CORRECTION TO “DIMENSION DES FIBRES DE SPRINGER AFFINES POUR LES GROUPES”

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Abstract. We fill some gaps in [Bou15] and finish the proof of dimension formula for the affine Springer fibers studied in loc. cit.

1. Erratum to [Bou15]

1.1.

Proposition 2.22 and 3.6 are incorrect as stated: the map $\chi_+ : [V_G^{\text{reg}}/G] \rightarrow \mathfrak{C}_+$ is not a J -gerbe unless all simple factors of G are SL_2 . The reason is that the fiber of χ_+ may not consist of a single G -orbit. For example, by Proposition 2.9 and the remark after, G -orbits in $\mathcal{N}^{\text{reg}} = \chi_+^{-1}(0)$ correspond bijectively to Coxeter elements in the Weyl group.

To remedy this, one defines for each Coxeter element $w \in W$ a BJ -gerbe $[V_G^w/G]$ neutralized by the Steinberg section ϵ_+^w , as in the beginning of [Bou15, §2.3]. The stacks $[V_G^w/G]$ form an open cover of $[V_G^{\text{reg}}/G]$. In summary, the correct statement should be:

Proposition 1.1 (Replaces Proposition 2.22). *The stack $[V_G^{\text{reg}}/G]$ is a union of open substacks $\cup_w [V_G^w/G]$ over the set of Coxeter elements in W . Moreover, each open substack $[V_G^w/G]$ is a BJ -gerbe neutralized by the Steinberg section ϵ_+^w .*

Similarly Proposition 3.6 has to be replaced by the following statement:

Proposition 1.2 (Replaces Proposition 3.6). *The regular open subset $X_\gamma^{\lambda, \text{reg}}$ of the affine Springer fiber X_γ^λ is a union of open subsets $\cup_w X_\gamma^{\lambda, w}$ over the set of Coxeter elements in W . Moreover, each open subset $X_\gamma^{\lambda, w}$ is a $\mathcal{P}(J_a)$ -torsor.*

Also we remark that in [Bou15], the notations X_γ^λ and $\mathcal{M}_\lambda(a)$ are used interchangeably.

1.2.

Proposition 4.17 and hence its consequences Theorem 4.19 and Proposition 4.24 are incorrect. In general (when $\lambda \neq 0$) the complement of the regular open $X_\gamma^{\lambda, \text{reg}}$ may have the same dimension as X_γ^λ , see [Chi].

It is claimed that the proof of Proposition 4.17 is the same as the similar statement in Lie algebra case ([KL, §4, Lem.1]). However, when $\lambda \neq 0$, the proof does not carry over to group case. In [KL, §4, Lem.1], the argument relies on the fact that the $\mathcal{E}_\alpha$ defined in loc. cit. are line bundles. But in the setting of [Bou15], similar objects $\mathcal{E}_\alpha$ defined before Proposition 4.15 using Rees quotients of semigroups are no longer line bundles.

Instead of Proposition 4.24, the arguments in [Bou15] prove that

$$\dim X_\gamma^{\lambda, \text{reg}} = \langle \rho, \lambda \rangle + \frac{1}{2}[\delta(\gamma) - \text{def}(\gamma)].$$

Hence the proof of dimension formula in [Bou15] is not sufficient. It remains to show that

$$\dim X_\gamma^\lambda = \dim X_\gamma^{\lambda, \text{reg}}. \quad (1)$$

We will prove (1) in the present paper.

2. Finishing the proof of dimension formula

Instead of X_γ^λ , we also consider a larger affine Springer fiber

$$X_\gamma^{\leq \lambda} := \{g \in G(F)/G(\mathcal{O}) \mid g^{-1}\gamma g \in \overline{G(\mathcal{O})\pi^\lambda G(\mathcal{O})}\}$$

which contains X_γ^λ as an open subset. If we let $a = \chi(\gamma)$ be the characteristic polynomial of γ , we also denote the spaces X_γ^λ (resp. $X_\gamma^{\leq \lambda}$) by X_a^λ (resp. $X_a^{\leq \lambda}$) as their geometry only depends on $a = \chi(\gamma)$. Also it suffices to prove the dimension formula for the larger $X_a^{\leq \lambda}$ since then the complement of X_a^λ will have strictly smaller dimension. In other words, we need to prove

Theorem 2.1. *For $a \in \mathfrak{C}(F)^{\text{rs}}$ and $\lambda \in X_*(T)_+$ such that X_a^λ is nonempty, we have*

$$\dim X_a^{\leq \lambda} = \dim X_a^{\lambda, \text{reg}}.$$

Theorem 2.1 is a family of equalities depending on two parameters $a \in \mathfrak{C}(F)^{\text{rs}}$ and $\lambda \in X_*(T)_+$. We already know it in the following cases:

- (1) a is unramified (or split, since we assume the base field k to be algebraically closed) and any $\lambda \in X_*(T)_+$: this is done in [Bou15, §4.5], see also [Chi];
- (2) $\lambda = 0$ and any $a \in \mathfrak{C}(F)^{\text{rs}}$: the proof in this case is the same as in the Lie algebra case and is done in [Bou15, §4.2].

The goal is to deduce the general case from these two special cases, hence prove Theorem 2.1 and the dimension formula in general. Our method is global and is based on some results from [Bou15b]. Although [Bou15b] is a sequel to [Bou15], the results we use in loc. cit. will be independent of [Bou15].

2.1. Review of Hitchin–Frenkel–Ngô fibration

We collect some results on Hitchin–Frenkel–Ngô fibration that we are going to use. These results are established in [Bou15b] and are independent of [Bou15].

Let X be a smooth projective curve of genus g over k . Let x, x_0 be two distinct points and consider the $X_*(T)_+$ -valued divisor $\lambda x + \lambda_0 x_0$. Then one constructs affine spaces V_G^λ and $\mathfrak{C}_+^\lambda$ over the curve X by pulling back $[V_G/Z_+] \rightarrow [\mathfrak{C}_+/Z_+] \rightarrow [A_G/Z_+]$ along the map

$$-w_0(\lambda x + \lambda_0 x_0) : X \rightarrow [A_G/Z_+].$$

Note that we omit λ_0 from the notation since it is of auxiliary purpose. Similarly, one pulls back the discriminant divisor $\mathfrak{D}_+ \subset \mathfrak{C}_+$ to get the discriminant divisor $\mathfrak{D}_\lambda \subset \mathfrak{C}_\lambda$.

The Hitchin–Frenkel–Ngô moduli stack associated to the divisor $\lambda x + \lambda_0 x_0$ is the mapping stack

$$\overline{\mathcal{M}}_\lambda := \mathcal{H}om(X, [V_G^\lambda/G])$$

where G acts on V_G^λ by adjoint action. More concretely, this stack classifies pairs (E, φ) where E is a G -torsor on X and π is an automorphism of $E|_{X-\{x, x_0\}}$ such that $\text{inv}(\varphi_x) \leq \lambda$ and $\text{inv}(\varphi_{x_0}) \leq \lambda_0$.

Let $\mathcal{A}_\lambda$ be the affine space classifying sections of $\mathfrak{C}_+^\lambda$. Then we have the Hitchin–Frenkel–Ngô fibration

$$f_\lambda : \overline{\mathcal{M}}_\lambda \rightarrow \mathcal{A}_\lambda.$$

Theorem 2.2. *There is an open subspace $\mathcal{A}_\lambda^{\text{ani}} \subset \mathcal{A}_\lambda$ such that:*

- (1) $\overline{\mathcal{M}}_\lambda^{\text{ani}} := f_\lambda^{-1}(\mathcal{A}_\lambda^{\text{ani}})$ is a Deligne–Mumford stack;
- (2) The restriction $f_\lambda^{\text{ani}} : \overline{\mathcal{M}}_\lambda^{\text{ani}} \rightarrow \mathcal{A}_\lambda^{\text{ani}}$ is proper;
- (3) The codimension of the complement of $\mathcal{A}_\lambda^{\text{ani}}$ in $\mathcal{A}_\lambda$ grows with λ_0 . More precisely, for any integer $N > 0$, one can choose λ_0 sufficiently dominant regular such that the complement of $\mathcal{A}_\lambda^{\text{ani}}$ has codimension greater than N .

Proof. (1) is [Bou15b, Cor. 6.11].

(2) is [Bou15b, Thm. 8.1].

For (3), the same argument as in [Ngo, 6.3.5] shows that the complement of $\mathcal{A}_\lambda^{\text{ani}}$ in $\mathcal{A}_\lambda$ is the union of the image of $\mathcal{A}_\lambda^M$ for all proper Levi subgroups M . We know from [Bou15b, Lem. 5.10] that the codimension of $\mathcal{A}_\lambda^M$ in $\mathcal{A}_\lambda$ for any proper Levi subgroup M will grow linearly when λ_0 grows. $\square$

For each $b \in \mathcal{A}_\lambda$, let J_b be the smooth commutative group scheme over X obtained by pulling back the universal centralizer $J \rightarrow \mathfrak{C}_+$ along the section b . Let $\mathcal{P}_b$ be the Picard stack classifying J_b -torsors on X . As b varies, one obtains a Picard stack $\mathcal{P}$ over $\mathcal{A}_\lambda$ which is smooth. Over $\mathcal{A}_\lambda^{\text{ani}}$, $\mathcal{P}$ is moreover Deligne–Mumford, see [Bou15b, Cor. 6.11]

Finally we need the relation between the Hitchin–Frenkel–Ngô fibrations and the group version of the affine Springer fibers.

Let $b \in \mathcal{A}_\lambda(k)$ be generically regular semisimple. Then there exists a largest open subset $U_b \subset X$ where the section b intersects the discriminant divisor $\mathfrak{D}_\lambda$ transversally.

Theorem 2.3. *Suppose that $\overline{\mathcal{M}}_{\lambda,b} := f_{\lambda}^{-1}(b)$ is non-empty. Then we have*

$$\begin{aligned} \dim \overline{\mathcal{M}}_{\lambda,b} - \dim \mathcal{P}_b &= \sum_{v \in X - U_b} (\dim X_{b_v}^{\leq \lambda} - \dim P(J_{b_v})) \\ &= \sum_{v \in X - U_b} (\dim X_{b_v}^{\leq \lambda} - \dim X_{b_v}^{\lambda, \text{reg}}). \end{aligned}$$

Proof. This is a consequence of the product formula, see [Bou15b, Thm. 5.12]. $\square$

Remark 2.1. The nonemptiness assumption is satisfied when the T -torsor associated to $\lambda x + \lambda_0 x_0$ has a c -th root where $c := |Z_G|$ is the order of the finite group Z_G . In this case, one can define a global Steinberg section $\epsilon_{\lambda} : \mathcal{A}_{\lambda} \rightarrow \overline{\mathcal{M}}_{\lambda}$. We can achieve this by adding an extra divisor whose support is away from x, x_0 if necessary.

2.2. Specializing to unramified conjugacy class

Let X be a smooth projective curve. For any closed point $x \in X$, let F_x be the completion of the functions field of X at x and $\mathcal{O}_x \subset F_x$ its ring of integers. Let $\pi \in \mathcal{O}_x$ be a uniformizer.

We start from the local setting. Recall that

$$\mathfrak{C}(\mathcal{O}_x) = \mathcal{O}_x^r \subset \mathfrak{C}_+^{\lambda}(\mathcal{O}_x) = \bigoplus_{i=1}^r \pi^{\langle w_0(\lambda), \omega_i \rangle} \mathcal{O}_x \subset \mathfrak{C}(F) = F_x^r.$$

Let $a \in \mathfrak{C}_+^{\lambda}(\mathcal{O}_x) \mathcal{A} p \mathfrak{C}(F_x)^{\text{rs}}$. Suppose that a splits over a ramified extension of degree $e > 1$. We view a as a power series with coefficients in $\mathfrak{C}$.

We assume that

$$a(\pi^e) \in \mathfrak{C}_+^{\lambda}(\mathcal{O}_x). \quad (2)$$

By the equality $\mathfrak{C}_+^{\lambda}(\mathcal{O}_x) = \bigoplus_{i=1}^r \pi^{\langle w_0(\lambda), \omega_i \rangle} \mathcal{O}_x$ we see that this assumption is satisfied if λ is sufficiently dominant regular. Moreover if $a \in \mathfrak{C}(\mathcal{O}_x)$, then this assumption is satisfied for any $\lambda \in X_*(T)_+$. In the next section we will remove this assumption.

Consider the following one-parameter family: for $s \in k$, let

$$a_s := a(s\pi + (1-s)\pi^e).$$

Then $a_s \in \mathfrak{C}_+^{\lambda}(\mathcal{O}_x) \mathcal{A} p \mathfrak{C}(F_x)^{\text{rs}}$ for all s by the assumption (2). Also, it is clear that $a_1 = a$ and $a_0 = a(\pi^e)$ is unramified (or split).

By [Bou15b, Prop. 4.6], we can choose $N > 0$ large enough so that the affine Springer fiber $X_a^{\leq \lambda}$ only depends on $a \bmod \pi^N$. Then for all $s \neq 0$, the fibers $X_{a_s}^{\leq \lambda}$ are all isomorphic to X_a^{λ} since they are obtained from a by changing uniformizer; in particular they also only depend on $a_s \bmod \pi^N$. Enlarging N if necessary, we may assume that $X_{a_0}^{\leq \lambda}$ also only depends on $a \bmod \pi^N$.

The one-parameter family a_s defines an algebraic curve in the finite-dimensional vector space

$$\mathfrak{C}_+^{\lambda}(\mathcal{O}_x / \pi^N) = \bigoplus_{i=1}^r \pi^{\langle w_0(\lambda), \omega_i \rangle} \mathcal{O}_x / \pi^N \mathcal{O}_x.$$

Now we can globalize. Consider a curve X and 2 distinct points $x, x_0 \in X$. Consider the divisor $\lambda x + \lambda_0 x_0$ and the associated Hitchin–Frenkel–Ngô fibration $f : \overline{\mathcal{M}}_\lambda \rightarrow \mathcal{A}_\lambda$. We omit the divisor λ_0 from the notation since it is of auxiliary purpose. We can choose λ_0 large enough, so that the map

$$\mathrm{ev}_{(Nx)} : \mathcal{A}_\lambda \rightarrow \mathfrak{C}_+^\lambda(\mathcal{O}_x/\pi_x^N)$$

is surjective.

Let $U \subset \mathcal{A}_\lambda^{\mathrm{ani}}$ be the open subset consisting of those anisotropic a which intersect the discriminant divisor $\mathfrak{D}_\lambda$ transversally at x_0 . By Theorem 2.2(3), we can enlarge λ_0 if necessary so that the complement of U has a large codimension. Then we can find $b_0, b \in U$ that map to a_0, a under $\mathrm{ev}_{(Nx)}$ respectively. Choose a section of the linear surjection $\mathrm{ev}_{(Nx)}$ which sends a_0 to b_0 and a to b . The image of the algebraic curve a_s under this chosen section will be an algebraic curve in $\mathcal{A}_\lambda$ which generically lands in U . This curve is parametrized by $\mathbb{A}^1$ and we denote its closed points by b_s with $s \in k$. In particular, $b_1 = b$.

Define the finite set

$$S := \{s \in k \mid b_s \notin U\}.$$

Then $0, 1 \in \mathbb{A}^1 - S$. For $s \in \mathbb{A}^1 - S$ with $s \neq 0$, by the product formula 2.3

$$\dim \overline{\mathcal{M}}_{\lambda, b_s} - \dim \mathcal{P}_{b_s} = \dim X_{a_s}^{\leq \lambda} - \dim X_{a_s}^{\lambda, \mathrm{reg}} \quad (3)$$

since at points $y \neq x, x_0$, the divisor is 0 and we know that $\dim X_{(b_s)_y}^0 = \dim X_{(b_s)_y}^{0, \mathrm{reg}}$.

Since $X_{a_s}^{\leq \lambda}$ are all isomorphic for $s \neq 0$, $\dim \overline{\mathcal{M}}_{\lambda, b_s}$ is constant for all except finitely many $s \neq 0$. By the properness of the Hitchin fibration over $\mathcal{A}_\lambda^{\mathrm{ani}}$ (Theorem 2.2) and the semicontinuity of the fiber dimensions, we have

$$\dim \overline{\mathcal{M}}_{\lambda, b_s} \leq \dim \overline{\mathcal{M}}_{\lambda, b_0}$$

for almost all $s \neq 0$. Since $(b_0)_x = a_0$ is unramified, we know that Theorem 2.1 is true for $X_{a_0}^{\leq \lambda}$. Hence by Theorem 2.3 we have $\dim \overline{\mathcal{M}}_{\lambda, b_0} = \dim \mathcal{P}_{b_0}$. Combined with the smoothness of $\mathcal{P}$ and the fact that $\dim \overline{\mathcal{M}}_{\lambda, b_s} \geq \dim \mathcal{P}_{b_s}$, we get that for $s \in \mathbb{G}_m - S$, $\dim \overline{\mathcal{M}}_{\lambda, b_s} = \dim \mathcal{P}_{b_s}$ which implies that $\dim X_{a_s}^{\leq \lambda} = \dim X_{a_s}^{\lambda, \mathrm{reg}}$ by (3) and the fact that for any $s \neq 0$, $X_{a_s}^{\leq \lambda}$ is isomorphic to $X_a^{\leq \lambda}$.

2.3. The remaining cases

We see from the previous section that Theorem 2.1 is true for those a, λ that satisfy the assumption (2). In other words, for any $a \in \mathfrak{C}(F)^{\mathrm{rs}}$ we have

- Theorem 2.1 is true for X_a^λ when λ is sufficiently dominant regular.
- If $a \in \mathfrak{C}(\mathcal{O}_x)$, then Theorem 2.1 is true for any $\lambda \in X_*(T)_+$.

Now we reduce the remaining cases to these known cases.

Suppose $a \in \mathfrak{C}_+^\lambda(\mathcal{O}_x) \mathcal{A} p \mathfrak{C}(F)^{\mathrm{rs}}$ and $a(\pi^e) \notin \mathfrak{C}_+^\lambda(\mathcal{O}_x)$, where $e > 1$ is the degree of the ramified extension that splits a . There exists $\mu \in X_*(T)_+$ sufficiently dominant regular so that $a(\pi^e) \in \mathfrak{C}_+^{\lambda+\mu}(\mathcal{O}_x)$. By what we have proved above, it follows that the dimension formula is true for $X_a^{\lambda+\mu}$.

To globalize, first consider the divisor $\lambda x + \lambda_0 x_0$ as above, which defines the Hitchin fibration $f_{\leq \lambda} : \overline{\mathcal{M}}_\lambda \rightarrow \mathcal{A}_\lambda$. Choose $b \in \mathcal{A}_\lambda^{\text{ani}}$ as above that approximates a when evaluating at x .

Now consider a one-parameter family of Hitchin fibrations with moving divisor $\lambda x + \mu y + \lambda_0 x_0$. Here y is the moving point and we allow y to move in $X^0 := X - \{x_0\}$. Then we obtain a family of Hitchin fibrations $\overline{\mathcal{M}} \xrightarrow{f} \mathcal{A} \rightarrow X^0$ whose fiber over $y \in X^0$ is $\overline{\mathcal{M}}_{\lambda x + \mu y} \rightarrow \mathcal{A}_{\lambda x + \mu y}$. (Here again we omit the auxiliary coweight λ_0 from the notation.)

For each $y \in X^0$, we have a closed embedding $\mathcal{A}_\lambda \rightarrow \mathcal{A}_{\lambda x + \mu y}$. Together we get a closed embedding over X^0 :

$$\mathcal{A}_\lambda \times X^0 \rightarrow \mathcal{A}.$$

The image of $b \times X^0$ is a curve in $\mathcal{A}$. For each $y \neq x$, consider the fiber $\overline{\mathcal{M}}_{b,y} := f^{-1}(b, y)$. Since b comes from $\mathcal{A}_\lambda$ (which is defined by the divisor $\lambda x + \lambda_0 x_0$), $b_y \in \mathfrak{C}(\mathcal{O}_y)$ and thus Theorem 2.1 is true for $X_{b_y}^{\leq \mu}$. By Theorem 2.3, we have

$$\dim \overline{\mathcal{M}}_{b,y} - \dim \mathcal{P}_{b,y} = \dim X_a^{\leq \lambda} - \dim X_a^{\lambda, \text{reg}} \quad (4)$$

since $b_x = a$ and the points away from x do not contribute. In particular, $\dim \mathcal{M}_{b,y}$ is constant when $y \neq x$.

Now consider the fiber $\mathcal{M}_{b,x}$ which lives in the Hitchin space with divisor $(\lambda + \mu)x + \lambda_0 x_0$. Since $b_x = a$ and Theorem 2.1 is true for $X_a^{\leq (\lambda + \mu)}$, we see that

$$\dim \overline{\mathcal{M}}_{b,x} = \dim \mathcal{P}_{b,x}.$$

Finally, since $b \in \mathcal{A}_\lambda^{\text{ani}}$, the whole curve $b \times X^0$ lands in $\mathcal{A}^{\text{ani}}$ and by properness and semicontinuity, we have

$$\dim \mathcal{P}_{b,x} = \dim \overline{\mathcal{M}}_{a,x} \geq \dim \overline{\mathcal{M}}_{b,y} \geq \dim \mathcal{P}_{b,y}$$

for any $y \neq x$. As $\mathcal{P}$ is smooth over $\mathcal{A}$, we get $\dim \overline{\mathcal{M}}_{b,y} = \dim \mathcal{P}_{b,y}$ and hence $\dim X_a^\lambda = \dim X_a^{\leq \lambda, \text{reg}}$ by (4).

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