COMPLETING THE CLASSIFICATION OF REPRESENTATIONS OF SL_n WITH COMPLETE INTERSECTION INVARIANT RING

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Abstract. We present a full list of all representations of the special linear group SL_n over the complex numbers with complete intersection invariant ring of homological dimension greater than or equal to two, completing the classification of Shmelkin. For this task, we combine three techniques. Firstly, the graph method for invariants of SL_n developed by the author to compute invariants, covariants and explicit forms of syzygies. Secondly, a new algorithm for finding a monomial order such that a certain basis of an ideal is a Gröbner basis with respect to this order, in between usual Gröbner basis computation and computation of the Gröbner fan. Lastly, a modification of an algorithm by Xin for MacMahon partition analysis to compute Hilbert series.

Introduction

Let $A = \mathbb{C}[x_1, \dots, x_n]/I$ be an affine algebra. We say that A is regular if it is isomorphic to some polynomial ring, a hypersurface if I is a principal ideal, a *complete intersection* if I is minimally generated by $\text{hd}(A) = n - \dim(A)$ elements, where $\dim(A)$ is the Krull and $\text{hd}(A)$ the homological dimension of A .

Now let $\phi : SL_n \rightarrow SL(W)$ be a finite-dimensional representation of the special linear group SL_n . We denote by $\mathbb{C}[W]^{SL_n}$ the *ring of invariants* of this representation. It is a long-standing task in invariant theory to determine these representations that have a regular, hypersurface or complete intersection ring of invariants respectively. Representations of connected simple groups with regular ring of invariants have been classified in [12], [1], [26], while *irreducible* representations with complete intersection invariant rings can be found in [21]. *Reducible* representations of SL_2 with complete intersection invariant ring are classically known, see [15], while those for arbitrary SL_n have been classified by Shmelkin in [29] — all but six cases, for which it was still open if they are complete intersections or not.

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The task of the present paper is to address these remaining cases, three single ones and three series to be exact. Let V always denote the standard representation of SL_n and by Λ^k , S^k , Ad and S_λ denote the k th exterior power, k th symmetric power, adjoint representation and Schur module for a vector λ of natural numbers respectively. We denote reducible representations by sums of these symbols and the dual by a starred version respectively. The six left open cases are:

$$(\mathrm{SL}_5, 2\Lambda^2 + \Lambda^3 + V^*), \quad (\mathrm{SL}_5, 3\Lambda^2 + V^*), \quad (\mathrm{SL}_7, 3V + \Lambda^3 + V^*),$$

$$(\mathrm{SL}_n, 2\Lambda^2 + 4V^*), \quad (\mathrm{SL}_n, V + 2\Lambda^2 + 3V^*), \quad (\mathrm{SL}_n, S^2 + \Lambda^2 + 2V^*), \quad 5 \leq n \in 2\mathbb{Z} + 1.$$

We show in Sections 2 and 3 that all six representations have complete intersection invariant rings. Therefore, we can add to [29, Table 9] these six additional representations, yielding a complete classification of those representations of SL_n having a complete intersection invariant ring of homological dimension ≥ 2 :

Theorem 1. *A representation of SL_n has a complete intersection invariant ring of homological dimension ≥ 2 if and only if it or its dual is contained in the following table.*

no.	n	representation	homological dimension
1	≥ 3	$(n+1)V + rV^*$ $2 \leq r < n$	r
2	≥ 3	$S^2 + 2V + (n-2)V^*$	2
3	≥ 4	$\Lambda^2 + 2V + nV^*$	2
4	≥ 4	$\Lambda^2 + 4V + (n-2)V^*$	2
5	≥ 4	$2\Lambda^2 + kV + lV^*$, $k+l=4$	$n-1$ if $l=4$, $n=4,5$, 2 if $l=3$, $n=5$, m if $n=2m < 2l+2$, $n-l-1$ else
6	≥ 5	$\Lambda^2 + (\Lambda^2)^* + kV + lV^*$, $k+l=4$, $k \geq l$, $l \geq 1$ if n is odd	$\lfloor n/2 \rfloor$ if $(k,l) = (2,2)$, m if $(k,l) = (3,1)$, $n=2m$, $\lfloor n/2 \rfloor + 1$ else
7	≥ 4	$S^2 + \Lambda^2 + kV + lV^*$, $k+l=2$	2 if $n=4$, $(k,l) = (0,2)$, $n-l-1$ else
8	≥ 5	$S^2 + (\Lambda^2)^* + kV + lV^*$, $k+l=2$, $l \geq 1$ if n is odd	$\lfloor n/2 \rfloor$ if $(k,l) = (0,2)$, m if $(k,l) = (1,1)$, $n=2m$, $\lfloor n/2 \rfloor + 1$ else
9	2	$2S^3$	2
10	4	$S^2 + 3\Lambda^2$	2
11	4	$\mathrm{Ad} + 2\Lambda^2$	3
12	5	$2\Lambda^2 + \Lambda^3 + V^*$	2

13	5	$3\Lambda^2 + V^*$	3
14	6	$\Lambda^3 + kV + lV^*$, $(k, l) = (3, 3), (4, 1), (4, 2)$	2, if $(k, l) = (3, 3), (4, 1)$, 4, if $(k, l) = (4, 2)$
15	6	$\Lambda^3 + \Lambda^2 + 2V$	2
16	7	$\Lambda^3 + 3V + V^*$	2

The representations of SL_n with hypersurface invariant ring can be found in [29, Tables 1–8]. Generators and at least multidegrees of syzygies are also given there. Generators and syzygies for the six left open cases can be found in Propositions 3, 5, 9, 12, 14 of the present work and in the respective proofs.

The setup of the paper is as follows. In Section 1, we discuss the three main techniques involved, which are the graph approach for invariants of SL_n developed by the author in [4], a new algorithm for determining monomial orderings for a certain *desired* Gröbner basis (with *desired* leading terms) of an ideal, and a slight modification of an algorithm by Xin [32] for MacMahon partition analysis. We want to stress that all three techniques are applicable in greater generality and that the present work may serve as a blueprint for such applications.

Section 2 covers the three single cases while the main focus lies on the serial cases in Section 3, where we compute the invariant rings in terms of generators and syzygies. This is done with the help of covariant rings of subrepresentations and toric invariant rings thereof. The resulting algebras A are ‘almost’ toric — the ideal of syzygies being generated by binomials as well as trinomials, such that a torus of dimension $\dim(A) - 1$ acts on them, compare [3]. These almost toric algebras are contractions (i.e., special fibers of degenerations) of the original invariant rings and thus provide information on multidegrees of generators and syzygies. See [17] for a related approach that uses both a graphical method for the invariants and a toric degeneration of the invariant ring.

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1. The basic techniques

In the following, three different techniques are presented: the graph approach for SL_n -invariants from [4], a Gröbner basis algorithm for finding suitable monomial orderings and a modification of an algorithm for constant term evaluation of rational functions, see [32] for the original algorithm. In the first case, the work [4] is exhaustive in the sense that it discusses techniques for finding generators and syzygies for SL_n -invariant rings of antisymmetric tensors. What is new in the present work is that we include symmetric tensors in one case and also *covariants*. Another application of the graph method to covariants in the case of SL_4 can be found in the work [5] by the author.

1.1. The graph method for (anti-)symmetric tensors

The following is merely a summary of [4, Sect. 2, 3], while the basis for the methods therein are the (skew) brackets from [16]. Both should be considered for a deeper discussion. Let V denote the standard representation of SL_n with standard basis (e_i) and dual basis (e_i^*) . For nonnegative i, j, ι, γ we denote $\Lambda_{i,j} := \Lambda^i(V)$ and $S_{\iota,\gamma} := S^\iota(V)$. Let also n_i, m_ι be nonnegative, then we have a representation

$$W := \bigoplus_i \left(\bigoplus_{j=1}^{n_i} \Lambda_{i,j} \right) \oplus \bigoplus_\iota \left(\bigoplus_{\gamma=1}^{m_\iota} S_{\iota,\gamma} \right).$$

To a subspace $\Lambda_{i,j}, S_{\iota,\gamma}$ of W associate an infinite supply of letters $a_{ijk}, b_{\iota\gamma\kappa}$ respectively. A *bracket*, denoted by $[*]$, contains n of these letters. We consider the free algebra generated by these brackets, with respect to the following relations, and denote the resulting algebra by Bra .

(1) Inside a bracket, the letters behave commutatively for all combinations but two letters of type b , which behave anticommutatively, i.e.,

$$\begin{aligned} [*a_{ijk}a_{i'j'k'}*] &= [*a_{i'j'k'}a_{ijk}*], & [*a_{ijk}b_{\iota\gamma\kappa}*] &= [*b_{\iota\gamma\kappa}a_{ijk}*], \\ [*b_{\iota\gamma\kappa}b_{\iota'\gamma'\kappa'}*] &= -[*b_{\iota'\gamma'\kappa'}b_{\iota\gamma\kappa}*]. \end{aligned}$$

(2) If n is even, Bra is commutative in the brackets, while it is anticommutative for odd n .

(3) A monomial in brackets is nonzero only if a letter a_{ijk} ($b_{\iota\gamma\kappa}$) appears either zero or exactly i (ι) times inside the monomial.

(4) The Plücker relations (or *Exchange Lemma* [16, p. 60]):

$$\sum_{(u', u'') \vdash u} [u'v][u''w] = (-1)^{n-|w|} \sum_{(v', v'') \vdash v} [v'u][v''w],$$

where the notation $(u', u'') \vdash u$ means that we sum over all decompositions of u in two subwords u', u'' . Here a *subword* is obtained from u by deleting some of the letters and keeping the order of the remaining ones, while a *decomposition* means that u can be obtained from the word $u'u''$ by a permutation of letters. All occurring summands where a bracket contains more than n letters are set to zero. If u or v contain letters of type b , we get different presigns — see [16, Prop. 10] for the statement in whole generality — but we omit this here for simplicity as it is not needed.

Observe that the above Plücker relation does not include signs of the underlying permutations as the standard Plücker relations do. This is due to the construction in [16]. The signs will appear when we assign polynomials in $\mathbb{C}[W]^{\mathrm{SL}_n}$ to the bracket polynomials.

In order to do so, fix a total order on the letters $a_{ijk}, b_{\iota\gamma\kappa}$. We have a surjective linear map U from Bra to $\mathbb{C}[W]^{\mathrm{SL}_n}$ — called the umbral operator in [16] — defined by linearly extending the following definition for bracket monomials to polynomials. The image $U(\mathfrak{m})$ of a bracket monomial $\mathfrak{m}$ is the polynomial mapping an element $\sum t_{i,j} + \sum s_{\iota,\gamma}$ to the following:

- (1) Inside each bracket, arrange letters with respect to the chosen total order.
- (2) Take the tensor product of one tensor $\det := e_1^* \wedge \cdots \wedge e_n^*$ for each bracket and one tensor $t_{i,j}(s_{\iota,\gamma})$ for each letter $a_{ijk}(b_{\iota\gamma\kappa})$ appearing in the monomial.
- (3) The image under $U(\mathbf{m})$ is the complete contraction, where the ν th index of $t_{i,j}(s_{\iota,\gamma})$ and μ th index of some $\det$ are contracted if and only if the ν th appearance of the corresponding $a_{ijk}(b_{\iota\gamma\kappa})$ is at the μ th position of the respective bracket.

In [29], tensor contractions are used to compute invariants without utilizing brackets. Compare also [24, Sect. 9.5]. The content of the graph method from [4] is to use an algebra of hypergraphs instead of the bracket algebra Bra by identifying a bracket monomial with a graph that has a vertex for each bracket and an i -hyperedge of *color* j and *shading* k for each letter a_{ijk} that is connected to a vertex if and only if the letter turns up in the corresponding bracket. As the shading only affects the presign, we often omit it and denote only the color of hyperedges. We will also omit the 'hyper' and only speak of graphs and k -edges. Moreover, we will often refuse to distinguish between the graph and the associated invariant when the meaning is clear. We say that an edge is *looping* if it connects to the same vertex with both ends. A somehow dual approach for invariants of binary forms can be found in [18].

We want to extend this graphical method on the one hand to representations containing symmetric powers of V and on the other hand to covariants—accordingly to how this is done for brackets in [16]. The first task means that we have to represent letters of type $b_{\iota\gamma\kappa}$ —we do this by assigning jagged ι -edges to such letters.

Concerning the second task, in analogy to [16, §4.5], covariant graphs will be similar to invariant graphs but can in addition contain looping *dummy* k -edges, behaving as if they correspond to additional copies of $\Lambda^k V$. We will depict these dummy edges in a grey color. In fact, the algebra of covariants is isomorphic to the algebra of invariants $\mathbb{C}[W]^{U(G)}$ for a maximal unipotent subgroup $U(G)$ of G , see [30]. The isomorphism is given—if we choose the upper triangular matrices for $U(G)$ —by evaluation at $e_1 \wedge \cdots \wedge e_k$ for each dummy k -edge.

Remark 1. From the definition of the umbral operator U , it can be seen that the *multidegree* of an invariant corresponding to a graph is given by the number of edges corresponding to a subrepresentation of W . Consequently, the total degree is given by the total number of edges. The *weight* of a covariant is given by $\sum_{d \in D} k_d$, where D is the set of dummy edges and any $d \in D$ is a k_d -edge.

Formal sums of these graphs together with multiplication given by writing one next to the other—which gives a disconnected graph—form the algebra of *graphsums* G and we have a natural surjection $\gamma : G \rightarrow \text{Bra}$ from this algebra to the bracket algebra. Then denote the algebra of equivalence classes of graphsums by $\mathcal{G} := G / \ker(\gamma)$. We will loosely speak of *graphs* when we mean such equivalence classes. See [4, §3] for more details.

Now obviously the Plücker relation from above becomes a relation between graphs in $\mathcal{G}$. We will from now on say that we *apply the Plücker relation to some edges* if these edges correspond to the word u in the original Plücker relation for brackets. If it is not clear to which vertex the bracket with the word v corresponds,

we say we *pull the edges to the respective vertex*. A disconnected graph is the product of its connected components, and we say that a graph is *reducible*, if it equals a sum of disconnected graphs. We say that two irreducible graphs are *reducibly equivalent*, if their difference is reducible. A set of irreducible graphs is said to be *reducibly independent*, if the only linear combination of them that is reducible is the trivial one. A minimal generating set of the ring of invariants $\mathbb{C}[W]^{\mathrm{SL}_n}$ is in one-to-one-correspondence to a maximal set of reducibly independent irreducible graphs, see [4, Thm. 3.8].

The advantage of the graph method is twofold: firstly, one can use graph theoretical methods and secondly, it is somehow easier to see the consequences of Plücker relations or how they can be applied reasonably. The following example gives a blueprint of how these advantages interact.

Example 1. Consider invariants of SL_4 acting on antisymmetric 1-, 2-, and 3-tensors. So we have to find a generating set of the algebra of 4-regular graphs with 1-, 2-, and 3-edges. In particular, such a generating set consists of irreducible graphs. In [4, Lem. 4.3], such a generating set was determined. It consists of the following types of graphs:

This set is not minimal, but a first step in the direction of a minimal one that is determined in [4, Prop. 4.1]. We recall the proof of [4, Lem. 4.3] in the following.

Let Γ be an irreducible 4-regular graph with 1-, 2-, and 3-edges. If Γ has only 1-edges, then it is connected only if it has one vertex. This is nothing new, but we get the type 1 from above, corresponding to the determinant.

Now observe that by [4, Lem. 3.9] if we choose a non-looping edge, we can pull it over to one vertex and make it looping. That means Γ equals a sum of graphs, where the respective edge is a looping edge at the same vertex. Moreover, this does not harm looping edges at other vertices. Thus we can always assume that at each vertex we have a looping edge and we can choose these looping edges as we want.

So if Γ has only 3-edges, we can assume that at every vertex there is a looping 3-edge. Thus three such vertices with a connecting 3-edge must make up a connected component of Γ . Since Γ is irreducible, it is of type 2.

Now let Γ contain only 2-edges. If it has one vertex, it is of type 3 of course. Otherwise, we can assume that every vertex has a looping 2-edge, so when we forget the looping edges, Γ is a *connected simple 2-regular graph*, which is nothing else than a *cycle*. So we get type 4.

Now if Γ has 1- and 3-edges, we can assume that at every vertex we have a looping 3-edge. Otherwise, if there is no such vertex, there are two possibilities. Either it has no connection to a 3-edge at all, then it makes up a connected component of type 1. Or it has a connection to a 3-edge, then we can make this

edge looping at the vertex. So forgetting the looping edges, we have a 1-regular graph—that is all vertices are of degree one—with 1 and 3-edges. A connected component of such a graph is either of type 2 or of type 5.

If Γ has 1- and 2-edges, it either has one vertex and is of type 1, 3, or 6, or it has more than one vertex. In this case, with the same arguments as above for 3-edges, we can assume that at every vertex there is a looping 2-edge. It is possible for such a vertex to have a looping 1-edge in addition. So forgetting the looping edges again, we have vertices of degree 1 and 2. Connected components of such a graph are *chains*. We get type 7.

Similar arguments as in the last paragraph lead to the chains of type 8 and 9.

1.2. The Crosshair-sieve-algorithm

The Crosshair-sieve-algorithm 11 is developed in Section 3 to show that a certain ideal basis is a Gröbner basis in fact. We give a general but very rough account of this algorithm here. It lies between standard Gröbner basis computations with usually a limited amount of available monomial orders and the computation of the Gröbner fan, introduced in [20]. In particular, if we want to find a Gröbner basis with certain good properties (for instance a particular set of leading monomials), then it is in general not very likely that the standard monomial orders such as lexicographic, total degree reverse lexicographic, et cetera, produce such basis. This problem is addressed by Gröbner fan computations, see [14]. On the other hand, the computation of the whole Gröbner fan encoding all possible Gröbner bases of one ideal or a universal—i.e., with respect to all monomial orders—Gröbner basis, see [31], can be too complex, see [19] for such problems in applied Gröbner basis theory. This is in particular the case if some parameters are involved as in our three series of invariant rings. Moreover, even the computation of a single Gröbner basis for a badly suited monomial order can be too complex in such cases.

So assume we are in the following situation: we have a finite set of polynomials f_i with monomials m_i appearing in the f_i and we want to find a monomial order such that the f_i have leading monomials $\text{lm}(f_i) = m_i$. If for example the m_i pairwise have no variable in common, then the f_i are already a Gröbner basis of the ideal generated by themselves. This follows directly from the Buchberger algorithm [9, Thm. 3.3] as all S -polynomials reduce to zero.

Any monomial order on s variables can be expressed as a matrix order, see [25], [13], with a matrix $\mathcal{M} \in \text{GL}_s(\mathbb{R})$ where degrees are given by the first row of $\mathcal{M}$ with ties broken by the second row, et cetera. That means if two monomials have the same degree with respect to the first row, one has to compare their degrees with respect to the second row and so on. The degree of a monomial $x_1^{a_1} \cdots x_s^{a_s}$ with respect to a row $(b_1, \dots, b_s)$ of $\mathcal{M}$ is just $\sum_{i=1}^s a_i b_i$. Associating a degree vector $v := \sum_{i=1}^s a_i e_i \in \mathbb{R}^s$ to a monomial, we can denote rows of $\mathcal{M}$ by $\mathcal{M}_\nu := \sum_{i=1}^s b_i e_i^* \in (\mathbb{R}^s)^*$. Then we can write the degree of $x_1^{a_1} \cdots x_s^{a_s}$ as $\mathcal{M}_\nu(v)$.

It is easy to see that in fact we require a matrix $\mathcal{M}$ with s columns, an arbitrary number of rows and rank s . Matrices in $\mathcal{M} \in \text{GL}_s(\mathbb{R})$ meet this requirement, but any Matrix with s columns can of course be expanded by adding more rows to one of full rank s . What we want to do is build up a matrix $\mathcal{M}$ such that the induced monomial order gives the required leading terms.

Algorithm 2 (General Crosshair-sieve). In order to do so, let d be the maximal total degree of all f_i and denote the variables by $x_1, \dots, x_s$. Set $x_0 := 1$.

To each row $\mathcal{M}_\nu$ of $\mathcal{M}$ associate a symmetric tensor $S_\nu \in S^d(\mathbb{R}^{s+1})$, the so-called *sieve*, collecting the degrees associated by $\mathcal{M}_\nu$ to all monomials in the x_i of degree $\leq d$. In the entry $(S_\nu)_{i_1, \dots, i_d}$ stands the degree of $x_{i_1} \cdots x_{i_d}$.

The degrees of all monomials of one f_i are a collection of entries $(S_\nu)_{i_1, \dots, i_d}$. Now we 'target' a monomial m_i by increasing the coefficient b_j in $\mathcal{M}_\nu$ for at least one x_j occurring in m_i . We do this for every m_i in such way that $\deg_\nu(m_i) \geq \deg_\nu(m'_i)$ for all monomials m'_i of f_i . It is necessary for the algorithm to work (and for the m_i to be a possible set of leading monomials) that such way exists. This is the case for example if all f_i are homogeneous and we set $b_j = 1$ for all x_j occurring in some m_i . But this condition is not sufficient of course. At least for one single i , we require $\deg_\nu(m_i)$ to be truly greater than $\deg_\nu(m'_i)$ for all other monomials m'_i of f_i . We say m_i is *filtered out* by S_ν . Which means we do not have to target this very m_i in $S_{\nu+1}$ any more and thus get less *unwanted interferences* of b_j leading to high degrees of unwanted monomials. It is clear that this algorithm terminates if and only if at each step—for each row of $\mathcal{M}$ —at least one m_i is filtered out.

Remark 2. The success of this approach of course relies heavily on the structure of the f_i , i.e., how the degrees of their monomials are distributed over S_ν . In practice, for example in invariant theory, the f_i will most likely inherit some symmetry like weighted homogeneity and a symmetric distribution over S_ν .

Example 2. We give a short account of how such properties fit together very well in the serial cases considered in Section 3. For details we refer to Algorithm 11. Due to weighted homogeneity, even if the m_i are not of maximal total degree under all monomials in the f_i , there will be some variables occurring in the m_i but not in the m'_i of higher total degree. Targeting these variables at first will filter out monomials of the same total degree as the m_i . Now we can arrange the variables in such way that degrees of one f_i lie on counterdiagonals in the 2-tensors S_ν (seen as symmetric matrices). Moreover, as we have some freedom in the choice of the m_i , we can arrange them on diagonals. Now targeting them by setting $b_j = 1$ for each x_j in some m_i that has not yet been filtered out, we see that at least the right- and lowermost entry of S_ν will be filtered out at each step. The picture of S_ν with one m_i targeted in this way is exactly that of a *crosshair*.

1.3. Hilbert series by MacMahon partition analysis

We compute Hilbert series of the three single cases in Section 2 and of low-dimensional representatives of the serial cases in Section 3 by the following method.

According to [11, Sect. 4.6] the (univariate) Hilbert series of the invariant ring of a representation of a reductive algebraic group can be expressed as the *constant term* in l variables $z_1, \dots, z_l$ of a rational function of the form

$$\frac{f(z_1, \dots, z_l)}{(1 - f_1(z_1, \dots, z_l)t) \cdots (1 - f_k(z_1, \dots, z_l)t)}.$$

It is one of the main objectives of *MacMahon partition analysis* to compute constant terms of so called *Elliot-rational* functions, with various applications and

implementations, see [2], [32], [33]. Constant terms in more than one variable $z_1, \dots, z_l$ are treated as iterated constant terms in one variable z_i . The constant term can be directly read off if one has a partial fraction decomposition of the rational function. This is the approach of the algorithm E11 developed in [32] with an implementation in Maple. But there are two bottlenecks: firstly, if one of the denominator factors is not *linear* in z_i and secondly, if two of the denominator factors are not relatively prime. These problems lead to a very significant increase in runtime, which makes it practically impossible to address complicated problems. In theory, the first problem can be solved by introducing roots of unity, the second one by introducing 'slack variables' s_i , one for each denominator factor, so that the function will change to

$$\frac{f(z_1, \dots, z_l)}{(1 - f_1(z_1, \dots, z_l)ts_1) \cdots (1 - f_k(z_1, \dots, z_l)ts_k)}.$$

This is the approach of [33]. But the output of such an algorithm is a large sum of rational functions—most of them with poles at some $s_i = 1$ —which must be simplified to a single rational function where we can set $s_i = 1$. So the problem of computational complexity is only shifted. In [33], MacMahon partition analysis then is used again to compute constant terms in $v_i = s_i - 1$ for each of the output functions, which may by far be the part of the computation with the highest complexity, see [33, Sect. 5.3].

In the Hilbert series computations of the present paper, we did not succeed in reasonable time with any of the mentioned algorithms. So we propose the following modified version: assume the only denominator factors that are not relatively prime are equal. Instead of introducing a slack variable for each factor, we introduce *one* slack variable s and take different powers of s for each member of a collection of identical denominator factors. A function of the form

$$\frac{f}{(1 - f_1t)^{n_1} \cdots (1 - f_kt)^{n_k}}$$

will thus become

$$\frac{f}{(1 - f_1t) \cdots (1 - f_1ts^{n_1-1}) \cdots (1 - f_kt) \cdots (1 - f_kts^{n_k-1})}.$$

The output of the algorithm from [32] applied to such function will be a large sum of rational functions with no chance to be simplified by, e.g., standard Maple simplification. This is where the second aspect of our modification comes into play: most of the rational functions will have a pole in $s = 1$, but of different *order*. Let $c \in \mathbb{N}$ be the highest pole order. Since we know that the resulting function has no pole at $s = 1$, the sum of all functions with a pole of order c must have a pole of order $c - 1$ at most. Since it is likely that such sum consists of not nearly as many terms as the whole sum, it might be possible to compute it and thus reduce the highest overall pole order. Iterating this procedure will of course finally result in the sum of all rational functions, our desired output.

The Maple-worksheets for the Hilbert series computations of the SL_5 representations $2\Lambda^2 + \Lambda^3 + V^*$, $3\Lambda^2 + V^*$ and $2\Lambda^2 + 4V^*$ will be available as supplementary material [6]. We claim that the described modification is useful for at least problems of certain complexity.

Remark 3. The above algorithms may analogically be used to compute multivariate Hilbert series, i.e., not with respect to total degree but multigradings. As such Hilbert series contain more information, compare Section 3; they may be more useful in some situations than the usual univariate ones, for example if one is searching for generators and relations. But then on the other hand, one has to compute multivariate Hilbert series of ideals given by generators and multidegrees. To our knowledge, there is no such implementation available, though it is analogous to computation of univariate Hilbert series by means of Gröbner bases. A basic implementation in Maple, applied to computing the Hilbert series of the algebra A_3 from Lemma 15, is provided as supplementary material as well [6].

2. The single cases

Consider the SL_5 -representations $U_1 := 2\Lambda^2 + \Lambda^3 + \Lambda^4$, $U_2 := 3\Lambda^2 + \Lambda^4$ and the SL_7 -representation $U_3 := 3V + \Lambda^3 + \Lambda^6$. These are the three single cases that Shmelkin left open in [29]. We prove that all three are complete intersections in the following.

2.1. The case (SL_5, U_1)

Proposition 3. *The ring of invariants is a complete intersection of homological dimension two minimally generated by the images of the following graphs, where the occurring 2-edges of colors 1 and 2 correspond to the first and second copy of Λ^2 , and the 3- and 4-edges correspond to the copies of Λ^3 and Λ^4 in U_1 respectively.*

$$\begin{aligned}
 g_{122} &= \text{graph with 2 loops on edge 1, 2 loops on edge 2}, & g_{211} &= \text{graph with 2 loops on edge 2, 1 loop on edge 1}, & g_{111} &= \text{graph with 1 loop on edge 1, 2 loops on edge 1}, & g_{222} &= \text{graph with 2 loops on edge 2, 2 loops on edge 2}, & g_{112} &= \text{graph with 1 loop on edge 1, 1 loop on edge 2}, \\
 g &= \text{graph with 1 loop on edge 1, 1 loop on edge 2}, & g_1 &= \text{graph with 1 loop on edge 1}, & g_2 &= \text{graph with 1 loop on edge 2}, & g_{1112122} &= \text{graph with 1 loop on edge 1, 1 loop on edge 2, 1 loop on edge 1, 2 loops on edge 2}, & g_{11122} &= \text{graph with 1 loop on edge 1, 1 loop on edge 2, 1 loop on edge 1, 1 loop on edge 2}, \\
 g_{21222} &= \text{graph with 1 loop on edge 2, 1 loop on edge 1, 1 loop on edge 2, 2 loops on edge 2}, & g_{12111} &= \text{graph with 1 loop on edge 1, 1 loop on edge 2, 1 loop on edge 1, 1 loop on edge 1}, & g_{21} &= \text{graph with 1 loop on edge 2, 1 loop on edge 1, 1 loop on edge 2}.
 \end{aligned}$$

Proof. The list of generators can be extracted from [4, Thm. 1.3]. Since the Krull dimension of the ring of invariants is 11, the homological dimension is two. Due to [23, Rem. to Prop. 1.5], if $\mathbb{C}[V]^G$ is of homological dimension two for G connected and semisimple, then $\mathbb{C}[V]^G$ is a complete intersection. Thus $\mathbb{C}[U_1]^{\mathrm{SL}_5}$ is so. We also computed the Hilbert series of $\mathbb{C}[U_1]^{\mathrm{SL}_5}$; it is

$$\frac{(1-t^{10})(1-t^{12})}{(1-t^2)^2(1-t^3)(1-t^4)^5(1-t^5)(1-t^6)^3(1-t^7)}.$$

Moreover, as can be seen by comparing Hilbert series, generators of the ideal of relations are

$$\begin{aligned}
 &16g_{11222}g_2g_{211} - 6g_2^2g_{211}^2 - 3g_1g_{11222}g_{122} - 8g_{111}g_{122}^2 - 8g_{12}g_{122}g_{211} \\
 &\quad - 3g_{1121222}g_{21} - 16g_{1122}^2 - 16g_{1211}g_{2122},
 \end{aligned}$$

$$2gg_{112122} - 4g_{12}g_{1122} - 2g_{2122}g_{11} - 2g_{1211}g_{22} \\ - g_{122}g_{2}g_{11} + g_{211}g_1g_{22} - 2g_{211}g_2g_{12}.$$

□

2.2. The case (SL_5, U_2)

Proposition 4. *The ring of invariants is a complete intersection of homological dimension three minimally generated by the images of the following graphs, where the occurring 2-edges of colors 1, 2, and 3 correspond to the first, second, and third copy of Λ^2 in U_2 respectively, and the 4-edges correspond to the copy of Λ^4 in U_1 .*

$$f_{abcde} = \begin{array}{c} \text{a} \quad \text{d} \\ \text{b} \quad \text{e} \\ \text{c} \end{array}, abcde \in \{12311, 12322, 13233, 11233, 11322, 22133\};$$

$$f_{abc} = \begin{array}{c} \text{a} \quad \text{b} \\ \text{c} \end{array}, abc \in \{122, 322, 233, 211, 311, 133, 123, 213\}.$$

Proof. The list of generators can be extracted from [4, Thm. 1.3, Prop. 5.1]. Since the Krull dimension of the ring of invariants is 11, the homological dimension is three. We need to show that the ideal of relations is generated by three polynomials. The Hilbert series of $\mathbb{C}[W]^{\mathrm{SL}_5}$ is

$$\frac{(1-t^9)^3}{(1-t^4)^8(1-t^5)^6}$$

and the following three relations hold between the invariants:

$$\begin{aligned} & -2f_{122}f_{13233} - 2f_{322}f_{11233} + f_{233}f_{11322} - f_{133}f_{12322} + 2f_{123}f_{22133}, \\ & 2f_{211}f_{13233} - 2f_{311}f_{22133} + f_{133}f_{11322} - f_{233}f_{12311} + 2f_{213}f_{11233}, \\ & 2f_{311}f_{12322} - 2f_{211}f_{22133} + f_{122}f_{11233} + f_{322}f_{12311} - 2(f_{213} + f_{123})f_{11322}. \end{aligned}$$

The Hilbert series of $\mathbb{C}[f_{abcde}, f_{abc}]/I$ with the ideal I generated by these three polynomials and the above one of $\mathbb{C}[W]^{\mathrm{SL}_5}$ coincide, thus the assertion follows.

□

2.3. The case (SL_7, U_3)

Proposition 5. *The ring of invariants is a complete intersection of homological dimension two minimally generated by the images of the following graphs, where the occurring 1-edges of colors 1, 2, and 3 correspond to the first, second, and third copy of V in U_3 , the occurring 3-edges correspond to the copy of Λ^3 , and the 6-edges correspond to the copy of Λ^6 in U_1 respectively.*

$$h_a = \begin{array}{c} \text{b} \\ \text{a} \end{array}, a = 1, 2, 3; \quad h_{ab} = \begin{array}{c} \text{b} \\ \text{a} \end{array}, ab \in \{11, 12, 13, 22, 23, 33\};$$

$$\tilde{h}_{ab} = \begin{array}{c} \text{b} \\ \text{a} \end{array}, ab \in \{12, 13, 23\}; \quad h = \begin{array}{c} \text{b} \\ \text{a} \end{array}, \quad h_{123} = \begin{array}{c} \text{b} \\ \text{a} \end{array},$$

$$\tilde{h} = \begin{array}{c} \text{b} \\ \text{a} \end{array}, \quad h_{123123} = \begin{array}{c} \text{b} \\ \text{a} \end{array}, \quad \tilde{h}_{123} = \begin{array}{c} \text{b} \\ \text{a} \end{array}.$$

Proof. The Krull dimension of the ring of invariants is 15. The first 15 invariants can be extracted from [29, Table 7]. Also from there, we know that an irreducible graph with five 3-edges and six 1-edges—two of each color 1, 2, 3—must exist. Since such a graph can not be equivalent to a sum of disconnected graphs due to the lack of appropriate graphs with fewer vertices, any graph with such edges either evaluates to zero or it is irreducible and the irreducible ones are pairwise reducibly equivalent. The penultimate one from the above list is such an irreducible graph.

We have 16 irreducible graphs so far. Remaining graphs from a maximal set of reducibly independent irreducible ones must now contain at least one 3-, one 6-, and one 1-edge of each possible color. The last one from the above list is such a graph. Now we want to show that these 17 invariants generate $\mathbb{C}[W]^{\text{SL}_7}$, from which directly follows that it is a complete intersection due to [23, Rem. to Prop. 1.5].

In order to do so, we follow the outline of Shmelkin [29, pp. 221, 227]. We know that the invariant h from the proposition is a generator of $\mathbb{C}[W]^{\text{SL}_7}$. Let

$$z := e_1 \wedge e_2 \wedge e_5 + e_3 \wedge e_4 \wedge e_6 + e_1 \wedge e_3 \wedge e_7 + e_2 \wedge e_4 \wedge e_7.$$

The isotropy group of z is $G_2 \subseteq \text{SL}_7$ and $\text{SL}_7 z = V(h)$, where h is the invariant from the proposition.

Consider the natural $\mathbb{Z}^5$ and $\mathbb{Z}^4$ -gradings of $\mathbb{C}[W]$ and $\mathbb{C}[4\mathbb{C}^7] = \mathbb{C}[3V + \Lambda^6]$ respectively. Let $f_i \in \mathbb{C}[W]^{\text{SL}_7}$, $i \in I$ be a set of invariants, different from h . Denote by g_i their restrictions to $3V + z + \Lambda^6$. Then the proof of [28, Prop. 4.5] says that if the Hilbert series of the algebras $\mathbb{C}[4\mathbb{C}^7]^{G_2}$ and $\mathbb{C}[g_i, i \in I]$ are identical, then $\mathbb{C}[W]^{\text{SL}_7} = \mathbb{C}[h, f_i, i \in I]$.

Due to [27], the multivariate Hilbert series of $\mathbb{C}[4\mathbb{C}^7]^{G_2}$ is

$$\frac{(1 - t_1^2 t_2^2 t_3^2 t_4^2)}{\prod_{1 \leq i \leq j \leq 4} (1 - t_i t_j) \prod_{1 \leq k \leq 4} (1 - t_1 t_2 t_3 t_4 t_k^{-1}) (1 - t_1 t_2 t_3 t_4)}.$$

Consider all invariants from the proposition but h —a total of 16. The ideal of relations of the restrictions—denoted by Fraktur letters—of the 16 invariants to $3V + z + \Lambda^6$ is generated by

$$\begin{aligned} & \mathfrak{h}_{13}^2 \mathfrak{h}_{22} - 2\mathfrak{h}_{12} \mathfrak{h}_{13} \mathfrak{h}_{23} + \mathfrak{h}_{11} \mathfrak{h}_{23}^2 + 2\mathfrak{h}_{12}^2 \mathfrak{h}_{33} - 2\mathfrak{h}_{11} \mathfrak{h}_{22} \mathfrak{h}_{33} - 8\mathfrak{h}_{123}^2, \\ & \mathfrak{h}_3^2 \mathfrak{h}_{12}^2 - 2\mathfrak{h}_2 \mathfrak{h}_3 \mathfrak{h}_{12} \mathfrak{h}_{13} + \mathfrak{h}_2^2 \mathfrak{h}_{13}^2 - \mathfrak{h}_3^2 \mathfrak{h}_{11} \mathfrak{h}_{22} + 2\mathfrak{h}_1 \mathfrak{h}_3 \mathfrak{h}_{13} \mathfrak{h}_{22} + 2\mathfrak{h}_2 \mathfrak{h}_3 \mathfrak{h}_{11} \mathfrak{h}_{23} \\ & \quad - 2\mathfrak{h}_1 \mathfrak{h}_3 \mathfrak{h}_{12} \mathfrak{h}_{23} - 2\mathfrak{h}_1 \mathfrak{h}_2 \mathfrak{h}_{13} \mathfrak{h}_{23} + \mathfrak{h}_1^2 \mathfrak{h}_{23}^2 - 2\mathfrak{h}_2^2 \mathfrak{h}_{11} \mathfrak{h}_{33} + 4\mathfrak{h}_1 \mathfrak{h}_2 \mathfrak{h}_{12} \mathfrak{h}_{33} \\ & \quad - 2\mathfrak{h}_1^2 \mathfrak{h}_{22} \mathfrak{h}_{33} - 4\mathfrak{h}_{23} \tilde{\mathfrak{h}}_{12} \tilde{\mathfrak{h}}_{13} + 2\mathfrak{h}_{22} \tilde{\mathfrak{h}}_{13}^2 + 4\mathfrak{h}_{13} \tilde{\mathfrak{h}}_{12} \tilde{\mathfrak{h}}_{23} - 4\mathfrak{h}_{12} \tilde{\mathfrak{h}}_{13} \tilde{\mathfrak{h}}_{23} + 2\mathfrak{h}_{11} \tilde{\mathfrak{h}}_{23}^2 \\ & \quad - 8\mathfrak{h}_3 \tilde{\mathfrak{h}}_{12} \mathfrak{h}_{123} + 8\mathfrak{h}_2 \tilde{\mathfrak{h}}_{13} \mathfrak{h}_{123} - 8\mathfrak{h}_1 \tilde{\mathfrak{h}}_{23} \mathfrak{h}_{123} + 4\tilde{\mathfrak{h}}_{12}^2 \mathfrak{h}_{33} + 32\mathfrak{h}_{123123} \tilde{\mathfrak{h}} - 16\tilde{\mathfrak{h}}_{123}^2. \end{aligned}$$

The SINGULAR code of the computation is provided as supplemental material [6]. As both have a quadratic term in the two different variables $\mathfrak{h}_{123}$ and $\tilde{\mathfrak{h}}_{123}$ respectively, they form a Gröbner basis. Therefore, the Hilbert series of $\mathbb{C}[\mathfrak{h}_1, \dots, \tilde{\mathfrak{h}}_{123}]$ coincides with the one of $\mathbb{C}[4\mathbb{C}^7]^{G_2}$. By [28, Prop. 4.5], the algebra of invariants

of (SL_7, U_3) is generated by the invariants from the proposition. Moreover, by suitably multiplying with $\tilde{h}$ and h_{33} and adding hh_{123123} , we get the relations

$$\begin{aligned} & h_{13}^2 h_{22} - 2h_{12} h_{13} h_{23} + h_{11} h_{23}^2 + h_{33} h_{12}^2 - h_{11} h_{22} h_{33} - 8h_{123}^2 + hh_{123123}, \\ & h_3^2 h_{12}^2 - 2h_2 h_3 h_{12} h_{13} + h_2^2 h_{13}^2 - h_3^2 h_{11} h_{22} + 2h_1 h_3 h_{13} h_{22} \\ & \quad + 2h_2 h_3 h_{11} h_{23} - 2h_1 h_3 h_{12} h_{23} - 2h_1 h_2 h_{13} h_{23} + h_1^2 h_{23}^2 \\ & \quad - 2h_2^2 h_{11} h_{33} + 4h_1 h_2 h_{12} h_{33} - 2h_1^2 h_{22} h_{33} - 4h_{23} \tilde{h}_{12} \tilde{h}_{13} \\ & \quad + 2h_{22} \tilde{h}_{13}^2 + 4h_{13} \tilde{h}_{12} \tilde{h}_{23} - 4h_{12} \tilde{h}_{13} \tilde{h}_{23} + 2h_{11} \tilde{h}_{23}^2 - 8h_3 \tilde{h}_{12} h_{123} \\ & \quad + 8h_2 \tilde{h}_{13} h_{123} - 8h_1 \tilde{h}_{23} h_{123} + 4\tilde{h}_{12}^2 h_{33} - 16\tilde{h}_{123}^2 + 32h_{123123} \tilde{h}, \end{aligned}$$

that generate the ideal of relations of the ring of invariants. $\square$

3. The serial cases

In this section, let always $n = 2p + 1$. We consider the three SL_n -representations $V_1 := \Lambda^2 V + 2\Lambda^{2p} V$, $V_2 := V + \Lambda^2 V + \Lambda^{2p} V$, $V_3 := S^2 V$. The serial cases that were left open in [29] are

$$W_1 := 2V_1, \quad W_2 := V_1 + V_2, \quad W_3 := V_1 + V_3.$$

Our approach is similar in all cases and involves three steps, which we describe in the following. For the first step, let us denote the maximal unipotent subgroup of SL_n of lower triangular matrices by U , the *opposite* maximal unipotent subgroup of upper triangular matrices by U^o and the normalizing maximal torus by T . By Theorem 0.2 of [22], the algebra $\mathbb{C}[X \times Y]^{\mathrm{SL}_n}$ is a deformation of $(\mathbb{C}[X]^U \otimes \mathbb{C}[Y]^{U^o})^T$ for affine varieties X, Y and both algebras share the same Hilbert series with respect to a common SL_n -stable grading. We explicitly compute the algebra $A_i := (\mathbb{C}[V_1]^U \otimes \mathbb{C}[V_i]^{U^o})^T$ and its Hilbert series for $i = 1, 2, 3$. In order to do this, in Lemmata 10, 13, 15 we apply our graph method to algebras of covariants.

Now the explicit form of the A_i not only provides us degrees of potential syzygies for $\mathbb{C}[W_i]^{\mathrm{SL}_n}$, but also *very important parts* of them, because the syzygies of A_i turn out to be—as one would expect—contractions of syzygies holding in $\mathbb{C}[W_i]^{\mathrm{SL}_n}$. Not all of them deform to *generators* of the ideal of syzygies of $\mathbb{C}[W_i]^{\mathrm{SL}_n}$ —those that do not are responsible for A_i not being a complete intersection. But those that do represent *important parts*, because they have no variables in common and suggest a way to prove the complete intersection property.

In step two, we find explicit forms of syzygies for $\mathbb{C}[W_i]^{\mathrm{SL}_n}$, which is made possible by our graph theoretic method.

Finally in step three, we prove that the syzygies we found in step two generate the ideal of syzygies. In all three cases, we find a suitable monomial matrix order such that these syzygies are a Gröbner basis with respect to this order and derive the complete intersection property. The respective leading monomials also occur in the contracted versions of the syzygies in A_i . A posteriori, we see that A_i is just not the *best* contraction of $\mathbb{C}[W_i]^{\mathrm{SL}_n}$ for proving the complete intersection property: we can deform A_i to some algebra B_i that on the one hand lacks all the superfluous equations and generators of A_i and on the other hand keeps the important parts of the syzygies of $\mathbb{C}[W_i]^{\mathrm{SL}_n}$.

3.1. Rings of Covariants

Lemma 6. *The algebra of covariants $\text{Cov}(\text{SL}_{2p+1}, V_1)$ for $p \geq 2$ is minimally generated by the images of the following $4p + 3$ graphs, where black filled edges correspond to copies of Λ^{2p} in V_1 and grey edges correspond to tensors $e_1 \wedge \cdots \wedge e_k$, where k is such that every vertex is of degree $2p + 1$, that is k is the weight of the respective covariant.*

$$\begin{aligned}
 c_1 &= \text{graph with a loop on a grey vertex}, & c_2 &= \text{graph with two loops on a grey vertex}, & \dots, & c_p &= \text{graph with } p \text{ loops on a grey vertex}, \\
 c_1^{(a)} &= \text{graph with a loop on a black vertex labeled } a, & c_2^{(a)} &= \text{graph with a loop on a black vertex labeled } a \text{ and a grey edge}, & c_3^{(a)} &= \text{graph with a loop on a black vertex labeled } a \text{ and a grey loop}, & \dots, & c_{p+1}^{(a)} &= \text{graph with a loop on a black vertex labeled } a \text{ and a grey loop with } p \text{ sub-loops}, & a = 1, 2; \\
 c_1^* &= \text{graph with two black vertices connected by a grey edge}, & c_2^* &= \text{graph with two black vertices connected by a grey edge and a loop on the first}, & c_3^* &= \text{graph with two black vertices connected by a grey edge and a grey loop on the first}, & \dots, & c_{p+1}^* &= \text{graph with two black vertices connected by a grey edge and a grey loop with } p \text{ sub-loops on the first}.
 \end{aligned}$$

Moreover, the ideal of relations is generated by

$$c_p c_1^* - p c_1^{(1)} c_{p+1}^{(2)} + p c_1^{(2)} c_{p+1}^{(1)}.$$

Remark 4. It may look as if the series from the above theorem are not of a very intuitive — *serial* - form. But the only rule is that when the subscript of c increases by one, there is one 2-edge added. So, at some point it may happen that in order to do so, an additional vertex is required. This may look *unserial* though it is not. The same holds for the series in the following Lemmata 7 and 8.

Proof of Proposition 6. The multidegrees of the generating covariants can be deduced from [8, Table 1]. The explicit forms of the graphs for these covariants are then obvious.

Since the Krull dimension of $\mathbb{C}[W]^{U(G)}$ is $4p + 2$, see [8], we have one syzygy. We obtain this syzygy by considering the graph

$$\text{graph with two black vertices connected by a grey edge, with a loop on the first black vertex and a loop on the second black vertex}.$$

Applying the Plücker relation to the left non-looping 2-edge gives:

$$(p-1) \text{graph} = \text{graph with a loop on the first black vertex} + \text{graph with a loop on the second black vertex} - \text{graph with a loop on both black vertices}.$$

Now in the last graph on the right, we apply the Plücker relation to the non-looping 2-edge of color one and get:

$$(p-1) \text{graph} = \text{graph with a loop on the first black vertex} + \text{graph with a loop on the second black vertex} - \frac{1}{p} \text{graph with a loop on both black vertices}.$$

But due to the isomorphism from above, we can evaluate at e_1 and $e_1 \wedge e_2$ for the dummy 1- and 2-edge respectively and see that the two leftmost graphs disappear. We arrive at the desired relation. $\square$

Remark 5. Observe that in Lemma 6, we did not always choose the graph with a looping dummy edge for the invariant, but it is easy to deduce one version from another by applying the Plücker relation to the dummy edge.

Lemma 7. *The algebra of covariants $\text{Cov}(\text{SL}_{2p+1}, V_2)$ for $p \geq 2$ is polynomial, generated by the images of $c_1, \dots, c_p$ and $c_1^{(1)}, \dots, c_{p+1}^{(1)}$ from Lemma 6 and in addition by the images of the following graphs, where black filled edges correspond to copies of Λ^{2p} in V_1 and grey edges correspond to tensors $e_1 \wedge \dots \wedge e_k$, where k is such that every vertex is of degree $2p+1$.*

$$\begin{aligned} c_1^\circ &= \text{grey circle} \text{---} 1, & c_2^\circ &= \text{grey circle with loop} \text{---} 1, \dots, & c_{p+1}^\circ &= \text{grey circle with } p \text{ loops} \text{---} 1, \\ c_1^\diamond &= \text{black circle} \text{---} 1, & c_2^\diamond &= \text{black circle} \text{---} 1 \text{---} \text{grey circle}, & c_3^\diamond &= \text{black circle} \text{---} 1 \text{---} \text{grey circle with loop}, \dots, \\ & & c_p^\diamond &= \text{black circle} \text{---} 1 \text{---} \text{grey circle with } p-1 \text{ loops}. \end{aligned}$$

Proof. Due to the equivalence with the algebra of invariants of $U(\text{SL}_{2p+1})$, we get multidegrees of covariants from [8, Table 1], which in addition gives polynomiality. It is straightforward to find the only possible covariants of the matching multi-degrees. $\square$

Lemma 8. *The algebra of covariants $\text{Cov}(\text{SL}_n, V_3)$ is polynomial, generated by the images of the n graphs, where the jagged 2-edges correspond to the copy of S^2 in V_3 and grey edges correspond to tensors $e_1 \wedge \dots \wedge e_k$, where k is such that every vertex is of degree n .*

$$c_1^\triangle = \text{grey circle} \text{---} \text{jagged 2-edge} \text{---} \text{grey circle}, \quad c_2^\triangle = \text{grey circle with loop} \text{---} \text{jagged 2-edge} \text{---} \text{grey circle}, \dots, \quad c_n^\triangle = \text{grey circle with } n-1 \text{ loops} \text{---} \text{jagged 2-edge} \text{---} \text{grey circle}.$$

Proof. Here, we get polynomiality and multidegrees of covariants from [7, Thm. 3] and proceed as in Lemma 7. $\square$

3.2. The case (SL_{2p+1}, W_1)

Proposition 9. *The ring of invariants $\mathbb{C}[W_1]^{\text{SL}_{2p+1}}$ is a complete intersection. Its homological dimension is four if $p = 2$ and $2p - 4$ if $p \geq 3$. It is minimally generated by the images of the following graphs, where black filled edges correspond to copies of Λ^{2p} in W_1 .*

$$\begin{aligned} i_{abc} &= \text{graph with two black circles connected by a black edge labeled } c, \text{ and two grey circles connected by a jagged 2-edge labeled } b, \text{ with } ab \in \{12, 13, 14, 23, 24, 34\}, c \in \{1, 2\}; \\ j_{abc} &= \text{graph with a black circle at the bottom connected to a grey circle above it by a black edge labeled } c, \text{ and the grey circle connected to } a \text{ black circles above it by } a \text{ jagged 2-edges labeled } 1, \text{ and } b-1 \text{ jagged 2-edges labeled } 2, \text{ with } a, b \geq 1, c \in \{1, 2, 3, 4\}; \end{aligned}$$

$$\mathfrak{k}_{abc} = \begin{array}{c} \overbrace{1 \dots 1}^a \quad \overbrace{2 \dots 2}^{b-3} \\ \begin{array}{c} \text{diagram of a tree with root } e \text{ and children } d, f, \text{ and } e \text{ itself} \\ \text{with weights } 1, 2, 2, 2 \text{ on the edges} \end{array} \end{array}, \quad a, b \geq 3, def \in \{234, 134, 124, 123\}, c = \{1, 2, 3, 4\} \setminus \{d, e, f\}.$$

Lemma 10. Denote by x the images of the covariants c from Lemma 6 in $\mathbb{C}[V_1]^U$ and by y those in $\mathbb{C}[V_1]^{U^\circ}$. The algebra $A_1 = (\mathbb{C}[V_1]^U \otimes \mathbb{C}[V_1]^{U^\circ})^T$ is minimally generated by x_2^*, y_2^* and the entries of the matrices

$$f^{(1)} := \begin{bmatrix} x_2^{(1)} \\ x_2^{(2)} \end{bmatrix} \cdot \begin{bmatrix} y_1^{(1)} & y_1^{(2)} & y_p \end{bmatrix}, \quad f^{(2)} := \begin{bmatrix} x_1 \\ x_3^* \end{bmatrix} \cdot \begin{bmatrix} y_{p+1}^{(1)} & y_{p+1}^{(2)} & y_1^* \end{bmatrix},$$

$$g^{(1)} := \begin{bmatrix} y_2^{(1)} \\ y_2^{(2)} \end{bmatrix} \cdot \begin{bmatrix} x_1^{(1)} & x_1^{(2)} & x_p \end{bmatrix}, \quad g^{(2)} := \begin{bmatrix} y_1 \\ y_3^* \end{bmatrix} \cdot \begin{bmatrix} x_{p+1}^{(1)} & x_{p+1}^{(2)} & x_1^* \end{bmatrix},$$

$$h^{(k)} := \begin{bmatrix} x_{2+k} \\ x_{4+k}^* \end{bmatrix} \cdot \begin{bmatrix} y_{p-k}^{(1)} & y_{p-k}^{(2)} \end{bmatrix}, \quad i^{(k)} := \begin{bmatrix} y_{2+k} \\ y_{4+k}^* \end{bmatrix} \cdot \begin{bmatrix} x_{p-k}^{(1)} & x_{p-k}^{(2)} \end{bmatrix}, \quad k = 0, \dots, p-3.$$

The ideal of syzygies is generated by all 2×2 -minors of the matrices $f^{(j)}$, $g^{(j)}$, $h^{(k)}$, $i^{(k)}$ and

$$f_{a3}^{(1)} f_{b3}^{(2)} - p f_{a1}^{(1)} f_{b2}^{(2)} + p f_{a2}^{(1)} f_{b1}^{(2)}, g_{a3}^{(1)} g_{b3}^{(2)} - p g_{a1}^{(1)} g_{b2}^{(2)} + p g_{a2}^{(1)} g_{b1}^{(2)}, \quad a, b \in \{1, 2\}.$$

Proof. Under the isomorphism between $\text{Cov}(\text{SL}_n, V_1)$ and $\mathbb{C}[V_1]^U$, the images of the covariants from Lemma 6 have weights according to [8, Table 1]. From this, we directly deduce generators of the ring of invariants and the ideal of syzygies, similarly as in [3, Proof of Thm. 1.8]. $\square$

Proof of Proposition 9. The list of generators can be extracted from [29, Lem. 2.5], it is minimal due to [29, Proof of Thm. 0.2]. We come to the syzygies. Let us first have a look at the Hilbert series of A_1 with respect to the $\mathbb{Z}^6$ -grading of W_1 , where variables t_i correspond to copies of $\Lambda^2 V$ and s_i to $\Lambda^{n-1} V$ respectively. Lemma 10 provides us with generators and syzygies. We have

$$\begin{aligned} A_1 &= \mathbb{C}[x_2^*, y_2^*] \otimes \mathbb{C}[x_1, x_3^*, x_2^{(1)}, x_2^{(2)}, y_p, y_1^*, y_1^{(1)}, y_1^{(2)}, y_{p+1}^{(1)}, y_{p+1}^{(2)}]^T \\ &\quad \otimes \mathbb{C}[y_1, y_3^*, y_2^{(1)}, y_2^{(2)}, x_p, x_1^*, x_1^{(1)}, x_1^{(2)}, x_{p+1}^{(1)}, x_{p+1}^{(2)}]^T \\ &\quad \otimes \bigotimes_{k=0}^{p-3} \mathbb{C}[x_{2+k}, x_{4+k}^*, y_{p-k}^{(1)}, y_{p-k}^{(2)}]^T \\ &\quad \otimes \bigotimes_{k=0}^{p-3} \mathbb{C}[y_{2+k}, y_{4+k}^*, x_{p-k}^{(1)}, x_{p-k}^{(2)}]^T. \end{aligned}$$

The first three algebras have the following Hilbert series, in which we put

$$m_{ijk} := 1 - t_i s_j s_k : \quad (*)$$

$$\frac{1}{m_{112}m_{234}},$$

$$\frac{(1 - t_1^2 t_2^p s_1 s_3 s_4)(1 - t_1^2 t_2^p s_2 s_3 s_4)}{m_{113}m_{114}m_{123}m_{124}(1 - t_1 t_2^p s_1)(1 - t_1 t_2^p s_2)(1 - t_1 t_2^p s_3)(1 - t_1 t_2^p s_4)m_{134}},$$

$$\frac{(1 - t_1^p t_2^2 s_1 s_2 s_3)(1 - t_1^p t_2^2 s_1 s_2 s_4)}{m_{213}m_{223}m_{214}m_{224}(1 - t_1^p t_2 s_3)(1 - t_1^p t_2 s_4)(1 - t_1^p t_2 s_1)(1 - t_1^p t_2 s_2)m_{212}},$$

while the k th factors of the fourth and fifth algebra have the respective Hilbert series

$$\frac{(1 - t_1^{5+2k} t_2^{2p-2k-2} s_1 s_2 s_3 s_4)}{(1 - t_1^{2+k} t_2^{p-k-1} s_3)(1 - t_1^{2+k} t_2^{p-k-1} s_4)(1 - t_1^{3+k} t_2^{p-k-1} s_1 s_2 s_3 s_4)(1 - t_1^{3+k} t_2^{p-k-1} s_1 s_2 s_4)},$$

$$\frac{(1 - t_2^{5+2k} t_1^{2p-2k-2} s_1 s_2 s_3 s_4)}{(1 - t_2^{2+k} t_1^{p-k-1} s_1)(1 - t_2^{2+k} t_1^{p-k-1} s_2)(1 - t_2^{3+k} t_1^{p-k-1} s_1 s_3 s_4)(1 - t_2^{3+k} t_1^{p-k-1} s_2 s_3 s_4)}.$$

Now we distinguish between the cases $p = 2, 3 \geq 4$ as they behave differently. For $p = 2$, the Hilbert series of $\mathbb{C}[W_1]^{\text{SL}_5}$ is the product of those of the first three algebras from above. Each factor in the denominator corresponds to one generator. Now consider the graphs

$$\begin{array}{c} \text{a} \quad \text{b} \quad \text{c} \\ \text{---} \text{1} \text{---} \text{2} \text{---} \text{2} \text{---} \text{1} \text{---} \\ \text{---} \text{1} \end{array}, abc \in \{234, 134, 124, 123\}.$$

Applying the Plücker relation to the two edges of color 1, we get

$$2 \begin{array}{c} \text{a} \quad \text{b} \quad \text{c} \\ \text{---} \text{1} \text{---} \text{2} \text{---} \text{2} \text{---} \text{1} \text{---} \\ \text{---} \text{1} \end{array} = - \begin{array}{c} \text{a} \quad \text{b} \quad \text{c} \\ \text{---} \text{2} \text{---} \text{1} \text{---} \text{2} \text{---} \text{1} \text{---} \\ \text{---} \text{1} \end{array} + \begin{array}{c} \text{a} \quad \text{b} \quad \text{c} \\ \text{---} \text{2} \text{---} \text{1} \text{---} \text{2} \text{---} \text{1} \text{---} \\ \text{---} \text{1} \end{array}.$$

On the other hand, if we apply the Plücker relation to the right edge of color 2, we get

$$\begin{array}{c} \text{a} \quad \text{b} \quad \text{c} \\ \text{---} \text{1} \text{---} \text{2} \text{---} \text{2} \text{---} \text{1} \text{---} \\ \text{---} \text{1} \end{array} = - \begin{array}{c} \text{a} \quad \text{b} \quad \text{c} \\ \text{---} \text{1} \text{---} \text{2} \text{---} \text{1} \text{---} \text{2} \text{---} \\ \text{---} \text{2} \end{array} + \begin{array}{c} \text{a} \quad \text{b} \quad \text{c} \\ \text{---} \text{1} \text{---} \text{2} \text{---} \text{1} \text{---} \text{2} \text{---} \\ \text{---} \text{2} \end{array} - \begin{array}{c} \text{a} \quad \text{b} \quad \text{c} \\ \text{---} \text{1} \text{---} \text{2} \text{---} \text{1} \text{---} \text{2} \text{---} \\ \text{---} \text{2} \end{array}.$$

Finally applying the Plücker relation to the vertical edge of color 2 in the first graph on the right and on the two edges of the same color in the other ones, we get the syzygy

$$i_{ab1}j_{12c} - i_{ac1}j_{12b} + i_{bc1}j_{12a} + i_{ab2}j_{21c} - i_{ac2}j_{21b} + i_{bc2}j_{21a}.$$

The Hilbert series of the ideal generated by the resulting four syzygies for all combinations of a, b, c coincides with that of A_1 , so we are done with the case $p = 2$.

In the case $p = 3$, since the homological dimension is two, the complete intersection property follows by [29, Lem. 5.1], concrete syzygies can be obtained in the same way as for $p \geq 4$ in the following. So consider $p \geq 4$. Here in the decomposition of A_1 from above, apart from the first three algebras, we have one factor for each $k = 0, \dots, p - 3$ in the fourth and fifth one. In the resulting Hilbert series, the factors $(1 - t_i^p t_j^2 s_a s_b s_c)$ in the denominator and numerator cancel out and in the reduced fraction, again all denominator factors correspond to generators. So we need syzygies corresponding to $(1 - t_1^k t_2^{2p+3-k} s_1 s_2 s_3 s_4)$, $k = 4, \dots, 2p - 1$. One of k and $2p + 3 - k$ is always greater than $p + 1$. Let us assume that $k > p + 1$ in the following. Consider the graphs

These graphs are decomposable in *two ways*. The first one is applying the Plücker relation on all two-edges of color 2 but those two going up. In the resulting sum, all graphs are either disconnected or evaluate to zero, since they have a vertex with p looping two-edges of color 1 and one additional non-looping two-edge of color 1. For the second one, we observe that we can move a looping two-edge of color 1 from the rightmost to the central vertex by first applying the Plücker relation to all edges connected to the rightmost vertex — now the central and rightmost vertex are connected by a two-edge of color 1 — and then applying it to this connecting edge and all looping edges of color 1 at the central vertex. Iterating this procedure gives us a sum of disconnected graphs plus the graph

Now finally applying in this graph the Plücker relation to all two-edges of color 1 looping at the central vertex and the one that connects to the $(n - 1)$ -edge of color 1, we see that it is decomposable as well. The resulting relation has the form

$$f_k := \left(\sum_{i=1}^4 \sum_{\substack{a+c=k \\ b+d=2p+3-k}} j_{abi} f_{cdi} \right) + \left(\sum_{(ij,l,m) \vdash (1234)} \sum_{\substack{e=1,2 \\ a+c=k-\delta_{e1} \\ b+d=2p+3-k-\delta_{e2}}} i_{ije} j_{abl} j_{cdm} \right),$$

where we ignore coefficients as they are not important for what follows. The notation $(ij, k, l) \vdash (1234)$ means that we sum over all subdivisions of the word 1234 in two words of length one and one of length two, i.e., we have twelve summands.

The same procedure applies for $2p + 3 - k > p + 1$ by interchanging two-edges of colors 1 and 2, yielding respective syzygies in this case as well.

What remains is to show that the f_k are a Gröbner basis for the ideal of syzygies of $\mathbb{C}[W_1]^{\text{SL}_n}$. Consider the $\mathbb{Z}^{4n}$ grading of $\mathbb{C}[W_1]^{\text{SL}_n}$ given by sending $i_{abc}, j_{abc}, \mathfrak{k}_{abc}$ to respective basis elements $e_{abc}^i, e_{abc}^j, e_{abc}^{\mathfrak{k}}$. Denote the dual basis of $(\mathbb{Z}^{4n})^*$ by $x_{abc}^i, x_{abc}^j, x_{abc}^{\mathfrak{k}}$.

Now we need to find a monomial grading so that the leading monomials of each f_k have no variable in common. Let us construct a matrix $\mathcal{M}$ with $4n$ columns and of rank $4n$, so that the associated grading meets our requirements. Our first goal is that the monomials $j_{abi}\mathfrak{k}_{cdi}$ are greater than the $i_{ije}j_{abk}j_{cdl}$. This is achieved by setting the first row $\mathcal{M}_1$ of $\mathcal{M}$ to

$$\mathcal{M}_1 := \sum x_{abc}^i + \sum x_{abc}^j + \sum 3x_{abc}^{\mathfrak{k}}.$$

Now we present an algorithm determining a monomial grading satisfying our needs.

Algorithm 11 (Crosshair-sieve). We see each row of the matrix $\mathcal{M}$ as a sieve (of increasing fineness), that filters out some of the monomials we want to be the leading ones of the f_k .

Now for some row $\mathcal{M}_\nu$ consider a $(p-1) \times (p-4) \times 4$ -matrix (or tensor if you want) $\mathcal{S}_\nu$, where in the (r, s, t) th entry stands the degree of $j_{r(p+1-r)t}\mathfrak{k}_{(s+2)(p-s)t}$. We say that the first entry gives the *row*, the second one the *column* and the third one the *level* of the *sieve* $\mathcal{S}_\nu$. So entries in the same level and row correspond to monomials sharing some $j_{r(p+1-r)t}$ and such in the same level and column some $\mathfrak{k}_{(s+2)(p-s)t}$ respectively. Observe that the degrees of all monomials of some f_k stand in the counterdiagonal of $\mathcal{S}_\nu$ given by $r + s + 2 = k$ (remember $k = 4, \dots, 2p-1$).

We begin with $\mathcal{M}_\nu = 0$. Now if we want a monomial $j_{abi}\mathfrak{k}_{cdi}$ to be filtered out (i.e., to be the one of highest degree of some f_{a+c}), we target this monomial setting

$$\mathcal{M}_\nu := \mathcal{M}_\nu + x_{abi}^j + x_{cdi}^{\mathfrak{k}}.$$

At the t th level of $\mathcal{S}_\nu$, this looks like a crosshair. Exemplarily for the case $p = 9$, we have this picture:

0	0	0	1	0
0	0	0	1	0
1	1	1	2	1
0	0	0	1	0
0	0	0	1	0
0	0	0	1	0
0	0	0	1	0
0	0	0	1	0

Now $j_{abi}\mathfrak{k}_{cdi}$ is the leading monomial—with degree two—of f_{a+c} , but we have many leading monomials (of degree one) of other f_k 's that share a variable, which

we do not want. So we have to target all other desired leading monomials in the same way. If this happens at another level of $\mathcal{S}_\nu$, all is fine, but if two desired leading monomials are at the same level—and this must happen for $p \geq 3$, since we only have four levels but need $2p - 4$ leading monomials—we get unwanted crossings, as for example the italic ones in:

1	0	0	1	0
1	0	0	1	0
<i>2</i>	1	1	2	1
1	0	0	1	0
2	1	1	<i>2</i>	1
1	0	0	1	0
1	0	0	1	0
1	0	0	1	0

So it may happen that in an $\mathfrak{f}_k$ some unwanted monomials are of the same degree as the desired leading polynomials with respect to $\mathcal{M}_\nu$, so the sieve $\mathcal{S}_\nu$ is *too coarse*. But as long as it filters out at least one desired leading monomial, we do not have to target this one in the following sieve $\mathcal{S}_{\nu+1}$, so we get lesser unwanted crossings and may be able to filter out another desired leading monomial. *So if at each stage we can filter out at least one, the algorithm will terminate with the desired set of leading monomials.* If the resulting matrix is not of full rank, we add rows to achieve full rank and find the desired grading.

The termination of the algorithm obviously depends on the choice of the desired leading monomials, which results in a choice of arrangements of crosshairs eventually leading to a sieve that does not filter any desired leading monomial.

So we need to find a set of monomials for which the algorithm terminates. We do this by induction in p . Let $\mathcal{M}^p$ be the grading matrix for respective p with first row $\mathcal{M}_1^p := \sum x_{abc}^i + \sum x_{abc}^j + \sum 3x_{abc}^k$ as defined above. Let $\mathcal{S}^p$ be the associated collection of sieves with $\mathcal{S}_i^p$ corresponding to $\mathcal{M}_i^p$. For $p = 4$, apart from permuting the last index, we only have one choice for the leading monomials lm :

$$\begin{aligned} \text{lm}(\mathfrak{f}_4) &= j_{1p1}k_{3(p-1)1}, & \text{lm}(\mathfrak{f}_5) &= j_{2(p-1)2}k_{3(p-1)2}, & \text{lm}(\mathfrak{f}_6) &= j_{3(p-2)3}k_{3(p-1)3}, \\ & & \text{lm}(\mathfrak{f}_7) &= j_{4(p-3)4}k_{3(p-1)4}. \end{aligned}$$

Applying Algorithm 11 to this choice, the sieve $\mathcal{S}_2^4$ becomes

2	1	1	1
1	2	1	1
1	1	2	1
1	1	1	2

This sieve already filters out our desired monomials. We describe the first iteration step in detail: for $p = 5$, the sieves have one additional column and row, so they are of the form

Observe that all entries on the counterdiagonal planes correspond to monomials of one f_k . For $f_4, \dots, f_7$, we keep the choice of leading monomials, while for f_8 and f_9 we choose

$$\text{lm}(f_8) = j_{4(p-3)1} \mathbf{e}_{4(p-2)1}, \quad \text{lm}(f_9) = j_{5(p-4)2} \mathbf{e}_{4(p-2)2},$$

which results in $\mathcal{S}_2^5$ (with italic unwanted highest degrees) becoming

We see that $\text{lm}(f_9) = j_{5(p-4)2} \mathbf{e}_{4(p-2)2}$ and $\text{lm}(f_4) = j_{1p1} \mathbf{e}_{3(p-1)1}$ are filtered out and so in the next sieve $\mathcal{S}_3^5$ we do not have to target them, which results in

So $\mathcal{S}_3^5$ filters out the remaining leading monomials and we are done. Now for general p , choose the leading monomials for $f_4, \dots, f_{2p-3}$ as for $p-1$ and

$$\text{lm}(f_{2p-2}) = j_{(p-1)21} \mathbf{e}_{(p-1)31}, \quad \text{lm}(f_{2p-1}) = j_{p12} \mathbf{e}_{(p-1)32}.$$

With this choice, the desired leading monomial of f_{2p-1} (among others, but this is just a bonus) is obviously filtered out by $\mathcal{S}_2^p$. In $\mathcal{S}_3^p$, it must not be targeted any more, and since the corresponding entry is the only one right and below the one of $\text{lm}(f_{2p-2})$, at least this will now be filtered out by $\mathcal{S}_3^p$. So every $\mathcal{S}_\nu^p$ will filter out at least $\text{lm}(f_{2p-\nu+1})$, which means that we are done. The f_k with respective leading monomials are a Gröbner basis of the ideal of relations of $\mathbb{C}[W_1]^{\text{SL}_n}$ and thus it is a complete intersection. $\square$

Remark 6. In fact, we only need one row of $\mathcal{M}$ to achieve our desired leading monomials by subsequently scaling the $\mathcal{M}_\nu$ with highest index ν by a factor smaller than one and adding it to $\mathcal{M}_{\nu-1}$. With factor $1/2$, in the case $p = 5$ from above, the one remaining sieve then becomes

2	$\frac{5}{2}$	$\frac{3}{2}$	1	$\frac{3}{2}$	0	$\frac{3}{2}$	0
1	$\frac{3}{2}$	$\frac{3}{2}$	3	$\frac{5}{2}$	0	$\frac{3}{2}$	0
1	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	1	3	$\frac{3}{2}$	0
$\frac{5}{2}$	3	$\frac{3}{2}$	$\frac{3}{2}$	1	$\frac{3}{2}$	0	3
1	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	2	$\frac{3}{2}$	0	0

The grading matrix may then be filled up arbitrarily. Of course this gives a different grading but the leading monomials stay the same.

3.3. The case $(\mathrm{SL}_{2p+1}, W_2)$

Proposition 12. *The ring of invariants $\mathbb{C}[W_2]^{\mathrm{SL}_{2p+1}}$ is a complete intersection. Its homological dimension is two for $p = 2$ and $2p - 3$ for $p \geq 3$. It is minimally generated by the invariants*

$$\mathfrak{h}_{abc}, a, b \in \{12, 13, 23\}, c \in \{1, 2\}; \quad \mathfrak{j}_{abc}, a, b \geq 1, c \in \{1, 2, 3\}; \quad \mathfrak{k}_{ab4}, a, b \geq 3,$$

from Proposition 9 and in addition by the images of the following graphs, where black filled edges correspond to copies of Λ^{2p} in W_2 .

$$\begin{aligned} \mathfrak{h}_a &= \begin{array}{c} a \\ \bullet \\ | \\ 1 \end{array} \quad a \in \{1, 2, 3\}; \quad \mathfrak{l}_{ab} = \begin{array}{c} \overbrace{1 \quad 1}^a \quad \overbrace{2 \quad 2}^b \\ \vdots \\ \bullet \\ | \\ 1 \end{array} \quad , a, b \geq 0; \\ \mathfrak{m}_{abc} &= \begin{array}{c} \overbrace{1 \quad 1}^a \quad \overbrace{2 \quad 2}^{b-2} \\ \vdots \\ \bullet \quad \bullet \quad \bullet \\ | \quad | \quad | \\ d \quad 1 \quad e \end{array} \quad , a, b \geq 2, de \in \{23, 13, 12\}, c = \{1, 2, 3\} \setminus \{d, e\}. \end{aligned}$$

Lemma 13. *Denote by x the images of the c in $\mathbb{C}[V_1]^U$ and by y those in $\mathbb{C}[V_2]^{U^\circ}$. The algebra $A_2 = (\mathbb{C}[V_1]^U \otimes \mathbb{C}[V_2]^{U^\circ})^T$ is minimally generated by $x_2^*, y_{p+1}^\circ, y_1^\diamond$ and the entries of the matrices*

$$\begin{aligned} d &:= \begin{bmatrix} y_p^\circ \\ y_{p+1}^{(1)} \end{bmatrix} \cdot [x_1 \quad x_3^*], & e &:= \begin{bmatrix} y_p \\ y_1^{(1)} \end{bmatrix} \cdot \begin{bmatrix} x_2^{(1)} & x_2^{(2)} \end{bmatrix}, \\ f &:= \begin{bmatrix} y_1^\circ \\ y_2^{(1)} \end{bmatrix} \cdot \begin{bmatrix} x_1^{(1)} & x_1^{(2)} & x_p \end{bmatrix}, & g &:= \begin{bmatrix} y_1 \\ y_2^\diamond \end{bmatrix} \cdot \begin{bmatrix} x_{p+1}^{(1)} & x_{p+1}^{(2)} & x_1^* \end{bmatrix}, \end{aligned}$$

$$h^{(k)} := \begin{bmatrix} x_{2+k} \\ x_{4+k}^* \end{bmatrix} \cdot \begin{bmatrix} y_{p-k+1}^\circ & y_{p-k}^{(1)} \end{bmatrix}, i^{(k)} := \begin{bmatrix} x_{p-k}^{(1)} \\ x_{p-k}^{(2)} \end{bmatrix} \cdot [y_{2+k} \ y_{3+k}^\diamond], \quad k = 0, \dots, p-3.$$

The ideal of syzygies is generated by all 2×2 -minors of the matrices d , e , f , g , $h^{(k)}$, $i^{(k)}$ and

$$f_{a3}g_{b3} - pf_{a1}g_{b2} + pf_{a2}g_{b1}, \quad a, b \in \{1, 2\}.$$

Proof. We proceed exactly as in Lemma 10. $\square$

Proof of Proposition 12. The list of generators can be extracted from [29, Le. 2.5]; it is minimal due to [29, Proof of Thm. 0.2]. We come to the syzygies. Let us first have a look at the Hilbert series of A_2 with respect to the $\mathbb{Z}^6$ -grading of W_2 , where variables t_i , s_i and r correspond to copies of $\Lambda^2 V$, $\Lambda^{n-1} V$ and V respectively. Lemma 13 provides us with generators and syzygies. We have

$$\begin{aligned} A_2 = & \mathbb{C}[x_2^*, y_{p+1}^\circ, y_1^\diamond] \otimes \mathbb{C}[y_{p+1}^{(1)}, y_p^\circ, y_1^*, x_1, x_3^*]^T \otimes \mathbb{C}[y_1^{(1)}, y_p, x_2^{(1)}, x_2^{(2)}]^T \\ & \otimes \mathbb{C}[y_2^{(1)}, y_1^\circ, x_1^{(1)}, x_1^{(2)}, x_p, y_1, y_2^\diamond, x_{p+1}^{(1)}, x_{p+1}^{(2)}, x_1^*]^T \\ & \otimes \bigotimes_{k=0}^{p-3} \mathbb{C}[x_{2+k}, x_{4+k}^*, y_{p-k+1}^\circ, y_{p-k}^{(1)}]^T \\ & \otimes \bigotimes_{k=0}^{p-3} \mathbb{C}[y_{2+k}, y_{3+k}^\diamond, x_{p-k}^{(1)}, x_{p-k}^{(2)}]^T. \end{aligned}$$

The first four algebras have the following Hilbert series (in which we use the above notation $(*)$):

$$\begin{aligned} & \frac{1}{m_{112}(1-t_2^p r)(1-s_3 r)}, \\ & \frac{(1-t_1^3 t_2^{2p-1} s_1 s_2 s_3 r)}{(1-t_1 t_2^{p-1} r)(1-t_1 t_2^p s_3)(1-t_1^2 t_2^{p-1} s_1 s_2 r)(1-t_1^2 t_2^p s_1 s_2 s_3)}, \\ & \frac{(1-t_1^2 t_2^p s_1 s_2 s_3)}{(1-t_1 t_2^p s_1)(1-t_1 t_2^p s_2) m_{113} m_{123}}, \\ & \frac{(1-t_1^p t_2^2 s_1 s_2 s_3)(1-t_1^p t_2 s_1 s_2 r)}{(1-s_1 r)(1-s_2 r)(1-t_1^p r) m_{213} m_{223} (1-t_1^p t_2 s_3)(1-t_1^p t_2 s_1)(1-t_1^p t_2 s_2) m_{212}}, \end{aligned}$$

whereas the ones of the k th factors of the fifth and sixth algebra are

$$\begin{aligned} & \frac{(1-t_1^{2k+5} t_2^{2p-2k-3} s_1 s_2 s_3 r)}{(1-t_1^{2+k} t_2^{p-k-1} s_3)(1-t_1^{3+k} t_2^{p-k-1} s_1 s_2 s_3)(1-t_1^{2+k} t_2^{p-k-2} r)(1-t_1^{3+k} t_2^{p-k-2} s_1 s_2 r)}, \\ & \frac{(1-t_1^{2p-2k-2} t_2^{2k+4} s_1 s_2 s_3 r)}{(1-t_1^{p-k-1} t_2^{2+k} s_1)(1-t_1^{p-k-1} t_2^{2+k} s_2)(1-t_1^{p-k-1} t_2^{2+k} s_1 s_3 r)(1-t_1^{p-k-1} t_2^{2+k} s_2 s_3 r)}. \end{aligned}$$

As in the proof of Proposition 9, we distinguish between $p = 2, 3, \geq 4$. For $p = 2$, we have homological dimension two and according to the Hilbert series and the given 18 generators, there are two syzygies corresponding to the factors $(1 - t_1^2 t_2^2 s_1 s_2 s_3)$ and $(1 - t_1^3 t_2^3 s_1 s_2 s_3 r)$. The first one already occurred in $\mathbb{C}[W_1]^{\text{SL}_5}$; it is

$$i_{121}j_{123} - i_{131}j_{122} + i_{231}j_{121} + i_{122}j_{213} - i_{132}j_{212} + i_{232}j_{211}.$$

For an explicit form of the second one, consider the graphs

Applying the Plücker relation to the diagonal edges in each of both graphs gives a relation, which we can write—ignoring coefficients—as

$$\sum_{(i,j,k) \vdash (123)} h_{ij} j_{12j} j_{21k} + \sum_{(ij,k) \vdash (123)} (i_{ij1}(j_{12k} l_{11} + j_{21k} l_{02}) + i_{ij2}(j_{12k} l_{20} + j_{21k} l_{11})).$$

Now consider the case $p = 3$. Here as with W_1 , we encounter all types of invariants but $\mathfrak{k}_{ab4}$. We need three relations corresponding to $(t_1^k t_2^{8-k} s_1 s_2 s_3 r)$ for $k = 3, 4, 5$. Consider the graphs

Applying Plücker relations to them results—for $k = 3, 4, 5$ —in

$$\mathfrak{g}_k = \sum_{(i,j,l) \vdash (123)} \sum h_{ij} j_{abj} j_{cdl} + \sum_{(ij,l) \vdash (123)} \left(\sum j_{abl} \mathfrak{m}_{cdl} + \sum i_{ije} j_{abl} l_{cd} \right),$$

where in the second and fourth sum, we sum over $a + c = k, b + d = 2p + 2 - k$ and in the last one over $e = 1, 2$ and $a + c = k - \delta_{e1}, b + d = 2p + 2 - k - \delta_{e2}$. In the terms $j_{abl} \mathfrak{m}_{cdl}$, we observe that $a = b = 2$ must hold, so $\mathfrak{g}_3, \mathfrak{g}_4, \mathfrak{g}_5$ contain terms $j_{221} \mathfrak{m}_{13l}, j_{22l} \mathfrak{m}_{22l}, j_{22l} \mathfrak{m}_{31l}$ respectively. We apply the Crosshair-sieve-algorithm 11. First we make the $\mathfrak{m}$ the variables of highest degree and second, we filter out $j_{221} \mathfrak{m}_{131}, j_{222} \mathfrak{m}_{222}, j_{223} \mathfrak{m}_{313}$ as leading monomials of $\mathfrak{g}_3, \mathfrak{g}_4, \mathfrak{g}_5$, which shows that they form a Gröbner basis and that $\mathbb{C}[W_2]^{\text{SL}_7}$ is a complete intersection.

Now for $p \geq 4$, the situation is similar, but the relations $\mathfrak{g}_k$ now have additional terms:

$$\mathfrak{g}_k = \sum \mathfrak{k}_{ab4} l_{cd} + \sum_{(i,j,l) \vdash (123)} h_{ij} j_{abj} j_{cdl} + \sum_{(ij,l) \vdash (123)} \left(\sum j_{abl} \mathfrak{m}_{cdl} + \sum i_{ije} j_{abl} l_{cd} \right).$$

We apply the Crosshair-sieve-algorithm 11—in a preliminary step $\mathcal{M}_1$ making variables $\mathfrak{m}$ and $\mathfrak{k}$ the ones of highest degree—with the following sieves: $\mathcal{S}_\nu$ now

comprises two parts — a $p \times (p-2) \times 3$ -matrix $\mathcal{S}_{\nu,1}$ and a $(p+1) \times (p-3)$ -matrix $\mathcal{S}_{\nu,2}$, where in the (r, s, t) th entry of $\mathcal{S}_{\nu,1}$ stands the degree of $j_{r(p+1-r)t} m_{(s+1)(p-s)t}$ and in the (r, s) th entry of $\mathcal{S}_{\nu,2}$ the degree of $k_{(s+2)(p-s)4} l_{(r-1)(p+1-r)}$. So for $p = 4$, the sieves are of the form

Again, monomials occuring in one $\mathfrak{g}_k$ lie on the counterdiagonals. For $\mathfrak{g}_3, \mathfrak{g}_4, \mathfrak{g}_5$, we choose the same leading monomials as in the case $p = 3$, while for $\mathfrak{g}_6, \mathfrak{g}_7$, we take

$$\text{lm}(\mathfrak{g}_6) = l_{31} k_{334}, \quad \text{lm}(\mathfrak{g}_7) = j_{411} m_{321}.$$

With this choice $\mathcal{S}_2$ filters out all leading monomials but the ones of $\mathfrak{g}_4$ and $\mathfrak{g}_6$, which is done by $\mathcal{S}_3$. If for general p , we choose the same leading monomials as for $p-1$ and for the two additional relations we take

$$\text{lm}(\mathfrak{g}_{2p}) = j_{(p-1)22} m_{(p-3)42}, \quad \text{lm}(\mathfrak{g}_{2p-1}) = j_{p33} m_{(p-3)43},$$

by the same argument as in the proof of Proposition 9, each $\mathcal{S}_\nu^p$ filters out at least the leading monomial of the remaining $\mathfrak{g}_k$ with highest index, so the Crosshair-sieve-algorithm terminates, and the $\mathfrak{g}_k$ are a Gröbner basis and $\mathbb{C}[W_2]^{\text{SL}_n}$ is a complete intersection. $\square$

3.4. The case (SL_{2p+1}, W_3)

Let from now on all two-edges be of color one.

Proposition 14. *The ring of invariants $\mathbb{C}[W_3]^{\text{SL}_{2p+1}}$ is a complete intersection. Its homological dimension is $2p-2$. It is minimally generated by the invariant i_{121} from Proposition 9 and in addition by the images of the following graphs, where black filled edges correspond to copies of Λ^{2p} and jagged 2-edges correspond to copies of S^2 in W_2 respectively.*

Lemma 15. Denote by x the images of the c in $\mathbb{C}[V_1]^U$ and by y those in $\mathbb{C}[V_3]^{U^\circ}$. The algebra $A_3 = (\mathbb{C}[V_1]^U \otimes \mathbb{C}[V_3]^{U^\circ})^T$ is minimally generated by x_2^*, y_{2p+1}^Δ and the entries on and above the diagonal of the matrices

$$f := y_1^\Delta \cdot \begin{bmatrix} x_1^{(1)2} & x_1^{(1)}x_1^{(2)} & x_1^{(1)}x_p \\ x_1^{(1)}x_1^{(2)} & x_1^{(2)2} & x_1^{(2)}x_p \\ x_1^{(1)}x_p & x_1^{(2)}x_p & x_p^2 \end{bmatrix}, \quad g := y_2^\Delta \cdot \begin{bmatrix} x_{p+1}^{(1)2} & x_{p+1}^{(1)}x_{p+1}^{(2)} & x_{p+1}^{(1)}x_1^* \\ x_{p+1}^{(1)}x_{p+1}^{(2)} & x_{p+1}^{(2)2} & x_{p+1}^{(2)}x_1^* \\ x_{p+1}^{(1)}x_1^* & x_{p+1}^{(2)}x_1^* & x_1^{*2} \end{bmatrix},$$

$$h^{(k)} := y_{2k+1}^\Delta \cdot \begin{bmatrix} x_{p-k}^2 & x_{p-k}x_{p+2-k}^* \\ x_{p-k}x_{p+2-k}^* & x_{p+2-k}^2 \end{bmatrix}, \quad k = 1, \dots, p-1,$$

$$i^{(k)} := y_{2k+2}^\Delta \cdot \begin{bmatrix} x_{p+1-k}^{(1)2} & x_{p+1-k}^{(1)}x_{p+1-k}^{(2)} \\ x_{p+1-k}^{(1)}x_{p+1-k}^{(2)} & x_{p+1-k}^{(2)2} \end{bmatrix}, \quad k = 1, \dots, p-1.$$

The ideal of relations is generated by all 2×2 -minors of the matrices $f, g, h^{(k)}, i^{(k)}$ with at most one entry below the diagonal and in addition

$$f_{a3}g_{b3} - pf_{a2}g_{1b} + pf_{1a}g_{2b}, \quad a, b \in \{1, 2, 3\}.$$

Proof. We proceed exactly as in Lemmata 10 and 13. $\square$

Proof of Prop. 14. We have

$$A_3 = \mathbb{C}[x_2^*, y_{2p+1}^\Delta] \otimes \mathbb{C}[y_1^\Delta, y_2^\Delta, x_1^{(1)}, x_1^{(2)}, x_p, x_{p+1}^{(1)}, x_{p+1}^{(2)}, x_1^*]^T \\ \otimes \bigotimes_{k=1}^{p-1} \mathbb{C}[y_{2k+1}^\Delta, x_{p-k}, x_{p+2-k}^*]^T \otimes \bigotimes_{k=1}^{p-1} \mathbb{C}[y_{2k+2}^\Delta, x_{p+1-k}^{(1)}, x_{p+1-k}^{(2)}]^T.$$

With variables t, s and r_i corresponding to copies of $\Lambda^2 V$, $S^2 V$ and $\Lambda^{n-1} V$ respectively, the first two algebras and the k th factors of the third and fourth algebra have respectively the following Hilbert series:

$$\frac{1}{(1 - tr_1 r_2)(1 - s^{2p+1})}, \\ \frac{1 - s^3 t^2 p r_1^2 r_2^2}{(1 - st^{2p})(1 - sr_2 t^p)(1 - sr_2^2)(1 - sr_1 t^p)(1 - sr_1 r_2)(1 - sr_1^2)}, \\ \frac{1 - s^{4k+2} t^{4p-4k+2} r_1^2 r_2^2}{(1 - s^{2k+1} t^{2p-2k})(1 - s^{2k+1} t^{2p-2k+1} r_1 r_2)(1 - s^{2k+1} t^{2p-2k+2} r_1^2 r_2^2)}, \\ \frac{1 - s^{4k+4} t^{4p-4k} r_1^2 r_2^2}{(1 - s^{2k+2} t^{2p-2k} r_1^2)(1 - s^{2k+2} t^{2p-2k} r_1 r_2)(1 - s^{2k+2} t^{2p-2k} r_2^2)}.$$

By [29, Proof of Thm. 0.4], a minimal system of generators has the same set of multidegrees as the system of generators of A_3 from Lemma 15, where those generators must be checked for reducibility and in case omitted, for which A_3 has a syzygy of smaller or equal multidegree. This is the case for all entries of g and for $h_{2,2}^{(1)}$, where invariants having the same multidegree can be shown to be reducible by applying Plücker relations to two-edges. We arrive at the proposed minimal system of generators of $\mathbb{C}[W_3]^{\text{SL}_{2p+1}}$. Concerning the syzygies, the Hilbert series tells us to search for ones corresponding to $(1 - s^{4k+2}t^{4p-4k+2}r_1^2r_2^2)$ and $(1 - s^{4k+4}t^{4p-4k}r_1^2r_2^2)$. Consider the reducible graphs

for $a = 2k + 1$ in the left and $a = 2k + 2$ in the right one. For both types, there are *two ways* to transform them into a sum of disconnected ones, so we get a syzygy for each of them. These two ways are the following: for the left one, consider the Plücker relation applied to the two-edges connecting the upper and lower part, but for the first way pulling both to the upper vertices and for the second way pulling one to the upper and one to the lower vertices. This results in two equivalent sums of reducible graphs, containing $\mathfrak{r}_a^2$ and $\mathfrak{q}_a \mathfrak{t}_a$ respectively. We do not list the other summands but remark that this is exactly the part of the relation that reflects in the algebra A_3 as the determinant of $h^{(k)}$.

For the right graph, consider for the first way the Plücker relation applied to the two-edges connecting the upper and middle part, and for the second way, which is more important for us, the Plücker relation applied to the horizontal non-looping $2p$ -edges, both pulled towards the middle. The resulting syzygy contains — resulting from the second way — $\mathfrak{s}_{a12}^2$ and $\mathfrak{s}_{a11}\mathfrak{s}_{a22}$, which again are exactly the part of the relation reflecting in A_3 , but now as the determinant of $i^{(k)}$.

Finally, we apply the Crosshair-sieve-algorithm 11 to show that $\mathbb{C}[W_3]^{\text{SL}_n}$ is a complete intersection. This is done by choosing the $\mathfrak{r}_a^2$ and $\mathfrak{s}_{a12}^2$ as leading monomials. The sieves here can be viewed as part-matrices up and above the diagonal, which contain in the (i, j) th entry the degree of $\mathfrak{r}_{2+i}\mathfrak{r}_{2+j}$, $\mathfrak{r}_{2+i}\mathfrak{s}_{(2+j)12}$, $\mathfrak{s}_{(2+i)12}\mathfrak{r}_{2+j}$ and $\mathfrak{s}_{(2+i)12}\mathfrak{s}_{(2+j)12}$ for $(i, j) \in \{(2\mathbb{Z}+1)^2, (2\mathbb{Z}+1) \times 2\mathbb{Z}, 2\mathbb{Z} \times (2\mathbb{Z}+1), 2\mathbb{Z}^2\}$ respectively. By the multidegrees of the syzygies, monomials that may occur in one relation lie on a counterdiagonal containing an entry on the diagonal (corresponding to a square monomial), as is shown in the following example picture:

Now as the targeted monomials lie on the diagonal, $\mathcal{S}_1$ filters out $\mathfrak{s}_3^2$ and $\mathfrak{s}_{(2p)12}^2$, then $\mathcal{S}_2$ filters out $\mathfrak{s}_{412}^2$ and $\mathfrak{r}_{(2p-1)}^2$, et cetera. So we found a Gröbner basis for the ideal of relations of $\mathbb{C}[W_3]^{\mathrm{SL}_n}$ and it is a complete intersection. $\square$

Proof of Theorem 1. By Propositions 3, 4, 5, 9, 12, 14, the six representations

$$(\mathrm{SL}_5, 2\Lambda^2 + \Lambda^3 + V^*), \quad (\mathrm{SL}_5, 3\Lambda^2 + V^*), \quad (\mathrm{SL}_7, 3V + \Lambda^3 + V^*), \\ (\mathrm{SL}_n, 2\Lambda^2 + 4V^*), \quad (\mathrm{SL}_n, V + 2\Lambda^2 + 3V^*), \quad (\mathrm{SL}_n, S^2 + \Lambda^2 + 2V^*), \quad 5 \leq n \in 2\mathbb{Z} + 1,$$

have a complete intersection invariant ring. We denote them by $r^1, \dots, r^6$, in the order from above. The table from Theorem 1 contains all representations from [29, Table 9] plus the ones from above, integrated in the following way: r^1 is number 12 of the table, while r^2 and r^3 are numbers 13 and 16 respectively. The representations r^4 and r^5 are cases $k = 0$ and $k = 1$ of number 5 respectively (these cases have been excluded in [29, Table 9]). Finally, the representation r^6 is case $k = 0$ for n odd of number 7; this was excluded in [29, Table 9] as well. $\square$

References

- [1] О. М. Адамович, Е. О. Головина, *Простые линейные группы, имеющие свободную алгебру инвариантов*, Вопросы теории групп и гомолог. алгебры, Ярославль, 1979, 3–41. Engl. transl.: O. M. Adamovich, E. O. Golovina, *Simple linear Lie groups having a free algebra of invariants*, Selected reprints, Selecta Math. Soviet. **3** (1983/84), no. 2, 183–220.
- [2] G. E. Andrews, P. Paule, A. Riese, *MacMahon’s partition analysis: the Omega package* European J. Combin. **22** (2001), no. 7, 887–904.
- [3] I. Arzhantsev, L. Braun, J. Hausen, M. Wrobel, *Log terminal singularities, platonic tuples and iteration of Cox rings*, Eur. J. Math. **4** (2018), no. 1, 242–312.
- [4] L. Braun, *Invariant rings of sums of fundamental representations of SL_n and colored hypergraphs*, [arXiv:1807.09642](https://arxiv.org/abs/1807.09642) (2018).
- [5] L. Braun, *Covariants of $\mathrm{SL}_4(\mathbb{C})$ and incidence geometry of $\mathbb{P}^3(\mathbb{C})$* , J. Phys.:Conf. Ser. **1194** (2019), 012015.
- [6] L. Braun, *Supplemental material to Completing the classification of representations of SL_n with complete intersection invariant ring*, available as ancillary files at [arXiv:1812.01978](https://arxiv.org/abs/1812.01978) (2018).
- [7] M. Brion, *Invariants d’un sous-groupe unipotent maximal d’un groupe semi-simple*, Ann. Inst. Fourier (Grenoble) **33** (1983), no. 1, 1–27.
- [8] M. Brion, *Représentations exceptionnelles des groupes semi-simples*, Ann. Sci. École Norm. Sup. (4) **18** (1985), no. 2, 345–387.
- [9] B. Buchberger, *A theoretical basis for the reduction of polynomials to canonical forms*, ACM SIGSAM Bull. **10** (1976), no. 3, 19–29.
- [10] W. Decker, G.-M. Greuel, G. Pfister, H. Schönemann, *SINGULAR 4-1-2—A computer algebra system for polynomial computations*, <http://www.singular.uni-kl.de> (2019).
- [11] H. Derksen, G. Kemper, *Computational Invariant Theory*, Encyclopaedia of Mathematical Sciences, Vol. 130, Subseries: *Invariant Theory and Algebraic Transformation Groups*, Vol. I, Springer-Verlag, Berlin, 2002.

- [12] V. G. Kac, V. L. Popov, E. B. Vinberg, *Sur les groupes linéaires algébriques dont l'algèbre des invariants est libre*, C. R. Acad. Sci. Paris Sér. A-B **283** (1976), no. 12, A1, A875–A878.
- [13] G. Kemper, N. Viet Trung, *Krull dimension and monomial orders*, J. Algebra **399** (2014), 782–800.
- [14] K. Fukuda, A. N. Jensen, R. R. Thomas, *Computing Gröbner fans*, Math. Comp. **76** (2007), no. 260, 2189–2212.
- [15] J. H. Grace, A. Young, *The Algebra of Invariants*, Cambridge Library Collection, Cambridge University Press, Cambridge, 2010.
- [16] F. D. Grosshans, G. C. Rota, J. A. Stein, *Invariant Theory and Superalgebras*, CBMS Regional Conference Series in Mathematics, Vol. 69, American Mathematical Society, Providence, RI, 1987.
- [17] B. Howard, J. Millson, A. Snowden, R. Vakil, *The ideal of relations for the ring of invariants of n points on the line*, J. Eur. Math. Soc. **14** (2012), no. 1, 1–60.
- [18] P. Olver, Ch. Shakiban, *Graph theory and classical invariant theory*, Adv. Math. **75** (1989), no. 2, 212–245.
- [19] V. Larsson, M. Oskarsson, K. Åström, A. Wallis, Z. Kukelova, T. Pajdla, *Beyond Gröbner bases: basis selection for minimal solvers*, [arXiv:1803.04360](https://arxiv.org/abs/1803.04360) (2018).
- [20] T. Mora, L. Robbiano, *The Gröbner fan of an ideal*, Computational aspects of commutative algebra, J. Symbolic Comput. **6** (1988), no. 2-3, 183–208.
- [21] H. Nakajima, *Representations of a reductive algebraic group whose algebras of invariants are complete intersections*, J. Reine Angew. Math. **367** (1986), 115–138.
- [22] D. Panyushev, *On deformation method in invariant theory*, Ann. Inst. Fourier (Grenoble) **47** (1997), no. 4, 985–1012.
- [23] В. Л. Попов, *Сизигии в теории инвариантов*, Изв. АН СССР. Сер. матем. **47** (1983), вып. 3, 544–622. Engl. transl.: V. L. Popov, *Syzgies in the theory of invariants*, Math. USSR-Izv. **22** (1984), no. 3, 507–585.
- [24] Э. Б. Винберг, В. Л. Попов, *Теория инвариантов*, Алгебраическая геометрия–4, Итоги науки и техн., Соврем пробл. матем., Фундам. направления, том 55, ВИНТИ, М., 1989, стр. 137–309. Engl. transl.: V. L. Popov, E. B. Vinberg, *Invariant Theory*, in: Encyclopaedia of Mathematical Sciences, Vol. 55, *Algebraic Geometry IV*, Springer-Verlag, Berlin, 1994, pp. 123–284.
- [25] L. Robbiano, *Term orderings on the polynomial ring*, EUROCAL '85, Vol. 2 (Linz, 1985), Lecture Notes in Comput. Sci. **204** (1985), 513–517.
- [26] G. W. Schwarz, *Representations of simple Lie groups with regular rings of invariants*, Invent. Math. **49** (1978), no. 2, 167–191.
- [27] G. W. Schwarz, *Invariant theory of G_2 and Spin_7* , Comment. Math. Helv. **63** (1988), no. 4, 624–663.
- [28] D. A. Shmelkin, *On algebras of invariants and codimension 1 Luna strata for non-connected groups*, Geom. Dedicata **72** (1998), no. 2, 189–215.
- [29] D. A. Shmelkin, *On representations of SL_n with algebras of invariants being complete intersections*, J. Lie Theory **11** (2001), no. 1, 207–229.
- [30] D. A. Shmelkin, *First fundamental theorem for covariants of classical groups*, Adv. Math. **167** (2002), no. 2, 175–194.

- [31] B. Sturmfels, *Gröbner Bases and Convex Polytopes*, University Lecture Series, Vol. 8, American Mathematical Society, Providence, RI, 1996.
- [32] G. Xin, *A fast algorithm for MacMahon's partition analysis*, Electron. J. Combin. **11** (2004), no. 1, research paper 58, 20 pp.
- [33] G. Xin, *A Euclid style algorithm for MacMahon's partition analysis*, J. Combin. Theory Ser. A **131** (2015), 32–60.

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