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CROSS RATIOS ON BOUNDARIES OF SYMMETRIC SPACES AND EUCLIDEAN BUILDINGS

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Abstract. We generalize the natural cross ratio on the ideal boundary of a rank one symmetric space, or even CAT(−1) space, to higher rank symmetric spaces and (non-locally compact) Euclidean buildings. We obtain vector valued cross ratios defined on simplices of the building at infinity. We show several properties of those cross ratios; for example that (under some restrictions) periods of hyperbolic isometries give back the translation vector. In addition, we show that cross ratio preserving maps on the chamber set are induced by isometries and vice versa, — motivating that the cross ratios bring the geometry of the symmetric space/Euclidean building to the boundary.

Introduction

Cross ratios on boundaries are a crucial tool in hyperbolic geometry and more general negatively curved spaces. In this paper we show that we can generalize these cross ratios to (the non-positively curved) symmetric spaces of higher rank and thick Euclidean buildings with many of the properties of the cross ratio still valid.

On the boundary $\partial_\infty \mathbb{H}^2$ of the hyperbolic plane $\mathbb{H}^2$ there is naturally a multiplicative cross ratio defined by

$$\mathrm{cr}_{\mathbb{H}^2}(z_1, z_2, z_3, z_4) = \frac{z_1 - z_2}{z_1 - z_4}\frac{z_3 - z_4}{z_3 - z_2}$$

when considering $\mathbb{H}^2$ in the upper half space model, i.e., $\partial_\infty \mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$. This cross ratio plays an essential role in hyperbolic geometry. For example it characterizes the isometry group by its boundary action and therefore allows us to study the geometry of the space from its boundary; which is an important perspective in hyperbolic geometry.

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This cross ratio can be generalized in a far broader context, namely CAT(-1) spaces [7]: Let $\partial_\infty Y$ be the ideal boundary of a CAT(-1) space Y, $x, y \in \partial_\infty Y$ and $o \in Y$. Then the *Gromov product* $(\cdot \mid \cdot)_o : \partial_\infty Y^2 \to [0, \infty]$ is defined by $(x \mid y)_o = \lim_{t\to\infty} t - \frac{1}{2} d(\gamma_{ox}(t), \gamma_{oy}(t))$, where γ_{ox}, γ_{oy} are the unique unit speed geodesics from o to x, y, respectively. Then an *additive cross ratio* $\mathrm{cr}_{\partial_\infty Y} : \mathcal{A} \subset \partial_\infty Y^4 \to [0, \infty]$ is defined by

$$\mathrm{cr}_{\partial_\infty Y}(x, y, z, w) := -(x \mid y)_o - (z \mid w)_o + (x \mid w)_o + (z \mid y)_o$$

for all $(x, y, z, w) \in \partial_\infty Y^4$ with no entry occurring three or four times; which is independent of the basepoint. For the hyperbolic plane the additive cross ratio corresponds to $\log |\mathrm{cr}_{\mathbb{H}^2}|$. By construction $\mathrm{cr}_{\partial_\infty Y}$ has several symmetries with respect to $(\mathbb{R}, +)$. In analogy to the hyperbolic plane, maps $f : \partial_\infty Y \to \partial_\infty Y$ that leave $\mathrm{cr}_{\partial_\infty Y}$ under the diagonal action invariant are called *Moebius maps.* It follows from the definition of the cross ratio together with the basepoint independence that isometries are Moebius maps when restricted to the boundary.

The cross ratios $\mathrm{cr}_{\partial_\infty Y}$ and Moebius maps have been proven to be very useful in hyperbolic geometry. For example Bourdon [8] has shown that Moebius maps of rank one symmetric spaces extend uniquely to isometric embeddings of the interior, and with this he gave a new proof of Hamenstädt's 'entropy against curvature' theorem [15]. Otal [28] has (implicitly) shown that Moebius bijections on boundaries of universal covers of closed negatively-curved surfaces can be uniquely extended to isometries; which yields that marked length spectrum rigidity holds for those manifolds, a prominent conjecture formulated in [10]. See [12], [19], [20] for more results in that context. Moreover, there is a close relation between the cross ratio on the boundary of the universal cover of a closed negatively curved manifold and the quasi-conformal structure on the boundary, and to dynamical properties of the geodesic flow; see, e.g., [26].

On the boundary $\partial_\infty \widetilde{S}$ of the universal cover of a closed surface S there are many other cross ratios, besides the above constructed one, that parametrize classical objects associated to the surface; such as simple closed curves, measured laminations, points of Teichmüller space [6], Hitchin representations [25] and positively ratioed representations [27][2], to name a few.

This prominence and importance of cross ratios in negative curvature motivates us to ask if such objects also exists for non-positively curved spaces and how much information about the geometry they carry.

There is already some work done in this context. In [11] a coarse cross ratio for arbitrary CAT(0) spaces on some subset of the boundary has been constructed. In [3] there is a cross ratio defined on the Roller boundary of a CAT(0) cube complex, using essentially the combinatorial structure of the space. In those works Moebius (respectively quasi-Moebius) bijections are connected to isometries (respectively quasi-isometries).

In this paper we will construct cross ratios for symmetric spaces and Euclidean buildings, which will generalize the cross ratios of CAT(-1) spaces. There is little

[2]We will see that the cross ratios associated to Hitchin representations and positively ratioed representations arise as pullbacks (under the natural boundary map) of cross ratios that we construct in this paper.

need to explain the importance of symmetric spaces in differential geometry and related areas. However, we want to point out that the study of symmetric spaces has recently gained renewed prominence in the active field of research of Anosov representations and Anosov subgroups (e.g., [24], [18], [14] and many more). We will see that the cross ratios we construct are connected to the study of those (e.g., [25], [27]) and hence we hope for applications of our work in this area.

Euclidean buildings arise in many different areas of mathematics. See [17] for an overview of some applications. Probably most prominently they arise in the study of algebraic groups and geometric group theory; they have also been a crucial tool in the proof of quasi-isometric rigidity of symmetric spaces [22] (extending Mostow–Prasad rigidity), to name a few.

We will denote by M either a symmetric space or a thick Euclidean building. It is well known that the ideal boundary $\partial_\infty M$ has naturally the structure of a spherical building $\Delta_\infty M$. Therefore there is a *type map* $\mathrm{typ} : \partial_\infty M \to \sigma$ with σ the closed fundamental chamber of the spherical Coxeter complex associated to M. Then we show that to each *type* $\xi \in \sigma$ there is $\iota\xi \in \sigma$ such that the Gromov product (defined exactly as for CAT(-1) spaces) restricted to the set $\mathrm{typ}^{-1}(\xi)\times\mathrm{typ}^{-1}(\iota\xi)$ is generically finite. Thus we get a generically defined *additive* cross ratio on $(\mathrm{typ}^{-1}(\xi) \times \mathrm{typ}^{-1}(\iota\xi))^2$ in the same way as for CAT(-1) spaces. We can show that this cross ratio is independent of the choice of basepoint; and denote it by cr_ξ.

Let τ be a face of the simplex σ, $\mathrm{int}(\tau)$ the interior of τ and $\xi \in \mathrm{int}(\tau)$. Moreover, we denote by $\mathrm{Flag}_\tau(M) \subset \Delta_\infty M$ the set of simplices of the building at infinity of type τ (i.e., those simplices that are mapped to τ under typ); in particular $\mathrm{Flag}_\sigma(M)$ is the chamber set of the building at infinity. Then one can naturally identify $\mathrm{typ}^{-1}(\xi)$ with $\mathrm{Flag}_\tau(M)$ and in the same way $\mathrm{typ}^{-1}(\iota\xi)$ with $\mathrm{Flag}_{\iota\tau}(M)$. This yields a cross ratio $\mathrm{cr}_\xi : \mathcal{A}_\tau \subset (\mathrm{Flag}_\tau(M) \times \mathrm{Flag}_{\iota\tau}(M))^2 \to [-\infty, \infty]$, which by construction has similar symmetries as the additive one on CAT(-1) spaces; for $\mathcal{A}_\tau$ see equation (1), for the symmetries see equation (3).

Clearly, we get a whole collection of cross ratios defined on the set $\mathcal{A}_\tau$ which is parametrized by $\xi \in \mathrm{int}(\tau)$. Then we show that we can put together this collection to a single vector valued cross ratio cr_τ with the same symmetries, and values in the Coxeter complex associated to M. We will see that the vector valued cross ratio is the natural object to consider; we can connect the so-called *period* $\mathrm{cr}_\sigma(g^-, g \cdot x, g^+, x)$ of a hyperbolic element $g \in \mathrm{Iso}(M)$ (with attractive and repulsive fixed points $g^\pm \in \mathrm{Flag}_\sigma(X)$ and generic $x \in \mathrm{Flag}_\sigma(X)$) to the translation vector of g along the unique maximal flat joining g^- and g^+, and we give a 'nice' geometric interpretation of the vector valued cross ratio.

Let M_1, M_2 be either two symmetric spaces or two thick Euclidean buildings. Let σ_1, σ_2 be the respective fundamental chambers of the spherical Coxeter complexes and let $\xi_i \in \mathrm{int}(\sigma_i)$ be two types. Let $f : \mathrm{Flag}_\sigma(M_1) \to \mathrm{Flag}_\sigma(M_2)$ be surjective. If $\mathrm{cr}_{\xi_1}(x, y, z, w) = \mathrm{cr}_{\xi_2}(f(x), f(y), f(z), f(w))$ for all $(x, y, z, w) \in \mathcal{A}_{\sigma_1}$, f is called ξ_1-*Moebius bijection*; if $\mathrm{cr}_{\sigma_1}(x, y, z, w) = \mathrm{cr}_{\sigma_2}(f(x), f(y), f(z), f(w))$ for all $(x, y, z, w) \in \mathcal{A}_{\sigma_1}$ f is called σ_1-*Moebius bijection*. Moreover, we call a locally compact Euclidean building with discrete translation group a *combinatorial Euclidean building* and a Euclidean building *thick* if and only if the building at infinity is thick. Then we show the following:

Theorem A. *Let M_1, M_2 be either symmetric spaces or thick combinatorial Euclidean buildings and $\xi_1 \in \mathrm{int}(\sigma_1)$. If M_1, M_2 are irreducible, then every ξ_1-Moebius bijection $f\colon \mathrm{Flag}_\sigma(M_1) \to \mathrm{Flag}_\sigma(M_2)$ can be extended to an isometry $F : M_1 \to M_2$. If none of the spaces is a Euclidean cone over a spherical building, then this extension is unique. If M_1, M_2 are reducible one can rescale the metric of M_1 on irreducible factors — denote this space by $\widehat{M_1}$, such that f can be extended to an isometry $F : \widehat{M_1} \to M_2$.*

Theorem B. *Let E_1, E_2 be thick (non-locally compact) Euclidean buildings. Then for every σ_1-Moebius bijection $f\colon \mathrm{Flag}_\sigma(E_1) \to \mathrm{Flag}_\sigma(E_2)$ one can rescale the metric of E_1 on irreducible factors — denote this space by $\widehat{E}_1$ — such that f can be extended to an isometry $F\colon \widehat{E}_1 \to E_2$. If none of the irreducible factors is a Euclidean cone over a spherical building, then f can be extended to an isometry $F\colon E_1 \to E_2$ (without rescaling the metric).*

We remark that essentially by definition of the cross ratio every isometry gives rise to a Moebius bijection. Therefore these theorems show that the cross ratios, at least for the chamber set of the building at infinity, carry a lot of the geometric information of the space, as they characterize isometries by their boundary action. In this spirit we hope that those cross ratios will be a valuable tool in the studies of symmetric spaces and Euclidean buildings.

We want to refer the reader to Section 4 for slightly more results in this spirit, e.g., when we get a one-to-one correspondence of Moebius bijections and isometries, and also an analysis of situations in which the rescaling of the metric is really necessary.

Concerning the proofs of those theorems: First we show that Moebius bijections split as products of Moebius bijections of irreducible factors; and that Moebius bijections can be extended to building isomorphisms. For rank one symmetric spaces and rank one thick Euclidean buildings it is already known that Moebius bijections extend to isometries. For irreducible thick combinatorial Euclidean buildings it will be enough that Moebius maps are restrictions of building isomorphisms to the chamber set. For symmetric spaces and (general) thick Euclidean buildings, we derive additional properties of the building map, using the cross ratio. Those properties will allow us to use theorems (essentially due to Tits) showing that the respective maps can be extended to isometries.

The structure of this paper is as follows. In the preliminaries we recall well known facts of symmetric spaces and Euclidean buildings (we assume the reader to be familiar with those objects) and show basic lemmas we need later on. In Section 3 we define $\mathbb{R}$-valued cross ratios, show basic properties, and illustrate the objects with two examples. In Section 4 we show that the collections of $\mathbb{R}$-valued cross ratios fit together to vector valued cross ratios and suggest that these are the natural objects to consider. In the last section, Section 5, we show that Moebius maps on the chamber set extend to isometries.

Related work. In [21] I. Kim constructed a cross ratio very similar to our $\mathbb{R}$-valued cross ratio (Definition 2.5). Labourie [25] has given one of the cross ratios in

Example 2.11 ad hoc and used it as a tool to understand Hitchin representations. Martone and Zhang [27] have constructed cross ratios on boundaries of surface groups, which in particular for $\mathrm{SL}(n,\mathbb{R})$-Hitchin representations coincide with the pullback under the boundary map of some of the cross ratios in Example 2.11. In [30] (see also [5]) there is a Gromov product defined, which is closely related to ours.

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1. Preliminaries

We use the notation that M is either a symmetric space of non-compact type or a thick Euclidean building, X is a symmetric space of non-compact type and E is a thick Euclidean building. In the case of a symmetric space when writing *affine apartment* we mean a maximal flat.

A reference for symmetric spaces of non-compact type is, e.g., [13]; for Euclidean buildings we refer to [23], [29], [32] and also [22].[3]

Coxeter complex and spherical buildings ([1]). Let W be a finite Coxeter group and S the standard set of generators consisting of involutions. Then W can be realized as a reflection group along hyperplanes in $\mathbb{R}^r$ with $r = |S|$. The hyperplanes decompose $\mathbb{R}^r$ and the unit sphere S^{r-1} into (cones over) simplicial cells. The maximal, i.e., r-dimensional, closed cells in $\mathbb{R}^r$ are called *Weyl sectors*. Lower dimensional cells will be called *conical cells*. The maximal, i.e., $r-1$-dimensional, closed simplicial cells in S^{r-1} are called *Weyl chambers*. The set S corresponds to exactly the hyperplanes bounding a Weyl sector. This Weyl sector will be called the *positive sector*, the corresponding chamber in S^{r-1} will be called the *positive chamber*. We can give each simplex adjacent to the positive chamber or positive sector a different label. Then the action of W on the simplicial complex induces a unique labeling for all simplices. A fixed label will be called a *type*.

In this paper we refer to $(\mathbb{R}^r, W)$ as the *Coxeter complex* and to (S^{r-1}, W) as the *spherical Coxeter complex*.

A *spherical building* is a simplicial complex B together with a collection of subcomplexes $\mathrm{Apt}(B)$, called *apartments*, which are isomorphic to a fixed spherical Coxeter complex (S^{r-1}, W), such that the following holds:

(1) For any two simplices $a, b \in B$ there is an apartment $A \in \mathrm{Apt}(B)$ with $a, b \in A$.
(2) If A, A' are apartments containing the simplices a, b, then there is a type preserving simplicial isomorphism $A \to A'$ fixing a, b.

We say that the building is *modelled over the spherical Coxeter complex* (S^{r-1}, W).

A spherical building is called *thick* if each non-maximal simplex is contained in at least three chambers. A (spherical) Coxeter complex is called *irreducible* if the

[3]We will use the definition due to [32], which is equivalent to the axioms in [23] and [29], while the definition in [22] would additionally assume metrical completeness.

Coxeter group can not be written as a product $W = W_1 \times W_2$ of two nontrivial Coxeter groups. A spherical building is called *irreducible* if the spherical Coxeter complex over which it is modelled is irreducible. If a building B is reducible, i.e., modelled over the spherical Coxeter complex $W_1 \times W_2$, then it can be written as the spherical join of two buildings, i.e., $B = B_1 \circ B_2$ for two spherical buildings B_1, B_2 modelled over W_1, W_2 respectively and $\circ$ being the spherical join [22, Sec. 3.3].

Given a simplex $x \in B$ with B a thick spherical building. The *residue of* x is given by $\mathrm{Res}(x) := \{y \in B \mid x \subsetneq y\}$. Let A be an apartment containing x, i.e., a Coxeter complex containing x. Let W be the Coxeter group of A and denote by W_x the stabilizer of x under W. If x is not a chamber then $\mathrm{Res}(x)$ is itself a spherical building modeled over the Coxeter complex to W_x [33, 3.12].

Euclidean buildings ([23], [29], [32], [22]). Let $\widehat{W}$ be an affine Coxeter group, i.e., $\widehat{W}$ can be realized as a subgroup of the isometry group of $\mathbb{R}^r$ and can be decomposed as a semi-direct product $\widehat{W} = W \ltimes T_W$, where W is a finite reflection group and $T_W < \mathbb{R}^r$ is a co-bounded subgroup of translations. Here we assume $r = |S|$, where S is the standard generating set of W. Moreover, let (E, d) be a metric space. A *chart* is an isometric embedding $\phi : \mathbb{R}^r \to E$, and its image is called an*affine apartment*; the image of a Weyl sector and *conical cells* are again called *Weyl sectors* and *conical cells*. Two charts ϕ, ψ are called $\widehat{W}$*-compatible* if $Y = \phi^{-1}\psi(\mathbb{R}^r)$ is convex in the Euclidean sense and if there is an element $w \in \widehat{W}$ such that $\psi \circ w|_Y = \phi|_Y$. A metric space E together with a collection of charts $\mathcal{C}$, called an *apartment system*, is called a *Euclidean building (modelled over the Coxeter group* $\widehat{W}$*)* if it has the following properties:

1. (1) For all $\phi \in \mathcal{C}$ and $w \in \widehat{W}$, the composition $\phi \circ w$ is in $\mathcal{C}$.
2. (2) Any two points $p, q \in E$ are contained in some affine apartment.
3. (3) The charts are $\widehat{W}$-compatible.
4. (4) If $a, b \subset E$ are Weyl sectors, then there exists an affine apartment A such that the intersections $A \cap a$ and $A \cap b$ contain Weyl sectors.
5. (5) If A is an affine apartment and $p \in A$ a point, then there is a 1-Lipschitz retraction $\rho : E \to A$ with $d(p, q) = d(p, \rho(q))$ for all $q \in E$.

From these properties it follows that the metric space E is necessarily CAT(0). The dimension of $\mathbb{R}^r$ is called the *rank* of E, i.e., $\mathrm{rk}(E) = r$. While the definition depends on a fixed set of affine apartments, there is always a unique maximal set of affine apartments, called the *complete apartment system.* A set is an affine apartment in the complete apartment system if and only if it is isometric to $\mathbb{R}^r$. In the ongoing *we will always consider* E *with its complete apartment system.* If the subgroup of translations T_W is discrete and E is locally compact we call E a *combinatorial* Euclidean building.

Symmetric spaces ([13, Chap. 2]). Let X be a symmetric space. We will always assume that X is *of non-compact type* and be $d : X \times X \to [0, \infty)$ the natural metric. Moreover, be $G = \mathrm{Iso}_0(X)$, i.e., the connected component of the identity of the isometry group.

Let $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the Cartan decomposition. Fixing a maximal flat F in X together with a basepoint $o \in F$ yields the identification $T_oM \cong \mathfrak{p}$. This identification is such that $T_oF \cong \mathfrak{a}$ where $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$. The restricted root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$ defines hyperplanes in $\mathfrak{a}$ — namely the zero sets of the restricted roots. The Weyl group W of X is the group generated by the reflections along those hyperplanes with respect to the metric that $\mathfrak{a}$ inherits from $T_oF \subset T_oX$. Hence we can associate to X a Coxeter complex $(\mathfrak{a}, W)$. Let $\mathfrak{a}_1$ be the unit sphere in $\mathfrak{a}$, then we also get a spherical Coxeter complex $(\mathfrak{a}_1, W)$. It is well known that up to isometry the Coxeter complex is independent of the choices. We fix a Weyl sector in $\mathfrak{a}$ which we denote by $\mathfrak{a}^+$ and call a *positive sector*. Then $\mathfrak{a}_1^+$ will be called the *positive chamber*.[4] The *rank* of X is the usual rank and equals $\mathrm{rk}(X) = \dim \mathfrak{a}$. To keep the notation consistent with buildings we will call maximal flats in X *affine apartments*.

The ideal boundary and Busemann functions ([9, Part II, Chap. 8]). We denote by $\partial_\infty M$ the ideal boundary; equipped with the cone topology $\partial_\infty M$ is naturally a topological space. For every $o \in M$ and every $x \in \partial_\infty M$ we denote by γ_{ox} the unique unit-speed geodesic ray joining o to x, i.e., $\gamma_{ox}(0) = o$ and γ_{ox} in the class of x. For $o, p, q \in M$ the *Gromov product on M* is defined by

$$(p \,|\, q)_o = \tfrac{1}{2}(d(o,p) + d(o,q) - d(p,q)).$$

Let $o \in M$ and $x, y \in \partial_\infty M$. Then $(\cdot \,|\, \cdot)_o : \partial_\infty M \times \partial_\infty M \to [0, \infty]$, the *Gromov product* with respect to o, is given by

$$(x \,|\, y)_o = \lim_{t\to\infty} (\gamma_{ox}(t) \,|\, \gamma_{oy}(t))_o = \lim_{t\to\infty} t - \tfrac{1}{2} d(\gamma_{ox}(t) \,|\, \gamma_{oy}(t)).$$

We remark that the convexity of the distance function guarantees the existence of the limit in $[0, \infty]$.

Given $x \in \partial_\infty M$ the *Busemann function with respect to x*, which will be denoted by $b_x : M \times M \to (-\infty, \infty)$, is defined by

$$b_x(o,p) = \lim_{t\to\infty} d(o, \gamma_{px}(t)) - d(p, \gamma_{px}(t)) = \lim_{t\to\infty} d(o, \gamma_{px}(t)) - t.$$

It holds that $-d(o,p) \leq b_x(o,p) = -b_x(p,o) \leq d(o,p)$ and $b_x(o,p) + b_x(p,q) = b_x(o,q)$ for $o, p, q \in M$. Moreover, it follows directly that $b_x(o, \gamma_{ox}(s)) = s$ for all $s \geq 0$ and for all $s \in \mathbb{R}$ if γ_{ox} is extended bi-infinitely.

An easy argument in Euclidean geometry yields that the level sets of Busemann functions in $\mathbb{R}^n$ with respect to x in the boundary sphere are affine hyperplanes orthogonal to the direction x. In general Busemann level sets with respect to one coordinate are called *horospheres* and the collection of horospheres is independent of the choice of the other coordinate.

The isometry group $\mathrm{Iso}(M)$ acts naturally by homeomorphisms on $\partial_\infty M$, since they map equivalence classes of geodesic rays to equivalence classes of geodesic rays. Moreover, by definition of the Busemann function, it follows $b_x(o,p) = b_{g\cdot x}(g\cdot o, g\cdot p)$ for every $g \in \mathrm{Iso}(M)$.

[4] Usually $\mathfrak{a}^+$ is called a positive Weyl chamber. However, as we will consider Euclidean buildings and symmetric spaces at the same time and we want to distinguish between spherical chambers and cones, we change the usual notation.

The building at infinity ([13, Chap. 3], [23], [29], [32], [22]). Let M now be either a symmetric space or a Euclidean building. To keep notation simple, we will denote by $(\mathfrak{a}, W)$ also the Coxeter complex over which a Euclidean building is modeled. Moreover, $\mathfrak{a}_1$ is the unit sphere in $\mathfrak{a}$ and hence $(\mathfrak{a}_1, W)$ a spherical Coxeter complex. We fix a positive Weyl sector $\mathfrak{a}^+ \subset \mathfrak{a}$ and the respective positive chamber $\mathfrak{a}_1^+ = \mathfrak{a}_1 \cap \mathfrak{a}^+$. Let S denote the generating set of W consisting of reflections along the walls of $\mathfrak{a}^+$. By definition we have $\mathrm{rk}(M) = \dim \mathfrak{a}$.

The ideal boundary $\partial_\infty M$ carries naturally the structure of a spherical building $\Delta_\infty M$ modeled over the spherical Coxeter complex $(\mathfrak{a}_1, W)$. The building $\Delta_\infty M$ will be called the *building at infinity.*

For a Euclidean building E the building at infinity arises as follows: Let $A \subset E$ be an affine apartment. Then A being the image of $(\mathfrak{a}, W)$ under a chart implies that A is decomposed into conical cells. Each conical cell defines a simplex in $\partial_\infty E$ by taking the geodesic rays contained in the cell for all times. One can show that two conical cells define the same set in $\partial_\infty E$ if and only if they have finite Hausdorff distance. In the latter case we say the conical cells are equivalent. Taking all conical cells in E modulo the equivalence relation yields a simplicial structure on $\partial_\infty E$; which can be shown to be a spherical building over the spherical Coxeter complex $(\mathfrak{a}_1, W)$.

In a very similar way we get the building at infinity of symmetric spaces X: Every maximal flat F with fixed basepoint can be isometrically identified with $\mathfrak{a}$. Then the conical cells of $\mathfrak{a}$ descend to conical cells in $F \subset X$. Again taking all conical cells in X modulo the equivalence relation of finite Hausdorff distance gives $\partial_\infty X$ a simplicial structure, which yields a spherical building modeled over $(\mathfrak{a}_1, W)$.

Apartments in $\Delta_\infty M$ correspond to the ideal boundaries of affine apartments of M. It is well known that $\Delta_\infty X$ is a thick building. We call a Euclidean building *thick* if in the case $\mathrm{rk}(E) \geq 2$ we have that $\Delta_\infty E$ is thick, and in the case $\mathrm{rk}(E) = 1$ we have that $|\partial_\infty E| \geq 3$, i.e., $E \neq \mathbb{R}$.

In particular the following important property holds: To every two points $p, q \in M \cup \partial_\infty M$ we find an affine apartment A in M such that $p, q \in A \cup \partial_\infty A$. We say that A *joins* p and q.

Given two affine apartments A, A' in a Euclidean building E that have a common chamber at infinity, i.e., $c \in \Delta_\infty E$ such that $c \subset \partial_\infty A$ and $c \subset \partial_\infty A'$, then the intersection $A \cap A'$ contains a Weyl sector with c being its boundary at infinity. Such a Weyl sector is called a *common subsector* of A and A'.

The type map ([22, Sec. 4.2.1], [18, Sec. 2.4]). To the visual boundary $\partial_\infty M$ with the building structure $\Delta_\infty M$ there exists a map $\mathrm{typ} : \partial_\infty M \to \mathfrak{a}_1^+$, called *type map*. Given $x \in \partial_\infty M$ there is a chamber $c_x \in \Delta_\infty M$ with $x \in c_x$ and an affine apartment A with $c_x \subset \partial_\infty A$. Then this yields an isometry from c_x to $\mathfrak{a}_1^+$ with respect to the Tits metric on c_x and the angular metric on $\mathfrak{a}_1^+$. In this way we can assign to each element of $\partial_\infty M$ an unique element of $\mathfrak{a}_1^+$. It can be shown that the image is independent of the chamber and the apartment chosen, hence we get a well-defined map $\mathrm{typ} : \partial_\infty M \to \mathfrak{a}_1^+$. The type map is consistent with the types of the spherical building $\Delta_\infty M$, i.e., two simplices of $\Delta_\infty M$ are of the same type if and only if they are mapped to the same face of $\mathfrak{a}_1^+$ under typ. Hence we also call the faces of $\mathfrak{a}_1^+$ *types* ($\mathfrak{a}_1^+$ will be a face of itself). When speaking of types we denote $\sigma = \mathfrak{a}_1^+$, i.e., a simplex of $\Delta_\infty M$ is a chamber if and only if it is of type σ. Faces of σ will usually be denoted by τ. The set of simplices in $\Delta_\infty M$ of type τ will be denoted by $\mathrm{Flag}_\tau(M)$, or just by Flag_τ if M is clear from the context and will be called *flag space*. If we consider chambers we denote this by Flag_σ and call it *full flag space*.

We (ambiguously) call elements in $\xi \in \sigma = \mathfrak{a}_1^+$ *types*. However, from the context it is clear if an element or a simplex is meant. We denote by $\mathrm{int}(\tau)$ the interior of a simplex (and set the interior of a point to be the point itself). Given a simplex $x \in \mathrm{Flag}_\tau(M)$ and $\xi \in \tau$, we denote by x_ξ the unique point in $x \subset \partial_\infty M$ of type ξ.

Let $F : M_1 \to M_2$ be an isometry between either two symmetric spaces or two thick Euclidean buildings. Restricting F to the ideal boundary $\partial_\infty M_1$ induces a building isomorphism $F_\infty : \Delta_\infty M_1 \to \Delta_\infty M_2$. The map F_∞ is in general not type preserving. However, that M_1, M_2 are isometric implies that they are modeled over the same Coxeter complex and hence have the same fundamental chamber σ. Then we can associate to F a type map $F_\sigma : \sigma \to \sigma$ such that $\mathrm{typ}(F_\infty(x)) = F_\sigma(\mathrm{typ}(x))$ for every $x \in \partial_\infty M_1$ and F_σ is an isometry with respect to the angular metric. Moreover, $F(\mathrm{Flag}_\tau(M_1)) = \mathrm{Flag}_{F_\sigma(\tau)}(M_2)$.

The G-action and flag manifolds ([13, Chap. 3], [18, Sec. 2.4]). Let X be a symmetric space and $G = \mathrm{Iso}_0(X)$. Then the cone topology on $\partial_\infty X$ induces a topology on $\Delta_\infty X$ such that all flag spaces are compact. Moreover, given $x \in \mathrm{Flag}_\tau(X)$, let P_x denote the stabilizer of x under the G-action. Then we can identify $\mathrm{Flag}_\tau(X) \simeq G/P_x$ with the identification being G-equivariant and homeomorphic; the group P_x is a parabolic subgroup of G and G/P_x is equipped with the quotient topology of the topological group G. Moreover, $\mathrm{Flag}_\tau(X) \simeq G/P_x$ yields a smooth structure on $\mathrm{Flag}_\tau(X)$ (inherited from G/P_x) making it a compact connected manifold. The spaces G/P_x are called *Furstenberg boundaries* or *flag manifolds* (motivating our notion of flag space). Let K be a maximal compact subgroup of G. Then already K acts transitively on the flag manifolds and given $x \in \mathrm{Flag}_\tau(X)$ we can identify $\mathrm{Flag}_\tau(X) \simeq K/K_x$ K-equivariant and homeomorphically, where $K_x = \mathrm{stab}_K(x)$. Moreover, we remark that the G-action is type preserving, i.e., $g_\sigma = \mathrm{id}$ for all $g \in G$.

The opposition involution. An important map for us will be the *opposition involution* $\iota : \mathfrak{a} \to \mathfrak{a}$, which is given by $\iota = -\mathrm{id} \circ w_0$ with $w_0 \in W$ the maximal element of the Coxeter group with respect to the generating set S. If W is an irreducible Weyl group, then $\iota = id$ if and only if W is not of type A_n with $n \geq 2$, D_{2n+1} with $n \geq 2$ or E_6 [33, 2.39]. Moreover, we remark that we can restrict $\iota : \mathfrak{a}_1^+ \to \mathfrak{a}_1^+$ and that ι is an isometry with respect to the angular metric.

Opposite simplices ([18, Sec. 2.2, 2.4]). There is a natural notion of *opposition* in spherical buildings. This corresponds to the following: Let $x, y \in \Delta_\infty M$ and let A_∞ be an apartment in $\Delta_\infty M$ such that $x, y \in A_\infty$. Since A_∞ can be identified with the unit sphere $\mathfrak{a}_1$, there is a natural map $-\mathrm{id} : A_\infty \to A_\infty$. Then x is the *opposite* of y, denoted by $x \;\mathsf{op}\; y$, if and only if $x = -\mathrm{id}(y)$. The action of the spherical Coxeter group W leaves the type invariant. Therefore, assume for the moment that W is modeled in A_∞ and x is a face of the positive chamber. Denote by $w_0 : A_\infty \to A_\infty$ the maximal element of W. Then $w_0(y)$ is a face of the positive chamber and of the same type as y and hence y is of type $-\mathrm{id} \circ w_0(x) = \iota x$. Hence all simplices opposite of elements in Flag_τ are contained in $\mathrm{Flag}_{\iota\tau}$. For later use we denote

$$\begin{aligned} \mathcal{A}_\tau^{\mathtt{op}} &:= \{(x_1, y_1, x_2, y_2) \in (\mathrm{Flag}_\tau \times \mathrm{Flag}_{\iota\tau})^2 \,|\, x_1, x_2 \;\mathtt{op}\; y_1, y_2\}, \\ \mathcal{A}_\tau &:= \{(x_1, y_1, x_2, y_2) \in (\mathrm{Flag}_\tau \times \mathrm{Flag}_{\iota\tau})^2 \,|\, x_i \,\mathtt{op}\, y_i \text{ or } x_i \,\mathtt{op}\, y_j, i, j = 1, 2, i \neq j\}. \end{aligned} \tag{1}$$

Opposition of simplices has the following important connection to bi-infinite geodesics: Let $z_1, z_2 \in \partial_\infty M$ and $A \subset M$ an affine apartment with $z_1, z_2 \in \partial_\infty A$. Then one can show that there exists a bi-infinite geodesics joining z_1 and z_2 if and only if there exists one in A. From Euclidean geometry it follows that the z_i can be joined by a bi-infinite geodesic in A if and only if $z_1 = -\mathrm{id}(z_2)$ with $-\mathrm{id} : \partial_\infty A \to \partial_\infty A$ as before. This can easily be seen to be equivalent to the unique simplices $\tau_{z_i} \in \Delta_\infty M$ containing the z_i in its interior being opposite, i.e., $\tau_{z_1} \;\mathtt{op}\; \tau_{z_2}$, and $\mathrm{typ}(z_1) = \iota\mathrm{typ}(z_2)$.

We will call points $z_1, z_2 \in \partial_\infty M$ *opposite* if they can be joined by a bi-infinite geodesic and denote this also by $z_1 \;\mathtt{op}\; z_2$. Moreover, for every $\xi \in \tau$ and $(x, y) \in \mathrm{Flag}_\tau \times \mathrm{Flag}_{\iota\tau}$ with $x \;\mathtt{op}\; y$, it follows that x_ξ is opposite to $y_{\iota\xi}$.

Symmetric spaces, Langlands decomposition ([13, Sec. 2.17], [18, Sec. 2.10]). In case of a symmetric space X and given $x \in \mathrm{Flag}_\tau(X)$, the set of simplices opposite to x is an open and dense subset of $\mathrm{Flag}_{\iota\tau}(X)$ (which can be deduced from the Bruhat decomposition of G/P). Moreover, for $(x, y) \in \mathrm{Flag}_\tau(X) \times \mathrm{Flag}_{\iota\tau}(X)$ we have $x \;\mathtt{op}\; y$ if and only if the pair is in the unique open and dense G-orbit in $\mathrm{Flag}_\tau(X) \times \mathrm{Flag}_{\iota\tau}(X)$. In particular, it follows in this case that $\mathcal{A}_\tau$ and $\mathcal{A}_\tau^{\mathtt{op}}$ are open and dense subsets of $(\mathrm{Flag}_\tau \times \mathrm{Flag}_{\iota\tau})^2$.

Every parabolic subgroup P_x has a natural decomposition $P_x = K_x A_x N_x$ called the *Langlands decomposition*, where K_x is compact and N_x is nilpotent. The group N_x is called *horospherical subgroup* and is unique, while K_x and A_x are not. The horospherical subgroup has several important properties; it leaves the Busemann function with respect to $x_\xi \in x \in \mathrm{Flag}_\tau(X)$ invariant, i.e., $b_{x_\xi}(o, p) = b_{x_\xi}(n \cdot o, p) = b_{x_\xi}(o, n \cdot p)$ for all $n \in N_x$ and $\xi \in \tau$; given a geodesic ray γ_{x_ξ} with endpoint in $x \subset \partial_\infty X$, we have $d(\gamma_{x_\xi}(t), n \cdot \gamma_{x_\xi}(t)) \to 0$ for $t \to \infty$ for all $n \in N_x$; moreover, N_x acts simply transitive on the set of simplices opposite to x. If x is a chamber, i.e., $x \in \mathrm{Flag}_\sigma(M)$, then N_x acts simply transitive on the set of maximal flats containing x in its boundary.

Parallel sets ([13, Sec. 2.11, 2.20], [18, Sec. 2.4], [22, Sec. 4.8]). Let (x, y) be a point of $\mathrm{Flag}_\tau(M) \times \mathrm{Flag}_{\iota\tau}(M)$ with $x \;\mathtt{op}\; y$ and let ξ be an element of $\mathrm{int}(\tau)$. Then the *parallel set* with respect to x, y, denoted by $P(x, y)$, is the set of all points that lie on a bi-infinite geodesic joining x_ξ to $y_{\iota\xi}$.

The parallel sets split metrically as products, i.e., $P(x, y) \simeq F_{xy} \times CS(x, y)$, where F_{xy} is an isometrically embedded $\mathbb{R}^n$ such that $x, y \subset \partial_\infty F_{xy}$ and x, y are simplices of maximal dimension in the sphere $\partial_\infty F_{xy}$, in particular the dimension of the spherical simplices x, y equals $n - 1$. Then it follows that the parallel set is independent of the choice of type $\xi \in \mathrm{int}(\tau)$, as for each type $\xi \in \mathrm{int}(\tau)$ geodesics in M joining $x_\xi, y_{\iota\xi}$ are of the form $(\gamma_{x_\xi y_{\iota\xi}}(t), p)$ with $\gamma_{x_\xi y_{\iota\xi}}$ a geodesic in F_{xy} joining $x_\xi, y_{\iota\xi}$ and p is a point in $CS(x,y)$.

The space $CS(x,y)$ is called a *cross section*. In the case of a symmetric space X the cross section is itself a symmetric space without Euclidean de Rham factors; in the case of a Euclidean building the cross section is again a Euclidean building. In both cases the rank is given by $\mathrm{rk}(CS(x,y)) = \mathrm{rk}(M) - \dim F_{xy}$.

Let τ be a face of $\sigma = \mathfrak{a}_1$. Let $\mathfrak{a}_\tau$ be the subspace of $\mathfrak{a}$ defined by τ, i.e., the smallest subspace of $\mathfrak{a}$ containing τ and 0. Let $\xi_1, \dots, \xi_k \in \mathfrak{a}$ be the corners of the spherical simplex τ. Then $\mathfrak{a}_\tau = \mathrm{span}_{i=1,\dots,k}\, \xi_i$. It is immediate that we can also identify $P(x,y) \simeq \mathfrak{a}_\tau \times CS(x,y)$. We can additionally impose that this identification is such that $x \simeq \partial_\infty \mathfrak{a}_\tau^+$ where $\mathfrak{a}_\tau^+ := (\mathfrak{a}_\tau \cap \mathfrak{a}^+)$.

Lemma 1.1. *Let $(x,y) \in \mathrm{Flag}_\tau \times \mathrm{Flag}_{\iota\tau}$ with x op y and be $p,q \in P(x,y)$. Let $\pi : P(x,y) \simeq \mathfrak{a}_\tau \times CS(x,y) \to \mathfrak{a}_\tau$ be the projection to the first factor. Then for each $\xi \in \tau$ we have that $b_{x_\xi}(p,q) = (b_{x_\xi})|_{\mathfrak{a}_\tau}(\pi(p), \pi(q))$, i.e., the Busemann function is independent of the second factor of the product.*

Proof. Let γ_{qx_ξ} denote the geodesic ray from q to x_ξ. Moreover, be $q = (q_1, q_2)$ under the identification $P(x,y) \simeq \mathfrak{a}_\tau \times CS(x,y)$. Then we have that $\gamma_{qx_\xi} \simeq (\gamma_{q_1 x_\xi}, q_2)$ where $\gamma_{q_1 x_\xi}$ is the geodesic ray in $\mathfrak{a}_\tau$ from q_1 to x_ξ. Using that metrically $P(x,y) \simeq \mathfrak{a}_\tau \times CS(x,y)$ and $p = (p_1, p_2)$ we derive the equality$d(p, \gamma_{qx_\xi}(t)) = \sqrt{d(p_1, \gamma_{q_1 x_\xi}(t))^2 + d(p_2, q_2)^2)}$. If we set $K_2 := d(p_2, q_2)^2$, then we have $b_{x_\xi}(p,q) = \lim_{t\to\infty} \sqrt{d(p_1, \gamma_{q_1 x_\xi}(t))^2 + K_2} - t$. As $p_1, \gamma_{q_1 x_\xi}(t) \in \mathfrak{a}_\tau$, it reduces to Euclidean geometry, i.e., $d(p_1, \gamma_{q_1 x_\xi}(t)) = \sqrt{b_{x_\xi}(p_1, \gamma_{q_1 x_\xi}(t))^2 + K_1}$ with K_1 the squared distance from p_1 to the (now) bi-infinite geodesic $\gamma_{q_1 x_\xi}$. It follows that we have $b_{x_\xi}(p_1, \gamma_{q_1 x_\xi}(t)) = t + b_{x_\xi}(p_1, q_1)$. Using a substitution $t = s^{-1}$ and a Taylor series for the root expression below yields

$$\begin{aligned} b_{x_\xi}(p,q) &= \lim_{t\to\infty} \sqrt{(t + b_{x_\xi}(p_1,q_1))^2 + K_1 + K_2} - t \\ &= \lim_{s\to 0} s^{-1}\left(\sqrt{(1 + 2s b_{x_\xi}(p_1,q_1) + s^2(b_{x_\xi}(p_1,q_1)^2 + K_1 + K_2)} - 1\right) \\ &= b_{x_\xi}(p_1, q_1). \qquad \square \end{aligned}$$

We will also need the following lemma.

Lemma 1.2. *Let $(x,y) \in \mathrm{Flag}_\tau \times \mathrm{Flag}_{\iota\tau}$ with x op y and $\xi \in \tau$. Moreover let $p_1, p_2 \in P(x,y)$. Then $b_{x_\xi}(p_1,p_2) = -b_{y_{\iota\xi}}(p_1,p_2)$.*

Proof. Let γ_i, $i = 1,2$ be bi-infinite geodesics with $\gamma_i(0) = p_i$, $\gamma_i(+\infty) = x_\xi$ and $\gamma_i(-\infty) = y_{\iota\xi}$, which exists by assumption. The γ_i are parallel and denote by C their distance. Then the Flat Strip Theorem (see, e.g., [9]) implies that the convex hull of $\gamma_1(\mathbb{R}) \cup \gamma_2(\mathbb{R})$ is isometric to a flat strip $\mathbb{R} \times [0,C] \subset \mathbb{R}^2$ with γ_i identified with $\mathbb{R} \times 0$, $\mathbb{R} \times C$ respectively.

It follows that the level sets of the Busemann function $b_{x_\xi}(\cdot, p_2)$ in $\mathbb{R} \times [0,C]$ are given by hyperplanes orthogonal to γ_i, i.e., are of the form $s \times [0,C]$ and the same holds for $b_{y_{\iota\xi}}(\cdot, p_2)$. In addition, γ_i joining x_ξ to $y_{\iota\xi}$ implies $b_{x_\xi}(\cdot, p_2)|_{\gamma_i} = -b_{y_{\iota\xi}}(\cdot, p_2)|_{\gamma_i}$. Then the claim is a direct consequence. $\square$

Retracts ([29]). Lastly, we need to introduce the notion of retracts of M to affine apartments with respect to chambers at infinity. For the construction we will distinguish between Euclidean buildings and symmetric spaces.

Let E be a Euclidean building. Let $A \subset E$ be an affine apartment and $x \subset \partial_\infty A$ a chamber of the building at infinity. Then there exists a 1-Lipschitz map $\rho_{x,A} : E \to A$ which is an isometry when restricted to any affine apartment A' with $x \subset \partial_\infty A'$ (i.e., any affine apartment that contains the chamber x in its boundary), and the identity on A [29, Prop.1.20]. We call this map a *(horospherical) retract* with respect to x. Horospherical retracts have the following important property:

Lemma 1.3. *Let $\rho_{x,A} : E \to A$ be a horospherical retract with respect to $x \in \mathrm{Flag}_\sigma(E)$. Then $b_{x_\xi}(o,p) = b_{x_\xi}(\rho_{x,A}(o),p) = b_{x_\xi}(o,\rho_{x,A}(p))$ for all $o,p \in E$ and $\xi \in \sigma$.*

Proof. To $o \in E$ there exists an affine apartment A_o containing o and $x \subset \partial_\infty A_o$. As mentioned, the horospheres with respect to x_ξ in A_o are hyperplanes orthogonal to the direction x_ξ.

By construction, the two affine apartments A, A_o have the same chamber in its boundary, which implies that they have a common subsector. Hence $\rho_{x,A}$ is the identity on the non-empty intersection $A \cap A_o$. Moreover, $\rho_{x,A}$ is an isometry when restricted to A_o. Since $\rho_{x,A}$ leaves each horosphere intersecting $A \cap A_o$ invariant, it has to map the level set of $b_{x_\xi}(\cdot\,,p)$ in A_o to the corresponding level set in A. The other equality follows, for example, from the symmetry $b_{x_\xi}(o,p) = -b_{x_\xi}(p,o)$. □

Let X be a symmetric space, $A \subset X$ be a maximal flat (an affine apartment for us) and $x \subset \partial_\infty A$ a chamber at infinity. To any $o \in X$ there exists a unique maximal flat A_o with $o \in A_o$ and $x \subset \partial_\infty A_o$. Then we define $\rho_{x,A}(o) := n_{x,A_o} \cdot o$ for n_{x,A_o} the unique element in N_x that maps A_o to A. Again we call $\rho_{x,A} : X \to A$ a (*horospherical*) *retract.*

For later reference: To every affine apartment $A \subset M$ and a chamber $x \subset \partial_\infty A$ we have a well-defined map $\rho_{x,A} : M \to A$ such that

$$b_{x_\xi}(o,p) = b_{x_\xi}(\rho_{x,A}(o),p) = b_{x_\xi}(o,\rho_{x,A}(p)) \tag{2}$$

for all $o,p \in M$ and $\xi \in \sigma$. Moreover, it is known that two opposite chambers $x, y \in \mathrm{Flag}_\sigma$ are contained in an unique apartment A_∞ of $\Delta_\infty M$ and this corresponds to an unique affine apartment $A_{xy} \subset M$. Hence to $x, y \in \mathrm{Flag}_\sigma$ with $x \;\mathtt{op}\; y$ we set $\rho_{x,y} := \rho_{x,A_{xy}}$.

Lemma 1.4. *Let $x, y \in \mathrm{Flag}_\tau$ with $x \;\mathtt{op}\; y$ and $o \in M$. Then for all $\xi \in \tau$ we have that $\rho_{c_x,c_y}(\gamma_{ox_\xi}(t))$ is a geodesic in $P(x,y)$, where $c_x, c_y \in \mathrm{Flag}_\sigma$ such that x is a face of c_x, y is a face of c_y and $c_x \;\mathtt{op}\; c_y$.*

We remark that $x \;\mathtt{op}\; y$ implies that such $c_x, c_y \in \mathrm{Flag}_\sigma$ always exist. Namely, take an apartment containing x and y. Take $c_x \in \mathrm{Flag}_\sigma$ such that x is a face of c_x. Take $c_y \in \mathrm{Flag}_\sigma$ the unique opposite chamber in the apartment. Then $x \;\mathtt{op}\; y$ implies that y is a face of c_y.

Proof. For a symmetric space X this follows since ρ_{c_x,c_y} is the same element of G for all points $\gamma_{ox_\xi}(t)$ and that $G < \mathrm{Iso}(X)$. Hence $\rho_{c_x,c_y}(\gamma_{ox_\xi}(t))$ is the image of a geodesic under an isometry. The image $\rho_{c_x,c_y}(\gamma_{ox_\xi}(t))$ is a geodesic ray with endpoint x_ξ in an affine apartment joining x and y. Then $y \;\mathtt{op}\; x$ implies that if we extend $\rho_{c_x,c_y}(\gamma_{ox_\xi}(t))$ bi-infinitely it joins x_ξ to $y_{\iota\xi}$, i.e., this geodesic is contained in $P(x,y)$.

Consider a Euclidean building E. Denote by A_{xy} the unique affine apartment joining c_x and c_y. Let A be an affine apartment containing o and $c_x \subset \partial_\infty A$. Then it follows that $\gamma_{ox_\xi}(t) \in A$ for all $t \in \mathbb{R}_+$. As ρ_{c_x,c_y} is an isometry on affine apartments containing c_x, it follows that $\rho_{c_x,c_y}(\gamma_{ox_\xi}(t)) \subset A_{xy}$ is the image of a geodesic under an isometry. Since one of the endpoints is x_ξ, we can extend the geodesic in A_{xy} uniquely to a bi-infinite geodesic joining x_ξ and $y_{\iota\xi}$. Thus $\rho_{c_x,c_y}(\gamma_{ox_\xi}(t)) \subset P(x,y)$. □

2. Cross ratios

Let M be a symmetric space of non-compact type or a thick Euclidean building. Let σ be the fundamental chamber of the associated spherical Coxeter complex and τ a face of σ. For any type $\xi \in \sigma$ such that $\xi \in \mathrm{int}(\tau)$ and any $o \in M$ we define a *Gromov product* $(\,\cdot\,|\,\cdot\,)_{o,\xi} : \mathrm{Flag}_\tau(M) \times \mathrm{Flag}_{\iota\tau}(M) \to [0,\infty]$ *with base-point* o by

$$(x|y)_{o,\xi} := \lim_{t\to\infty} t - \tfrac{1}{2} d(\gamma_{ox_\xi}(t), \gamma_{oy_{\iota\xi}}(t))$$

for $(x,y) \in \mathrm{Flag}_\tau(M) \times \mathrm{Flag}_{\iota\tau}(M)$ and $\gamma_{ox_\xi}(t), \gamma_{oy_{\iota\xi}}(t)$ the unit speed geodesics from o to $x_\xi, y_{\iota\xi}$, respectively. Using this we define the (*additive*) *cross ratio* $\mathrm{cr}_{o,\xi} : \mathcal{A}_\tau \to [-\infty,\infty]$ *with respect to* (o,ξ) by

$$\mathrm{cr}_{o,\xi}(x_1,y_1,x_2,y_2) := -(x_1|y_1)_{o,\xi} - (x_2|y_2)_{o,\xi} + (x_1|y_2)_{o,\xi} + (x_2|y_1)_{o,\xi}$$

where $\mathcal{A}_\tau$ is the set of quadruples $(x_1,y_1,x_2,y_2) \subset (\mathrm{Flag}_\tau(M) \times \mathrm{Flag}_{\iota\tau}(M))^2$ as in equation (1). If $\xi \in \mathrm{int}(\tau)$, we also denote $\mathcal{A}_\xi := \mathcal{A}_\tau$. By definition $\mathrm{cr}_{o,\xi}$ has the following symmetries, whenever all factors are defined,

$$\begin{aligned}
\mathrm{cr}_{o,\xi}(x_1,y_1,x_2,y_2) &= -\mathrm{cr}_{o,\xi}(x_1,y_2,x_2,y_1) = -\mathrm{cr}_{o,\xi}(x_2,y_1,x_1,y_2)\\
\mathrm{cr}_{o,\xi}(x_1,y_1,x_2,y_2) &= \mathrm{cr}_{o,\xi}(x_1,y_1,w,y_2) + \mathrm{cr}_{o,\xi}(w,y_1,x_2,y_2)\\
\mathrm{cr}_{o,\xi}(x_1,y_1,x_2,y_2) &= \mathrm{cr}_{o,\xi}(x_1,y_1,x_2,v) + \mathrm{cr}_{o,\xi}(x_1,v,x_2,y_2).
\end{aligned} \tag{3}$$

The last two symmetries are called *cocycle identities.*

Notation: Let τ be the face of σ and be $\xi \in \partial\tau$. Then we drop for any $(x,y) \in \mathrm{Flag}_\tau \times \mathrm{Flag}_{\iota\tau}$ the projection maps in the Gromov product (and in the cross ratio) for notational reasons, i.e., $(x|y)_{o,\xi} := (\pi_\xi(x), \pi_{\iota\xi}(y))_{o,\xi}$, where τ_ξ is the face of τ containing ξ in its interior and $\pi_\xi : \mathrm{Flag}_\tau \to \mathrm{Flag}_{\tau_\xi}$, $\pi_{\iota\xi} : \mathrm{Flag}_{\iota\tau} \to \mathrm{Flag}_{\iota\tau_\xi}$ are the obvious projection maps.

Proposition 2.1. *Let M be a symmetric space or thick Euclidean building, $o \in M$, $(x,y) \in \mathrm{Flag}_\tau(M) \times \mathrm{Flag}_{\iota\tau}(M)$ with $x \;\mathtt{op}\; y$ and $c_x, c_y \in \mathrm{Flag}_\sigma(M)$ such that x is a face of c_x, y is a face of c_y and $c_x \;\mathtt{op}\; c_y$. Then for every $\xi \in \tau$*

$$(x\,|\,y)_{o,\xi} = \tfrac{1}{2} b_{x_\xi}(o, \rho_{c_y,c_x}(o)) = \tfrac{1}{2} b_{y_{\iota\xi}}(o, \rho_{c_x,c_y}(o)).$$

Proof. In the case of a symmetric space let N_x be the horospherical subgroup of $P_x = \mathrm{stab}(x)$ and be $n_x(o,y) \in N_x$ the unique element such that $n_x(o,y) \cdot o \in P(x,y)$: Extend γ_{ox} bi-infinitely and let $z \in \mathrm{Flag}_{\iota\tau}$ be such that $\gamma_{ox}(-\infty) \in z$. Then $n_x(o,y) \in N_x$ is the unique element with $n_x(o,y)(z) = y$. By construction we have $n_x(o,y) \cdot o \in P(x,y)$.

We define in the same way $n_y(o,x) \in N_y$ and set $\gamma_{xy}(t) := n_x(o,y) \cdot \gamma_{ox_\xi}(t)$ and $\gamma_{yx}(t) := n_x(o,y) \cdot \gamma_{oy_{\iota\xi}}(t)$. Then γ_{xy}, γ_{yx} are geodesics in $P(x,y)$ with the same (un-ordered) end points. Hence they are parallel. Moreover, $n_x(o,y) \in N_x$ implies that $d(\gamma_{ox_\xi}(t), \gamma_{xy}(t)) \to 0$ for $t \to \infty$ and similarly $d(\gamma_{oy_{\iota\xi}}(t), \gamma_{yx}(t)) \to 0$.

The triangle inequality yields that $(x|y)_{o,\xi} = \lim_{t\to\infty} t - \frac{1}{2} d(\gamma_{xy}(t), \gamma_{yx}(t))$. By construction γ_{xy}, γ_{yx} are parallel geodesics; hence by the Flat Strip Theorem (see, e.g., [9]) the distance $d(\gamma_{xy}(t), \gamma_{yx}(t))$ decomposes into a part parallel to the geodesics and the distance of the images of the geodesics, which is a constant and will be denoted by C.

The part parallel to the geodesics is $b_{x_\xi}(\gamma_{yx}(t), \gamma_{xy}(t))$—or in the same way $b_{y_{\iota\xi}}(\gamma_{xy}(t), \gamma_{yx}(t))$. Using that we have geodesics asymptotic to x_ξ we derive that $b_{x_\xi}(\gamma_{yx}(t), \gamma_{xy}(t))) = 2t + b_{x_\xi}(\gamma_{yx}(0), \gamma_{xy}(0))$. Altogether

$$\begin{aligned} (x|y)_{o,\xi} &= \lim_{t\to\infty} t - \tfrac{1}{2} d(\gamma_{ox_\xi}(t), \gamma_{oy_{\iota\xi}}(t)) = \lim_{t\to\infty} t - \tfrac{1}{2} d(\gamma_{xy}(t), \gamma_{yx}(t)) \\ &= \lim_{t\to\infty} t - \tfrac{1}{2}\left(\sqrt{(2t + b_{x_\xi}(\gamma_{yx}(0), \gamma_{xy}(0)))^2 + C^2}\right) \\ &= -\tfrac{1}{2} b_{x_\xi}(\gamma_{yx}(0), \gamma_{xy}(0)) = \tfrac{1}{2} b_{x_\xi}(\gamma_{xy}(0), \gamma_{yx}(0)), \end{aligned} \tag{4}$$

while the second to last equality follows using Taylor series at $s = 0$ after substituting $s = t^{-1}$ (see also the calculations in Example 2.6).

In the case of a Euclidean building E, let A_o be an affine apartment containing $\gamma_{ox_\xi}(t)$, let $d_x \in \mathrm{Flag}_\sigma$ be such that $d_x \subset \partial_\infty A_o$ and $x \subset d_x$. Moreover, be $d_y \in \mathrm{Flag}_\sigma$ a chamber opposite to d_x such that y is a face of d_y and let A_{xy} be the unique affine apartment that d_x and d_y define.

Then the affine apartments A_o and A_{xy} have a common subsector. Hence there exists $T_x \geq 0$ such that for $t \geq T_x$ the geodesic $\gamma_{ox_\xi}(t)$ is parallel to a geodesic γ_{xy} in the subsector, denote the distance of the geodesic rays by C_x; Extend γ_{xy} bi-infinite in A_{xy} such that it is in the same horosphere with respect to x_ξ as $\gamma_{ox_\xi}(t)$ for all (positive) time. That γ_{xy} is in A_{xy} with one endpoint being x_ξ implies that γ_{xy} joins x_ξ and $y_{\iota\xi}$ and hence $\gamma_{xy} \subset P(x,y)$.

In the same way we construct $\gamma_{yx} \subset P(x,y)$ to $\gamma_{oy_{\iota\xi}}$ such that those geodesics are parallel for $t \geq T_y$—denote the distance by C_y. Since γ_{xy}, γ_{yx} join the same points at infinity, they are parallel— denote the distance by C_0. Then the triangle inequality together with the Flat Strip theorem yields for $t \geq \max\{T_x, T_y\}$ that $d(\gamma_{ox_\xi}(2t), \gamma_{oy_{\iota\xi}}(2t))$ is smaller than or equal to

$$\begin{aligned} &d(\gamma_{ox_\xi}(2t), \gamma_{xy}(t)) + d(\gamma_{xy}(t), \gamma_{yx}(t)) + d(\gamma_{yx}(t), \gamma_{oy_{\iota\xi}}(2t)) \\ &\qquad = \sqrt{t^2 - C_x^2} + \sqrt{b_{x_\xi}(\gamma_{yx}(t), \gamma_{xy}(t))^2 + C_0^2} + \sqrt{t^2 - C_y^2}. \end{aligned}$$

Since γ_{xy} and γ_{yx} are asymptotic to x_ξ, we derive that $b_{x_\xi}(\gamma_{yx}(t), \gamma_{xy}(t))) = 2t + b_{x_\xi}(\gamma_{yx}(0), \gamma_{xy}(0))$. Therefore

$$(x \mid y)_{o,\xi} \geq \lim_{t\to\infty} 2t - \tfrac{1}{2}\Big(\sqrt{t^2 - C_x^2} + \sqrt{(2t + b_{x_\xi}(\gamma_{yx}(0), \gamma_{xy}(0)))^2 + C_0^2} + \sqrt{t^2 - C_y^2}\Big).$$

We substitute $t = s^{-1}$. Then a Taylor expansion for the root expressions at $s = 0$ yields that $(x|y)_{o,\xi} \geq -\frac{1}{2} b_{x_\xi}(\gamma_{yx}(0), \gamma_{xy}(0)) = \frac{1}{2} b_{x_\xi}(\gamma_{xy}(0), \gamma_{yx}(0))$.

We claim that $\lim_{t\to\infty} b_{x_\xi}(\gamma_{yx}(t), \gamma_{xy}(t))) - b_{x_\xi}(\gamma_{oy_{\iota\xi}}(t), \gamma_{ox_\xi}(t)) = 0$: By construction $b_{x_\xi}(\gamma_{xy}(t), \gamma_{ox_\xi}(t)) = 0$. As we have $b_z(p,q) + b_z(q,o) = b_z(p,o)$, it is enough to show that $\lim_{t\to\infty} b_{x_\xi}(\gamma_{oy_{\iota\xi}}(t), \gamma_{yx}(t)) = 0$.

By construction we have that the geodesic γ_{yx} joins x_ξ and $y_{\iota\xi}$. Therefore $b_{x_\xi}(\gamma_{oy_{\iota\xi}}(t), \gamma_{yx}(t)) = \lim_{s\to\infty} d(\gamma_{oy_{\iota\xi}}(t), \gamma_{yx}(t-s)) - s$. Moreover,

$$d(\gamma_{oy_{\iota\xi}}(t), \gamma_{yx}(t-s)) \leq d(\gamma_{oy_{\iota\xi}}(t), \gamma_{yx}(T_y)) + |t - s - T_y|.$$

Applying the Flat Strip Theorem with an according Taylor expansion as before, we derive that $\lim_{t\to\infty} d(\gamma_{oy_{\iota\xi}}(t), \gamma_{yx}(T_y)) - t \to -T_y$. In particular,

$$\lim_{t\to\infty} b_{x_\xi}(\gamma_{oy_{\iota\xi}}(t), \gamma_{yx}(t)) \leq \lim_{t\to\infty}\big(\lim_{s\to\infty} d(\gamma_{oy_{\iota\xi}}(t), \gamma_{yx}(T_y)) - t + s + T_y - s\big) = 0.$$

It follows from the definition of Busemann functions that if $q \in M$ lies on a bi-infinite geodesics joining $z, w \in \partial_\infty M$, then $b_z(p,q) + b_w(p,q) \geq 0$. Hence we derive $b_{x_\xi}(\gamma_{oy_{\iota\xi}}(t), \gamma_{yx}(t)) + b_{y_{\iota\xi}}(\gamma_{oy_{\iota\xi}}(t), \gamma_{yx}(t)) \geq 0$. Since by construction we have $b_{y_{\iota\xi}}(\gamma_{yx}(t), \gamma_{oy_{\iota\xi}}(t)) = 0$, it follows $b_{x_\xi}(\gamma_{oy_{\iota\xi}}(t), \gamma_{yx}(t)) \geq 0$; which yields the claim.

We have $d(\gamma_{oy_{\iota\xi}}(t), \gamma_{ox_\xi}(t)) \geq b_{x_\xi}(\gamma_{oy_{\iota\xi}}(t), \gamma_{ox_\xi}(t)) \to b_{x_\xi}(\gamma_{yx}(t), \gamma_{xy}(t))$, for $t \to \infty$. Thus

$$(x|y)_{o,\xi} \leq \lim_{t\to\infty} t - \tfrac{1}{2} b_{x_\xi}(\gamma_{yx}(t), \gamma_{xy}(t)) = \tfrac{1}{2} b_{x_\xi}(\gamma_{xy}(0), \gamma_{yx}(0)).$$

Altogether $(x|y)_{o,\xi} = \frac{1}{2} b_{x_\xi}(\gamma_{xy}(0), \gamma_{yx}(0))$.

Consider a symmetric space or a Euclidean building M and let γ_{xy}, γ_{yx} be the accordingly constructed geodesics. Then $b_{x_\xi}(\gamma_{xy}(0), \gamma_{ox_\xi}(0)) = 0$ while $\gamma_{ox_\xi}(0) = o$ and also $b_{y_{\iota\xi}}(\gamma_{yx}(0), o) = 0$. For notational reasons set $\rho_x := \rho_{c_x, c_y}$ and $\rho_y := \rho_{c_y, c_x}$ Then $\rho_y(o), \gamma_{yx}(0) \in P(x,y)$. Together with equation (2) and Lemma 1.2 this yields

$$\begin{aligned} b_{x_\xi}(\gamma_{xy}(0), \gamma_{yx}(0)) =& b_{x_\xi}(\gamma_{xy}(0), \rho_x(o)) + b_{x_\xi}(\rho_x(o), \rho_y(o)) + b_{x_\xi}(\rho_y(o), \gamma_{yx}(0)) \\ =& b_{x_\xi}(o, \rho_y(o)) - b_{y_{\iota\xi}}(\rho_y(o), \gamma_{yx}(0)) = b_{x_\xi}(o, \rho_y(o)). \end{aligned}$$

In a similar way it follows also that $b_{x_\xi}(\gamma_{xy}(0), \gamma_{yx}(0)) = b_{y_{\iota\xi}}(o, \rho_x(o))$. Finally the equality $(x|y)_{o,\xi} = \frac{1}{2} b_{x_\xi}(\gamma_{xy}(0), \gamma_{yx}(0))$ implies the claim. $\square$

Corollary 2.2. *Let* $(x,y) \in \mathrm{Flag}_\tau \times \mathrm{Flag}_{\iota\tau}$ *and* $o \in M$. *Then* $(x|y)_{o,\xi} = \infty \iff x \not{\mathrm{op}}\, y$.

Proof. Let $(x,y) \in \mathrm{Flag}_\tau \times \mathrm{Flag}_{\iota\tau}$ be such that $x \not\mathtt{op}\, y$. Let A be an affine apartment containing x, y in its boundary. Let $p \in A$ and γ_{px_ξ}, $\gamma_{py_{\iota\xi}}$ be the unit speed geodesics joining p to $x_\xi, y_{\iota\xi}$, respectively. A straightforward argument in Euclidean geometry yields that $d(\gamma_{px_\xi}(t), \gamma_{py_{\iota\xi}}(t)) = 2\alpha t$ with α depending on the angle of the geodesics. Then $x \not\mathtt{op}\, y$ implies that $\gamma_{px_\xi}(t) \neq \gamma_{py_{\iota\xi}}(-t)$ and hence $\alpha < 1$, i.e., $(x|y)_{p,\xi} = \infty$.

Now let $\gamma_{ox_\xi}, \gamma_{oy_{\iota\xi}}$ be the unit speed geodesics joining o to $x_\xi, y_{\iota\xi}$, respectively. Since γ_{ox_ξ} and γ_{px_ξ} define the same point in the ideal boundary, we can derive by the convexity of the distance functions along geodesics in non-positive curvature that $d(\gamma_{ox_\xi}(t), \gamma_{px_\xi}(t)) \leq d(o,p)$ for all $t \geq 0$. Thus

$$\begin{aligned}(x|y)_{o,\xi} &= \lim_{t\to\infty} t - \tfrac{1}{2} d(\gamma_{ox_\xi}(t), \gamma_{oy_{\iota\xi}}(t)) \\ &\geq \lim_{t\to\infty} t - \tfrac{1}{2} d(\gamma_{px_\xi}(t), \gamma_{py_{\iota\xi}}(t)) - d(o,p) = \infty.\end{aligned}$$

Let $(x,y) \in \mathrm{Flag}_\tau \times \mathrm{Flag}_{\iota\tau}$ be such that $x \;\mathtt{op}\; y$. Then by the above proposition $(x|y)_{o,\xi} = \frac{1}{2} b_{x_\xi}(o, \rho_{c_x,c_y}(o)) \leq d(o, \rho_{c_x,c_y}(o))$, i.e., $(x|y)_{o,\xi} < \infty$. □

The above corollary implies that $\mathcal{A}_\xi$ is the maximal domain of definition for $\mathrm{cr}_{o,\xi}$. As mentioned, in the case of a symmetric space X is the set $\mathcal{A}_\xi$ is an open and dense subset of $(\mathrm{Flag}_\tau(X) \times \mathrm{Flag}_{\iota\tau}(X))^2$, i.e., the cross ratio is generically defined.

Proposition 2.3. *Let* $o, \widehat{o} \in M$, $(x,y) \in \mathrm{Flag}_\tau \times \mathrm{Flag}_{\iota\tau}$ *and* $\xi \in \tau$. *Then* $(x|y)_{o,\xi} = (x|y)_{\widehat{o},\xi} + \frac{1}{2} b_{x_\xi}(o,\widehat{o}) + \frac{1}{2} b_{y_{\iota\xi}}(o,\widehat{o})$.

Proof. If $x \not\mathtt{op}\, y$, then by the above corollary $(x|y)_{o,\xi} = \infty = (x|y)_{\widehat{o},\xi}$.

If $x \;\mathtt{op}\; y$, let $\rho_{x,y}, \rho_{y,x}$ be any horospherical retracts as in Proposition 2.1. Then

$$b_{x_\xi}(o, \rho_{y,x}(o)) = b_{x_\xi}(o, \widehat{o}) + b_{x_\xi}(\widehat{o}, \rho_{y,x}(\widehat{o})) + b_{x_\xi}(\rho_{y,x}(\widehat{o}), \rho_{y,x}(o)).$$

By construction $\rho_{y,x}(o), \rho_{y,x}(\widehat{o}) \in P(x,y)$. Moreover x, y are opposite and hence by Lemma 1.2 and equation (2)

$$b_{x_\xi}(\rho_{y,x}(\widehat{o}), \rho_{y,x}(o)) = -b_{y_{\iota\xi}}(\rho_{y,x}(\widehat{o}), \rho_{y,x}(o)) = -b_{y_{\iota\xi}}(\widehat{o}, o) = b_{y_{\iota\xi}}(o, \widehat{o}).$$

Together with Proposition 2.1 the claim follows. □

Proposition 2.4. *Let* $o, \widehat{o} \in M$. *Then* $\mathrm{cr}_{o,\xi}(x_1,y_1,x_2,y_2) = \mathrm{cr}_{\widehat{o},\xi}(x_1,y_1,x_2,y_2)$ *for all* $(x_1,y_1,x_2,y_2) \in \mathcal{A}_\xi$.

Proof. Plugging in the above proposition in the definitions of $\mathrm{cr}_{o,\xi}$ and $\mathrm{cr}_{\widehat{o},\xi}$ yields directly the result. □

Definition 2.5. *Given* $(x_1,y_1,x_2,y_2) \in \mathcal{A}_\xi$, *we define the* cross ratio with respect to $\xi \in \sigma$ *to be* $\mathrm{cr}_\xi(x_1,y_1,x_2,y_2) = \mathrm{cr}_{o,\xi}(x_1,y_1,x_2,y_2)$ *for some* $o \in M$.

Example 2.6 (see also [21]). Consider the symmetric space $X = \mathbb{H}^2 \times \mathbb{H}^2$, where $\mathbb{H}^2$ is the hyperbolic plane. The ideal boundary $\partial_\infty(\mathbb{H}^2 \times \mathbb{H}^2)$ can be identified with $S^1 \times S^1 \times [0, \pi/2]$ — this is in such a way that the unit-speed geodesic ray from a base-point $(o_1, o_2) \in \mathbb{H}^2 \times \mathbb{H}^2$ to the point in $(x_1, x_2, \alpha) \in S^1 \times S^1 \times [0, \pi/2] \cong \partial_\infty(\mathbb{H}^2 \times \mathbb{H}^2)$ is given by $(\gamma_{o_1x_1}(\cos(\alpha)t), \gamma_{o_2x_2}(\sin(\alpha)t))$.

The types are exactly determined by the angle α and the opposition involution equals the identity. In particular every type is self-opposite.

Fix $o = (o_1, o_2) \in \mathbb{H}^2 \times \mathbb{H}^2$ and $x = (x_1, x_2, \alpha), y = (y_1, y_2, \alpha) \in \partial_\infty(\mathbb{H}^2 \times \mathbb{H}^2)$ and set $\gamma_1 := \gamma_{o_1x_1}, \widehat{\gamma}_1 := \gamma_{o_1y_1}, \gamma_2 := \gamma_{o_2x_2}$ and $\widehat{\gamma}_2 := \gamma_{o_2y_2}$. Then

$$(x|y)_{o,\alpha} = \lim_{t\to\infty} t - \tfrac{1}{2}\sqrt{|\gamma_1(\cos(\alpha)t)\widehat{\gamma}_1(\cos(\alpha)t)|^2 + |\gamma_2(\sin(\alpha)t)\widehat{\gamma}_2(\sin(\alpha)t)|^2}.$$

Using $\lim_{t\to\infty} |\gamma_1(\cos(\alpha)t)\widehat{\gamma}_1(\cos(\alpha)t)| - 2\cos(\alpha)t = -2(x_1|y_1)_{o_1}$, if $\alpha \neq \pi/2$

$$(x|y)_{o,\alpha} = \lim_{t\to\infty} t - \sqrt{(-(x_1|y_1)_{o_1} + \cos(\alpha)t)^2 + (-(x_2|y_2)_{o_2} + \sin(\alpha)t)^2}$$

$$= \lim_{t\to\infty} t - \sqrt{t^2 - 2t(\cos(\alpha)(x_1|y_1)_{o_1} + \sin(\alpha)(x_2|y_2)_{o_2}) + (x_1|y_1)_{o_1}^2 + (x_2|y_2)_{o_2}^2}.$$

We substitute $t = s^{-1}$. Then a Taylor expansion for the root expression at $s = 0$ yields that

$$\begin{aligned}(x|y)_{o,\alpha} &= \lim_{s\to 0} \frac{1}{s}(1 - (1 - s(\cos(\alpha)(x_1|y_1)_{o_1} + \sin(\alpha)(x_2|y_2)_{o_2}) + o(s)) \\ &= \cos(\alpha)(x_1|y_1)_{o_1} + \sin(\alpha)(x_2|y_2)_{o_2}.\end{aligned}$$

Therefore $\mathrm{cr}_\alpha = \cos(\alpha)\log|\mathrm{cr}_{\mathbb{H}^2}| + \sin(\alpha)\log|\mathrm{cr}_{\mathbb{H}^2}|$, where $\mathrm{cr}_{\mathbb{H}^2}$ is the usual *multiplicative* cross ratio on $\partial_\infty\mathbb{H}^2$.

Lemma 2.7. *Let X be a symmetric space. Then for every $o \in X$ the Gromov product $(\cdot\,|\,\cdot)_{o,\xi} : \mathrm{Flag}_\tau(X) \times \mathrm{Flag}_{\iota\tau}(X) \to [0, \infty]$ is continuous. In particular also cr_ξ is continuous.*

Proof. Since $\mathrm{Flag}_\tau(X), \mathrm{Flag}_{\iota\tau}(X)$ are manifolds it is enough to consider sequential continuity. Therefore let $(x, y) \in \mathrm{Flag}_\tau(X) \times \mathrm{Flag}_{\iota\tau}(X)$ and let $x_i \to x$ and $y_i \to y$.

If $x \not\mathsf{op}\, y$, we have $(x|y)_{o,\xi} = \infty$. We set $(x|y)_{o,\xi}(t) := (\gamma_{ox_\xi}(t)|\gamma_{oy_{\iota\xi}}(t))_o$ with the Gromov product on the right-hand side the usual Gromov product on the metric space (X, d). As X is non-positively curved, the function $t \mapsto (x|y)_{o,\xi}(t)$ is monotone increasing. Let $C > 0$ be given. Then there is $t_C \in \mathbb{R}_+$ such that $(x|y)_{o,\xi}(t_C) \geq C + 2$. Since the topology on $\mathrm{Flag}_\tau(X)$ is induced by the cone topology, we have that $(x_i)_\xi \to x_\xi$ in the cone topology and similarly for y_i and y. Hence we find $L \in \mathbb{N}$ such that $d(\gamma_{o(x_i)_\xi}(t_C), \gamma_{ox_\xi}(t_C)) < 1$ and $d(\gamma_{o(y_i)_{\iota\xi}}(t_C), \gamma_{oy_{\iota\xi}}(t_C)) < 1$ for all $i \geq L$. Hence by the triangle inequality $(x_i|y_j)_{o,\xi}(t_C) > (x|y)_{o,\xi}(t_C) - 2 > C$ for all $i, j \geq L$. As C was arbitrary, this yields $\lim_{i,j\to\infty}(x_i|y_j)_{o,\xi} = \infty$, which proves continuity for $x \not\mathsf{op}\, y$.

Assume $x \,\mathsf{op}\, y$. Let $K = \mathrm{stab}_G(o)$. We know that K acts transitively on $\mathrm{Flag}_\tau(X)$ and we have a K-equivariant, homeomorphic identification $\mathrm{Flag}_\tau(X) \simeq K/K_x$. Therefore $x_i \to x$ implies that we find $k_i \in K$ such that $k_i x_i = x$ and $k_i \to e \in G$. Now, $x \,\mathsf{op}\, y$ and opposition being an open condition, together with $y_i \to y$ and $k_i \to e$, imply that there exists $L \in \mathbb{N}$ such that $k_i y_j \,\mathsf{op}\, x$ for all $i, j \geq L$. Thus there exists a unique $n_{ij} \in N_x$ such that $n_{ij}k_i y_j = y$ for $i, j \geq L$. From $k_i \to e$ and $y_j \to y$ it follows $n_{ij} \to e \in G$ for $i, j \to \infty$. We set $g_{ij} := n_{ij}k_i$ and by construction $g_{ij} \to e$, $g_{ij}x_i = x$, $g_{ij}y_j = y$. Hence $(x_i|y_j)_{o,\xi} = (x|y)_{g_{ij}o,\xi}$. Proposition 2.3 and $g_{ij} \to e$ yield that $(x_i\,|\,y_j)_{o,\xi} \to (x\,|\,y)_{o,\xi}$. □

Lemma 2.8. *Let $(x,y) \in \mathrm{Flag}_\tau \times \mathrm{Flag}_{\iota\tau}$ and $x \,\mathsf{op}\, y$. Moreover, let $\xi_i \in \tau$ be a sequence with $\xi_i \to \xi_0 \in \tau$. Then we have $(x|y)_{o,\xi_i} \to (x|y)_{o,\xi_0}$. In particular, $\mathrm{cr}_{\xi_i}(x,y,z,w) \to \mathrm{cr}_{\xi_0}(x,y,z,w)$ for all $(x,y,z,w) \in \mathcal{A}_\tau^{\mathsf{op}}$.*

Proof. Let $c_x, c_y \in \mathrm{Flag}_\sigma$ such that $c_x \,\mathsf{op}\, c_y$, x is a face of c_x and y is a face of c_y. Then Proposition 2.1 and equation (2) imply $(x|y)_{o,\xi} = \frac{1}{2} b_{x_\xi}(\rho_{c_x,c_y}(o), \rho_{c_y,c_x}(o))$ for all $\xi \in \tau$. Denote $p_x := \rho_{c_x,c_y}(o)$, $p_y := \rho_{c_y,c_x}(o)$ and by A_{xy} the unique affine apartment with $c_x, c_y \subset \partial_\infty A_{xy}$.

Every affine apartment can be isometrically identified with $\mathbb{R}^r$ where r is the rank of M. We identify A_{xy} with $\mathbb{R}^r$ such that $0 \simeq p_x$. Let $v_\xi \in A_{xy} \simeq \mathbb{R}^r$ be of norm one and such that the line from 0 through v_ξ is the geodesic ray in A_{xy} from p_x to x_ξ. Then Euclidean geometry yields that $b_{x_\xi}(p_x, p_y) = \langle v_\xi, p_y \rangle$. In particular, we get

$$(x|y)_{o,\xi_i} = \tfrac{1}{2} \langle v_{\xi_i}, p_y \rangle. \tag{5}$$

Moreover $\xi_i \to \xi_0$ implies that $v_{\xi_i} \to v_{\xi_0}$ and hence the claim follows. □

The assumption of opposition in the above lemma is needed, since there are $(x,y) \in \mathrm{Flag}_\tau \times \mathrm{Flag}_{\iota\tau}$ with $x \not\!\mathsf{op}\, y$ but there are faces x_0 of x and y_0 of y with $x_0 \,\mathsf{op}\, y_0$. Then if $\xi_i \in \mathrm{int}(\tau)$ converge to ξ_0 such that $\xi_0 \in \mathrm{int}(\tau_0)$ and τ_0 is the type of x_0, we get $(x|y)_{o,\xi_i} = \infty \not\to (x_0|y_0)_{o,\xi_0}$ (as the latter is finite).

We recall that any isometry $F : M_1 \to M_2$ induces a building isomorphism $F_\infty : \Delta_\infty M_1 \to \Delta_\infty M_2$ together with a type map $F_\sigma : \sigma_1 \to \sigma_2$ with the property that $F(\mathrm{Flag}_\tau(M_1)) = \mathrm{Flag}_{F_\sigma(\tau)}(M_2)$.

Proposition 2.9. *Let $F : M_1 \to M_2$ be an isometry between either symmetric spaces or thick Euclidean buildings, $F_\infty : \Delta_\infty M_1 \to \Delta_\infty M_2$ the induced building isomorphism and $\xi \in \sigma_1$. Then*

$$\mathrm{cr}_{\xi_1}(x_1,y_1,x_2,y_2) = \mathrm{cr}_{F_\sigma(\xi_1)}(F_\infty(x_1), F_\infty(y_1), F_\infty(x_2), F_\infty(y_2))$$

for all $(x_1,y_1,x_2,y_2) \in \mathcal{A}_{\xi_1}$. Equivalently, $\mathrm{cr}_{\xi_1} = F_\infty^* \mathrm{cr}_{F_\sigma(\xi_1)}$ with F_∞^* denoting the pullback under F_∞.

Proof. Let $\xi_1 \in \tau$ and $(x,y) \in \mathrm{Flag}_\tau(M_1) \times \mathrm{Flag}_{\iota\tau}(M_1)$. Since the Gromov product $(\cdot\,|\,\cdot)_{o,\xi_1}$ is defined in terms of a limit of distances involving unit speed geodesics and isometries leave those invariant, it follows that $(x\,|\,y)_{o,\xi_1} = (F_\infty(x)|F_\infty(y))_{F(o),F_\sigma(\xi_1)}$. Therefore $(x_1,y_1,x_2,y_2) \in \mathcal{A}_{\xi_1}$ implies $(F_\infty(x_1), F_\infty(y_1), F_\infty(x_2), F_\infty(y_2)) \in \mathcal{A}_{F_\sigma(\xi_1)}$ by Corollary 2.2. Finally, $\mathrm{cr}_{\xi_1} = \mathrm{cr}_{o,\xi_1} = F_\infty^* \mathrm{cr}_{F(o),F_\sigma(\xi_1)} = F_\infty^* \mathrm{cr}_{F_\sigma(\xi_1)}$ by Proposition 2.4. □

Corollary 2.10. *Let $g \in \mathrm{Iso}(M)$ and ξ_0 be the center of gravity of σ with respect to the angular metric. Then $\mathrm{cr}_{\xi_0} = g^* \mathrm{cr}_{\xi_0}$. In case of a symmetric space X and $g \in G$ we have $\mathrm{cr}_{\xi,X} = g^* \mathrm{cr}_{\xi,X}$ for all $\xi \in \sigma$.*

Proof. For the center of gravity $\xi_0 \in \sigma$ we have $g_\sigma(\xi_0) = \xi_0$ for all $g \in \mathrm{Iso}(M)$, as $g_\sigma : \sigma \to \sigma$ is an isometry with respect to the angular metric. Then the first claim follows. In case of a symmetric space and $g \in G$, we know $g_\sigma = \mathrm{id}_\sigma$, which implies the second claim. □

Example 2.11. We want to determine the Gromov products and cross ratios of the symmetric spaces $X(n) := \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$. For a deeper description of the symmetric space $X(n)$ see [13].

The ideal boundary $\partial_\infty X(n)$ can be identified with eigenvalue flag pairs (λ, F), where $F = (V_1, \dots, V_l)$ is a flag in $\mathbb{R}^n$, i.e., the V_i are subspaces of $\mathbb{R}^n$ with $V_i \subsetneq V_{i+1}$, $V_l = \mathbb{R}^n$, and $\lambda = (\lambda_1, \dots, \lambda_l) \in \mathbb{R}^l$ such that $\lambda_i > \lambda_{i+1}$, $\sum_{i=1}^l m_i\lambda_i = 0$ for $m_i = \dim V_i - \dim V_{i-1}$ and $\sum_{i=1}^l m_i\lambda_i^2 = 1$. In particular, $2 \leq l \leq n$. The action of $g \in \mathrm{SL}(n,\mathbb{R})$ on an eigenvalue flag pair is given by $g\cdot(\lambda,F) = (\lambda, g\cdot F)$, where $g\cdot(V_1,\dots,V_l) = (g\cdot V_1, \dots, g\cdot V_l)$ and $F = (V_1,\dots,V_l)$.

The "eigenvalues" λ in the eigenvalue flag pairs (λ, F) determine the type of any point in the ideal boundary. Namely, the set of pairs $(\lambda_1,\dots,\lambda_l), (m_1,\dots,m_l)$, $\lambda_i \in \mathbb{R}, m_i \in \mathbb{N}\backslash\{0\}$ with $\lambda_i > \lambda_{i+1}$, $\sum_{i=1}^l m_i\lambda_i = 0$, $\sum_{i=1}^l m_i\lambda_i^2 = 1$ and $\sum_{i=1}^l m_i = n$ parametrize the Weyl chamber σ. We have that $\lambda = (\lambda_1,\dots,\lambda_l)$ is in the interior of the chamber if and only if $l = n$.

Faces of σ can be characterized in the following way: Two pairs as above $(\lambda_1,\dots,\lambda_l),(m_1,\dots,m_l)$ and $(\lambda'_1,\dots,\lambda'_l),(m'_1,\dots,m'_l)$ are in the interior of the same face if and only if $m_i = m'_i$ for all $i = 1,\dots,l$. In particular we can identify the set of faces of σ with $\{(m_1,\dots,m_l) \in \mathbb{N}^l \mid l \geq 2, m_i \neq 0, \sum_{i=1}^l m_i = n\}$. For $\tau \simeq (m_1,\dots,m_l)$ we have $\mathrm{Flag}_\tau = \{(V_1,\dots,V_l) \mid V_i \subsetneq V_{i+1}, \dim V_i - \dim V_{i-1} = m_i\}$. The action of the opposition involution is given by $\iota(\lambda_1,\dots,\lambda_l) = (-\lambda_l,\dots,-\lambda_1)$ and $\iota(m_1,\dots,m_l) = (m_l,\dots,m_1)$. Hence, if $V = (V_1,\dots,V_l) \in \mathrm{Flag}_\tau$ and $W = (W_1,\dots,W_l) \in \mathrm{Flag}_{\iota\tau}$, then $\dim V_i + \dim W_{l-i} = n$. In this situation $V \;\mathsf{op}\; W \iff V_i \oplus W_{l-i} = \mathbb{R}^n$ for all $i = 1,\dots,l-1$.

Let $V = (V_1,\dots,V_l), Y = (Y_1,\dots,Y_l) \in \mathrm{Flag}_\tau$ and $W = (W_1,\dots,W_l), Z = (Z_1,\dots,Z_l) \in \mathrm{Flag}_{\iota\tau}$ such that $V, Y \;\mathsf{op}\; W, Z$. Let $i_j = \dim V_j$. Then fix a basis $(v_1,\dots,v_n)$ such that $V_j = \mathrm{span}\{v_1,\dots,v_{i_j}\}$. In the same way we fix a basis $(w_1,\dots \dots, w_n), (y_1,\dots,y_n)$ and $(z_1,\dots,z_n)$ for W, Y, Z, respectively. Additionally, fix an identification $\wedge^n\mathbb{R}^n \cong \mathbb{R}$. We set $V_j \wedge W_{l-j} := v_1 \wedge \dots \wedge v_{i_j} \wedge w_1 \wedge \dots \wedge w_{n-i_j}$ (we have $W_{l-j} = \mathrm{span}\{w_1,\dots,w_{n-i_j}\}$) and in the same way for the other flags. Then the term $(V_j \wedge W_{l-j})(Y_j \wedge Z_{l-j})(V_j \wedge Z_{l-j})^{-1}(Y_j \wedge W_{l-j})^{-1}$ can be shown to be independent of all choices for all $j = 1,\dots,l-1$; compare, e.g., [27].

Let V, W, Y, Z be as before and $\lambda = (\lambda_1,\dots,\lambda_l)$ a type with $\lambda \in \mathrm{int}(\tau)$. Then

$$\mathrm{cr}_\lambda(V,W,Y,Z) = n\sum_{j=1}^{l-1}(\lambda_j - \lambda_{j+1})\log\Big(\Big|\frac{V_j \wedge W_{l-j}}{V_j \wedge Z_{l-j}}\frac{Y_j \wedge Z_{l-j}}{Y_j \wedge W_{l-j}}\Big|\Big),$$

using the above conventions — see the appendix for a proof. We remark that some specifics of those cross ratios are known already and have been used for analysing Hitchin representations and more general Anosov representations (see, e.g., [25], [27]).

Let $M = M_1 \times \dots \times M_k$ be a product of either symmetric spaces or Euclidean buildings. Then the building at infinity $\Delta_\infty M$ is the spherical join of the buildings $\Delta_\infty M_i$ [22, Sec. 4.3]. In particular, the Weyl chamber σ decomposes as a spherical join $\sigma = \sigma_1 \circ \dots \circ \sigma_k$. Hence we get a surjective map

$$\pi : \sigma_1 \times \dots \times \sigma_k \times S_k^+ \to \sigma, \tag{6}$$

where $S_k^+ := \{\mu = (\mu_1, \dots, \mu_k) \in [0,1]^k \mid \Sigma_1^k \mu_i^2 = 1\}$. We remark that π is in general not injective, since it is independent of the exact choice of the type $\xi_i \in \sigma_i$ if $\mu_i = 0$.

Let $\xi = \pi(\xi_1, \dots, \xi_k, \mu)$ with $\mu = (\mu_1, \dots, \mu_k) \in S_k^+$ and let $x = (x_1, \dots, x_k) \in \mathrm{Flag}_\tau(M) \simeq \mathrm{Flag}_{\tau_1}(M_1) \times \dots \times \mathrm{Flag}_{\tau_k}(M_k)$[5] such that $\xi \in \mathrm{int}(\tau)$ and $\xi_i \in \mathrm{int}(\tau_i)$. For simplicity we assume $\mu_i \neq 0$ for all $1 \leq i \leq k$; if some $\mu_i = 0$ essentially the same formula holds, but the factor $\mathrm{Flag}_{\tau_i}(M_i)$ is not apparent in the decomposition of $\mathrm{Flag}_\tau(M)$.

We remark that the unit-speed geodesic from some point $(o_1, \dots, o_k) \in M$ to x_ξ is of the form $(\gamma_{o_1 x_{\xi_1}}(\mu_1 t), \dots, \gamma_{o_k x_{\xi_k}}(\mu_k t))$, where $\gamma_{o_i x_{\xi_i}}$ denote the unit speed geodesics in the factors M_i joining o_i to $(x_i)_{\xi_i}$; cp. also Example 2.6.

Let $y = (y_1, \dots, y_k) \in \mathrm{Flag}_{\iota\tau}(M) \simeq \mathrm{Flag}_{\iota\tau_1}(M_1) \times \dots \times \mathrm{Flag}_{\iota\tau_k}(M_k)$ and be x and ξ as above. Then similar calculations as in Example 2.6, yield that

$$(x \mid y)_{(o_1,\dots,o_k),\pi(\xi_1,\dots,\xi_k,\mu)} = \mu_1 (x_1 \mid y_1)_{o_1,\xi_1} + \dots + \mu_k (x_k \mid y_k)_{o_k,\xi_k}.$$

Proposition 2.12. *Notations as before. Moreover, let* $z \in \mathrm{Flag}_\tau(M)$ *and* $w \in \mathrm{Flag}_{\iota\tau}(M)$*. Then*

$$\mathrm{cr}_{\pi(\xi_1,\dots,\xi_k,\mu)}(x, y, z, w) = \mu_1 \mathrm{cr}_{\xi_1}(x_1, y_1, z_1, w_1) + \dots + \mu_k \mathrm{cr}_{\xi_k}(x_k, y_k, z_k, w_k)$$

for $(x, y, z, w) \in \mathcal{A}_{\pi(\xi_1,\dots,\xi_k,\mu)}$.

3. Vector valued cross ratios

So far, we have constructed families of cross ratios on subsets of the spaces $(\mathrm{Flag}_\tau \times \mathrm{Flag}_{\iota\tau})^2$ which are parametrized by $\xi \in \mathrm{int}(\tau)$. In this section we show that such a family gives rise to a single vector valued cross ratio containing all the information of the family. The vector valued cross ratio has the same symmetries as the usual cross ratios (cp. equations (3)) justifying the name cross ratio.

We recall that $\sigma = \mathfrak{a}_1^+$; hence every type can be viewed as vector in $\mathfrak{a}$ of norm one.

Lemma 3.1. *Let* τ *be a face of* σ *and* $\xi_0, \xi_1, \dots, \xi_j \in \tau$ *such that there exist* $a_i \in \mathbb{R}$ *with* $\xi_0 = \sum_{i=1}^j a_i \xi_i$*. Then for* $(x, y) \in \mathrm{Flag}_\tau \times \mathrm{Flag}_{\iota\tau}$ *with* x op y *we have* $(x \mid y)_{o,\xi_0} = \sum_{i=1}^j a_i (x \mid y)_{o,\xi_i}$*. In particular,* $\mathrm{cr}_{\xi_0}(x, y, z, w) = \sum_{i=1}^j a_i \mathrm{cr}_{\xi_i}(x, y, z, w)$ *for all* $(x, y, z, w) \in \mathcal{A}_\tau^{\mathrm{op}}$.

Proof. Let $c_x, c_y \in \mathrm{Flag}_\sigma$ such that c_x op c_y, x is a face of c_x and y is a face of c_y. We recall the notation of the proof of Lemma 2.8: We denote $p_x := \rho_{c_x,c_y}(o)$, $p_y := \rho_{c_y,c_x}(o)$ and by A_{xy} the unique apartment with $c_x, c_y \subset \partial_\infty A_{xy}$. Moreover, let $A_{xy} \simeq \mathbb{R}^r$ such that $p_x \simeq 0$, in particular A_{xy} inherits an inner product. Let $v_\xi \in A_{xy} \simeq \mathbb{R}^r$ be of norm one and such that the line from $p_x \simeq 0$ through v_ξ is the geodesic ray in A_{xy} from p_x to x_ξ. Then we know from equation (5) that $(x|y)_{o,\xi_i} = \frac{1}{2}\langle v_{\xi_i}, p_y \rangle$.

[5] Actually we would have a spherical join instead of the product. However, we can naturally identify a simplex in a join with the product of the simplices in the different factors — and that is what we do here for simplicity.

By the definition of the v_{ξ_i} it is immediate that $v_{\xi_0} = \sum_{i=1}^{j} a_i v_{\xi_i}$, where we have the addition inherited to A_{xy} under the identification with $\mathbb{R}^r$ such that $p_x \simeq 0$. Hence

$$(x|y)_{o,\xi_0} = \tfrac{1}{2}\langle v_{\xi_0}, p_y\rangle = \sum_{i=1}^{j} \tfrac{1}{2} a_i \langle v_{\xi_i}, p_y\rangle = \sum_{i=1}^{j} a_i (x|y)_{o,\xi_i}. \qquad \square$$

Let $\xi_1, \dots, \xi_r \in \mathfrak{a}$ be the corners of $\sigma = \mathfrak{a}_1^+$. Then every subset $J \subset \{1, \dots, r\}$ defines a simplex in σ, i.e., a face τ of σ. In the same way every simplex $\tau \subset \sigma$ gives a subset $J_\tau \subset \{1, \dots, r\}$.

Given a simplex τ we recall that $\mathfrak{a}_\tau = \mathrm{span}_{j \in J_\tau} \xi_j \subset \mathfrak{a}$. Moreover, we define $\alpha_j^\tau \in \mathfrak{a}_\tau$ for $j \in J_\tau$ by $\langle \alpha_j^\tau, \xi_i\rangle = \delta_{ij}$ for all $i \in J_\tau$; this yields well defined vectors, as the ξ_i with $i \in J_\tau$ form a basis of $\mathfrak{a}_\tau$. We recall that $\mathfrak{a}$ was naturally equipped with an inner product.

The ξ_j correspond to normalized fundamental weights of the root system and the α_j^σ to possibly rescaled roots.

Definition 3.2. *Let τ be a face of σ and J_τ, α_j^τ as above. Then we define a (vector valued) cross ratio* $\mathrm{cr}_\tau : \mathcal{A}_\tau \to \mathfrak{a}_\tau \cup \{\pm\infty\}$ *by*

$$\mathrm{cr}_\tau(x, y, z, w) := \sum_{i \in J_\tau} \mathrm{cr}_{\xi_i}(x, y, z, w)\alpha_i^\tau.$$

Here we set $\mathrm{cr}_\tau(x, y, z, w) := -\infty$ *if $x \not\!\!\mathrm{op}\, y$ or $z \not\!\!\mathrm{op}\, w$ and* $\mathrm{cr}_\tau(x, y, z, w) := \infty$ *if $x \not\!\!\mathrm{op}\, w$ or $z \not\!\!\mathrm{op}\, y$.*

It is straightforward to see that cr_τ has the same symmetries as in equations (3), where the addition is now in the vector space $\mathfrak{a}_\tau$.

The vector valued cross ratio contains the full information of the collection of cross ratios from the previous section:

Lemma 3.3. *Let $\xi \in \mathrm{int}(\tau)$. Then we have $\langle \mathrm{cr}_\tau(x, y, z, w), \xi\rangle = \mathrm{cr}_\xi(x, y, z, w)$ for $(x, y, z, w) \in \mathcal{A}_\tau^{\mathrm{op}}$ and $\mathrm{cr}_\tau(x, y, z, w) = \pm\infty = \mathrm{cr}_\xi(x, y, z, w)$ for $(x, y, z, w) \in \mathcal{A}_\tau \backslash \mathcal{A}_\tau^{\mathrm{op}}$.*

Proof. If $(x, y, z, w) \in \mathcal{A}_\tau \backslash \mathcal{A}_\tau^{\mathrm{op}}$, then the equality is immediate. Hence assume $(x, y, z, w) \in \mathcal{A}_\tau^{\mathrm{op}}$. Then

$$\langle \mathrm{cr}_\tau(x, y, z, w), \xi\rangle = \sum_{i \in J_\tau} \mathrm{cr}_{\xi_i}(x, y, z, w)\langle \alpha_i^\tau, \xi\rangle.$$

Since $\langle \alpha_j^\tau, \xi_i\rangle = \delta_{ij}$ for all $i \in J_\tau$, we derive that $\langle \sum_{i \in J_\tau} \langle \alpha_i^\tau, \xi\rangle \xi_i, \alpha_j^\tau\rangle = \langle \xi, \alpha_j^\tau\rangle$ for all in $j \in J_\tau$. Moreover, it is immediate that the α_j^τ form a base of $\mathfrak{a}_\tau$. Thus we get that $\sum_{i \in J_\tau} \langle \alpha_i^\tau, \xi\rangle \xi_i = \xi$. Therefore Lemma 3.1 implies $\sum_{i \in J_\tau} \langle \alpha_i^\tau, \xi\rangle \mathrm{cr}_{\xi_i}(x, y, z, w) = \mathrm{cr}_\xi(x, y, z, w)$. $\square$

The above lemma also holds for $\xi \in \partial\tau$ as long as $(x, y, z, w) \in \mathcal{A}_\tau^{\mathrm{op}}$, but does not hold for general $(x, y, z, w) \in \mathcal{A}_\tau$; in this case $\mathrm{cr}_\xi(x, y, z, w)$ might be finite while $\mathrm{cr}_\tau(x, y, z, w)$ is not (compare the discussion just after Lemma 2.8).

The following corollary captures the topological properties of cr_τ in case of symmetric spaces. It is an immediate consequence of the lemma above and Lemma 2.7.

Corollary 3.4. *Let X be a symmetric space. The map cr_τ restricted to $\mathcal{A}_\tau^{\mathrm{op}}$ is continuous and for all $\xi \in \mathrm{int}(\tau)$ the map $\langle \mathrm{cr}_\tau(\cdot), \xi\rangle : \mathcal{A}_\tau \to \mathbb{R} \cup \{\pm\infty\}$ is continuous.*

Let $\pi_\tau : \mathfrak{a} \to \mathfrak{a}_\tau$ be the orthogonal projection. Then it is straightforward to show that $\pi_\tau(\alpha_i^\sigma) = \alpha_i^\tau$ for all $i \in J_\tau$ and $\pi_\tau(\alpha_j^\sigma) = 0$ for all $j \notin J_\tau$. Then we can derive that $\mathrm{cr}_\tau(x, y, z, w) = \pi_\tau(\mathrm{cr}_\sigma(x, y, z, w))$ for all $(x, y, z, w) \in \mathcal{A}_\sigma^{\mathrm{op}}$.

Translation vectors and periods. We assume for this section that τ is *self-opposite*, i.e., $\tau = \iota\tau$. Moreover denote by $\mathrm{Iso}_e(M)$ the subgroup of $\mathrm{Iso}(M)$ such that $g_\sigma = \mathrm{id}$ for all $g \in \mathrm{Iso}_e(M)$—in particular $G = \mathrm{Iso}_e(X)$ for a symmetric space X. Let $g \in \mathrm{Iso}_e(M)$ such that g stabilizes two points $g^\pm \in \mathrm{Flag}_\tau$ with $g^- \ \mathtt{op}\ g^+$. Since g is an isometry, it maps every geodesic connecting points of the interior of g^- and g^+ to another geodesic connecting the same points. In particular g stabilizes $P(g^-, g^+)$ set-wise.

In the preliminaries we have seen that $P(g^-, g^+)$ splits as a product $\mathfrak{a}_\tau \times CS(g^-, g^+)$ such that $g^\pm$ are identified with the positive and negative, respectively, maximal dimensional simplices in $\mathfrak{a}_\tau$, i.e., $g^+ \simeq \partial_\infty \mathfrak{a}_\tau^+$ where $\mathfrak{a}_\tau^+ := \mathfrak{a}_\tau \cap \mathfrak{a}^+$. Note that g descends to an isometry $g_{\mathfrak{a}_\tau}$ of $\mathfrak{a}_\tau$. Since $\mathfrak{a}_\tau$ is Euclidean and $g_{\mathfrak{a}_\tau}$ stabilizes each boundary point of $\mathfrak{a}_\tau$, $g_{\mathfrak{a}_\tau}$ acts as a translation on $\mathfrak{a}_\tau$. More precisely, there exists a *translation vector* $\ell_g^\tau \in \mathfrak{a}_\tau$ such that $g_{\mathfrak{a}_\tau}(p) = p + \ell_g^\tau$ for all $p \in \mathfrak{a}_\tau$.

Proposition 3.5. *Let $g \in \mathrm{Iso}_e(M)$ such that $g^\pm \in \mathrm{Flag}_\tau$ with $g^- \ \mathtt{op}\ g^+$ are stabilized by g. Let ℓ_g^τ denote the translation vector along the first factor of $P(g^-, g^+) \simeq \mathfrak{a}_\tau \times CS(g^-, g^+)$. Then $\mathrm{cr}_\tau(g^-, g \cdot x, g^+, x) = \frac{1}{2}(\ell_g^\tau + \iota\ell_g^\tau)$, for any $x \in \mathrm{Flag}_\tau$ with $x\ \mathtt{op}\ g^\pm$.*

Proof. We remark that $\mathrm{cr}_\tau(g^-, g \cdot x, g^+, x)$ is independent of the choice of $x\ \mathtt{op}\ g^\pm$; this follows from the symmetries of cr_τ together with Proposition 2.9. Therefore, we fix one $x \in \mathrm{Flag}_\tau$ with $x\ \mathtt{op}\ g^\pm$.

Let $o \in P(g^-, g^+)$ and ξ_i with $i \in J_\tau$ be the corners of τ. By assumption $x\ \mathtt{op}\ g^\pm$ and hence $g \cdot x\ \mathtt{op}\ g^\pm$. Then Proposition 2.3 yields

$$(g^\pm \,|\, g \cdot x)_{o,\xi_i} = (g^\pm \,|\, x)_{g^{-1}\cdot o,\xi_i} = (g^\pm \,|\, x)_{o,\xi_i} + \tfrac{1}{2} b_{g^\pm_{\xi_i}}(g^{-1} \cdot o, o) + \tfrac{1}{2} b_{x_{\iota\xi_i}}(g^{-1} \cdot o, o).$$

Moreover, we have $b_{g^\pm_{\xi_i}}(g^{-1} \cdot o, o) = b_{g^\pm_{\xi_i}}(o, g \cdot o)$. Plugging this in the definition of cr_{ξ_i} several terms cancel such that $\mathrm{cr}_{\xi_i}(g^-, g\cdot x, g^+, x) = \frac{1}{2} b_{g^+_{\xi_i}}(o, g\cdot o) - \frac{1}{2} b_{g^-_{\xi_i}}(o, g\cdot o)$. Since $o, g\cdot o \in P(g^-, g^+)$ and $g^+_{\iota\xi_i} \in g^+$ is the point opposite to $g^-_{\xi_i} \in g^-$, Lemma 1.2 implies $b_{g^-_{\xi_i}}(o, g \cdot o) = -b_{g^+_{\iota\xi_i}}(o, g \cdot o)$. Altogether we get that $\mathrm{cr}_{\xi_i}(g^-, g \cdot x, g^+, x) = \frac{1}{2} b_{g^+_{\xi_i}}(o, g \cdot o) + \frac{1}{2} b_{g^+_{\iota\xi_i}}(o, g \cdot o)$.

Since o was arbitrary in $P(g^-, g^+)$ we can assume that its first coordinate under the identification $P(g^-, g^+) \simeq \mathfrak{a}_\tau \times CS(g^-, g^+)$ is $0 \in \mathfrak{a}_\tau$. Moreover, we can use Lemma 1.1 to see that only the first factor matters for the Busemann functions $b_{g_{\xi_i}}, b_{g_{\iota\xi_i}}$. As g acts as a translation on $\mathfrak{a}_\tau$, we have that $g \cdot 0 = \ell_g^\tau$. Therefore $b_{g^+_{\xi_i}}(o, g \cdot o) = \langle \xi_i, \ell_g^\tau\rangle$ (cp. the arguments around equation (5)). By assumption $\tau = \iota\tau$, hence ι restricts to an isometry $\iota : \mathfrak{a}_\tau \to \mathfrak{a}_\tau$. Together with $\iota^2 = id$, this yields $\langle \iota\xi_i, \ell_g^\tau\rangle = \langle \xi_i, \iota\ell_g^\tau\rangle$. Altogether we derive

$$\mathrm{cr}_\tau(g^-, g\cdot x, g^+, x) = \sum_{i\in J_\tau} \tfrac{1}{2}(\langle \xi_i, \ell_g^\tau\rangle + \langle \xi_i, \iota\ell_g^\tau\rangle)\alpha_i^\tau.$$

It is immediate that $\langle \mathrm{cr}_\tau(g^-, g\cdot x, g^+, x), \xi_i\rangle = \frac{1}{2}(\langle \xi_i, \ell_g^\tau\rangle + \langle \xi_i, \iota\ell_g^\tau\rangle)$ for all $i \in J_\tau$. Since the ξ_i with $i \in J_\tau$ form a basis of τ, it follows that $\mathrm{cr}_\tau(g^-, g\cdot x, g^+, x) = \frac{1}{2}(\ell_g^\tau + \iota\ell_g^\tau)$. □

Let $g \in \mathrm{Iso}_e(M)$ be as before. Then the term $\mathrm{cr}_\tau(g^-, g\cdot x, g^+, x)$ is also called *period* — in analogy to rank one spaces. In particular, the periods give rise to the translation vector of the first factor of the parallel set if $\iota = \mathrm{id}$.

Geometric interpretation of the cross ratio. Let $x, z \in \mathrm{Flag}_\tau$ and $y, w \in \mathrm{Flag}_{\iota\tau}$ with $x, z \ \mathtt{op}\ y, w$. Pick $c_x, c_z, d_y, d_w, d'_w \in \mathrm{Flag}_\sigma$ such that x is a face of c_x and accordingly the other chambers and that $c_x \ \mathtt{op}\ d_y, d_w$ as well as $c_z \ \mathtt{op}\ d_y, d'_w$. Then we use the following notations for the horospherical retracts $\rho_x := \rho_{c_x,d_y}$, $\rho_w := \rho_{d_w,c_x}$, $\rho_z := \rho_{c_z,d'_w}$ and $\rho_y := \rho_{d_y,c_z}$.

Lemma 3.6. *Let $(x,y,z,w) \in \mathcal{A}_\tau^{\mathrm{op}}$ and let ρ_x, ρ_w, ρ_z and ρ_y as above. Moreover, be o in the unique affine apartment joining c_x and d_y. Then for all $i \in J_\tau$ we have $2\mathrm{cr}_{\xi_i}(x,y,z,w) = b_{x_{\xi_i}}(o, \rho_x\rho_w\rho_z\rho_y(o))$.*

Proof. Denote by A_{xy} the unique affine apartment joining c_x and d_y. Then ρ_{d_y,c_x} restricted to A_{xy} is the identity, i.e., $\rho_{d_y,c_x}(o) = o$. Therefore Proposition 2.1 implies that $2(x\,|\,y)_{o,\xi_i} = b_{x_{\xi_i}}(o,o) = 0$.

By definition $\rho_y(o)$ is contained in the unique affine apartment joining c_z and d_y. Then in the same way it follows that $(z\,|\,y)_{\rho_y(o),\xi_i} = 0$. Moreover, equation (2) yields $b_{y_{\iota\xi_i}}(o, \rho_y(o)) = b_{y_{\iota\xi_i}}(o,o) = 0$.

We can use Proposition 2.3 and again equation (2) to derive that

$$2(z\,|\,y)_{o,\xi_i} = 2(z\,|\,y)_{\rho_y(o),\xi_i} + b_{z_{\xi_i}}(o, \rho_y(o)) + b_{y_{\iota\xi_i}}(o, \rho_y(o)) = b_{z_{\xi_i}}(o, \rho_z\rho_y(o)).$$

In a very similar way we get

$$\begin{aligned} 2(z\,|\,w)_{o,\xi_i} &= b_{z_{\xi_i}}(o, \rho_z\rho_y(o)) + b_{w_{\iota\xi_i}}(o, \rho_w\rho_z\rho_y(o)) \\ 2(x\,|\,w)_{o,\xi_i} &= b_{x_{\xi_i}}(o, \rho_x\rho_w\rho_z\rho_y(o)) + b_{w_{\iota\xi_i}}(o, \rho_w\rho_z\rho_y(o)). \end{aligned}$$

Using that $\mathrm{cr}_{\xi_i}(x,y,z,w) = -(x\,|\,y)_{o,\xi_i} - (z\,|\,w)_{o,\xi_i} + (x\,|\,w)_{o,\xi_i} + (z|y)_{o,\xi_i}$, we get $2\mathrm{cr}_{\xi_i}(x,y,z,w) = b_{x_{\xi_i}}(o, \rho_x\rho_w\rho_z\rho_y(o))$. □

Proposition 3.7. *Let ρ_x, ρ_w, ρ_z and ρ_y as before. Let o be in the unique affine apartment joining c_x, d_y such that we have under the identification $P(x,y) \simeq \mathfrak{a}_\tau \times CS(x,y)$ that $\pi(o) = 0 \in \mathfrak{a}_\tau$, where π is the projection to the first factor (also assume $x \simeq \mathfrak{a}_\tau^+$). Then $2\mathrm{cr}_\tau(x,y,z,w) = \pi(\rho_x\rho_w\rho_z\rho_y(o))$.*

Proof. By construction we have that $o, \rho_x\rho_w\rho_z\rho_y(o)$ are in the unique affine apartment joining c_x and d_y. Then by Lemma 1.1 and from similar arguments as around equation (5) we can derive that $b_{x_{\xi_i}}(o, \rho_x\rho_w\rho_z\rho_y(o)) = \langle \xi_i, \pi(\rho_x\rho_w\rho_z\rho_y(o))\rangle$ for all $i \in J_\tau$. Together with Lemma 3.6 and the definition of cr_τ we get

$$2\mathrm{cr}_\tau(x, y, z, w) = \sum_{i\in J_\tau} \langle \xi_i, \pi(\rho_x\rho_w\rho_z\rho_y(o))\rangle \alpha_i^\tau.$$

The $\xi_i \in \mathfrak{a}_\tau$ for $i \in J_\tau$ form a basis of $\mathfrak{a}_\tau$. Moreover, for all $i \in J_\tau$ we have that $\langle 2\mathrm{cr}_\tau(x, y, z, w), \xi_i\rangle = \langle \xi_i, \pi(\rho_x\rho_w\rho_z\rho_y(o))\rangle$. Thus it follows that $2\mathrm{cr}_\tau(x, y, z, w) = \pi(\rho_x\rho_w\rho_z\rho_y(o))$. □

4. Cross ratio preserving maps

We assume in this section that τ *is self-opposite*, i.e., $\tau = \iota\tau$.

Definition 4.1. *Let M_i, $i = 1, 2$ be either both symmetric spaces or thick Euclidean buildings. A map $f : \mathrm{Flag}_{\tau_1}(M_1) \to \mathrm{Flag}_{\tau_2}(M_2)$ is called* ξ_1-Moebius map (*or* cross ratio preserving) *if there exists $\xi_i \in \mathrm{int}(\tau_i)$ such that $\mathrm{cr}_{\xi_1}(x, y, z, w) = \mathrm{cr}_{\xi_2}(f(x), f(y), f(z), f(w))$ for all $(x, y, z, w) \in \mathcal{A}_{\tau_1}$, we in particular assume that $f(\mathcal{A}_{\tau_1}) \subset \mathcal{A}_{\tau_2}$.*

If f is a ξ_1-Moebius map with respect to ξ_1, ξ_2, we also denote this by $\mathrm{cr}_{\xi_1} = f^*\mathrm{cr}_{\xi_2}$. If ξ_1 is clear from the context, we sometimes call f just a *Moebius map.* Moreover, for any map $f : \mathrm{Flag}_{\tau_1}(M_1) \to \mathrm{Flag}_{\tau_2}(M_2)$ we denote $f^*\mathrm{cr}_{\xi_2}(x, y, z, w) := \mathrm{cr}_{\xi_2}(f(x), f(y), f(z), f(w))$ for $x, y, z, w \in \mathrm{Flag}_{\tau_1}(M_1)$.

Lemma 4.2. *Let $x, y \in \mathrm{Flag}_\tau$. Then there exists $z \in \mathrm{Flag}_\tau$ with z $\mathtt{op}$ x, y.*

Proof. We take $c_x, c_y \in \mathrm{Flag}_\sigma$ such that x is a face of c_x and y is a face c_y. Then there exists $c_z \in \mathrm{Flag}_\sigma$ with c_z $\mathtt{op}$ c_x, c_y [2, 5.1]. Be z the face of c_z which is of type τ. Then $z \in \mathrm{Flag}_\tau$ with z $\mathtt{op}$ x, y. □

Lemma 4.3. *Let $f : \mathrm{Flag}_{\tau_1}(M_1) \to \mathrm{Flag}_{\tau_2}(M_2)$ be a ξ_1-Moebius map. Then for $x, y \in \mathrm{Flag}_{\tau_1}(M_1)$ we have that x $\mathtt{op}$ y if and only if $f(x)$ $\mathtt{op}$ $f(y)$.*

Proof. Let $x, y \in \mathrm{Flag}_{\tau_1}(M_1)$ be given. Choose $z_1, z_2, z_3 \in \mathrm{Flag}_{\tau_1}(M_1)$ such that z_3 $\mathtt{op}$ x; z_2 $\mathtt{op}$ y, z_3 and z_1 $\mathtt{op}$ x, z_2. From Corollary 2.2 we know that $\mathrm{cr}_{\xi_1}(x,y,z_2,z_3) = r$ and $\mathrm{cr}_{\xi_1}(x, z_1, z_2, z_3) \neq \pm\infty$, i.e., x $\mathtt{op}$ $y \iff r \neq -\infty$. Since $\mathrm{cr}_{\xi_1} = f^*\mathrm{cr}_{\xi_2}$, we can derive that $f(z_2)$ $\mathtt{op}$ $f(z_3)$ and therefore we have $f(x)$ $\mathtt{op}$ $f(y) \iff r \neq -\infty$. In particular, $f(x)$ $\mathtt{op}$ $f(y) \iff x$ $\mathtt{op}$ y. □

A map $f : \mathrm{Flag}_{\tau_1}(M_1) \to \mathrm{Flag}_{\tau_2}(M_2)$ such that for all $x, y \in \mathrm{Flag}_{\tau_1}(M_1)$ it holds that x $\mathtt{op}$ y if and only if $f(x)$ $\mathtt{op}$ $f(y)$ is called *opposition preserving.*

Lemma 4.4. *Let $f : \mathrm{Flag}_{\tau_1}(M_1) \to \mathrm{Flag}_{\tau_2}(M_2)$ be a ξ_1-Moebius map. Then f is injective.*

Proof. Assume there exist $x \neq y \in \mathrm{Flag}_{\tau_1}(M_1)$ with $f(x) = f(y)$. Take $a \in \mathrm{Flag}_{\tau_1}(M_1)$ with a $\mathtt{op}$ x and a $\not\mathtt{op}$ y: For example take an apartment which contains x and y. Take a opposite of x in this apartment. Then $x \neq y$ implies that a $\not\mathtt{op}$ y— opposite points are unique in apartments.

In addition, choose $z, w \in \mathrm{Flag}_{\tau_1}(M_1)$ such that z $\mathtt{op}$ a and w $\mathtt{op}$ z, x. Then $\mathrm{cr}_{\xi_1}(x, a, z, w) \neq \pm\infty$ and $\mathrm{cr}_{\xi_1}(y, a, z, w) = -\infty$ or is not defined; but

$$\mathrm{cr}_{\xi_1}(x, a, z, w) = f^*\mathrm{cr}_{\xi_2}(x, a, z, w) = f^*\mathrm{cr}_{\xi_2}(y, a, z, w) = \mathrm{cr}_{\xi_1}(y, a, z, w),$$

contradicting $\mathrm{cr}_{\xi_1}(x, a, z, w) \neq \mathrm{cr}_{\xi_1}(y, a, z, w)$. Hence $f(x) \neq f(y)$ if $x \neq y$. □

Definition 4.5. *A surjective ξ_1-Moebius map is called a ξ_1-*Moebius bijection*.*

When restricting to the full flag space we can apply the following result due to Abramenko and van Maldeghem.[6]

Proposition 4.6 (Corollary 5.2 of [2])**.** *Let $f : \mathrm{Flag}_\sigma(M_1) \to \mathrm{Flag}_\sigma(M_2)$ be a surjective map that preserves opposition. Then f extends in a unique way to an automorphism of the building $f : \Delta_\infty M_1 \to \Delta_\infty M_2$.*

Lemma 4.7. *Let $B = B_1 \circ \cdots \circ B_k$ and $B' = B'_1 \circ \cdots \circ B'_{k'}$ be joins of irreducible thick spherical buildings. Moreover, be $f : B \to B'$ a building isomorphism. Then $k = k'$ and there exists a permutation s on k numbers such that $f = f_1 \times \cdots \times f_k$ with $f_i : B_i \to B'_{\mathrm{s}(i)}$ building isomorphisms.*

Proof. That f is a building isomorphism implies that B and B' are modeled over the same spherical Coxeter complex, i.e., over the Coxeter complex to $W = W_1 \times \cdots \times W_k$, where W_i are irreducible Coxeter groups. The irreducibility of the buildings B_i, B'_i yields then that $k = k'$.

Assume without loss of generality that $|W_1| \leq |W_i|$ for all $i = 1, \dots, k$. Let x_1 be a chamber in B_1. Then x_1 is a simplex in B. We know that $\mathrm{Res}(x_1)$ is a spherical building over the spherical Coxeter complex to $W_2 \times \cdots \times W_k$. As f is a building isomorphism, we derive that $f(\mathrm{Res}(x_1)) = \mathrm{Res}(f(x_1))$ is a spherical building over $W_2 \times \cdots \times W_k$. If $f(x_1)$ would not correspond to a chamber in an irreducible factor B'_i, then there would be a subgroup W' of W isomorphic to $W_2 \times \cdots \times W_k$ such that the projection of W' to each W_i is non-trivial (as W_1 is minimal). This would yield a decomposition of $W_2 \times \cdots \times W_k$ into k Coxeter groups, which contradicts the irreducibility of the factors. In particular, up to reordering $\mathrm{Res}(f(x_1))$ is a spherical building over $W_1 \times W_3 \times \cdots \times W_k$ and W_1 is isomorphic to W_2. Thus $f(x_1) = y_2$ for a chamber $y_2 \in B'_2$. Since f is a building isomorphism it maps all simplices of the same type as x_1 to simplices of the same type as y_2 i.e., it maps the chambers of B_1 to chambers of B'_2. In particular, f induces a building isomorphism $f_1 = f|_{B_1} : B_1 \to B'_2$ (B_1 is naturally a subset of B, namely the set of simplices of B fully contained in B_1) and thus $f = f_1 \times f_0$ for a building isomorphism $f_0 : B_2 \circ \cdots \circ B_k \to B'_1 \circ B'_3 \ldots \circ B'_k$. A straightforward induction yields the result. □

We remark that multiplying the metric of a space M by some positive constant α, yields that the Gromov product on $\mathrm{Flag}_\tau(\alpha M)$ is given by $(\cdot|\cdot)_{\xi,\alpha M} = \alpha(\cdot|\cdot)_{\xi,M}$ and hence also $\mathrm{cr}_{\xi,\alpha M} = \alpha \mathrm{cr}_{\xi,M}$. Moreover, there is a natural identification of $\mathrm{Flag}_\tau(\alpha M)$ with $\mathrm{Flag}_\tau(M)$.

Lemma 4.8. *Let $M_i = M_i^1 \times \cdots \times M_i^k$ be products of either irreducible symmetric spaces or irreducible thick Euclidean buildings. Moreover, be $f : \mathrm{Flag}_\sigma(M_1) \to \mathrm{Flag}_\sigma(M_2)$ a ξ_1-Moebius bijection. Then there exists a permutation s on k numbers such that $f = f_1 \times \cdots \times f_k$ with $f_i : \mathrm{Flag}_\sigma(\widehat{M_1^i}) \to \mathrm{Flag}_\sigma(M_2^{\mathrm{s}(i)})$ a ξ_1^i-Moebius*

[6]We remark that every spherical building is 2-*spherical* as in the notation of [2]. Moreover, the buildings at infinity of symmetric spaces and thick Euclidean building are thick; hence we can apply their result.

bijection and $\widehat{M_1^i}$ is the space M_1^i with its metric rescaled (for the types ξ_1^i see the proof).

Proof. Let $f : \Delta_\infty M_1 \to \Delta_\infty M_2$ be the building isomorphism from Proposition 4.6. From Lemma 4.7 we get a permutation s on k letters and building isomorphisms $f_i : \Delta_\infty M_1^i \to \Delta_\infty M_2^{\mathrm{s}(i)}$ such that

$$f = f_1 \times \cdots \times f_k : \Delta_\infty M_1^1 \circ \cdots \circ \Delta_\infty M_1^k \to \Delta_\infty M_2^{\mathrm{s}(1)} \circ \cdots \circ \Delta_\infty M_2^{\mathrm{s}(k)}.$$

Moreover, we know from Proposition 2.12 that $\mathrm{cr}_{\xi_i} = \mu_i^1 \mathrm{cr}_{\xi_i^1} + \cdots + \mu_i^k \mathrm{cr}_{\xi_i^k}$ with $\xi_i^j \in \sigma_i^j$ for $i = 1, 2$ and $j = 1, \ldots, k$ and $\mu_i \in S_k^+$ such that $\xi_i = \pi_i(\xi_i^1, \ldots, \xi_i^k, \mu_i)$ with π_i as in the proposition (the numbers in the exponent are for indexing, not powers). Fix $(x_0, y_0, z_0, w_0) \in \mathrm{Flag}_{\sigma_2}(M_1^2) \circ \cdots \circ \mathrm{Flag}_{\sigma_k}(M_1^k)$ with x_0, z_0 $\mathtt{op}$ y_0, w_0. Then for any $(x_1, y_1, z_1, w_1) \in \mathcal{A}_{\sigma_1}$ we get

$$\begin{aligned}&\mu_1^1 \mathrm{cr}_{\xi_1^1}(x_1, y_1, z_1, w_1) + (\mu_1^2 \mathrm{cr}_{\xi_1^2} \ldots + \mu_1^k \mathrm{cr}_{\xi_1^k})(x_0, y_0, z_0, w_0) \\ &\quad = \mu_2^{\mathrm{s}(1)} f_1^* \mathrm{cr}_{\xi_2^{\mathrm{s}(1)}}(x_1, y_1, z_1, w_1) + f_0^*(\mu_2^{\mathrm{s}(2)} \mathrm{cr}_{\xi_2^{\mathrm{s}(2)}} \cdots + \mu_2^{\mathrm{s}(k)} \mathrm{cr}_{\xi_2^{\mathrm{s}(2)}})(x_0, y_0, z_0, w_0)\end{aligned}$$

with $f_0 = f_2 \times \cdots \times f_k$. The equality also holds when we replace (x_0, y_0, z_0, w_0) with (z_0, y_0, x_0, w_0). Moreover, we have that $(\mu_1^2 \mathrm{cr}_{\xi_1^2} \cdots + \mu_1^k \mathrm{cr}_{\xi_1^k})(x_0, y_0, z_0, w_0) = -(\mu_1^2 \mathrm{cr}_{\xi_1^2} \cdots + \mu_1^k \mathrm{cr}_{\xi_1^k})(z_0, y_0, x_0, w_0)$. Hence we derive that

$$\mu_1^1 \mathrm{cr}_{\xi_1^1}(x_1, y_1, z_1, w_1) = \mu_2^{\mathrm{s}(1)} f_1^* \mathrm{cr}_{\xi_2^{\mathrm{s}(1)}}(x_1, y_1, z_1, w_1).$$

As (x_1, y_1, z_1, w_1) was arbitrary in $\mathcal{A}_{\sigma_1}$ we get $\mu_1^1 \mathrm{cr}_{\xi_1^1} = \mu_2^{\mathrm{s}(1)} f_1^* \mathrm{cr}_{\xi_2^{\mathrm{s}(1)}}$. In the same way it follows for all $i = 1, \ldots, k$ that $\mu_1^i \mathrm{cr}_{\xi_1^i} = \mu_2^{\mathrm{s}(i)} f_i^* \mathrm{cr}_{\xi_2^{\mathrm{s}(i)}}$.

If we rescale the metric on M_1^i by $\mu_2^{\mathrm{s}(i)}/\mu_1^i$ — denote this space by $\widehat{M_1^i}$ — then $f_i : \Delta_\infty \widehat{M_1^i} \to \Delta_\infty M_2^{\mathrm{s}(i)}$ restricts to a Moebius bijection on the chamber sets, i.e., we get a Moebius bijection $f_i : \mathrm{Flag}_\sigma(\widehat{M_1^i}) \to \mathrm{Flag}_\sigma(M_2^{\mathrm{s}(i)})$. □

We will need the following fact:

Theorem 4.9 ([4]). *Let T_1, T_2 be geodesically complete trees with $|\partial_\infty T_i| \geq 3$. Then every isometry from T_1 to T_2 restricted to the boundary is a Moebius bijection and every Moebius bijection $f : \partial_\infty T_1 \to \partial_\infty T_2$ can be uniquely extended to an isometry.*

Let T be a rank one thick Euclidean building; in particular T is a tree. Then every geodesic segment in T lies in an affine apartment, i.e., in a bi-infinite geodesic. This means that T is geodesically complete (in the notation of [4]). Moreover, by definition of thickness for rank one Euclidean buildings we have that $|\partial_\infty T| \geq 3$.

We remark that $\mathrm{rk}(T) = 1$ implies that the positive chamber of the Coxeter complex σ_T consists of a single point. Thus $\Delta_\infty T = \mathrm{Flag}_\sigma(T) = \partial_\infty T$. Hence there is a unique Gromov product $(\cdot, \cdot)_{o_T}$ for any $o_T \in T$ on $\partial_\infty T^2$ and a unique cross ratio cr_T on $\mathcal{A}_T \subset \partial_\infty T^4$.

Recall that a locally compact Euclidean building with discrete translation group is called a *combinatorial Euclidean building.* Moreover, given a metric realization (B, d_B) of a spherical building as a CAT(1) space, the *cone* E_B *over* B is the quotient of $B \times [0, \infty)/\!\sim$ for the equivalence relation $(b_1, t) \sim (b_2, s) \iff s = 0 = t$ with $b_i \in B$ and $s, t \in [0, \infty)$. The metric on E_B is given by $d_{E_B}((b_1, t), (b_2, s)) = s^2 + t^2 - 2st\cos(d_B(b_1, b_2))$.

Proposition 4.10. *Let E_1, E_2 be irreducible thick combinatorial Euclidean buildings. Then every Moebius bijection $f : \mathrm{Flag}_\sigma(E_1) \to \mathrm{Flag}_\sigma(E_2)$ is the restriction of an isometry $F : \widehat{E}_1 \to E_2$ to the boundary where $\widehat{E}_1$ is E_1 with its metric rescaled. If E_1 is not the cone over a spherical building, then F is unique.*

Proof. If the rank is one, then the result follows from the theorem above.

If the rank is at least 2, Struyve has shown in [31] that every isometry between $\partial_\infty E_1$ and $\partial_\infty E_2$ with respect to the Tits metric is induced by an isometry after rescaling the metric on E_1. The isometry is unique if E_1 is not the cone over a spherical building. We know that f induces a building isomorphism $f : \Delta_\infty E_1 \to \Delta_\infty E_2$ and this yields an isometry $f : \partial_\infty E_1 \to \partial_\infty E_2$ with respect to the Tits metric when viewing simplices as subsets of $\partial_\infty E_i$. Hence we can apply the result of Struyve. □

The non-uniqueness for cones over spherical buildings arises, for example, as follows: Let E_B be a cone over a spherical building B. Then clearly the identity map $\mathrm{id} : \mathrm{Flag}_\sigma(E_B) \to \mathrm{Flag}_\sigma(E_B)$ is a Moebius bijection. However, every homothety of E_B, i.e., every map $F_\lambda : E_B \to E_B$, $(b_1, t) \mapsto (b_1, \lambda t)$ for $\lambda \in (0, \infty)$, is an isometry from $F_\lambda : \lambda^2 E_B \to E_B$, where $\lambda^2 E_B$ is the space E_B with its metric rescaled by λ^2. In particular, every F_λ extends the map $\mathrm{id} : \mathrm{Flag}_\sigma(E_B) \to \mathrm{Flag}_\sigma(E_B)$ as an isometry after rescaling the metric on the domain of F_λ by λ^2.

Corollary 4.11. *Let E_1 and E_2 be combinatorial Euclidean buildings and let $f : \mathrm{Flag}_\sigma(E_1) \to \mathrm{Flag}_\sigma(E_2)$ be a Moebius bijection. Then one can rescale the metric of E_1 on irreducible factors — denote this space by $\widehat{E}_1$ — such that f is the restriction of an isometry $F : \widehat{E}_1 \to E_2$ to the boundary. If none of the irreducible factors is a cone over a spherical building the isometry F is unique.*

Proof. This follows from Lemma 4.8 and the proposition above. □

Symmetric spaces. We want to show that the above proposition and corollary hold in a similar way for symmetric spaces. We will see that we essentially only need to show that Moebius bijections are homeomorphisms. Therefore we analyze some topological properties of Moebius bijections for the case of symmetric spaces.

In this section we only consider symmetric spaces X. For $r \in \mathbb{R}$, $\xi \in \mathrm{int}(\tau)$ and $x_2, y_1, y_2 \in \mathrm{Flag}_\tau(X)$ we define

$$B^+_{r,\xi}(y_1, x_2, y_2) := \{x_1 \in \mathrm{Flag}_\tau(X) \mid (x_1, y_1, x_2, y_2) \in \mathcal{A}_\xi,\ \mathrm{cr}_\xi(x_1, y_1, x_2, y_2) > r\},$$
$$B^-_{r,\xi}(y_1, x_2, y_2) := \{x_1 \in \mathrm{Flag}_\tau(X) \mid (x_1, y_1, x_2, y_2) \in \mathcal{A}_\xi,\ \mathrm{cr}_\xi(x_1, y_1, x_2, y_2) < r\}.$$

Those sets are open by the continuity of cr_ξ and the fact that $\mathcal{A}_\xi$ is open. However, it can happen that they are empty — which holds if $x_2 \not\mathrm{op}\ y_1, y_2$.

Proposition 4.12. *Let X be a symmetric space. The sets $B^-_{r,\xi}(y_1, x_2, y_2)$ varying over all $r \in \mathbb{R}$ and all $x_2, y_1, y_2 \in \mathrm{Flag}_\tau$ form a subbase of the topology on $\mathrm{Flag}_\tau(X)$*

Proof. As mentioned, those sets are open. Thus it is enough to show that any open neighborhood U of a point $x \in \mathrm{Flag}_\tau(X)$ contains an open neighborhood V which can be written as a finite intersection of sets of the form $B^-_{r,\xi}(y_1, x_2, y_2)$.

Let $x \in \mathrm{Flag}_\tau(X)$ and let any neighborhood U of x be given. We set $K := \mathrm{Flag}_\tau \backslash U$. Then K is compact and $x \notin K$.

For any $a \in K$, choose $y_a \in \mathrm{Flag}_\tau(X)$ such that $y_a \,\mathtt{op}\, a$ and $y_a \,\not\!\mathtt{op}\, x$. In addition, choose $w_a, z_a \in \mathrm{Flag}_\tau(X)$ such that $w_a \,\mathtt{op}\, a, x$ and $z_a \,\mathtt{op}\, y_a, w_a$. This yields $\mathrm{cr}_\xi(x, y_a, z_a, w_a) = -\infty$ and $\mathrm{cr}_\xi(a, y_a, z_a, w_a) > r_a$ for some $r_a \in \mathbb{R}$ and hence $x \in B^-_{r_a,\xi}(y_a, z_a, w_a)$, $x \notin B^+_{r_a,\xi}(y_a, z_a, w_a)$, $a \in B^+_{r_a,\xi}(y_a, z_a, w_a)$.

Varying over all $a \in K$ the sets $B^+_{r_a,\xi}(y_a, z_a, w_a)$ cover K and by compactness we find a finite number of points $a_i \in K$, $i = 1, \ldots, l$ such that the according sets already cover K. We set $V := \bigcap_{a_i : i=1,\ldots,l} B^-_{r_{a_i},\xi}(y_{a_i}, z_{a_i}, w_{a_i})$. As a finite intersection of open sets, V is open. Furthermore, $x \in V$ and hence V is non-empty. By construction $V \subset K^C$ and hence $V \subset U$. □

Lemma 4.13. *Let $f : \mathrm{Flag}_{\tau_1}(X_1) \to \mathrm{Flag}_{\tau_2}(X_2)$ be a ξ_1-Moebius bijection. Then f is a homeomorphism.*

Proof. Since f leaves the cross ratio invariant and is a bijection, it is immediate that $f(B^-_{r,\xi_1}(y, z, w)) = B^-_{r,\xi_2}(f(y), f(z), f(w))$. This means that f yields a bijection of subbases of the topology and hence f is a homeomorphism. □

As mentioned, for a symmetric space X the boundary $\mathrm{Flag}_\tau(X)$ can be identified homeomorphically with G/P_x for $P_x = \mathrm{stab}(x)$ and $x \in \mathrm{Flag}_\tau(X)$. Hence $\mathrm{Flag}_\tau(X)$ can be given the structure of compact connected manifold (without boundary)— inherited from G/P_x. Using this there is a different way to characterize Moebius bijections captured in the following lemma.

Lemma 4.14. *Let X_1, X_2 be symmetric spaces. Assume that $\dim \mathrm{Flag}_{\tau_1}(X_1) = \dim \mathrm{Flag}_{\tau_2}(X_2)$ and let $f : \mathrm{Flag}_{\tau_1}(X_1) \to \mathrm{Flag}_{\tau_2}(x_2)$ be a continuous ξ_1-Moebius map. Then f is a homeomorphism, in particular f is a ξ_1-Moebius bijection.*

Proof. Since f is a ξ_1-Moebius map and hence injective, $f : \mathrm{Flag}_{\tau_1}(X_1) \to \mathrm{Im}(f)$ is a bijection, with $\mathrm{Im}(f)$ denoting the image. Moreover, $f^*\mathrm{cr}_{\xi_2} = \mathrm{cr}_{\xi_1}$ implies $f(B^-_{r,\xi_1}(y, z, w)) = B^-_{r,\xi_2}(f(y), f(z), f(w)) \cap \mathrm{Im}(f)$. Then Proposition 4.12 yields that f maps a subbase of the topology on $\mathrm{Flag}_{\tau_1}(X_1)$ into a subbase of the topology on $\mathrm{Im}(f)$ equipped with the subset topology. Hence $f : \mathrm{Flag}_{\tau_1}(X_1) \to \mathrm{Im}(f)$ is open and therefore a homeomorphism.

We derive that $\mathrm{Im}(f)$ is compact connected submanifold of $\mathrm{Flag}_{\tau_2}(X_2)$ of the same dimension. However, $\mathrm{Flag}_{\tau_2}(X_2)$ is a compact connected manifold without boundary and hence the only such submanifold is $\mathrm{Flag}_{\tau_2}(X_2)$ itself, i.e., $\mathrm{Im}(f) = \mathrm{Flag}_{\tau_2}(X_2)$, which proves the claim. □

Theorem 4.15. *Let X_1, X_2 be symmetric spaces of rank at least two with no rank one de Rham factors and let $f : \mathrm{Flag}_\sigma(X_1) \to \mathrm{Flag}_\sigma(X_2)$ be a ξ_1-Moebius bijection. Then one can multiply the metric of X_1 by positive constants on de Rham factors — denote this space by $\widehat{X}_1$ — such that f is the restriction of a unique isometry $F : \widehat{X}_1 \to X_2$ to $\mathrm{Flag}_\sigma(X_1)$.*

Proof. We know that a ξ_1-Moebius bijection $f : \mathrm{Flag}_\sigma(X_1) \to \mathrm{Flag}_\sigma(X_2)$ can uniquely be extended to a building isomorphism $f : \Delta_\infty X_1 \to \Delta_\infty X_2$. Moreover, f is a homeomorphism on the chamber sets $\mathrm{Flag}_\sigma(X_i)$ by Lemma 4.13. Then for such maps the result is known [13, Sec. 3.9]. □

Actually all we need for the above result is that $f : \mathrm{Flag}_\sigma(X_1) \to \mathrm{Flag}_\sigma(X_2)$ is opposition preserving and a homeomorphism. However, when dealing also with rank one factors we really need Moebius maps.

Corollary 4.16. *Let X_1 and X_2 be symmetric spaces of non-compact type and let $f : \mathrm{Flag}_\sigma(X_1) \to \mathrm{Flag}_\sigma(X_2)$ be a Moebius bijection. Then one can rescale the metric of X_1 on de Rham factors — denote this space by $\widehat{X}_1$ — such that f is the restriction of an unique isometry $F : \widehat{X}_1 \to X_2$ to the boundary.*

Proof. This follows from Lemma 4.8 together with the theorem above and the fact that Moebius bijections of rank one symmetric spaces can be uniquely extended to isometries. For the latter result see [8]. □

Rescaling on irreducible factors. In this generality it is not possible to drop the scaling on the irreducible factors in the Corollaries 4.11, 4.16 and Theorem 4.15. For example consider the following situation: Let M_0 be a symmetric space or a combinatorial Euclidean building. We set $M_1 := \mu_1^{-1} M_0$, $M_2 := \mu_2^{-1} M_0$ for $\mu_i > 0$ with $\mu_1^2 + \mu_2^2 = 1$ and $M := M_1 \times M_2$ — here $M_i = \mu_i^{-1} M_0$ means we take the space M_0 with its metric multiplied by μ_i^{-1}. Moreover, we define $f : \mathrm{Flag}_\sigma(M) \to \mathrm{Flag}_\sigma(M)$ by $f(x, y) := (y, x)$.

Let $\xi \in \mathrm{int}(\sigma_0)$ and σ_0 the fundamental of the space M_0. Consider the cross ratio $\mathrm{cr}_{\pi(\xi,\xi,(\mu_1,\mu_2)),M} = \mu_1 \mathrm{cr}_{\xi,M_1} + \mu_2 \mathrm{cr}_{\xi,M_2}$ — cp. Proposition 2.12. As mentioned, we have $\mu_1 \mathrm{cr}_{\xi,M_1} = \mathrm{cr}_{\xi,M_0} = \mu_2 \mathrm{cr}_{\xi,M_2}$ and hence f is a $\pi(\xi, \xi, (\mu_1, \mu_2))$-Moebius bijection.

We see that f is induced by a map $F := F_1 \times F_2 : M_1 \times M_2 \to M_2 \times M_1$, such that $F_i : \mathrm{Flag}_\sigma(M_i) \to \mathrm{Flag}_\sigma(M_j)$, $i \neq j$ is the identity (under the natural identification with $\mathrm{Flag}_\sigma(M_0)$). As F and hence the F_i shall be isometries, it follows that $F(p, q) = (q, p)$ and clearly F is an isometry only after rescaling on de Rham factors.

Let M_1 be a symmetric space or a combinatorial Euclidean building and assume that the image of cr_{σ,M_1} lies not in a proper subspace of $\mathfrak{a}_{M_1}$. Then the above situation is essentially the only possibility where rescaling can appear:

Let M_1, M_2 be irreducible. In addition, be $f : \mathrm{Flag}_\sigma(M_1) \to \mathrm{Flag}_\sigma(M_2)$ a ξ_1-Moebius bijection, i.e., $\mathrm{cr}_{\xi_1} = f^* \mathrm{cr}_{\xi_2}$. Then we know that we can rescale the metric on M_1 by some positive number μ_1, such that f is induced by an isometry $F : \mu_1 M_1 \to M_2$. Thus Proposition implies 2.9 $f^* \mathrm{cr}_{\xi_2} = \mathrm{cr}_{\xi_1', \mu_1 M_1} = \mu_1 \mathrm{cr}_{\xi_1', M_1}$ for $\xi_1' \in \sigma_1$ with $F_\sigma(\xi_1') = \xi_2$.

However, it follows from the assumption on cr_{σ,M_1} together with Lemma 3.3 that $\mathrm{cr}_\xi \neq \alpha \mathrm{cr}_{\xi'}$ for $\xi \neq \xi' \in \sigma_1$ and any $\alpha \in \mathbb{R}$. Therefore $\mathrm{cr}_{\xi_1,M_1} = f^* \mathrm{cr}_{\xi_2} = \mu_1 \mathrm{cr}_{\xi_1',M_1}$ implies $\xi_1 = \xi_1'$ and $\mu_1 = 1$ — in particular f is induced by an isometry without rescaling the metric.

We remark that for symmetric spaces with $\iota = \mathrm{id}$ the image of cr_σ is all of $\mathfrak{a}$. This follows from the fact that every vector of $\mathfrak{a}$ can be realized as a translation vector of a hyperbolic element in G. Then the periods of those elements in G are exactly those translation vectors, as seen in Proposition 3.5. Hence the above discussion applies.

Corollary 4.17. *Let M either be a symmetric space or a combinatorial Euclidean building with none of the irreducible factors being a cone over a spherical building. In addition, assume that the image of* cr_σ *is not contained in a proper subspace of* $\mathfrak{a}$. *Let $\xi_0 \in \sigma$ be the center of gravity of σ. Then there is a one-to-one correspondence between* $\mathrm{Iso}(M)$ *and ξ_0-Moebius bijections.*

Proof. Let $g \in \mathrm{Iso}(M)$ and $g_\sigma : \sigma \to \sigma$ the induced map. Then g_σ is an isometry with respect to the angular metric, hence g_σ stabilizes the center of gravity ξ_0 of σ. Therefore Proposition 2.9 yields a ξ_0-Moebius bijection for each $g \in \mathrm{Iso}(M)$.

On the other hand, by Corollaries 4.11 and 4.16, we know that each ξ_0-Moebius bijection is induced by a unique isometry — after possible rescaling on irreducible factors. However, following the above discussion we can exclude rescaling of the metric:

Let f be a ξ_0-Moebius bijection and let $f = f_1 \times \cdots \times f_k$ be the decomposition on irreducible factors $M_1, \ldots, M_k$ as in Lemma 4.8. Assume w.l.o.g. that $f_1 : \mathrm{Flag}_\sigma(M_1) \to \mathrm{Flag}_\sigma(M_2)$, i.e., M_1, M_2 are isometric after possibly rescaling the metric. From Proposition 2.12 we know $\mathrm{cr}_{\xi_0} = \mu_1 \mathrm{cr}_{\xi_1, M_1} + \mu_2 \mathrm{cr}_{\xi_2, M_2} + \cdots + \mu_k \mathrm{cr}_{\xi_k, M_k}$. However, $\xi_0 \in \sigma$ being the center of gravity of σ and M_1, M_2 isometric after possibly rescaling the metric implies $\mu_1 = \mu_2$ and $\xi_1 \simeq \xi_2$. Then f_1 is a ξ_1-Moebius bijection between irreducible spaces. From the above discussion it follows that it is induced by an isometry without rescaling the metrics. The same argument implies the result for all f_i and hence the claim follows. □

General Euclidean buildings. In this section we consider general Euclidean buildings, i.e., in particular non-locally compact ones. The goal is again to show that Moebius bijections are induced by isometries. However, now we will need the vector valued cross ratio cr_σ to derive such a result.

Let E be a thick Euclidean building considered with the complete apartment system. Let $x \in \mathrm{Flag}_\tau(E)$ and $y \in \mathrm{Flag}_{\iota\tau}(E)$ with $x \,\mathsf{op}\, y$ and τ is a codimension 1 face of σ — in this case x, y are called *panels* of the building $\Delta_\infty E$. Then metrically we have the splitting $P(x,y) \simeq \mathfrak{a}_\tau \times CS(x,y)$, where $CS(x,y)$ is a Euclidean building of rank $\mathrm{rk}(E) - \dim \mathfrak{a}_\tau = 1$, i.e., $CS(x,y)$ is an $\mathbb{R}$-tree. This tree is called a *wall tree* and will be denoted by T_{xy}. One can show that the isomorphism type of T_{xy} does not depend on the choice of $y \in \mathrm{Flag}_{\iota\tau}(E)$ with $y \,\mathsf{op}\, x$ [23]; hence the isomorphism class of T_{xy} will be denoted by T_x.

We recall that the *residue* of an element $z \in \Delta_\infty E$ is defined by $\mathrm{Res}(z) = \{w \in \Delta_\infty E \mid z \subsetneq w\}$. In case of a panel $x \in \Delta_\infty E$ we have that $\mathrm{Res}(x)$ consists of all the chambers in $\Delta_\infty E$ containing x.

It is known that one can naturally identify $\mathrm{Res}(x) \simeq \partial_\infty T_x$. Let us describe this identification: Fix $y \,\mathsf{op}\, x$ and consider T_{xy} in the isomorphism class T_x. Let $o \in P(x,y)$. Then we can identify $P(x,y) \simeq \mathfrak{a}_\tau \times T_{xy}$ such that $o \simeq (0, o_T)$ and $x \simeq \partial_\infty \mathfrak{a}_\tau^+$; recall that $\mathfrak{a}_\tau^+ = \mathfrak{a}_\tau \cap \mathfrak{a}^+$. Then there is a one-to-one correspondence of chambers in $\mathrm{Res}(x)$ with (specific) Weyl sectors in $P(x,y)$ with tip o [29, Cor. 1.9.].[7] The affine apartments in $P(x,y) \simeq \mathfrak{a}_\tau \times T_{xy}$ containing o are of the form $\mathfrak{a}_\tau \times \gamma$, where γ is a bi-infinite geodesic ray in T_{xy} passing through o_T (those are easily seen to be isometric to $\mathbb{R}^r$). By definition every Weyl sector is contained in an affine apartment; hence we can derive that every Weyl sector with tip o and boundary chamber $c \in \mathrm{Res}(x)$ is contained in $\mathfrak{a}_\tau^+ \times \gamma_{o_T z}$ where $\gamma_{o_T z}$ is a geodesic ray in T_{xy} from o_T to a boundary point $z \in \partial_\infty T_{xy}$. This yields a one-to-one correspondence of $\mathrm{Res}(x)$ with geodesic rays emanating from o_T. As those rays are in one-to-one correspondence with $\partial_\infty T_{xy}$, we get $\mathrm{Res}(x) \simeq \partial_\infty T_x$ as claimed.

Remark 4.18. It follows that for $z \in \partial_\infty T_{xy}, c \in \mathrm{Res}(x)$ and $d \in \mathrm{Res}(y)$ we have that $z \simeq c$ and $z \simeq d$ under $\mathrm{Res}(x) \simeq \partial_\infty T_{xy}$, $\mathrm{Res}(y) \simeq \partial_\infty T_{xy}$ respectively if and only if the Weyl sectors with tip $o = (0, o_T)$ defining c, d are contained in $\mathfrak{a}_\tau^+ \times \gamma_{o_T z}$, $\mathfrak{a}_\tau^- \times \gamma_{o_T z}$, respectively.

By definition $\mathrm{Res}(x)$ is the set of chambers that contain x. Hence there is a unique corner ξ_x of σ such that $c_{\xi_x} \notin x$ for every chamber $c \in \mathrm{Res}(x)$. In the same way we get a type from y and it is immediate that this type equals $\iota\xi_x$ — following for example from the fact that $x \in \mathrm{Flag}_\tau$ implies that $y \in \mathrm{Flag}_{\iota\tau}$.

Lemma 4.19. *Let x, y be opposite panels in $\Delta_\infty E$ and T_{xy} the associated tree. Let $z_c, z_d \in \partial_\infty T_{xy}$, $c \in \mathrm{Res}(x)$ such that $c \simeq z_c$ under $\mathrm{Res}(x) \simeq \partial_\infty T_{xy}$ and $d \in \mathrm{Res}(y)$ such that $d \simeq z_d$ under $\mathrm{Res}(y) \simeq \partial_\infty T_{xy}$. Then $(c|d)_{o,\xi_x} = \sin(\alpha)(z_c|z_d)_{o_T}$ where $o \simeq (0, o_T)$ under $P(x,y) \simeq \mathfrak{a}_\tau \times T_{xy}$ and $\alpha \in (0,\pi)$ does only depend on σ and the type of x.*

Proof. Let γ_c, γ_d be the geodesics in $P(x,y)$ from o to c_{ξ_x} and d_{ξ_y}, respectively. The splitting $P(x,y) \simeq \mathfrak{a}_\tau \times T_{xy}$ yields geodesics γ_x, γ_y in $\mathfrak{a}_\tau$ from 0 and $\gamma_{z_c}, \gamma_{z_d}$ in T_{xy} emanating from o_T such that $\gamma_c(t) = (\gamma_x(t), \gamma_{z_c}(t))$ and $\gamma_d(t) = (\gamma_y(t), \gamma_{z_d}(t))$ — while γ_c, γ_d are unit speed, the geodesics $\gamma_x, \gamma_y, \gamma_{z_c}$ and γ_{z_d} are not. It is clear that the geodesics γ_x, γ_y do not depend on the choice of c, d and are in opposite directions (since the γ_c, γ_d are): The geodesics γ_c, γ_d are along those corners of Weyl sectors that are not contained in $\mathfrak{a}_\tau$. Since Weyl sectors are isometric to convex subsets of $\mathbb{R}^r$, it reduces to Euclidean geometry; for example γ_x is the geodesic in $\mathfrak{a}_\tau$ from 0 to the point in x of type $\pi_{\tau_x}(\xi_x)$, where π_{τ_x} is the orthogonal projection from σ to τ_x and τ_x is the type of x.

Let now $\gamma_x, \gamma_y, \gamma_{z_c}$ and γ_{z_d} be the geodesics as above but reparametrized such that they are unit speed. Then the above discussion implies $d(\gamma_x(t), \gamma_y(t)) = 2t$. Let α be the angle of ξ_x and $\pi_{\tau_x}(\xi_x)$. Then $\gamma_c(t) = (\gamma_x(\cos(\alpha)t), \gamma_{z_c}(\sin(\alpha)t))$. Basic facts of trees imply that $d(\gamma_{z_c}(t), \gamma_{z_d}(t)) = 2t - 2(z_c|z_d)_{o_T}$ for $t \geq (z_c|z_d)_{o_T}$ (see, e.g., [4]). Altogether,

$$
\begin{aligned}
(c|d)_{o,\xi_x} &= \lim_{t\to\infty} t - \tfrac{1}{2}\sqrt{4\cos^2(\alpha)t^2 + (2\sin(\alpha)t - 2(z_c|z_d)_{o_T})^2} \\
&= \lim_{t\to\infty} t - \sqrt{t^2 - 2t\sin(\alpha)(z_c|z_d)_{o_T} + (z_c|z_d)_{o_T}^2} = \sin(\alpha)(z_c|z_d)_{o_T},
\end{aligned}
$$

while the last equality follows from a Taylor series in the same way as we have seen several times before. □

[7]Here *Weyl sector* includes also all translates in an affine apartment of the Weyl sectors we have considered so far.

Corollary 4.20. *The natural cross ratio on* $\partial_\infty T_{xy}$ *is given by*

$$\mathrm{cr}_{T_{xy}}(z_1, w_1, z_2, w_2) = \sin(\alpha)\mathrm{cr}_{\xi_x}(c_1, d_1, c_2, d_2)$$

where $\xi_x \in \sigma$ *is the corner not contained in* τ_x*, the type of* x*,* α *is the angle between* ξ_x *and* τ_x*,* $c_i \simeq z_i$ *under* $\mathrm{Res}(x) \simeq \partial_\infty T_{xy}$ *and* $d_i \simeq w_i$ *under* $\mathrm{Res}(y) \simeq \partial_\infty T_{xy}$*.*

The thickness of E means that $\Delta_\infty E$ is thick and therefore for every panel x we have that $|\partial_\infty T_x| \geq 3$ (as $\mathrm{Res}(x) \simeq \partial_\infty T_x$), i.e., T_x is thick and geodesically complete. Therefore Theorem 4.9 implies that the whole isometry class T_x has a natural cross ratio cr_{T_x}.

Definition 4.21. *Let* E_1, E_2 *be thick irreducible Euclidean buildings. A building isomorphism* $\phi : \Delta_\infty E_1 \to \Delta_\infty E_2$ *is called* tree-preserving *or* ecological*, if for every panel* $x \in \Delta_\infty E_1$ *we have that* $\phi|_{\mathrm{Res}(x)} : \mathrm{Res}(x) \to \mathrm{Res}(\phi(x))$ *is induced by an isometry* $\phi_x : T_x \to T_{\phi(x)}$ *— i.e.,* $(\phi_x)|_{\partial_\infty T_x} \simeq \phi|_{\mathrm{Res}(x)}$ *under the identification* $\mathrm{Res}(x) \simeq \partial_\infty T_x$

Theorem 4.22 (Tits, [32, Thm. 2])**.** *Let* E_1, E_2 *be two thick irreducible Euclidean buildings and* $\phi : \Delta_\infty E_1 \to \Delta_\infty E_2$ *an ecological isomorphism. Then* ϕ *extends to an isomorphism, i.e., an isometry after possibly rescaling the metric on* E_1*.*

In a similar way as before, we call a surjective map $f : \mathrm{Flag}_{\sigma_1}(E_1) \to \mathrm{Flag}_{\sigma_2}(E_2)$ such that $\mathrm{cr}_{\sigma_1}(x, y, z, w) = f^*\mathrm{cr}_{\sigma_2}(x, y, z, w)$ for all $(x, y, z, w) \in \mathcal{A}_{\sigma_1}$ a σ_1*-Moebius bijection.* We remark that to identify the image of cr_{σ_1} with the one of cr_{σ_2} it is already necessary that E_1 and E_2 are modeled over the same Coxeter complex, i.e., $\sigma_1 \simeq \sigma_2 =: \sigma$.

It is immediate that such a map is a ξ_0-Moebius map, for ξ_0 the center of gravity of σ. We assumed f to be surjective, hence f is a ξ_0-Moebius bijection and therefore f can be extended uniquely to a building isomorphism $f : \Delta_\infty E_1 \to \Delta_\infty E_2$ by Proposition 4.6.

We recall that the affine Weyl group $\widehat{W} = W \ltimes T_W$ of the Coxeter complex over which a Euclidean building is defined gives a collection of hyperplanes, namely the hyperplanes of the finite reflection group W together with all its translates under T_W. Each hyperplane defines two half spaces which we call *affine half apartments.* The image of an affine half apartment under a chart map is again called *affine half apartment.*

In spherical buildings the hyperplanes associated to the spherical Coxeter group define walls in apartments and those walls separate the apartments in two halfs, called *half apartments.* One can show that the boundary of an affine half apartment $H \subset E$ defines a half apartment in $H_\infty \subset \Delta_\infty E$ and to every half apartment in $H_\infty \subset \Delta_\infty E$ we find an affine half apartment $H \subset E$ which has H_∞ as its boundary.

Now, let $f : \Delta_\infty E_1 \to \Delta_\infty E_2$ be a building isomorphism and let x, y be opposite panels. The identifications $\partial_\infty T_{xy} \simeq \mathrm{Res}(x), \partial_\infty T_{xy} \simeq \mathrm{Res}(y)$ together with $f|_{\mathrm{Res}(x)} : \mathrm{Res}(x) \to \mathrm{Res}(f(x))$, $f|_{\mathrm{Res}(y)} : \mathrm{Res}(y) \to \mathrm{Res}(f(y))$ induce two maps $f_x, f_y : \partial_\infty T_{xy} \to \partial_\infty T_{f(x)f(y)}$.

Lemma 4.23. *Notations as above; in particular let x, y be opposite panels and $f_x, f_y : \partial_\infty T_{xy} \to \partial_\infty T_{f(x)f(y)}$ are induced by $f|_{\mathrm{Res}(x)} : \mathrm{Res}(x) \to \mathrm{Res}(f(x))$, $f|_{\mathrm{Res}(y)} : \mathrm{Res}(y) \to \mathrm{Res}(f(y))$. Then $f_x = f_y$.*

Proof. Let $z \in \partial_\infty T_{xy}$, i.e., z is an equivalence class of geodesic rays. Every ray γ_z in the class starting at a branching point defines an affine half apartment $\mathfrak{a}_\tau \times \gamma_z$ in E_1 and thus (the equivalence class of rays) defines a half apartment $H_\infty \subset \Delta_\infty E_1$. Then it follows from Remark 4.18 that $c \simeq z$ with $c \in \mathrm{Res}(x)$ if and only if c is contained in the half apartment H_∞ and in the same way $d \simeq z$ with $d \in \mathrm{Res}(y)$ if and only if d is contained in the half apartment H_∞. By assumption, f is a building isomorphism, i.e., $f(H_\infty) \subset \Delta_\infty E_2$ is a half apartment with $f(x), f(y) \in f(H_\infty)$. The metric splitting $P(f(x), f(y)) \simeq \mathfrak{a}_\tau \times T_{f(x)f(y)}$ yields that we find an affine half apartment $\mathfrak{a}_\tau \times \gamma_w$ with γ_w a geodesic ray in $T_{f(x)f(y)}$ and boundary point $w \in \partial_\infty T_{f(x)f(y)}$ such that the boundary of this affine half apartment is exactly $f(H_\infty)$. By definition $f(c), f(d) \in f(H_\infty)$. Hence from Remark 4.18 we get that $f(c) \simeq w \simeq f(d)$. Therefore $f_x(z) = w$ and $f_y(z) = w$. □

Theorem 4.24. *Let E_1, E_2 be thick irreducible Euclidean buildings and let the map $f : \mathrm{Flag}_\sigma(E_1) \to \mathrm{Flag}_\sigma(E_2)$ be a σ-Moebius bijection. Then the induced isomorphism $f : \Delta_\infty E_1 \to \Delta_\infty E_2$ is ecological and hence can be extended to an isomorphism $F : E_1 \to E_2$, i.e., an isometry after possibly rescaling the metric on E_1.*

Proof. What we need to show is, given a panel $x \in \Delta_\infty E_1$, the induced map $f_x : \partial_\infty T_x \to \partial_\infty T_{f(x)}$ is the restriction of an isometry. This implies that f is ecological and therefore by the Theorem of Tits induced by an isomorphism.

We fix $y \,\mathsf{op}\, x$ to get a tree T_{xy} in the class of T_x. Since we are considering isometry classes of trees, it is enough to show that $f_{xy} : \partial_\infty T_{xy} \to \partial_\infty T_{f(x)f(y)}$ is induced by an isometry.

Corollary 4.20 implies that for $z_1, w_1, z_2, w_2 \in \partial_\infty T_{xy}$ and $c_1, c_2 \in \mathrm{Res}(x)$, $d_1, d_2 \in \mathrm{Res}(y)$ with $z_i \simeq c_i$, $w_i \simeq d_i$ there is some $\alpha \in (0, \pi)$ with

$$\mathrm{cr}_{T_{xy}}(z_1, w_1, z_2, w_2) = \sin(\alpha)\mathrm{cr}_{\xi_x}(c_1, d_1, c_2, d_2) = \sin(\alpha) f^*\mathrm{cr}_{\xi_x}(c_1, d_1, c_2, d_2),$$

while the last equality follows from f being a σ-Moebius bijection. By construction $f_{xy} : \partial_\infty T_{xy} \simeq \mathrm{Res}(x) \to \partial_\infty T_{f(x)f(y)} \simeq \mathrm{Res}(f(x))$ is defined in the way that $f(c_1) \simeq f_{xy}(z_1)$ under $\partial_\infty T_{f(x)f(y)} \simeq \mathrm{Res}(f(x))$ and similar for c_2. In light of Lemma 4.23 we have that $f(d_i) \simeq f_{xy}(w_i)$. Applying again Corollary 4.20 this yields that $\sin(\alpha) f^*\mathrm{cr}_{\xi_x}(c_1, d_1, c_2, d_2) = f_{xy}^*\mathrm{cr}_{T_{f(x)f(y)}}(z_1, w_1, z_2, w_2)$; we remark that the α is the same as before as the simplices σ_1 and σ_2 coincide. Hence f_{xy} is a Moebius bijection. Since T_{xy} is a geodesically complete tree and the thickness of E_1 implies that $|\partial_\infty T_{xy}| \geq 3$, we can apply Theorem 4.9 to derive that f_{xy} is induced by an isometry. □

Corollary 4.25. *Let E_1, E_2 be thick Euclidean buildings and let $f : \mathrm{Flag}_\sigma(E_1) \to \mathrm{Flag}_\sigma(E_2)$ be a σ-Moebius bijection. Then we can rescale the metric on the irreducible factors of E_1 — denote this space by $\widehat{E}_1$ — such that f is the restriction of an isometry $F : \widehat{E}_1 \to E_2$ to the boundary.*

Proof. Since f can be extended to a building isomorphism (as we have seen before), f is opposition preserving for each type of simplex. This, together with Lemma 3.3 and f being a σ-Moebius bijection, yield that $f^*\mathrm{cr}_\xi = \mathrm{cr}_\xi$ for every type $\xi \in \sigma$.

Let $\sigma = \sigma_1 \circ \cdots \circ \sigma_k$ be the decomposition of σ corresponding to the decomposition of E_i into irreducible factors; the decompositions coincide as both buildings are thick and modeled over the same spherical Coxeter complex. Moreover, be $f = f_1 \times \cdots \times f_k$ the decomposition from Lemma 4.8.

Then $f^*\mathrm{cr}_\xi = \mathrm{cr}_\xi$ for all $\xi \in \sigma$ implies that each f_i is a σ_i-Moebius bijection. Thus the above theorem yields the claim. □

Corollary 4.26. *Let E_1, E_2 be thick irreducible Euclidean buildings. In addition, assume that there exists a wall tree T_x for a panel $x \in \Delta_\infty E_1$ which has more than one branching point. Let $f : \mathrm{Flag}_\sigma(E_1) \to \mathrm{Flag}_\sigma(E_2)$ be a σ-Moebius bijection. Then f can be extended to an isometry $F : E_1 \to E_2$ (without rescaling the metric). Moreover, if E_1 is not a Euclidean cone over a spherical building then every wall tree has more than one branching point.*

Proof. From Theorem 4.24 we know that we can rescale the metric by some $\mu \in \mathbb{R}_+$ such that f is induced by an isometry $F : \mu E_1 \to E_2$, where μE_1 is E_1 with the metric rescaled by μ. Let $x \in \Delta_\infty E_1$ be a panel such that the wall tree T_x has more than one branching point. Then clearly the wall tree of $x \in \Delta_\infty \mu E_1$ is μT_x. Let $f_x : \partial_\infty T_x \to \partial_\infty T_{f(x)}$ be the induced map from f on the wall tree. Since F restricted to the boundary is f, the map induced from F on $\partial_\infty \mu T_x$ equals f_x. Therefore we have $\mathrm{cr}_{T_x} = f_x^*\mathrm{cr}_{T_{f(x)}} = \mathrm{cr}_{\mu T_x} = \mu \mathrm{cr}_{T_x}$ (the first equality follows from f being a σ-Moebius bijection, the second from $f_x = F|_{\partial_\infty \mu T_x}$).

By assumption, T_x has two branching points. The distance of those two points can be given in terms of the cross ratio — i.e., let $p, q \in T_x$ be the branching points, then there exist $z_1, z_2, w_1, w_2 \in \partial_\infty T_x$ such that $d(p,q) = \mathrm{cr}_{T_x}(z_1, w_1, z_2, w_2)$ [4, Lem. 4.2]. Since this distance $d(p,q)$ is non-zero, we derive from $\mathrm{cr}_{T_x}(z_1, w_1, z_2, w_2) = \mu \mathrm{cr}_{T_x}(z_1, w_1, z_2, w_2)$ that $\mu = 1$. Hence F is an isometry without rescaling the metric on E_1.

The second claim is a direct consequence of Propositions 4.21 and 4.26 in [23]. □

The second claim of Theorem B follows now from the fact that every σ-Moebius bijection splits as a product of σ_i-Moebius bijections on irreducible factors, as in the proof of Corollary 4.25. The corollary above implies that those σ_i-Moebius bijections induce isometries without the need of rescaling.

5. Appendix

Here, we determine the cross ratios that we construct explicitly for the symmetric spaces $X(n) := \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$. We will use the notation as in Example 2.11.

The map $g \cdot \mathrm{SO}(n,\mathbb{R}) \mapsto gg^t$ yields an identification of $X(n)$ with the space $P_n = \{A \in \mathrm{Mat}(n \times n, \mathbb{R}) | A = A^t \wedge \det(A) = 1 \wedge A \text{ is positive definite}\}$. The action of $g \in \mathrm{SL}(n,\mathbb{R})$ on $A \in P_n$ is given by $g \cdot A = gAg^t$. By definition of the cross ratio, it will be enough to determine $(\cdot\,|\,\cdot)_{I_n,\lambda}$ with I_n being the identity matrix in P_n and $\lambda = (\lambda_1, \ldots, \lambda_l)$ being identified with some type.

Let $\tau = (i_1, \ldots, i_l)$, $i_j \in \{1, \ldots, n\}$ such that $i_l = n$, $i_j < i_m$ for $1 \le j < m \le l \le n$ and let S_τ be the corresponding standard flag, i.e., $\mathrm{S}_\tau = (V_{i_1}, \ldots, V_{i_l})$

for $V_{i_j} = \mathrm{span}\{e_1, e_2, \ldots, e_{i_j}\}$. Let $\mathrm{S}_{\iota\tau}$ be the standard opposite flag to S_τ, i.e., $\mathrm{S}_{\iota\tau} = (V^*_{i_{l-1}}, \ldots, V^*_{i_1}, \mathbb{R}^n)$ with $V^*_{i_j} = \mathrm{span}\{e_n, e_{n-1}, \ldots, e_{i_j+1}\}$. Furthermore, be $\lambda = (\lambda_1, \ldots, \lambda_l) \in \mathbb{R}^l$ such that $\lambda_j > \lambda_{j+1}$, $\sum_{j=1}^l m_j\lambda_j = 0$ for $m_j = \dim V_{i_j} - \dim V_{i_j-1}$ if $j > 1$, $m_1 = \dim V_{i_1}$ and $\sum_{j=1}^l m_j\lambda_j^2 = 1$.

Claim. *Notations as before; $k, h \in \mathrm{SO}(n,\mathbb{R})$ and denote by $\widehat{h}_i$ the i-th column of the matrix h and accordingly $\widehat{k}_i$. Then*

$$(k\mathrm{S}_\tau|h\mathrm{S}_{\iota\tau})_{I_n,\lambda} = n\sum_{j=1}^{l-1}(\lambda_{j+1} - \lambda_j)\log|\det(\,\widehat{k}_1\,|\,\cdots\,|\,\widehat{k}_{i_j}\,|\,\widehat{h}_1\,|\,\cdots\,|\,\widehat{h}_{n-i_j}\,)|.$$

Proof. We show the claim for types $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathrm{int}(\sigma)$ and the full standard flag $\mathrm{S} = (V_1, \ldots, V_n)$ where $V_i = \mathrm{span}\{e_1, \ldots, e_i\}$ (the e_i being the standard base of $\mathbb{R}^n$). The claim follows then in full generality from Lemma 2.8.

Since $(\cdot\,|\,\cdot)_{I_n,\lambda}$ is invariant under the $\mathrm{SO}(n,\mathbb{R})$ action, it is enough to determine $(k\mathrm{S}|\mathrm{S})_{I_n,\lambda}$ or $(\mathrm{S}|k\mathrm{S})_{I_n,\lambda}$ for arbitrary $k \in \mathrm{SO}(n,\mathbb{R})$. Proposition 2.1 implies that $(\mathrm{S}|k\mathrm{S})_{I_n,\lambda} = \frac{1}{2}b_{\mathrm{S}_\lambda}(I_n, n_{k\mathrm{S}}(I_n,\mathrm{S})\cdot I_n)$, where S_λ is a point in the ideal boundary $\partial_\infty X(n)$ determined by the eigenvalue flag pair (λ, S) and $n_{k\mathrm{S}}(I_n,\mathrm{S}) \in N_{k\mathrm{S}}$, i.e., the element in the horospherical subgroup to $k\mathrm{S}$ such that $n_{k\mathrm{S}}(I_n,\mathrm{S})\cdot I_n \in P(k\mathrm{S},\mathrm{S})$.

We first determine $n_{kS}(I_n,\mathrm{S})\cdot I_n$. Let $k_w \in \mathrm{SO}(n,\mathbb{R})$ be the standard antidiagonal matrix with -1 in the upper right corner. Then $k_w\mathrm{S} = W$ with W the standard opposite flag, i.e., $W = (V_1^*, \ldots, V_n^*)$ with $V_i^* = \mathrm{span}\{e_n, \ldots, e_{n-i+1}\}$. Since any $k \in \mathrm{SO}(n,\mathbb{R})$ stabilizes I_n, the maximal flat through $k\mathrm{S}$ and I_n is the unique maximal flat (i.e., affine apartment) that joins $k\mathrm{S}$ and $kW = kk_wS$. This yields $n_{k\mathrm{S}}(I_n,\mathrm{S}) = n_{k\mathrm{S}}(kk_w\mathrm{S},\mathrm{S})$; here $n_{k\mathrm{S}}(kk_w\mathrm{S},\mathrm{S}) \in N_{k\mathrm{S}}$ is the unique element mapping $kk_w\mathrm{S}$ to S.

We know $N_{k\mathrm{S}} = kN_\mathrm{S}k^{-1} = kN_\mathrm{S}k^t$ and N_S is the group of upper triangular matrices with ones on the diagonal. Thus we are looking for $n_\mathrm{S} \in N_\mathrm{S}$ such that $kn_\mathrm{S}k^tkk_w\mathrm{S} = \mathrm{S}$, i.e., $kn_\mathrm{S}k_w \in \mathrm{stab}(\mathrm{S})$; which is equivalent to $kn_\mathrm{S}k_w$ being upper triangular.

Let k_i denote the i-th row of k. Then it is straightforward to check that the $(n+1-j)$-th column of n_s is given by $\sum_{i=1}^j a_{i,n+1-j}k_i$, with $a_{i,n+1-j}$ such that

$$\begin{pmatrix} k_{1,n-j+1} & \cdots & k_{j,n-j+1} \\ k_{1,n-j+2} & \cdots & k_{j,n-j+2} \\ \vdots & \cdots & \vdots \\ k_{1,n} & \cdots & k_{j,n} \end{pmatrix}\begin{pmatrix} a_{1,n+1-j} \\ a_{2,n+1-j} \\ \vdots \\ a_{j,n+1-j} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{7}$$

We set $A := kn_s$. Then $n_{k\mathrm{S}}(I_n,\mathrm{S})\cdot I_n = (kn_\mathrm{S}k^t)\cdot I_n = kn_\mathrm{S}n_\mathrm{S}^tk^t = AA^t$.

The Busemann function on $X(n)$ is well known — see Lemmata 2.4, 2.5 in [16]. Namely, for $p \in P_n$ we have $b_{\mathrm{S}_\lambda}(p, I_n) = n\log(\prod_{j=1}^{n-1}(\det\Delta_j^-(p))^{\lambda_{n-j}-\lambda_{n+1-j}})$, where $\Delta_j^-(p)$ is the lower right $j\times j$-minor of p -, e.g., $\Delta_1^-(p) = p_{n,n}$. This yields $(\mathrm{S}|k\mathrm{S})_{I_n,\lambda} = \frac{n}{2}\sum_{j=1}^{n-1}(\lambda_{n+1-j} - \lambda_{n-j})\log\det(\Delta_j^-(AA^t))$.

Let $(J)_{i,j} = \delta_{i,n+1-j}$, where $\delta_{i,j}$ is Kronecker's delta, i.e., J is the antidiagonal. Then AJ is upper triangular with the diagonal of the form $a_{1,n}, \dots, a_{n,1}$. Then one can easily show that $\Delta_j^-(AA^t) = \Delta_j^-(AJ)\Delta_j^-(JA^t)$; and thus $\det \Delta_j^-(AA^t) = \det \Delta_j^-(AJ) \det \Delta_j^-(JA^t) = a_{n,1}^2 \cdots a_{n+1-j,j}^2$.

If we apply Cramer's rule to equation (7), we get

$$a_{j,n+1-j} = (-1)^{j+1} \det \begin{pmatrix} k_{1,n-j+2} & \cdots & k_{j-1,n-j+2} \\ \vdots & \cdots & \vdots \\ k_{1,n} & \cdots & k_{j-1,n} \end{pmatrix} \Bigg/ \det \begin{pmatrix} k_{1,n-j+1} & \cdots & k_{j,n-j+1} \\ k_{1,n-j+2} & \cdots & k_{j,n-j+2} \\ \vdots & \cdots & \vdots \\ k_{1,n} & \cdots & k_{j,n} \end{pmatrix}.$$

for $j \geq 2$ and $a_{1,n} = k_{1,n}^{-1}$. Thus $\det \Delta_{n-j}^-(AA^t) = \det(\, e_1 \,|\, \cdots \,|\, e_{n-j} \,|\, k_1 \,|\, \cdots \,|\, k_j \,)^2$. Let $\widehat{k}_i$ denote the i-th column of $k \in \mathrm{SO}(n,\mathbb{R})$. Then $(k\mathrm{S}|\mathrm{S})_{I_n,\lambda} = (\mathrm{S}|k^t\mathrm{S})_{I_n,\lambda} = n \sum_{j=1}^{n-1} (\lambda_{n+1-j} - \lambda_{n-j}) \log |\det(\, e_1 \,|\, \cdots \,|\, e_j \,|\, \widehat{k}_1 \,|\, \cdots \,|\, \widehat{k}_{n-j} \,)|,$.

Let $k, h \in \mathrm{SO}(n,\mathbb{R})$. Then the i-th column of $h^{-1}k$ is given by $h^{-1}k \cdot e_i = h^{-1}\widehat{k}_i$. Then

$$\det(\, e_1 \,|\, \cdots \,|\, e_j \,|\, h^{-1}\widehat{k}_1 \,|\, \cdots \,|\, h^{-1}\widehat{k}_{n-j} \,) = \det(\, \widehat{h}_1 \,|\, \cdots \,|\, \widehat{h}_j \,|\, \widehat{k}_1 \,|\, \cdots \,|\, \widehat{k}_{n-j} \,)$$

yields

$$(h^{-1}k\mathrm{S}|\mathrm{S})_{I_n,\lambda} = n \sum_{j=1}^{n-1} (\lambda_{n+1-j} - \lambda_{n-j}) \log |\det(\, \widehat{h}_1 \,|\, \cdots \,|\, \widehat{h}_j \,|\, \widehat{k}_1 \,|\, \cdots \,|\, \widehat{k}_{n-j} \,)|.$$

Therefore $(k\mathrm{S}|h\mathrm{S})_{I_n,\lambda} = n \sum_{j=1}^{n-1} (\lambda_{j+1} - \lambda_j) \log |\det(\, \widehat{k}_1 \,|\, \cdots \,|\, \widehat{k}_j \,|\, \widehat{h}_1 \,|\, \cdots \,|\, \widehat{h}_{n-j} \,)|$. □

Proposition 5.1. *Let $\lambda = (\lambda_1, \dots, \lambda_l)$ be a type, and τ such that $\lambda \in \mathrm{int}(\tau)$. Let $V = (V_1, \dots, V_l)$, $Y = (Y_1, \dots, Y_l) \in \mathrm{Flag}_\tau$ and let $W = (W_1, \dots, W_l)$, $Z = (Z_1, \dots, Z_l) \in \mathrm{Flag}_{\iota\tau}$. Then*

$$\mathrm{cr}_\lambda(V, W, Y, Z) = n \sum_{j=1}^{l-1} (\lambda_j - \lambda_{j+1}) \log \left(\left| \frac{V_j \wedge W_{l-j}}{V_j \wedge Z_{l-j}} \frac{Y_j \wedge Z_{l-j}}{Y_j \wedge W_{l-j}} \right| \right),$$

using the above conventions.

Proof. As mentioned in Example 2.11, the term is independent of the choices made. By the transitivity of the $\mathrm{SO}(n,\mathbb{R})$ action, we know that every flag $V \in \mathrm{Flag}_\tau$ can be written as $k\mathrm{S}_\tau$ for $\mathrm{S}_\tau \in \mathrm{Flag}_\tau$ the standard flag and some $k \in \mathrm{SO}(n,\mathbb{R})$. Then the columns $\widehat{k}_i$ are such that $V_j = \mathrm{span}\{\widehat{k}_1, \dots, \widehat{k}_{i_j}\}$. In the same way every flag $W \in \mathrm{Flag}_{\iota\tau}$ can be written as $h\mathrm{S}_{\iota\tau}$ for $\mathrm{S}_{\iota\tau} \in \mathrm{Flag}_{\iota\tau}$ the standard flag and some $h \in \mathrm{SO}(n,\mathbb{R})$.

Fixing the identification $\wedge^n\mathbb{R}^n \simeq \det$, we get $|\det(\, \widehat{k}_1 \,|\, \cdots \,|\, \widehat{k}_{i_j} \,|\, \widehat{h}_1 \,|\, \cdots \,|\, \widehat{h}_{n-i_j} \,)| = |V_j \wedge W_{l-j}|$. Thus the claim follows from the lemma above. □

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