ERRATUM TO “HOMOGENEOUS COMPACT GEOMETRIES”

L. KRAMER*

A. LYTCHAK**

Mathematisches Institut

Universität Münster

Einsteinstr. 62

48149 Münster, Germany

linus.kramer@uni-muenster.de

Mathematisches Institut

Universität Köln

Weyertal 86-90

50931 Köln, Germany

alytchak@math.uni-koeln.de

Abstract. We correct an error in our article [6], completing the classification of compact homogeneous geometries.

The present note contains a correction for our article [6] and some additional remarks. In [6] we classified compact homogeneous geometries. Kollross and Gorodski [2] discovered in 2015 that our classification as well as many related classifications, for instance [7], were incomplete: there exists a further example of a polar action on the Cayley plane which gives rise to an exceptional compact homogeneous geometry of type C_3 . In the present note we correct the mistake in our classification by showing that this is the only other example of a compact homogeneous geometry. The final result is that there exist exactly two homogeneous compact geometries of type C_3 which are not covered by buildings. These two geometries have been studied in a purely algebraic setting by Schillewaert and Struyve [9], who determined in particular their automorphism groups and showed that the underlying simplicial complexes are simply connected. The last claim was independently obtained by Pasini [11]. The main results of [6] should be modified as follows.

Theorem A in [6], revised. *Let Δ be a compact geometry of irreducible spherical type and rank at least 2, with connected panels. Assume that a compact group acts continuously and transitively on the chambers of Δ .*

If Δ is not of type C_3 , then there exists a simple noncompact Lie group S , a compact chamber-transitive subgroup $K \subseteq S$ and a K -equivariant 2-covering $\tilde{\Delta} \rightarrow \Delta$, where $\tilde{\Delta}$ is the canonical spherical building associated to S .

If Δ is of type C_3 , then either there exists a building $\tilde{\Delta}$ and a 2-covering $\tilde{\Delta} \rightarrow \Delta$ as in the previous case, or Δ is isomorphic to one of the two exceptional

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Corresponding Author: A. Lytchak, e-mail: alytchak@math.uni-koeln.de

homogeneous compact C_3 geometries which cannot be 2-covered by any building.

Theorem 5.4 in [6], revised. *Let M be a Coxeter matrix of spherical type and let (G, Δ) be a homogeneous compact geometry in $\mathbf{HCG}(M)$. Suppose that the Coxeter diagram of M has no isolated nodes. Then there exists a homogeneous compact geometry $(K, \tilde{\Delta})$ in $\mathbf{HCG}(M)$ which is a join of buildings associated to simple noncompact Lie groups and geometries of type C_3 which are isomorphic to the exceptional geometry in Section 3B in [6] or to the exceptional geometry described in this note, and a continuous 2-covering $\tilde{\Delta} \rightarrow \Delta$, which is equivariant with respect to a compact connected Lie group K acting transitively on the chambers of $\tilde{\Delta}$.*

Theorem 4.1 in [6], revised. *Let (G, Δ) be a homogeneous compact geometry of type C_3 with connected panels. Assume that G is compact and acts faithfully, and let $(\tilde{G}, \tilde{\Delta})$ denote the corresponding universal compact homogeneous geometry, as in 2.27 in [6]. Then $\tilde{\Delta}$ is either a building or the exceptional geometry described in Section 3B in [6] or the exceptional geometry described below in 6.8.*

Remark 3.20 in [6] should also be modified accordingly: the alternative approach to the exceptional geometries would require in addition the classification results in [2].

We note also that Problem 5 in [6] is answered in [9]: the two exceptional compact homogeneous geometries of type C_3 are simply connected, and their automorphism groups are compact. Similar geometries may be defined at least over other real closed fields. We finally mention that an affirmative answer to Problem 1 in [6] has been given in [10]. In particular, [10] shows that without additional topological assumptions there exist free constructions of homogeneous geometries of type C_3 .

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6. The corrections

We will freely use the terminology and the results from our article, which we will refer to as Part I. We first indicate where an error occurs in Part I. At the beginning of Section 4D, on the bottom of p. 841 in Part I we write: “By 2.5 we have a product action of M° on $\mathbb{S}^1 \times \mathbb{S}^n$.” But this conclusion is not implied by Lemma 2.5 in Part I.

In the relevant section we consider a compact homogeneous geometry (G, Δ) of type C_3 with parameters $(1, n)$. The group G is a compact connected Lie group. We recall the notions introduced in 4.10 in Part I. We fix a chamber $\{p, d, q\}$ in the geometry Δ . Then

$$\mathrm{lk}(\{p, d\}) \cong \mathbb{S}^1 \cong \mathrm{lk}(\{p, q\}) \text{ and } \mathrm{lk}(\{d, q\}) \cong \mathbb{S}^n.$$

The link of $\{p\}$ is the projective geometry of the real projective plane, the link of q is a generalized quadrangle with parameters $(1, n)$ and the link of $\{d\}$ is a

generalized digon of type $\mathbb{S}^1 \times \mathbb{S}^m$. We denote the kernels of the actions of G_p , G_q and G_d on the respective links by A, B, C , and we put

$$K = G_p/A, \quad L = G_q/B \quad \text{and} \quad M = G_d/C.$$

We know that $K = \mathrm{SO}(3)$ and that the stabilizers K_d and K_q act as $\mathrm{O}(2)$ on the 1-spheres $\mathrm{lk}(\{p, d\})$ and $\mathrm{lk}(\{p, q\})$. For $n > 1$, the group G_p is connected by Lemma 4.4 in Part I. We recall also from Lemma 2.8 in Part I that $A \cap B = \{1\}$. The relevant case of Lemma 2.5 in Part I reads as follows.

Lemma 6.1. *Consider the standard orthogonal action of $\mathrm{SO}(2) \times \mathrm{SO}(n+1)$ on $\mathbb{S}^1 \times \mathbb{S}^n$. Suppose that $H \subseteq \mathrm{SO}(2) \times \mathrm{SO}(n+1)$ is a transitive compact connected subgroup. Then H splits as a product $H = H_1 \times H_2$, where H_j is the image of H under the projection pr_j for the diagram*

$$\mathrm{SO}(2) \xleftarrow{\mathrm{pr}_1} \mathrm{SO}(2) \times \mathrm{SO}(n+1) \xrightarrow{\mathrm{pr}_2} \mathrm{SO}(n+1),$$

provided that either $H_2 = \mathrm{SO}(n+1)$ or that H_2 is a compact simple group.

Corollary 6.2. *If the group N induced by $(G_{d,q})^\circ$ on $\mathrm{lk}(\{d, q\}) \cong \mathbb{S}^n$ is either $\mathrm{SO}(n+1)$ or if N is simple and maximal among compact connected subgroups in $\mathrm{SO}(n+1)$, then the action of M° on $\mathrm{lk}(\{d\})$ is a product action. Under this additional hypothesis, all the classification results in Section 4D of Part I are valid.*

Proof. For the projection H_2 in Lemma 6.1 we have $N \subseteq H_2 \subseteq \mathrm{SO}(n+1)$. $\square$

Inspecting the classification of compact connected groups acting transitively on compact generalized quadrangles with parameters $(1, n)$ in [4], [5], we see that the only cases where the connected group induced on $\mathrm{lk}(\{d, q\})$ is not $\mathrm{SO}(n+1)$ are $n = 5, 6$, with $L^\circ = \mathrm{SO}(2) \times \mathrm{G}_2$ and $L^\circ = \mathrm{SO}(2) \cdot \mathrm{Spin}(7)$, respectively. In the case $n = 6$, the group induced by $(G_{d,q})^\circ$ on $\mathbb{S}^6$ is G_2 , which is simple and a maximal connected subgroup in $\mathrm{SO}(7)$, see, e.g., 95.12 in [8]. So this case is covered by Lemma 6.1 and hence Corollary 4.31 in Part I shows that this case cannot occur in the geometry Δ .

In the case $n = 5$, the group induced by $(G_{d,q})^\circ$ on $\mathbb{S}^5$ is $\mathrm{SU}(3)$. Here, we can embed a 1-torus T diagonally into $\mathrm{SO}(2) \times \mathrm{U}(3)$. Then it may happen that $H_2 = \mathrm{U}(3)$ and then the action of $K = T \cdot \mathrm{SU}(3)$ is not split. This case was overlooked in Part I and leads to a new compact homogeneous $\mathbf{C}_3$ -geometry.

Auxiliary results

We introduce some groups. We put

$$C_2 = \{\pm 1\}$$

and we let $Q \subseteq \mathrm{Spin}(3) \subseteq \mathbb{H}$ denote the quaternion group of order 8, i.e., the group generated by $i, j \in \mathbb{H}$. We put

$$J = (Q \times \mathrm{SU}(2))/\{\pm(1, 1)\}$$

and we note that $\pi_0(J) \cong C_2^2$. The group J can be realized as a subgroup of $\mathrm{SO}(4)$.

Lemma 6.3. *The group J is not isomorphic to $C_2^2 \times \mathrm{SU}(2)$.*

Proof. The center of J is C_2 and the center of $C_2^2 \times \mathrm{SU}(2)$ is C_2^3 . $\square$

Lemma 6.4. *Consider the action of the group $G_2 = \mathrm{Aut}(\mathbb{O})$ on the 7-dimensional space of all pure elements in $\mathbb{O}$. The subgroup of G_2 consisting of all elements h with $h(\mathbf{i}) = \pm \mathbf{i}$ and $h(\mathbf{j}) = \pm \mathbf{j}$ is isomorphic to the group J .*

Proof. The quaternion subalgebra $\mathbb{H} \subseteq \mathbb{O}$ is generated by $\mathbf{i}, \mathbf{j}$ and thus invariant under this group. Let $\ell \in \mathbb{O}$ be a pure element with $\ell^2 = -1$ which is orthogonal to $\mathbb{H}$. Then $\mathbf{i}, \mathbf{j}, \ell$ is a Cayley triple in $\mathbb{O}$ and $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\ell$. By Lemma 11.30 in [8], the subgroup of $\mathrm{Aut}(\mathbb{O})$ of all automorphisms which leave $\mathbb{H}$ invariant consists of all maps of the form

$$[a, b] : \mathbb{H} \oplus \mathbb{H}\ell \rightarrow \mathbb{H} \oplus \mathbb{H}\ell, \quad u + v\ell \mapsto au\bar{a} + (bv\bar{a})\ell,$$

where $a, b \in \mathrm{Spin}(3) \subseteq \mathbb{H}$ are unit quaternions. The claim follows. $\square$

Lemma 6.5. *Consider the polar representation of $\mathrm{O}(2) \times G_2 \subseteq \mathrm{O}(2) \times \mathrm{SO}(7)$ on $\mathbb{R}^{2 \times 7}$. The principal isotropy group is isomorphic to J .*

Proof. Let $e_1, \dots, e_7$ denote the standard basis of $\mathbb{R}^7$, and consider the 2×7 -matrices $P_1 = (2e_1, e_2)$, $P_2 = (e_1, 0)$ and $P_3 = (e_1, e_2)$. The $\mathrm{O}(2) \times \mathrm{SO}(7)$ -stabilizer of P_1 is principal and fixes P_2 and P_3 . It consist of all elements of the form

$$\begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix} \times \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_1 \varepsilon_2 A \end{pmatrix}$$

with $\varepsilon_1, \varepsilon_2 = \pm 1$ and $A \in \mathrm{SO}(5)$. We may put $\mathbf{i} = e_1$ and $\mathbf{j} = e_2$ and then apply Lemma 6.4. $\square$

We remark that Lemma 6.5 contradicts the formula for the principal isotropy group given on p. 54 in [2].

The missing case with $n = 5$ and $\mathfrak{q} = \mathbb{R} \oplus \mathfrak{g}_2$

The maps $L_p \rightarrow L$ and $M_p \rightarrow M$ are $(n-1)$ -equivalences. Since the groups induced by $K = G_p/A$ on the 1-spheres $\mathrm{lk}(\{p, q\})$ are in both cases $\mathrm{O}(2)$, we conclude that neither L nor M are connected. The group $K_{d,q}$ is finite and hence $\mathfrak{g}_{p,d,q} = \mathfrak{a} = \mathfrak{q}_{p,d}$, and $\mathfrak{b} = \mathfrak{c} = 0$. It follows that $A^\circ = (G_{p,d,q})^\circ = \mathrm{SU}(2)$.

We first inspect the diagrams for the Lie algebras of M, L and K . They look as follows:

$$\begin{array}{ccccc} \mathbb{R} \oplus \mathfrak{su}(3) & \longleftarrow & \mathfrak{su}(3) & \mathfrak{su}(3) & \longrightarrow & \mathbb{R} \oplus \mathfrak{g}_2 \\ \uparrow \varphi & & \uparrow & \uparrow & & \nearrow \\ \mathbb{R} \oplus \mathfrak{su}(2) & \longleftarrow & \mathfrak{su}(2) & \mathfrak{su}(2) & \longrightarrow & \mathbb{R} \oplus \mathfrak{su}(2) \\ & & & & & \nearrow \\ & \mathbb{R} & \longleftarrow & 0 & \longrightarrow & \mathbb{R} \\ & \downarrow & & & & \nearrow \\ & \mathfrak{so}(3) & & & & \end{array}$$

Except for φ , all arrows are canonical or determined by the diagrams for $\mathrm{lk}(p)$ and $\mathrm{lk}(q)$. For φ , there are two fundamentally different possibilities. If φ restricts an isomorphism between the $\mathbb{R}$ -factors, then the action of M° on $\mathbb{S}^1 \times \mathbb{S}^5$ is split. This case is covered by Section 4D in Part I. So we have to consider the case where the φ -image of $\mathbb{R}$ is diagonally embedded into $\mathbb{R} \times \mathfrak{su}(3)$. Up to automorphism, there is just one such map. (The φ -image of $\mathbb{R}$ cannot be contained in $\mathfrak{su}(3)$, since $M/M_p \cong \mathbb{S}^5$.) There is just one way to glue these data to a diagram $\mathrm{Lie}(\mathcal{G})$, which looks as follows:

$$\begin{array}{ccccc}
 \mathbb{R} \oplus \mathfrak{su}(3) & \longleftarrow & \mathfrak{su}(3) & \longrightarrow & \mathbb{R} \oplus \mathfrak{g}_2 \\
 \uparrow \varphi & & \uparrow & & \nearrow \\
 \mathbb{R} \oplus \mathfrak{su}(2) & \longleftarrow & \mathfrak{su}(2) & \longrightarrow & \mathbb{R} \oplus \mathfrak{su}(2) \\
 \downarrow & & \nearrow & & \\
 \mathfrak{so}(3) \oplus \mathfrak{su}(2) & & & &
 \end{array}$$

In all instances, the maps between the $\mathfrak{su}(2)$ factors are the identity maps. (The Lie algebras $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ are of course isomorphic, but naming them differently helps keeping track of the maps.) By Lemma 4.13 in Part I the diagram $\mathcal{G}^\circ$ of the connected components is uniquely determined by the diagram $\mathrm{Lie}(\mathcal{G})$ and the subdiagram

$$\begin{array}{ccc}
 G_{p,d} & \longleftarrow G_{p,d,q} & \longrightarrow G_{p,q} \\
 \downarrow & & \nearrow \\
 G_p & &
 \end{array} \tag{1}$$

We now study the stabilizers in more detail. The group G_p is connected, and hence

$$G_p = (\mathrm{Spin}(3) \times \mathrm{SU}(2))/E,$$

and $E \subseteq C_2 \times C_2$ is a central subgroup. Therefore $G_{p,d,q} = (Q \times \mathrm{SU}(2))/E$, where $Q \subseteq \mathrm{Spin}(3)$ denotes again the quaternion group of order 8.

We noted above that the group L is not connected. By [5] we have $[L : L^\circ] = 2$ and thus $L = \mathrm{O}(2) \times \mathrm{G}_2$, in its natural polar action on $\mathbb{R}^{2 \times 7}$. By Lemma 6.5 we have $L_{p,d} \cong J$. Since $L_{p,d} = G_{p,d,q}/B$, we deduce that $|E| \cdot |B| = 2$.

If $|B| = 2$, then we have by Lemma 6.3 that $B = \{\pm(1,1)\}$, because $(Q \times \mathrm{SU}(2))/B \cong J$. But $(-1, -1)$ acts also trivially on $\mathrm{lk}(p)$, contradicting the fact that $A \cap B$ is trivial. Hence B is trivial.

It follows from Lemma 6.3 that $E = \{\pm(1,1)\}$ and thus $G_p = (\mathrm{Spin}(3) \times \mathrm{SU}(2))/E \cong \mathrm{SO}(4)$. This determines the diagram (1) uniquely, and hence by Lemma 4.13 in Part I the diagram $\mathcal{G}^\circ$ of connected components.

It also follows that $G_q = L$ and that

$$\mathbb{Z}/2 \cong \pi_0(G_q) \cong \pi_0(G_{p,q}) \cong \pi_0(G_{d,p}) \cong \pi_0(G_d). \tag{2}$$

At this stage, the following subdiagram of $\mathcal{G}$ is uniquely determined:

$$\begin{array}{ccccc}
 & & G_{d,q} & \xrightarrow{\quad} & G_q \\
 & & \uparrow & & \nearrow \\
 G_{p,d} & \xleftarrow{\quad} & G_{p,d,q} & \xrightarrow{\quad} & G_{p,q} \\
 \downarrow & & & \nearrow & \\
 G_p & & & &
 \end{array} \tag{3}$$

Let $O(2) \subseteq G_{p,d} \subseteq G_p = SO(4)$ denote the subgroup which maps onto $K_d \cong O(2)$. We choose an element h in $O(2)$ which is a reflection and fixes $\{p, d, q\}$ (there are two such elements in $O(2)$). Then h is an involution and generates each of the cyclic groups in Equation (2). In particular we have

$$G_{d,q} = (G_{d,q})^\circ \rtimes \langle h \rangle \quad \text{and} \quad G_{p,d} = (G_{p,d})^\circ \rtimes \langle h \rangle.$$

This determines the conjugation action of h on $(G_d)^\circ = \langle (G_{p,d})^\circ \cup (G_{d,q})^\circ \rangle$, and hence the structure of $G_d = (G_d)^\circ \rtimes \langle u \rangle$ as well as the missing upper left corner in diagram (3). We have established the following result.

Proposition 6.6. *In the case where $\mathfrak{q} = \mathbb{R} \oplus \mathfrak{g}_2$ and where the action of M° is not a product action, there is at most one possibility for the diagram $\mathcal{G}$ and hence at most one universal compact homogeneous geometry of type C_3 .*

It remains to show the existence of the geometry. There are two ways for this. In Section 3.4 in [2], the construction of a polar action on the Cayley plane is shown, in such a way that the resulting geometry is of the type described above. Similarly to Section 3B in Part I, one could work out the precise orbit structure for this polar action. There is a second and very explicit way to construct the geometry in a purely algebraic way, as shown in [9]. We indicate it briefly at the end of this note.

Proposition 6.7. *Besides the geometry described in Section 3B in Part I there exist, up to isomorphism, exactly one compact homogeneous geometry of type C_3 which is not 2-covered by a building.*

Proof. Let (G, Δ) denote the exceptional compact homogeneous geometry of type C_3 which arises from the polar action on the Cayley plane as described in [2]. We claim first that Δ is not covered by a building Δ' . Otherwise, there would (by Theorem 2.22 in Part I and the subsequent remarks) exist a fiber bundle $\mathbb{S}^m \cong |\Delta'| \rightarrow |\Delta| \cong \mathbb{O}P^2$, which arises from a free action of a compact Lie group E on the building Δ' . However, such a fiber bundle $\mathbb{S}^m/E \rightarrow \mathbb{O}P^2$ does not exist. Its mapping cone would be a finite 7-connected CW complex with cohomology ring $\mathbb{Z}[x]/(x^4)$ for a generator x in degree 8, which is impossible by Adams' results on the Hopf invariant. Theorem 5.1. and the subsequent remarks in [1] show this explicitly.

Hence (G, Δ) is a compact homogeneous geometry of type C_3 which is not 2-covered by a building. By our classification of possible diagrams for such geometries, this geometry has to be in the category $\mathbf{HCG}_{\mathcal{G}}(C_3)$, where $\mathcal{G}$ is the unique simple complex of groups determined above. Let $(\widehat{G}, \widehat{\Delta}, \{\widehat{p}, \widehat{d}, \widehat{q}\}, \widehat{\psi})$ denote the universal homogeneous compact geometry in $\mathbf{HCG}_{\mathcal{G}}(C_3)$. The actions of K and L° on $\mathrm{lk}(\widehat{p})$ and $\mathrm{lk}(\widehat{q})$ are minimal by the classification in [4, 5]. Therefore the $\widehat{G}$ -action on $\widehat{\Delta}$ is minimal, i.e., $\widehat{G}$ has no proper compact connected chamber transitive subgroup. Hence there exists a continuous surjective group homomorphism $\widehat{G} \rightarrow G$ which induces an equivariant morphism $\widehat{\Delta} \rightarrow \Delta$. We have to show that the kernel $F \trianglelefteq \widehat{G}$ of the group homomorphism is trivial. We first show that the Lie algebra $\mathfrak{f}$ of F is trivial. We have $\mathrm{Lie}(\widehat{G}) \cong \mathrm{Lie}(G) \oplus \mathfrak{f}$, and $\mathrm{Lie}(G) \cong \mathfrak{so}(3) \oplus \mathfrak{g}_2$ by the results in [2]. From the polar action on the Cayley plane as given in [2] we know that $|\Delta| \cong \mathbb{O}P^2$ in the coarse topology.

We put $H_1 = \exp(\mathfrak{so}(3)) \subseteq \widehat{G}_p$ and $H_2 = \exp(\mathfrak{g}_2) \subseteq \widehat{G}_q$ and we note the following. The group H_1 and H_2 induce transitive groups on the links $\mathrm{lk}(\{\widehat{p}, \widehat{q}\})$, $\mathrm{lk}(\{\widehat{p}, \widehat{d}\})$ and $\mathrm{lk}(\{\widehat{d}, \widehat{q}\})$ in $\widehat{\Delta}$ and hence the abstract group $H = \langle H_1 \cup H_2 \rangle$ acts transitively on the chambers of $\widehat{\Delta}$. Moreover, H_1 and H_2 consist of commutators, hence H is a perfect group. Being path-connected, $H \hookrightarrow \widehat{G}$ is an analytic subgroup with a Lie algebra $\mathfrak{h}$, see [3]. Since H is perfect, $\mathfrak{h}$ is a compact semisimple Lie algebra and hence $H \subseteq \widehat{G}$ is compact. Since $\widehat{G}$ is minimal, $\widehat{G} = H$ and $\mathrm{Lie}(\widehat{G})$ is generated as a Lie algebra by $\mathrm{Lie}(H_1) + \mathrm{Lie}(H_2)$.

By the formula given before 4.3 in Part I we have

$$\dim \widehat{G} - \dim \widehat{G}_{\widehat{p}, \widehat{d}, \widehat{q}} \leq 6 \cdot 1 + 3 \cdot 5 = 21$$

and thus $\dim(F) \leq 7$. The projection

$$\mathrm{Lie}(\widehat{G}) \cong \mathfrak{so}(3) \oplus \mathfrak{g}_2 \oplus \mathfrak{f} \longrightarrow \mathfrak{so}(3) \oplus \mathfrak{f}$$

is surjective and annihilates $\mathrm{Lie}(H_2) \cong \mathfrak{g}_2$. Therefore $\mathrm{Lie}(H_1)$ maps onto $\mathfrak{so}(3) \oplus \mathfrak{f}$. But $\dim(\mathrm{Lie}(H_1)) = 3$ and thus $\mathfrak{f} = 0$. It follows that $|\widehat{\Delta}| \rightarrow |\Delta|$ is in the coarse topology a covering. Since $|\Delta| \cong \mathbb{O}P^2$ is simply connected, $|\widehat{\Delta}| = |\Delta|$ and $F = \{1\}$. Finally, the Cayley plane admits no fixed-point free homeomorphism. Therefore no group can act continuously and freely on $\mathbb{O}P^2$ and therefore the geometry (G, Δ) has no equivariant quotients. $\square$

Construction 6.8 (The exceptional geometry). Let $\mathbb{H}$ denote the quaternion algebra, and $\mathbb{O}$ the Cayley algebra. For $a \in \mathbb{H}$ with $a^2 = -1$ put $[a] = \{\pm a\}$. For $b \in \mathbb{H}$ and $c \in \mathbb{O}$ with $b^2 = c^2 = -1$ put $[b, c] = \{\pm(b, c)\}$. Let Δ be the simplicial complex whose maximal simplices are the sets $\{[a], [b, c], \delta\}$, where $\delta : \mathbb{H} \rightarrow \mathbb{O}$ is an algebra homomorphism, with the property that

$$a \perp b \quad \text{and} \quad \delta(b) = c.$$

The group $G = \mathrm{Aut}(\mathbb{H}) \times \mathrm{Aut}(\mathbb{O}) = \mathrm{SO}(3) \times G_2$ acts in the obvious way on Δ . It is shown in [9] that Δ is a simply connected geometry of type C_3 and one can check that (G, Δ) fits the diagram $\mathcal{G}$ above. For example, we could put $a = \mathbf{j}$, $b = \mathbf{i} = c$ and $\delta : \mathbb{H} \subseteq \mathbb{O}$. In this case it is not difficult to work out the stabilizers of $p = \delta$, $d = [\mathbf{i}, \mathbf{i}]$ and $q = [\mathbf{j}]$.

References

- [1] W. Browder, *Higher torsion in H -spaces*, Trans. Amer. Math. Soc. **108** (1963), 353–375.
- [2] C. Gorodski, A. Kollross, *Some remarks on polar actions*, Ann. Global Anal. Geom. **49** (2016), no. 1, 43–58.
- [3] M. Goto, *On an arcwise connected subgroup of a Lie group*, Proc. Amer. Math. Soc. **20** (1969), 157–162.
- [4] T. Grundhöfer, N. Knarr, L. Kramer, *Flag-homogeneous compact connected polygons*, Geom. Dedicata **55** (1995), no. 1, 95–114.
- [5] T. Grundhöfer, N. Knarr, L. Kramer, *Flag-homogeneous compact connected polygons II*, Geom. Dedicata **83** (2000), no. 1–3, 1–29.
- [6] L. Kramer, A. Lytchak, *Homogeneous compact geometries*, Transform. Groups **19** (2014), no. 3, 793–852.
- [7] F. Podestà, G. Thorbergsson, *Polar actions on rank-one symmetric spaces*, J. Diff. Geom. **53** (1999), no. 1, 131–175.
- [8] H. Salzmann et al., *Compact Projective Planes*, de Gruyter Expositions in Mathematics, Vol. 21, de Gruyter, Berlin, 1995.
- [9] J. Schillewaert, K. Struyve, *On exceptional homogeneous compact geometries of type C_3* , Groups Geom. Dyn. **11** (2017), no. 4, 1377–1399.
- [10] K. Struyve, *Free construction of geometries of Coxeter type*, arXiv:1505.05475 (2015).
- [11] A. Pasini, *On two non-building but simply connected compact Tits geometries of type C_3* , Innov. in Incidence Geometry, to appear.