in G(K) \setminus \{e\}$. Let $H = G(\mathbb{C})^{r+s}$ and let $G(\mathcal{O}_K)$ be embedded into H via the diagonal embedding of $G(K)$ into $G(\mathbb{R})^r \times G(\mathbb{C})^s \subset G(\mathbb{C})^{r+s}$. Let Γ be a discrete subgroup of H which is commensurable to $G(\mathcal{O}_K)$. Then Γ is a tame discrete subset of the complex manifold H .

Proof. Let U denote the Zariski closure of $\{u^n : n \in \mathbb{Z}\}$. Since u is a K -rational unipotent element, U is a one-dimensional unipotent K -group. Consider the quotient morphism: $\tau: H \rightarrow H/U = V_0$. Note that V_0 is a quasi-affine variety defined over K . Let $j: V_0 \rightarrow V$ be a dense open embedding into an affine K -variety V . Observe that $j(\tau(G(\mathcal{O}_K)))$ is a discrete subset of $V(\mathbb{C})$ (cf. Fact 5). Since Γ and $G(\mathcal{O}_K)$ are commensurable, there exists a finite subset F of H with $\Gamma \subset F \cdot G(\mathcal{O}_K)$. Therefore $j(\tau(\Gamma)) \subset F \cdot \tau(G(\mathcal{O}_K))$ is likewise discrete in $V(\mathbb{C})$. Hence tameness of Γ follows from Corollary 10. $\square$

12. Lattices

A *lattice* Γ (in a Lie group G) is a discrete subgroup such that G/Γ has finite volume (with respect to the Haar measure). Lattices are also called discrete subgroups of *finite covolume*. Such a lattice Γ is called *cocompact* or *uniform* if the quotient G/Γ is compact.

A lattice Γ in a semisimple complex algebraic group S is called *irreducible* if there is no product decomposition $S = S_1 \cdot S_2$ with

- (1) Both S_i are normal semisimple subgroups of S with $0 < \dim S_i < \dim S$,
- (2) $S_1 \cap S_2$ is finite and
- (3) Γ and $(\Gamma \cap S_1) \cdot (\Gamma \cap S_2)$ are commensurable.

Theorem 24. Let Γ be a non-cocompact lattice in a semisimple complex linear algebraic group G . Then Γ is a tame discrete subset of the complex manifold G .

Proof. The general case is easily reduced to the case where Γ is an irreducible lattice. Hence from now on we assume that Γ is irreducible.

Next we recall that due to the theorem of Kazhdan–Margulis Γ contains a unipotent element (see [5]).

If G is of “rank one”, i.e., locally isomorphic to $\mathrm{SL}_2(\mathbb{C})$, tameness of Γ follows from Proposition 19 below.

Thus we may assume that G has rank at least two. Now we can apply the arithmeticity results of Margulis (see [7, Thm. (1), p. 2]). This implies that there is a number field K , a K -group H , a choice of complex embeddings σ_i of K and real embeddings τ_j such that Γ is commensurable to $\pi(\delta(H(\mathcal{O}_K)))$ where δ is the diagonal embedding of H into

$$S = (\Pi_i^{\sigma_i} H(\mathbb{C})) \times (\Pi_j^{\tau_j} H(\mathbb{R}))$$

and π denotes the quotient by the compact group

$$U = (\Pi_j^{\tau_j} H(\mathbb{R})).$$

However, since Γ and therefore also $\pi(\delta(H(\mathcal{O}_K)))$ is non-cocompact, there is a unipotent element. If $H(\mathbb{R})$ is compact, it cannot contain a unipotent element. Thus in our case the compact group U must be trivial. This allows us to invoke Theorem 23 in order to deduce tameness of Γ . $\square$

13. Discrete subgroups of $\mathrm{SL}_2(\mathbb{C})$

Proposition 19. *Let Γ be a discrete subgroup of $\mathrm{SL}_2(\mathbb{C})$ or $\mathrm{PSL}_2(\mathbb{C})$. If $\Gamma \setminus \{e\}$ contains a unipotent element, then Γ is a tame discrete subset.*

Proof. First we consider the case where the group is $\mathrm{SL}_2(\mathbb{C})$. Let u be a unipotent element of $\Gamma \setminus \{e\}$. All unipotent elements of $\mathrm{SL}_2(\mathbb{C})$ are conjugate, hence we may without loss of generality assume that

$$u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let $\pi: \mathrm{SL}_2(\mathbb{C}) \rightarrow \mathbb{C}^2$ denote the projection onto the first column. We claim that the image $\pi(\Gamma)$ is discrete in $\mathbb{C}^2$. To verify this, assume to the contrary that there exists a sequence $\gamma_n \in \Gamma$ such that $\pi(\gamma_n)$ is convergent in $\mathbb{C}^2$ with $\lim \pi(\gamma_n) = v$, but $\pi(\gamma_n) \neq v$. Let

$$\gamma_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}.$$

Our assumptions mean that $\lim(a_n, c_n) = (a, c)$ for some $(a, c) \in \mathbb{C}^2$, but $(a_n, c_n) \neq (a, c)$ for infinitely many $n \in \mathbb{N}$.

We calculate $\gamma_n \cdot u \cdot \gamma_n^{-1}$:

$$\gamma_n \cdot u \cdot \gamma_n^{-1} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} d_n & -b_n \\ -c_n & a_n \end{pmatrix} = \begin{pmatrix} 1 - a_n c_n & a_n^2 \\ -c_n^2 & 1 + a_n c_n \end{pmatrix}.$$

(Here we use that the determinant $\det(\gamma_n) = a_n d_n - b_n c_n$ equals 1.) It follows that

$$\lim_{n \rightarrow \infty} \gamma_n \cdot u \cdot \gamma_n^{-1} = \begin{pmatrix} 1 - ac & a^2 \\ -c^2 & 1 + ac \end{pmatrix}.$$

In particular, the sequence $\gamma_n \cdot u \cdot \gamma_n^{-1}$ is convergent. Since $\gamma_n \cdot u \cdot \gamma_n^{-1} \in \Gamma$ for all $n \in \mathbb{N}$, and Γ is discrete, it follows that

$$\gamma_n \cdot u \cdot \gamma_n^{-1} = \begin{pmatrix} 1 - ac & a^2 \\ -c^2 & 1 + ac \end{pmatrix}$$

for all sufficiently large n . In combination with

$$\pi(\gamma_n \cdot u \cdot \gamma_n^{-1}) = \begin{pmatrix} 1 - a_n c_n \\ -c_n^2 \end{pmatrix}$$

and

$$\pi(\gamma_n) = \begin{pmatrix} a_n \\ c_n \end{pmatrix}$$

it follows that

$$\pi(\gamma_n) \in \left\{ \begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} -a \\ -c \end{pmatrix} \right\}$$

for all sufficiently large n . Because $\lim_{n \rightarrow \infty} \pi(\gamma_n) = (a, c)$, this contradicts our assumption that $(a_n, c_n) \neq (a, c)$ for infinitely many $n \in \mathbb{N}$.

Thus we have established that $\pi(\Gamma)$ is discrete in $\mathbb{C}^2$.

Now the proof of tameness follows from Corollary 10.

Second, we discuss the case where the group is $\mathrm{PSL}_2(\mathbb{C})$. Let $\tau: \mathrm{SL}_2(\mathbb{C}) \rightarrow \mathrm{PSL}_2(\mathbb{C})$ be the quotient by the center $Z = \{I, -I\}$ of $\mathrm{SL}_2(\mathbb{C})$ and define $\tilde{\Gamma} = \tau^{-1}(\Gamma)$. By the arguments used above it follows that $\pi(\tilde{\Gamma})$ is discrete in $\mathbb{C}^2$. Now $\pi: \mathrm{SL}_2(\mathbb{C}) \rightarrow \mathbb{C}^2$ induces a map $\hat{\pi}: \mathrm{PSL}_2(\mathbb{C}) \rightarrow \mathbb{C}^2 / \sim$ with $v \sim w$ iff $v = \pm w$ for $v, w \in \mathbb{C}^2$. $\mathbb{C}^2 / \sim$ is an affine variety and by construction $\hat{\pi}(\Gamma)$ is discrete in $\mathbb{C}^2 / \sim$. Furthermore $\hat{\pi}$ is the quotient map of $\mathrm{PSL}_s(\mathbb{C})$ by a unipotent subgroup. Now tameness of Γ follows from Corollary 10. $\square$

13.1. Riemann surfaces

Let M' be a Riemann surface different from $\mathbb{P}_1$ and $\mathbb{C}$ and let $M = M' \setminus \{p\}$ for some point $p \in M'$. Then M' has the upper half plane $H^+ = \{z \in \mathbb{C} : \Im(z) > 0\}$ as universal covering and the fundamental group $\pi_1(M)$ embeds into $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{Aut}(H^+)$ as a discrete subgroup. The element of $\pi_1(M)$ corresponding to a circle around p is embedded into $\mathrm{PSL}_2(\mathbb{R})$ as a unipotent element. Therefore $\pi_1(M)$ becomes a tame discrete subset in $\mathrm{PSL}_2(\mathbb{C})$.

13.2. Hyperbolic manifolds

A real differentiable manifold M is called (real) hyperbolic if it admits a complete Riemannian metric with constant negative sectional curvature. Such a manifold has the hyperbolic n -space $\mathbb{H}_n$ as universal covering ($n = \dim_{\mathbb{R}}(M)$). Hence the fundamental group $\pi_1(M)$ of M acts by isometries on $\mathbb{H}_n$. Since the group of isometries of $\mathbb{H}_3$ is isomorphic to $\mathrm{PSL}_2(\mathbb{C})$, the fundamental group of a real hyperbolic 3-fold embeds into $\mathrm{PSL}_2(\mathbb{C})$ as a discrete subgroup Γ . A *knot* is a circle C embedded into the 3-sphere S^3 . It is known that the complement $S^3 \setminus C$ is a hyperbolic 3-fold with finite volume unless it is a “satellite knot” or a “torus knot”. The isotropy group H for the $\mathrm{SL}_2(\mathbb{C})$ -action on $\mathbb{H}_3$ is compact. Therefore

Γ is a discrete subgroup of finite covolume in $\mathrm{SL}_2(\mathbb{C})$. Since the knot complement $S^3 \setminus C$ is obviously non-compact, it is clear that $\mathrm{SL}_2(\mathbb{C})/\Gamma$ is likewise non-compact. By the theorem of Kazhdan–Margulis ([5]) it follows that Γ contains a unipotent element. Hence tameness of $\Gamma \simeq \pi_1(M)$ (as a discrete subset of $\mathrm{SL}_2(\mathbb{C})$) follows from Proposition 19.

14. $\mathbb{C}^*$ -factors

In many results we assume that we discuss a linear algebraic group which is *character-free*, i.e., without non-trivial morphism of algebraic groups with values in $\mathbb{C}^*$. The background for this assumption is the problem that $\mathbb{C}^*$ and its powers $(\mathbb{C}^*)^n$ have only few algebraic subgroups, namely only countably many. In particular there is no continuous family of projections from $(\mathbb{C}^*)^n$ to quotients by algebraic subgroups. That prohibits the usage of probabilistic methods which we employed in [14] in order to deduce Proposition 3, which is crucial for our instrument of “threshold sequences”.

To emphasize the problems one encounters in discussing tame discrete subsets of $(\mathbb{C}^*)^n$ we point out the proposition below.

Proposition 20. *Let $G = (\mathbb{C}^*)^n$ with $n \geq 2$. Let R_k be a strictly increasing sequence of positive real numbers. Let ρ be an exhaustion function on G . Then there exists a discrete subset D of G such that*

- (1) $\#\{x \in D : \rho(x) \leq R_k\} \leq k$.
- (2) *For every morphism of complex algebraic groups $F: G \rightarrow H$ with $\dim \ker F > 0$ the image $F(D)$ is dense in $F(G)$.*

Proof. Without loss of generality we may assume that F is surjective. The image $F(G) = H$ is connected, commutative and reductive and therefore isomorphic to some $(\mathbb{C}^*)^k$, $0 \leq k < n = \dim(G)$. Any such morphism F is determined by the induced homomorphism of fundamental groups, hence there are only countably many such morphisms $F: (\mathbb{C}^*)^n \rightarrow (\mathbb{C}^*)^k$. It suffices to consider only morphisms to $(\mathbb{C}^*)^{n-1}$, because every morphism of algebraic groups from $(\mathbb{C}^*)^n$ to $(\mathbb{C}^*)^k$ for $k < n$ fibers through $(\mathbb{C}^*)^{n-1}$.

We enumerate all surjective morphism of algebraic groups $F: (\mathbb{C}^*)^n \rightarrow (\mathbb{C}^*)^{n-1} = H$ as $(F_j)_{j \in \mathbb{N}}$. Now we fix a dense countable subset $S \subset H$. We fix a bijection $\zeta: \mathbb{N} \rightarrow \mathbb{N} \times S$ and choose elements $g_k \in G$ such that

- (i) $F_j(g_k) = s$ if $\zeta(k) = (j, s)$.
- (ii) $\rho(g_k) \geq \max\{k, R_k\}$.

Then $D = \{g_k : k \in \mathbb{N}\}$ has the desired properties:

- By construction we have $S \subset F_j(D)$ for every j . Therefore $F_j(D)$ is dense in $F_j(G) = H$.
- D is discrete, because $\rho(g_k) \geq k \forall k$.
- $\#\{x \in D : \rho(x) \leq R_k\} \leq k$ since $\rho(g_k) \geq R_k \forall k$. $\square$

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Funding Information Open Access funding provided by Projekt DEAL.

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