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HOMOGENEOUS HYPERCOMPLEX STRUCTURES I — THE COMPACT LIE GROUPS

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Abstract. We obtain a complete list of homogeneous hypercomplex structures on the compact Lie groups. The substantial results are formulated and proved entirely in terms of the structure theory of Lie groups and algebras.

1. Introduction

This paper is the first in a series of two (see also [DT2]), whose purpose is to give a description of compact hypercomplex homogeneous manifolds with a transitive action of a compact Lie group. The classification and proofs are entirely based on the structure theory of reductive Lie groups; it turns out that in the language of roots we get surprisingly clear answers to the natural questions.

We start with a complex manifold (M, I) with a transitive compact Lie group of biholomorphic automorphisms and look for another invariant complex structure J on M, such that $IJ = -JI$ (we say shortly that J *matches* I). We call the complex structure I *admissible* if there exists a matching J.

Our classification problem splits into two:

Problem A. In the class of compact complex homogeneous manifolds (M, I), discern those which are admissible.

Problem B. Given an admissible complex structure I on M, describe the class of all homogeneous hypercomplex structures on M (up to equivalence) of which I is one of the complex structures.

In the present paper we solve the above two problems in the case when $M = \mathbf{U}$ is a *compact Lie group*, whose Lie algebra we shall denote by $\mathfrak{u}$. In this case our two problems are easily reduced to determining the hypercomplex structures on the Lie algebra, which are integrable in the sense that the Nijenhuis tensors vanish.

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The corresponding solutions are formulated and proved in terms of a remarkable subset, *the stem* of the root system of the (semisimple part of) $\mathfrak{u}$ (see Definition 2.1). The stem combinatorics developed in Section 2 is used in the sections that follow for the calculation of Nijenhuis tensors, which results in solving Problem A (see Theorem 4.23 and corollaries) and Problem B (see Theorem 4.28). Theorems 4.23 and 4.28 are the main results of the paper.

A slightly more elaborate description of these results and their history follows:

It is well known ([Sam], [Wang]) that each compact even-dimensional Lie group carries a homogeneous complex structure. A comprehensive description of the regular homogeneous complex structures on reductive Lie groups (not necessarily compact) in terms of structure theory may be found in Snow [Snow]; the relevant facts may be found in Subsection 1.2 below.

We show that the group $\mathbf{U}$ carries an invariant hypercomplex structure if anf only if

$$\mathrm{rank}(\mathfrak{u}) = 2d + 4k,$$

where d is the number of the elements in the stem and k is a non-negative integer (see Corollary 4.24). If $\mathfrak{u}$ is semisimple, then $d \leq \mathrm{rank}(\mathfrak{u}) \leq 2d$. In particular, a compact simple Lie group $\mathbf{U}$ admits a left invariant hypercomplex structure if and only if $\mathrm{rank}(\mathfrak{u}) = 2d$ and this is the case only for $\mathbf{U} = \mathrm{SU}(2k+1),\ k \geq 1$ (see [DT1, Subsect. 2.9] for explicit descriptions of the stems of the irreducible reduced root systems).

We use the stem to define a class of Cayley transforms (see Subsection 2.4 and Section 3) of the Lie algebra $\mathfrak{u}$. When $\mathfrak{u}$ is "*nearest to semisimple*" (this is the case $2d = \mathrm{rank}(\mathfrak{u})$), then we show that all the complex structures matching a given admissible complex structure I are obtained by conjugation of I with the Cayley transforms. If we perceive a hypercomplex structure on $\mathfrak{u}$ as a representation of $\mathrm{SU}(2)$ on $\mathfrak{u}$, which splits into real 4-dimensional irreducible components, then an admissible complex structure on a nearest to semisimple $\mathfrak{u}$ determines (uniquely) the 4-dimensional subspaces and (up to rotation on a circle in $\mathrm{SU}(2)$) the action of $\mathrm{SU}(2)$ on each of these subspaces.

In the general case the complex structures matching a given admissible complex structure I are determined by the Cayley transforms and a random choice of a $(2k) \times (2k)$ complex matrix $\mathbf{b}$ satisfying $\overline{\mathbf{b}}\mathbf{b} = -1$.[2]

The idea to use a highest root to construct homogeneous "quaternionic" spaces goes back to Wolf [Wolf]. A wide class of examples of homogeneous hypercomplex structures was given by Spindel et al. [SSTP] and Joyce [Joy], where many ideas of the present paper may be traced in implicit form. One of them is the stem which is a maximal strongly orthogonal subset of the set of positive roots of a reduced root system (see, e.g., [AK]). The stem is determined by the root system Δ up to the action of the Weyl group (see Theorem 2.9). In Section 2 of this paper we study the properties of the stem, which give us the necessary language and facts to classify the hypercomplex structures on compact Lie groups. The stem decomposition is used also in the second paper of this study [DT2] to obtain a complete list

[2]This matrix $\mathbf{b}$ corresponds to a complex structure anti-commuting with I restricted to a certain real vector subspace of a Cartan subalgebra of $\mathfrak{u}$ (see (34) and Theorem 4.28).

of hypercomplex manifolds, on which a compact group of automorphisms acts transitively.

We are grateful to the referees, who informed us that the set we call the stem, was discerned and constructed earlier by Kostant and Joseph. Kostant has called it "the cascade" — unpublished but cited and used by other authors (see [Jos], [LW], [FHW], [Sm]), mainly in representation theory. Or rather, the stem is a special case of "Kostant's cascade construction". After the publication of this paper in `arXiv` as [DT1], professor Kostant published [K], where the cascade is defined and used to study the coadjoint structure of the nilradical of a Borel subgroup of a semisimple Lie group. There is some intersection with results in our section 2.

Less explicit results on classification of homogeneous hypercomplex manifolds, under additional constraints on the input data in the context of differential geometry have appeared in [BGP].

Remark 1.1. The hypercomplex structures appearing in the present study are not hyperkaehler. By the definition of hyperkaehler variety (see, e.g., [GHJ, p. 164]) it must be simply connected, and with this stipulation, it is well known that there are no homogeneous compact hyperkaehler varieties. Actually the simply connected hypercomplex manifolds in our list ($\mathrm{SU}(2k+1)$ and some of its factors; see also [DT2]) cannot be even Kaehler by topological reasons (or by comparing to the list in [Bor]). If we drop the simple connectedness condition from the definition of hyperkaehler, then by the results in [A], the only hyperkaehler manifolds in our list are flat tori.

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1.1. Conventions and notations

Here we fix notations and recall well-known facts, to be used throughout the paper.

We shall denote by $\mathfrak{u}$ a compact Lie algebra. Then the complexification $\mathfrak{u}^{\mathbb{C}} = \mathfrak{g} = \mathfrak{g}_s \oplus \mathfrak{c}$ is a reductive complex Lie algebra, whose semisimple ideal is $\mathfrak{g}_s$, and the center is $\mathfrak{c} \cong \mathbb{C}^r$. We denote by τ the conjugation of $\mathfrak{g}$ with respect to the real form $\mathfrak{u}$, so τ is an antilinear involution of $\mathfrak{g}$, such that $\mathfrak{u} = \mathfrak{g}^\tau = \mathfrak{u}_s \oplus \mathfrak{c}_u$. We denote by $\mathbf{U_s}$ and $\mathbf{G}_s$ the corresponding simply connected Lie groups, and by $\mathbf{U} = \mathbf{U_s} \times \mathbf{C}_u$, $\mathbf{G} = \mathbf{G_s} \times \mathbf{C}$ - the corresponding reductive Lie groups ($\mathbf{C}_u$ is a compact torus).

For $X, Y \in \mathfrak{g}$, we denote by $\langle X, Y \rangle$ an ad-invariant symmetric bilinear form such that its restriction to the compact real form $\mathfrak{u}$ is negative definite. We assume that $\langle \cdot, \cdot \rangle$ coincides with the Killing form on the semisimple part $\mathfrak{g}_s$. Such a bilinear form exists and it necessarily satisfies $\langle \mathfrak{c}, \mathfrak{g}_s \rangle = 0$.[3]

Let $\mathfrak{h}$ be a τ-stable Cartan subalgebra of $\mathfrak{g}$, then $\mathfrak{h} = \mathfrak{h}_s \oplus \mathfrak{c}$, where $\mathfrak{h}_s$ is a Cartan subalgebra of $\mathfrak{g}_s$. Let $\mathbf{H}$ be the corresponding Cartan subgroup of $\mathbf{G}$. We denote by Δ the set of roots of $\mathfrak{g}_s$ with respect to $\mathfrak{h}_s$ extended to the entire $\mathfrak{h}$ by zero on

[3]Recall that $\mathfrak{g}_s = [\mathfrak{g}, \mathfrak{g}]$ ([H, Prop. 6.6]).

$\mathfrak{c}$. For $\alpha \in \Delta$ we denote by h_α the element of $\mathfrak{h}$ determined by $\langle H, h_\alpha \rangle = \alpha(H)$ for all $H \in \mathfrak{h}$, and we denote[4]

$$H_\alpha = \frac{2}{\langle \alpha, \alpha \rangle} h_\alpha, \quad \mathfrak{g}(\alpha) = \{X \in \mathfrak{g} \mid \ \mathrm{ad} H(X) = \alpha(H)X, \ H \in \mathfrak{h}\}.$$

Further, for $\alpha, \beta \in \Delta$ we denote

$$C(\beta, \alpha) = \frac{2 \langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}, \quad s_\alpha(\beta) = \beta - C(\beta, \alpha)\alpha. \tag{1}$$

The map $\beta \mapsto s_\alpha(\beta)$ is the reflection along α (see, e.g., [H, Chap. III]).

By $\mathbf{Aut}(\Delta)$ we denote the group of all the elements in $GL(\mathfrak{h}^*_{\mathbb{R}})$ which leave the set $\Delta \subset \mathfrak{h}^*_s$ invariant and the center $\mathfrak{c}$ pointwise fixed.

The *Weyl group* $\mathbf{W} = \mathbf{W}(\Delta)$ is the (normal) subgroup of $\mathbf{Aut}(\Delta)$, which is generated by all reflections $s_\alpha, \alpha \in \Delta$. The Weyl group acts simply transitively on the set of all bases Π of Δ. For a fixed basis Π of the root system Δ we denote $\mathbf{Aut}_\Pi(\Delta) = \{\phi \in \mathbf{Aut}(\Delta) \mid \phi(\Pi) = \Pi\}$.

The adjoint action of the Weyl group $\mathbf{W}$ on $\mathfrak{h}$ is defined for $s \in \mathbf{W}$ by $\alpha(s(H)) = s^{-1}(\alpha)(H)$, $H \in \mathfrak{h}$. For any $\gamma \in \Delta$ we have $s_\gamma(H) = H - \gamma(H)H_\gamma$, $H \in \mathfrak{h}$. The normalizer $\mathbf{N} \subset \mathbf{G}$ of the Cartan subalgebra $\mathfrak{h}$ is $\mathbf{N} = \mathbf{N}(\mathfrak{h}) = \{g \in \mathbf{G} \mid \mathrm{Ad}(g)(\mathfrak{h}) = \mathfrak{h}\}$, $\mathbf{N}_u = \mathbf{N} \cap \mathbf{U}$. Let us denote by $\mathbf{H}_\mathfrak{u}$ the torus $\mathbf{H} \cap \mathfrak{u} = \mathbf{T} \times \mathbf{C}_\mathfrak{u}$ generated by $\mathfrak{h} \cap \mathfrak{u}$, where $\mathbf{T}$ is the maximal torus in $\mathbf{U_s}$ corresponding to $\mathfrak{h}_s \cap \mathfrak{u}_s$. The following exact sequence is a fundamental fact of structure theory (see, e.g., [H, the first paragraph on p. 300]):

$$1 \to \mathbf{H}_u \to \mathbf{N}_u \to \mathbf{W} \to 1. \tag{2}$$

Weyl–Chevalley basis. We (may) choose elements $E_\alpha \in \mathfrak{g}(\alpha)$, so that the structural constants are integers, i.e., for $\alpha, \beta, \alpha + \beta \in \Delta$:

$$\begin{gathered} [E_\alpha, E_{-\alpha}] = H_\alpha, \quad [E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}, \\ N_{\alpha,\beta} = -N_{-\alpha,-\beta}, \quad |N_{\alpha,\beta}| = 1 - p, \end{gathered} \tag{3}$$

where $\beta + n\alpha, \quad p \leq n \leq q$ is the α-series of β (see, e.g., [H, p. 195]).

It is convenient to extend (3) and *define the symbol* $N_{\alpha,\beta}$ for any couple of functionals $\alpha, \beta \in \mathfrak{h}^*$ by

$$N_{\alpha,\beta} = 0, \ \text{ if } \ \alpha \notin \Delta, \ \text{ or } \ \beta \notin \Delta, \ \text{ or } \ \alpha + \beta \notin \Delta. \tag{4}$$

In the above basis for $\mathfrak{g}_s$, the *contragredient* involution $\theta \in \mathbf{Aut}(\mathfrak{g})$ is the complex linear map determined by

$$\theta(E_\alpha) = -E_{-\alpha}, \ \theta(H) = -H, \quad \alpha \in \Delta, \ H \in \mathfrak{h}. \tag{5}$$

The conjugation τ is the *antilinear involution* given by

$$\tau(E_\alpha) = -E_{-\alpha} \quad \tau(H_\alpha) = -H_\alpha, \quad \tau((z_1, \ldots, z_r)) = (-\overline{z}_1, \ldots, -\overline{z}_r).$$

[4]Since $\langle \mathfrak{c}, \mathfrak{h}_s \rangle = 0$ and $\alpha \in \Delta$ vanishes on $\mathfrak{c}$ for each $\alpha \in \Delta$, the elements h_α lie in $\mathfrak{h}_s$. It follows also that $\mathfrak{g}(\alpha) \subset \mathfrak{g}_s$ for $\alpha \neq 0$ (recall that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}_s$).

where $(z_1, \dots, z_r) \in \mathfrak{c}$.

We have $\mathfrak{u} = \mathfrak{g}^\tau = \{X \in \mathfrak{g} \mid \tau(X) = X\}$. As $\mathfrak{h}$ is τ-invariant:

$$\alpha(\tau(H)) = -\overline{\alpha(H)}, \ \alpha \in \Delta, H \in \mathfrak{h}. \tag{6}$$

We now fix a *basis* $\Pi = \{\alpha_1, \dots, \alpha_l\}$ of Δ, which gives us a *system of positive roots* Δ^+. We denote

$$\mathfrak{n}^\pm = \bigoplus_{\alpha \in \Delta^\pm} \mathfrak{g}(\alpha), \quad \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-, \quad \mathfrak{b}^\pm = \mathfrak{h} \oplus \mathfrak{n}^\pm.$$

The *Borel subalgebra* $\mathfrak{b}^+$ is a maximal solvable subalgebra of $\mathfrak{g}$.

For any $\gamma \in \Delta$ we denote

$$sl_\gamma(2) = \mathrm{span}_{\mathbb{C}}\{E_\gamma, E_{-\gamma}, H_\gamma\} \subset \mathfrak{g}, \quad su_\gamma(2) = \mathfrak{u} \cap sl_\gamma(2).$$

Definition 1.1. A subalgebra $\mathfrak{a} \subset \mathfrak{g}$ is called $\mathfrak{h}$-*regular* if its normalizer $\mathfrak{n}(\mathfrak{a})$ contains a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. A subalgebra $\mathfrak{a}$ is called *regular* if it is $\mathfrak{h}$-regular for some Cartan subalgebra $\mathfrak{h}$.

It is well known that if $\mathfrak{a}$ is an $\mathfrak{h}$-regular subalgebra of $\mathfrak{g}$, then we have a decomposition (see, e.g., [Snow, Subsect. 2.1 on p. 199]):

$$\mathfrak{a} = (\mathfrak{h} \cap \mathfrak{a}) \oplus \bigoplus_{\alpha \in \Theta} \mathfrak{g}(\alpha), \text{ where } \quad \Theta = \{\alpha \in \Delta \mid \mathfrak{g}(\alpha) \subset \mathfrak{a}\}. \tag{7}$$

1.2. Complex structures on a compact Lie group

Any left invariant almost complex structure on the Lie group $\mathbf{U}$, determines (and is determined by) a complex structure $I : \mathfrak{u} \to \mathfrak{u}$.[5] The obvious condition for the existence of a complex structure on $\mathfrak{u}$ is even dimension, and this is the same as even rank.

In the present paper we determine all the operators $I : \mathfrak{u} \to \mathfrak{u}$, which correspond to admissible[6] left invariant integrable complex structures on $\mathbf{U}$ and for each such I we describe all the hypercomplex structures on $\mathbf{U}$ of which I is one of the complex structures. The equivalences among the admissible complex structures are easily seen on the universal covering group $\widetilde{\mathbf{U}} \cong \mathbf{U}_s \times \mathbb{R}^r$. We have $\mathbf{U} = \widetilde{\mathbf{U}}/\Lambda$, where Λ is some central lattice in $\widetilde{\mathbf{U}}$. It is well known that equivalent complex structures on $\widetilde{\mathbf{U}}$ may project to unequivalent complex structures on $\mathbf{U}$ (see the table in the end of the paper). The dependence on Λ is well understood in the literature (see, e.g., [M, Chap. 1], or [Weil, Chap. 6]).

Let I be any complex structure on $\mathfrak{u}$ (i.e., a linear operator whose square equals minus identity). We extend I to $\mathfrak{g}$ (and go on to denote the extension by I) setting $I(\mathrm{i}X) = \mathrm{i}IX$. Thus on $\mathfrak{g}$ we have $I \circ \tau = \tau \circ I$.

[5]By a complex structure on the Lie algebra $\mathfrak{u}$ we mean here a linear operator I with $I^2 = -\mathrm{Id}$. The integrability condition will be imposed later.

[6]See the introduction for the notion admissible.

Definition 1.2. Let I be a complex structure on $\mathfrak{u}$. We denote

$$\begin{aligned}\mathfrak{m}_I^+ &= \{X \in \mathfrak{g} \mid IX = \mathrm{i}X\} = \{X - \mathrm{i}IX \mid X \in \mathfrak{u}\};\\ \mathfrak{m}_I^- &= \{X \in \mathfrak{g} \mid IX = -\mathrm{i}X\} = \{X + \mathrm{i}IX \mid X \in \mathfrak{u}\} = \tau(\mathfrak{m}_I^+).\end{aligned}$$

In other words: $\mathfrak{m}_I^+$, $\mathfrak{m}_I^-$ are respectively the $(1,0)$ and $(0,1)$ components (with respect to the left invariant almost complex structure I) of the complexified tangent space to $\mathbf{U}$ at the unit element. It is also well known (and obvious) that

$$\mathfrak{g} = \mathfrak{m}_I^+ \oplus \mathfrak{m}_I^- . \tag{8}$$

If I is a complex structure on $\mathfrak{u}$ we define its *Nijenhuis tensor*:

$$N_I(X,Y) = [IX, IY] - I[IX,Y] - I[X,IY] - [X,Y], \quad X,Y \in \mathfrak{u}. \tag{9}$$

It is often convenient to "complexify" the Nijenhuis tensor by allowing X, Y in the above formula to vary in $\mathfrak{g} = \mathfrak{u}^{\mathbb{C}}$. We denote the complexified Nijenhuis tensor by the same letter.

The following proposition is well known (see, e.g., [Snow]).

Proposition 1.1. *The left invariant almost complex structure induced by I on $\mathbf{U}$ is a complex structure if and only if any one of the following conditions is satisfied:*

a) $\mathfrak{m}_I^+$ *is a subalgebra of* $\mathfrak{g}$*;*
b) $N_I \equiv 0$.

Definition 1.3. In this paper, a complex structure on the compact Lie algebra $\mathfrak{u}$ will be called *integrable*, if it satisfies the conditions from Proposition 1.1. Two complex structures I, I' on $\mathfrak{u}$ will be called *equivalent* if there exists an automorphism ξ of $\mathfrak{u}$ such that $\xi \circ I = I' \circ \xi$.

Definition 1.4. We shall say that a complex structure I on a Lie algebra $\mathfrak{u}$ is *regular* if $\mathfrak{m}_I^+$ is a regular subalgebra with respect to some τ-stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{u}^{\mathbb{C}}$.

Since $\mathbf{U}$ is compact, we may assume that I is a regular complex structure (see [Snow, Cor. in Subsect. 3.1 on p. 212]). Throughout the paper $\mathfrak{h}$ will denote a τ-stable Cartan subalgebra in the normalizer of $\mathfrak{m}_I^+$.

Let $\Delta \subset (\mathfrak{h})^*$ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. We have

Proposition 1.2. *An integrable complex structure I on $\mathfrak{u}$ determines a system of positive roots Δ^+, and a subspace $\mathfrak{h}^+ = \mathfrak{m}_I^+ \cap \mathfrak{h} \subset \mathfrak{h}$, such that*

$$\mathfrak{m}_I^+ = \mathfrak{h}^+ \oplus \mathfrak{n}^+, \quad \mathfrak{h} = \mathfrak{h}^+ \oplus \mathfrak{h}^-, \ \textit{where}\ \mathfrak{h}^- = \tau(\mathfrak{h}^+) = \mathfrak{m}_I^- \cap \mathfrak{h}.$$

In particular $\dim(\mathfrak{h}^+) = \dim(\mathfrak{h})/2$.

Proof. From the regularity of I we have the decomposition (7).

The Cartan subalgebra $\mathfrak{h}$ is τ-invariant, whence $\mathfrak{g} = \mathfrak{m}_I^+ \oplus \tau(\mathfrak{m}_I^+)$ implies $\mathfrak{h}^+ \oplus \mathfrak{h}^- = \mathfrak{h}$, whence the last statement of the lemma.

If $\alpha \in \Theta$ and $-\alpha \in \Theta$, then $H_\alpha = [E_\alpha, E_{-\alpha}] \in \mathfrak{m}_I^+$, but $\tau(H_\alpha) = -H_\alpha$, whence $\mathfrak{m}_I^+ \cap \tau(\mathfrak{m}_I^+) \neq \{0\}$, which contradicts (8). Because $\dim(\mathfrak{m}_I^+) = \dim(\mathfrak{g})/$, we conclude that Θ contains exactly one of the roots in each couple $\{\alpha, -\alpha\} \subset \Delta$. But $\mathfrak{m}_I^+$ is also a subalgebra, so $\Theta = \Delta^+$ for some basis of Δ (see, e.g., [Bou, Chap. VIII, Sect, 3, Prop. 7]). The lemma is proved. $\square$

Remark 1.2. If I is a regular complex structure on a noncompact reductive Lie algebra $\mathfrak{g}_0$, then the subalgebra $\mathfrak{m}_I^+$ may have a nontrivial Levi component (see, e.g., [Snow]).

Throughout this paper, given an integrable complex structure I on $\mathfrak{u} = \mathfrak{g}^\tau$ we shall denote the corresponding τ-invariant Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_I$, the subspace $\mathfrak{h}^+ = \mathfrak{h}_I^+$ with dimension $m = \dim(\mathfrak{h}^+) = \dim(\mathfrak{h})/2$, the Borel subalgebra $\mathfrak{b}^+ = \mathfrak{b}_I^+ = \mathfrak{h}_I \oplus \mathfrak{n}_I^+$, etc. When (we believe that) no confusion may arise, we shall omit the subscript I. When we have to refer to this connection between I and the structural data, we shall say briefly that *I is a $\mathfrak{b}^+$ complex structure.* In other words, a complex structure I on $\mathfrak{u}$, will be called a $\mathfrak{b}^+$*-complex structure* if anf only if $\mathfrak{b}^+$ is the normalizer of $\mathfrak{m}_I^+$.

It is well known that Ad$\mathfrak{u}$ acts transitively on the set of all Borel subalgebras of $\mathfrak{g}$, thus if we fix a Borel subalgebra $\mathfrak{b}^+$, then any integrable complex structure on $\mathfrak{u}$ is equivalent to a $\mathfrak{b}^+$-complex structure.

Remark 1.3. It is well known that a compact group $\mathbf{U}$ may have a left invariant complex structure I in such a way, that the simple factors are not complex submanifolds. Perhaps the best known semisimple example is a Calabi–Eckman invariant complex structure on $\mathrm{SU}(2) \times \mathrm{SU}(2)$ (see [CE]).

1.3. Left invariant almost hypercomplex structures

Definition 1.5. A *left invariant almost hypercomplex structure* on $\mathbf{U}$ is a couple of complex structures $I, J : \mathfrak{u} \to \mathfrak{u}$, which anti-commute, i.e., $I \circ J = -J \circ I$. An almost hypercomplex structure will be called a *hypercomplex structure* if both I, J are integrable.

Two hypercomplex structures $(I, J), (I', J')$ on $\mathfrak{u}$ will be called *equivalent* if there exists an automorphism ξ of $\mathfrak{u}$ such that $\xi \circ I = I' \circ \xi$, $\xi \circ J = J' \circ \xi$.

We use the same letters to denote the complexifications of the operators I, J, so we have two linear maps $I, J : \mathfrak{g} \longrightarrow \mathfrak{g}$, such that

$$IJ = -JI, \quad I^2 = J^2 = -1, \quad \tau \circ I = I \circ \tau, \quad \tau \circ J = J \circ \tau. \tag{10}$$

First we show

Lemma 1.3. *Let I, J be complex structures on $\mathfrak{u}$. Then $I \circ J = -J \circ I$ if and only if $I(\mathfrak{m}_J^+) = \mathfrak{m}_J^-$.*

Proof. If $I \circ J = -J \circ I$, then for $X \in \mathfrak{m}_J^+$ we have $JIX = -IJX = -\mathrm{i}IX$. If $I(\mathfrak{m}_J^+) = \mathfrak{m}_J^-$, then for $X \in \mathfrak{m}_J^+$ we have $JIX = -\mathrm{i}IX$ and $IJX = \mathrm{i}IX$, hence $(I \circ J)|_{\mathfrak{m}_J^+} = -(J \circ I)|_{\mathfrak{m}_J^+}$. Since I and J commute with τ and $\mathfrak{m}_J^+ \oplus \mathfrak{m}_J^- = \mathfrak{g}$, we have $I \circ J = -J \circ I$. $\square$

Definition 1.6. Let $\mathfrak{u}$ be a compact Lie algebra. Let I be a $\mathfrak{b}^+$ complex structure as described in Subsection 1.2. We shall say that a complex structure J on $\mathfrak{u}$ *matches* I if J is integrable and $IJ = -JI$. We call I *admissible* if there exists some J, which matches I.

Now we introduce more notation, which will be used throughout the paper. We are interested in hypercomplex structures, so from this moment *we assume that* *we have fixed a* $\mathfrak{b}^+$*-complex structure* I *on* $\mathfrak{u}$ and use the notations from subsection 1.2 and Proposition 1.2. Further, we assume that J is a complex structure on $\mathfrak{u}$, such that $JI = -IJ$, hence $J(\mathfrak{m}_I^+) = \mathfrak{m}_I^-$ (Lemma 1.3).

Definition 1.7. We fix a basis $U_1, \dots, U_m$ of $\mathfrak{h}^+$, then we define $V_k = \tau(U_k) \in \mathfrak{h}^-$ so that we have bases

$$\{E_\alpha \mid \alpha \in \Delta^+\} \cup \{U_1, \dots, U_m\} \text{ of } \mathfrak{m}_I^+;$$
$$\{E_{-\alpha} \mid \alpha \in \Delta^+\} \cup \{V_1, \dots, V_m\} \text{ of } \mathfrak{m}_I^-.$$

For $\alpha \in \Delta^+,\ q = 1, \dots, m$ we decompose the elements JE_α, JU_q as follows:

$$\begin{aligned} JE_\alpha &= \sum_{\beta \in \Delta^+} a_{\beta,\alpha} E_{-\beta} + \sum_{t=1}^m \xi_{t,\alpha} V_t; \\ JU_q &= \sum_{\beta \in \Delta^+} \eta_{\beta,q} E_{-\beta} + \sum_{t=1}^m b_{t,q} V_t. \end{aligned} \tag{11}$$

We introduce matrices with coefficients $a_{\alpha,\beta},\ b_{t,q},\ \xi_{t,\alpha},\ \eta_{\alpha,q}$ respectively:

$$\mathbf{a} \in \mathcal{M}(n \times n); \quad \mathbf{b} \in \mathcal{M}(m \times m); \quad \xi \in \mathcal{M}(m \times n); \quad \eta \in \mathcal{M}(n \times m).$$

Proposition 1.4. *Let* J *be a complex structure on* $\mathfrak{u}$*, such that* $J \circ I = -I \circ J$*. In the bases of Definition* 1.7 *the linear operator* J *has the matrix*

$$\mathsf{J} = \begin{bmatrix} 0 & 0 & \overline{\mathbf{a}} & -\overline{\eta} \\ 0 & 0 & -\overline{\xi} & \overline{\mathbf{b}} \\ \mathbf{a} & \eta & 0 & 0 \\ \xi & \mathbf{b} & 0 & 0 \end{bmatrix}, \qquad \begin{aligned} \overline{\eta}\xi - \overline{\mathbf{a}}\mathbf{a} &= \mathrm{I}_n, & \overline{\mathbf{b}}\xi - \overline{\xi}\mathbf{a} &= 0, \\ \overline{\mathbf{a}}\eta - \overline{\eta}\mathbf{b} &= 0, & \overline{\xi}\eta - \overline{\mathbf{b}}\mathbf{b} &= \mathrm{I}_m. \end{aligned} \tag{12}$$

Conversely, for any choice of $\mathbf{a}, \mathbf{b}, \xi, \eta$ *as in* (12)*, the operator given by the matrix* J *commutes with* τ *and defines a complex structure* J *on* $\mathfrak{u}$*, such that* $J \circ I = -I \circ J$*.*

Proof. Using $J \circ \tau = \tau \circ J$ and (11) we compute

$$JE_{-\alpha} = -\tau(JE_\alpha) = \sum_{\beta \in \Delta^+} \overline{a_{\beta,\alpha}} E_\beta - \sum_{t=1}^m \overline{\xi_{t,\alpha}} U_t,$$
$$JV_q = \tau(JU_q) = -\sum_{\beta \in \Delta^+} \overline{\eta_{\beta,q}} E_\beta + \sum_{t=1}^m \overline{b_{t,q}} U_t.$$

The equalities in (12) mean the same as $\mathsf{J}^2 = -\mathrm{I}$. $\square$

Obviously, many invariant almost hypercomplex structures on $\mathbf{U}$ exist if anf only if $\dim(\mathfrak{u})$ is divisible by 4.

2. Stems

The basic observation of the present paper is that a certain maximal strongly orthogonal subset of the set of positive roots of a reduced root system is crucial to describing all of the homogeneous hypercomplex structures on the compact Lie group and on its coset spaces. This subset of roots is present in the construction of hypercomplex structures by Spindel et al. and Joyce, and it has been constructed earlier by Kostant (unpublished) and by Joseph [Jos]. Kostant called it cascade (cited by other authors — see [LW], [FHW] [Sm]).

In this section we start with a neat definition of this set and derive the properties, which we need to solve our problems. We find it helpful to refer to this set of positive roots the "stem". As pointed out by our referees, some of these properties have been published (to the best of our knowledge for the first time) in [Jos] (compare with [Jos, Sect. 2]).

Throughout this section Δ is a reduced root system, Π is a basis of Δ, and Δ^+ is the corresponding subset of positive roots.

Definition 2.1. For any $\gamma \in \Delta^+$ we denote

$$\Phi^+_\gamma := \{\alpha \in \Delta^+ \mid \gamma - \alpha \in \Delta^+\}.$$

A subset $\Gamma \subset \Delta^+$ will be called a *stem of* Δ^+ if anf only if

$$\Delta^+ = \Gamma \cup \bigcup_{\gamma \in \Gamma} \Phi^+_\gamma, \qquad \text{disjoint union.} \tag{13}$$

If Γ is a stem of Δ^+ and $\gamma \in \Gamma$, we shall call Φ^+_γ the *branch* at γ.

We shall prove the existence and uniqueness of a stem for a reduced root system Δ with a fixed basis Π (hence fixed Δ^+). We also derive the properties of stems needed for applications to the existence and properties of hypercomplex structures. Next we give a list of notations related to a stem.

Definition 2.2. Let Γ be a stem of Δ^+. We denote

$$\begin{gathered} \Phi^-_\gamma = -\Phi^+_\gamma, \quad \Phi_\gamma = \Phi^+_\gamma \cup \Phi^-_\gamma, \\ \Phi^+ = \bigcup_{\gamma \in \Gamma} \Phi^+_\gamma, \quad \Phi^- = -\Phi^+, \quad \Phi = \Phi^+ \cup \Phi^-. \end{gathered} \tag{14}$$

So we have a disjoint union $\Delta^+ = \Gamma \cup \Phi^+$.

2.1. Existence and uniqueness of the stem

We start by recalling some terminology.

Definition 2.3. Let $\Delta = \Delta_1 \cup \cdots \cup \Delta_k$ be the decomposition of Δ into mutually orthogonal, irreducible root subsystems. A root $\gamma \in \Delta_j$ will be called *long root* if $\|\gamma\| \geq \|\alpha\|$ for each $\alpha \in \Delta_j$, and it will be called *maximal root* if γ is the highest root in Δ^+_j. Two roots $\alpha, \beta \in \Delta$ are *strongly orthogonal* if anf only if $\alpha \pm \beta \notin \Delta$.

Proposition 2.1. *Let γ be a maximal root and $\alpha \in \Delta$. Then*

a) *If* $\alpha \neq \pm\gamma$ *we have* $C(\alpha,\gamma) = 0$ *if and only if* γ *is strongly orthogonal to* α*, in particular,* $\langle\alpha,\gamma\rangle = 0$ *if and only if* γ *is strongly orthogonal to* α*, where* $C(\alpha,\gamma)$ *is as in the first equality of* (1)*.*
b) *If* $\alpha \in \Delta^+$ *and* $\alpha \neq \gamma$*, then* $0 \leq C(\alpha,\gamma) \leq 1$*.*
c) *Let* $\alpha \neq \gamma$*. Then* $\alpha \in \Phi_\gamma^+$ *if anf only if* $C(\alpha,\gamma) = 1$*.*

Proof. Claim b) is proved, e.g., in ([Bou, Chap. VI, Sect. 1.8]).

c) Recall that if the γ-series of α is $\{\alpha + n\gamma,\ p \leq n \leq q\}$ (it is an uninterrupted string), then $p + q = -C(\alpha,\gamma)$ (see, e.g., [Bou]).

If $C(\alpha,\gamma) = 1$, then by b) we get $\alpha \in \Delta^+$, and by the maximality of γ it follows that $p = -1$, $q = 0$, therefore, $\gamma - \alpha \in \Delta^+$.

Conversely, if $\gamma - \alpha \in \Delta^+$, then by the maximality of γ we have $\alpha \in \Delta^+$ and $p \leq -1, q \leq 0$. Now b) implies $C(\alpha,\gamma) = 1$.

Claim a) follows from b) and c). □

Proposition 2.2. *Let* γ *be a maximal root. Let* $\alpha \in \Phi_\gamma^+$*,* $\nu \in \Delta$*,* $\nu \neq \pm\gamma$*. Then*
a) *If* $\nu \in \Phi_\gamma^+$ *and* $\alpha + \nu \in \Delta$*, then* $\alpha + \nu = \gamma$*.*
b) *If* $\nu \notin \Phi_\gamma$ *and* $\alpha + \nu \in \Delta$*, then* $\alpha + \nu \in \Phi_\gamma^+$*.*

Proof. If $\alpha, \nu \in \Phi_\gamma^+$, then by Proposition 2.1c), we have

$$C(\alpha + \nu, \gamma) = C(\alpha,\gamma) + C(\nu,\gamma) = 2.$$

Now a) follows from Proposition 2.1b) and $\alpha + \nu \in \Delta^+$.

If $\alpha \in \Phi_\gamma^+$, $\nu \notin \Phi_\gamma \cup \{\pm\gamma\}$, then Proposition 2.1c) implies:

$$C(\alpha + \nu, \gamma) = C(\alpha,\gamma) + C(\nu,\gamma) = 1.$$

If $\alpha + \nu \in \Delta$, then again Proposition 2.1c) ensures $\alpha + \nu \in \Phi_\gamma^+$. □

Proposition 2.3. *Let* $\gamma \in \Delta^+$ *be a maximal root and let* Π *be our fixed basis of* Δ*. The set* $\Phi_\gamma^+ \cap \Pi$ *has at most two elements. Also*

$$\Phi_\gamma^+ = \varnothing \iff \gamma \in \Pi \iff \Phi_\gamma^+ \cap \Pi = \varnothing.$$

Proof. Without loss of generality we may assume that Δ is irreducible and γ is the highest root.

The first equivalence claimed is just the definition of a simple root. Any root $\gamma \in \Delta^+$ has a representation $\gamma = \beta_1 + \beta_2 + \cdots + \beta_k$ where all summands are simple roots and each partial sum is a root (see, e.g., [H, Chap. X, Lemma 3.10]). The last root in the sequence belongs to $\Phi_\gamma^+ \cap \Pi$, whence the second equivalence follows.

Now we treat the case when $\Phi_\gamma^+ \neq \varnothing$. We decompose

$$\gamma = \sum_{\alpha\in\Pi} n_\alpha(\gamma)\alpha, \tag{15}$$

where $n_\alpha(\gamma) \in \mathbb{N}$ for each $\alpha \in \Pi$. By Proposition 2.1c), we have

$$2 = C(\gamma,\gamma) = \sum_{\alpha\in\Pi} n_\alpha C(\alpha,\gamma) = \sum_{\alpha\in\Phi_\gamma^+\cap\Pi} n_\alpha.$$

So if $\xi \in \Phi_\gamma^+ \cap \Pi$, then either $n_\xi = 2$ and $\Phi_\gamma^+ \cap \Pi = \{\xi\}$, or $n_\xi = 1$, and there is exactly one element $\eta \in \Phi_\gamma^+ \cap \Pi$ with $\eta \neq \xi$. □

We shall need the following simple, but important, lemma:

Lemma 2.4. *Let γ be a maximal root in Δ^+ and let us denote $\widetilde{\Delta} = \Delta \setminus (\Phi_\gamma \cup \{\gamma, -\gamma\})$, then*

a) *$\widetilde{\Delta}$ is a reduced root system, $\widetilde{\Pi} = \Pi \cap \widetilde{\Delta}$ is a basis of $\widetilde{\Delta}$ with a subset of positive roots $\widetilde{\Delta}^+ = \Delta^+ \setminus (\Phi_\gamma^+ \cup \{\gamma\}) = \widetilde{\Delta} \cap \Delta^+$.*
b) *If $\alpha, \beta \in \widetilde{\Delta}$ and $\alpha + \beta \in \Delta$, then $\alpha + \beta \in \widetilde{\Delta}$.*
c) *For any $\alpha \in \widetilde{\Delta}^+$ we have $\Phi_\alpha^+ = \{\beta \in \Delta^+ \mid \alpha - \beta \in \Delta^+\} = \{\beta \in \widetilde{\Delta}^+ \mid \alpha - \beta \in \widetilde{\Delta}^+\}$.*

Proof. a) and b) follow from the fact (see Proposition 2.1) that $\widetilde{\Delta} = \{\alpha \in \Delta \mid \langle \alpha, \gamma \rangle = 0\}$.

c) From a) and b) it follows that $\widetilde{\Delta}^+$ consists of exactly those roots in Δ^+, which are in the linear span of $\widetilde{\Pi}$. If $\alpha \in \widetilde{\Delta}^+$, $\beta \in \Delta^+$, and $\alpha - \beta \in \Delta^+$, then by $\alpha = \beta + (\alpha - \beta)$ it follows that β, $(\alpha - \beta)$ lie in the linear span of $\widetilde{\Pi}$ as well, hence β, $(\alpha - \beta) \in \widetilde{\Delta}^+$.[7] □

The construction of a stem is contained in the following:

Proposition 2.5. *There exists a sequence $\Delta = \Delta_1 \supset \Delta_2 \supset \cdots \supset \Delta_d$ of closed root subsystems[8] with bases $\Pi_k = \Pi \cap \Delta_k$, corresponding sets of positive roots $\Delta_k^+ = \Delta^+ \cap \Delta_k$, and maximal roots γ_k of Δ_k^+, $\quad k = 1, \dots, d$, such that we have disjoint unions:*

$$\begin{gathered} \Delta_1^+ = \Phi_{\gamma_1}^+ \cup \{\gamma_1\} \cup \Delta_2^+, \quad \dots, \quad \Delta_{d-1}^+ = \Phi_{\gamma_{d-1}}^+ \cup \{\gamma_{d-1}\} \cup \Delta_d^+, \\ \Delta_d^+ = \Phi_{\gamma_d}^+ \cup \{\gamma_d\}. \end{gathered} \tag{16}$$

The set $\Gamma = \{\gamma_1, \dots, \gamma_d\}$ is a stem of Δ^+.

Proof. The construction goes by induction, taking at each step a maximal root $\gamma_k \in \Delta_k^+$ and defining $\Delta_{k+1} = \gamma_k^\perp = \{\alpha \in \Delta_k \mid \langle \alpha, \gamma_k \rangle = 0\}$. The point is to prove that for each $k = 1, \dots, d$ we have

$$\Phi_{\gamma_k}^+ = \{\alpha \in \Delta^+ \mid \gamma_k - \alpha \in \Delta^+\} = \{\alpha \in \Delta_k^+ \mid \gamma_k - \alpha \in \Delta_k^+\}. \tag{17}$$

The induction step is based on Lemma 2.4c). See the proof of [DT1, Prop. 2.9] for the details. □

Remark 2.1. As pointed out by our referees the idea for the construction in Proposition 2.5 can be seen in [Jos], [LW] and it is first given by Kostant.

We get some improvements of Propositions 2.1, 2.2, and Lemma 2.4.

Corollary 2.6. *Let $\{\gamma_k, \Delta_k,\}_{k=1}^d$ be as in Proposition 2.5. Then:*

a) *If $\alpha, \beta \in \Phi_{\gamma_k}^+$ and $\alpha + \beta \in \Delta$, then $\alpha + \beta = \gamma_k$.*
b) *If $\alpha \in \Phi_{\gamma_k}^+$, $\beta \in \Delta_{k+1}$ and $\alpha + \beta \in \Delta$, then $\alpha + \beta \in \Phi_{\gamma_k}^+$.*
c) *If $\alpha, \beta \in \Delta_k$ and $\alpha + \beta \in \Delta$, then $\alpha + \beta \in \Delta_k$.*

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[7] We thank the editor of the journal who pointed out the argument in the last sentence; our original proof of c) was longer.

[8] A root subsystem $\widetilde{\Delta} \subset \Delta$ is *closed* if the property in Lemma 2.4b) holds.

Proof. c) We apply induction using Lemma 2.4b).

a) We apply Proposition 2.2a) to Δ_k^+ using (17) and c).

b) Now we apply (17) and Proposition 2.2b) to Δ_k^+. $\square$

Corollary 2.7. *If $\gamma \in \Gamma$ and $\alpha \in \Phi_\gamma^+$, then $\alpha(H_\gamma) = C(\alpha, \gamma) = 1$.*

Proof. We apply Proposition 2.1c) recalling that $\gamma = \gamma_k$ is a maximal root in a root subsystem Δ_k and (17). $\square$

Corollary 2.8. *The stem Γ is a maximal strongly orthogonal subset of Δ^+.*

Proof. Let $\gamma_p, \gamma_q \in \Gamma, \quad p < q$, then by construction $\gamma_q \in \Delta_{p+1} \subset \gamma_p^\perp$ and γ_p is maximal in Δ_p^+, whence Proposition 2.1 a) imply strong orthogonality. From the definition of stem (formula (13)) it follows that no root may be strongly orthogonal to all $\gamma \in \Gamma$. $\square$

Now we can prove the following:

Theorem 2.9 (Existence and uniqueness)**.** *Let Δ be a reduced root system, let Π be a basis, and let Δ^+ be the corresponding set of positive roots. There exists exactly one stem of Δ^+.*

Proof. Let $\Gamma = \{\gamma_1, \ldots, \gamma_d\}$ be the stem of Δ^+ constructed in Proposition 2.5. Now we prove uniqueness.

Let Γ' be any stem of Δ^+. We have to prove that $\Gamma = \Gamma'$. It is sufficient to prove $\Gamma \subset \Gamma'$.

By maximality $\gamma_1 + \alpha$ is not a root for any $\alpha \in \Delta^+$, so $\gamma_1 \notin \Phi_\gamma^+$ for any $\gamma \in \Gamma'$, and because of (13) we have $\gamma_1 \in \Gamma'$.

Now assume that for some $k < d$ we have $\{\gamma_1, \ldots, \gamma_k\} \subset \Gamma'$. Assume that $\gamma_{k+1} \notin \Gamma'$. Since Γ' is a stem, there is an element $\delta \in \Gamma'$, such that $\gamma_{k+1} \in \Phi_\delta^+$. Now $\delta \notin \{\gamma_1, \ldots, \gamma_k\}$ (since Γ is a stem and $\gamma_{k+1} \notin \Phi_{\gamma_1}^+ \cup \cdots \cup \Phi_{\gamma_k}^+$). Furthermore, $\delta \notin \Phi_{\gamma_1}^+ \cup \cdots \cup \Phi_{\gamma_k}^+$ because Γ' is a stem. Therefore $\delta, \gamma_{k+1} \in \Delta_{k+1}^+$ and $\delta - \gamma_{k+1} \in \Delta^+$.

By Corollary 2.6, b) it follows that $\delta - \gamma_{k+1} \notin \Phi_{\gamma_i}^+$ for all $i \leq k$; we have also $\delta - \gamma_{k+1} \neq \gamma_i$ for $i \leq k$ (since γ_i is a maximal root in Δ_i), hence $\delta - \gamma_{k+1} \in \Delta_{k+1}^+$. This is impossible by Corollary 2.6, c) and since γ_{k+1} is a maximal root in Δ_{k+1}^+. So $\gamma_{k+1} \in \Gamma'$. $\square$

Example 1. The root system $\Delta = \mathfrak{D}_4$ is irreducible, and fixing Δ^+ we determine a highest root γ_1, while $\Delta_2 = \mathfrak{A}_1 \oplus \mathfrak{A}_1 \oplus \mathfrak{A}_1$, so we have $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$, where the last three roots are all maximal in Δ_2 and may come in any order.

From the construction in Proposition 2.5 we obtain a natural ordering of the stem Γ - there is a sequence $\Delta_1 \supset \Delta_2 \supset \cdots \supset \Delta_d$, which gives the indexation $\Gamma = \{\gamma_1, \ldots, \gamma_d\}$. The ordering is substantially *partial.* As Example 1 shows, each time when Δ_k is not irreducible we have to *choose* γ_{k+1} among the maximal roots of Δ_k^+. We shall give now the formal definition. First we have

Proposition 2.10. *Let Δ be a reduced root system, let Π be a basis, and let Δ^+ be the set of positive roots. Let Γ be the stem of Δ^+.*

For each $\gamma \in \Gamma$, there exists a unique irreducible closed subsystem of roots $\Theta_\gamma \subset \Delta$, such that the set $\Pi_\gamma = \Pi \cap \Theta_\gamma$ is a basis of Θ_γ and γ is the highest root for this basis.

The root subsystems $\{\Theta_\gamma \subset \Delta\}_{\gamma\in\Gamma}$ satisfy the following properties:

a) *The set of positive roots of the basis Π_γ is $\Theta_\gamma^+ = \Theta_\gamma \cap \Delta^+$ and*

$$\Phi_\gamma^+ = \{\alpha \in \Delta^+ \mid \gamma - \alpha \in \Delta^+\} = \{\alpha \in \Theta_\gamma^+ \mid \gamma - \alpha \in \Theta_\gamma^+\}.$$

b) *The stem of Θ_γ^+ is the subset $\Theta_\gamma \cap \Gamma$.*

c) *If $\delta \in \Theta_\gamma \cap \Gamma$, then $\Theta_\delta \subset \Theta_\gamma$.*

Proof. We look at the construction in Proposition 2.5. If $\gamma = \gamma_k$ in the construction there, then γ is a maximal root in the reduced closed root subsystem Δ_k, which means that γ is the highest root of exactly one irreducible component of Δ_k, which we denote by Θ_γ. Since $\Pi \cap \Delta_k$ is a basis of Δ_k, it follows that $\Pi_\gamma = \Pi \cap \Theta_\gamma$ is a basis of Θ_γ. If we decompose $\gamma = \sum_{\alpha\in\Pi} n_\alpha(\gamma)\alpha$, then obviously the basis Π_γ of Θ_γ is $\Pi_\gamma = \{\alpha \in \Pi \mid n_\alpha(\gamma) \neq 0\}$. Now the uniqueness of Θ_γ follows from the fact that it is a closed subsystem.

The verification of the properties a), b) and c) is easy by using Proposition 2.5 and its proof. $\square$

Definition 2.4. Let Δ be a reduced root system, Π be a basis and Δ^+ be the set of positive roots. Let Γ be the stem of Δ^+ and $\gamma, \delta \in \Gamma$. We shall write $\gamma \prec \delta$, if $\delta \in \Theta_\gamma$ (see Proposition 2.10).

In the following text, each time when we use *indexation* of Γ we shall assume that it is *compatible* with the *partial order* $\prec$, that is, when we write $\Gamma = \{\gamma_1, \ldots, \gamma_d\}$, *we assume that*

$$\gamma_k \prec \gamma_j \Longrightarrow k < j. \tag{18}$$

The order in Γ is important in the following useful corollary:

Corollary 2.11. *Let $\Gamma = \{\gamma_1, \ldots, \gamma_d\} \subset \Delta^+$ be the stem of Δ^+. Then for $i = 1, \ldots, d$ and $\alpha \in \Phi_{\gamma_i}^+$ we have*

$$1 \le p < i \Longrightarrow \alpha \pm \gamma_p \notin \Delta, \tag{19}$$

$$\alpha + \gamma_i \notin \Delta, \quad \alpha - \gamma_i = s_{\gamma_i}(\alpha) \in \Phi_{\gamma_i}^- \tag{20}$$

$$i < p \le d \text{ and } \beta \in \Delta_p \text{ and } \alpha + \beta \in \Delta \implies \alpha + \beta \in \Phi_{\gamma_i}^+. \tag{21}$$

Proof. All statements are direct consequences of the construction in Proposition 2.5 and the properties in Corollary 2.6. $\square$

Remark 2.2. Let Δ be irreducible, let Π be our fixed basis and let $\nu \in \mathbf{Aut}_\Pi(\Delta)$ be a diagram automorphism. So $\nu(\Delta^+) = \Delta^+$ and if Γ is the stem of Δ^+, then obviously $\nu(\Gamma)$ is also a stem. By uniqueness (see Theorem 2.9) we have $\nu(\Gamma) = \Gamma$. Also, because ν is an automorphism and $\nu(\Delta^+) = \Delta^+$, we have $\nu(\Phi_\gamma^+) = \Phi_{\nu(\gamma)}^+$. Moreover, if γ is the highest root, then $\nu(\gamma) = \gamma$.

Remark 2.3. A stem Γ is a *strongly orthogonal subset of* Δ^+ *with maximal number of elements*, that is, the number of elements of any strongly orthogonal subset $\Theta \subset \Delta$ is less than or equal to the number of elements of Γ. This fact is easy to prove and also easy to check comparing the list of stems of irreducible root systems with the list of maximal strongly orthogonal subsets of irreducible root systems in [AK]. We shall not use it in this paper.

It makes sense to notice that the converse is not true in general. For example, when $\Delta = \mathfrak{A}_n$, there are many different maximal strongly orthogonal subsets of Δ^+, one of them is the stem. Each of the others is the stem for some other choice of Weyl chamber.

On the other hand, if $\Delta = \mathfrak{C}_n$ then the stem is the set of all long roots in Δ^+. It is the unique strongly orthogonal subset of Δ^+ with maximal number of elements. In this case the same set Γ is the stem of Δ^+ for $n!$ different choices of the positive Weyl chamber. However, the stem Γ and the partial order $\prec$ in it (see Definition 2.4) determine Δ^+ completely. The same holds in general.

Theorem 2.12. *Let* Δ *be a reduced root system, let* Δ^+ *be a system of positive roots, let* Γ *be the stem of* Δ^+ *and let* $\prec$ *be the order in* Γ (*Definition* 2.4)*. Then the couple* $(\Gamma, \prec)$ *determines* Δ^+*. Hence the Weyl group acts simply transitively on the set of couples* $(\Gamma, \prec)$*.*

Proof. Let $\gamma_1, \dots, \gamma_d$ be any indexation of Γ compatible with $\prec$. The theorem follows if we show that:

$$\Delta^+ = \{\alpha \in \Delta \,|\, C(\alpha,\gamma_1) = \ldots = C(\alpha,\gamma_{k-1}) = 0,\ C(\alpha,\gamma_k) > 0 \text{ for some } k \in \{1,\dots,d\}\}.$$

Indeed, if $\alpha \in \pm\Gamma$ the above follows from strong orthogonality. If $\alpha \in \Phi$ by (13) there is exactly one $k \in \{1,\dots,d\}$, such that either $\alpha \in \Phi_k^+$ or $-\alpha \in \Phi_k^+$. By (19) and Proposition 2.1 c) (applied to Δ_j) we see that for $1 \le j < k$ we have $C(\alpha,\gamma_j) = 0$. Then using (20) we see that $\alpha \in \Phi_k^+ \subset \Delta_+$ if anf only if $C(\alpha,\gamma_k) > 0$. $\square$

The stem decomposition (14) determines a useful involution μ on Δ^+ such that $\alpha + \mu(\alpha) = \gamma$ for $\alpha \in \Phi_\gamma^+$. It is defined as follows:

$$\mu(\alpha) = \begin{cases} \alpha & \text{if } \alpha \in \Gamma, \\ -s_\gamma(\alpha) = \gamma - \alpha & \text{if } \alpha \in \Phi_\gamma^+,\ \gamma \in \Gamma. \end{cases} \tag{22}$$

2.2. The stem subalgebra

We fix notation for the Lie algebra entities which correspond to the root system combinatorics of the preceding subsection. So now $\mathfrak{u}$ is a compact Lie algebra, $\mathfrak{g} = \mathfrak{u}^{\mathbb{C}}$ is a reductive Lie algebra, $\mathfrak{h}$ is a τ-invariant Cartan subalgebra, Δ is the root system of $\mathfrak{g}$ w. r. to $\mathfrak{h}$, Π is a basis of Δ. So we have a fixed Δ^+, corresponding Borel subalgebra $\mathfrak{b}^+$, etc. (see Subsection 1.1). By $\Gamma = \{\gamma_1, \dots, \gamma_d\}$ we always denote the stem of Δ^+.

Definition 2.5. Let Γ be the stem of Δ^+. We denote

$$\begin{aligned} \mathcal{V}^{\pm}_{\gamma} &= \mathrm{span}_{\mathbb{C}}\{E_\alpha \mid \alpha \in \Phi^{\pm}_{\gamma}\}, \quad \mathcal{V}_\gamma = \mathcal{V}^+_\gamma \oplus \mathcal{V}^-_\gamma, \quad \mathcal{V}^{\mathfrak{u}}_\gamma = \mathcal{V}_\gamma \cap \mathfrak{u}; \\ \mathcal{V}^{\pm} &= \bigoplus_{\gamma\in\Gamma} \mathcal{V}^{\pm}_\gamma, \quad \mathcal{V} = \mathcal{V}^+ \oplus \mathcal{V}^-, \quad \mathcal{V}^{\mathfrak{u}} = \mathcal{V} \cap \mathfrak{u}, \\ \mathfrak{f} &= \bigoplus_{\gamma\in\Gamma} sl_\gamma(2), \quad \mathfrak{f}^{\pm} = \mathrm{span}_{\mathbb{C}}\{E_{\pm\gamma} \mid \gamma \in \Gamma\}, \quad \mathfrak{f}_{\mathfrak{u}} = \mathfrak{f} \cap \mathfrak{u}; \\ \mathfrak{o} &= \bigcap_{\gamma\in\Gamma} \{H \in \mathfrak{h} \mid \ \gamma(H) = 0\}, \quad \mathfrak{o}_s = \mathfrak{o} \cap \mathfrak{h}_s, \quad \mathfrak{o}_u = \mathfrak{o} \cap \mathfrak{u}. \end{aligned}$$

We shall call the subalgebra $\mathfrak{f}$ defined above *the stem subalgebra.* The corresponding subgroup of $\mathbf{G}_s$ will be denoted by $\mathbf{F}$ and will be called *the stem subgroup.*

If $\Gamma = \{\gamma_i\}_{i=1}^d$, in order to simplify notation we write sometimes

$$H_k = H_{\gamma_k},\ E_k = E_{\gamma_k},\ su_k(2) = su_{\gamma_k}(2),\ \mathcal{V}_k = \mathcal{V}_{\gamma_k}, \quad \text{etc.}$$

In the language of reductive Lie algebras, the stem decomposition (13) gives a decomposition of $\mathfrak{n}^+$ into two-step nilpotent subalgebras.

Definition 2.6. Let $\gamma \in \Gamma$. We denote $\mathfrak{heis}_\gamma = \mathfrak{g}(\gamma) \oplus \mathcal{V}^+_\gamma$. We shall call $\mathfrak{heis}_\gamma$ the γ*-component of* $\mathfrak{n}^+$.

Proposition 2.13. *Let $\gamma \in \Gamma$. Then $\mathfrak{heis}_\gamma$ is a Heisenberg algebra. We have a decomposition*

$$\mathfrak{n}^+ = \mathfrak{heis}_{\gamma_1} \oplus \cdots \oplus \mathfrak{heis}_{\gamma_d}.$$

Proof. By Corollary 2.6a), all brackets in $\mathfrak{heis}_\gamma$ vanish except

$$[E_\alpha, E_{\mu(\alpha)}] = N_{\alpha,\mu(\alpha)} E_\gamma.$$

The direct sum (of vector spaces) follows readily from (13). $\square$

Proposition 2.14. *Let $\gamma \in \Gamma,\ \alpha \in \Phi^+_\gamma,\ \beta = \gamma - \alpha$. Then*

$$|N_{\gamma,-\alpha}| = 1, \quad N_{\gamma,-\alpha} N_{\gamma,-\beta} = -1.$$

Proof. Formula (20) implies that $\alpha + \gamma \notin \Delta$, hence we have $p = 0$ in formula (3), whence the first equality.

The second equality follows from the first equality, the fact that $\|\alpha\| = \|s_\gamma(\alpha)\| = \|\beta\|$ (see Corollary 2.7), and the formula

$$\frac{N_{\alpha,\beta}}{\langle\gamma,\gamma\rangle} = \frac{N_{\beta,-\gamma}}{\langle\alpha,\alpha\rangle} = \frac{N_{-\gamma,\alpha}}{\langle\beta,\beta\rangle}, \qquad \alpha,\beta,\gamma \in \Delta, \quad \alpha+\beta=\gamma, \tag{23}$$

which follows from [H, Lem. 5.1, p. 171].[9] $\square$

Now we return to the stem subalgebra. Because Γ is strongly orthogonal, we have a decomposition $\mathfrak{f}_u = su_1(2) \oplus \cdots \oplus su_d(2)$ into commuting subalgebras. We introduce convenient bases for $\mathfrak{f}_u$.

[9]Our constants $N_{\alpha,\beta}$ are the $N_{\alpha,\beta}$ given there multiplied by $\frac{\sqrt{2\langle\alpha+\beta,\alpha+\beta\rangle}}{\sqrt{\langle\alpha,\alpha\rangle\langle\beta,\beta\rangle}}$.

Definition 2.7. For $\gamma \in \Gamma$ we choose a $\rho_\gamma \in \mathbb{C}$, $|\rho_\gamma| = 1$. We denote $\rho = \{\rho_\gamma \mid \gamma \in \Gamma\}$ and

$$\begin{gathered}
W_\gamma = \tfrac{\mathrm{i}}{2} H_\gamma, \quad X_\gamma(\rho) = \tfrac{1}{2}(\rho_\gamma E_\gamma - \overline{\rho_\gamma} E_{-\gamma}),\\
Y_\gamma(\rho) = X_\gamma(\mathrm{i}\rho_\gamma) = \tfrac{\mathrm{i}}{2}(\rho_\gamma E_\gamma + \overline{\rho_\gamma} E_{-\gamma});\\
\mathfrak{w} = \mathrm{span}_{\mathbb{R}}\{W_\gamma \mid \gamma \in \Gamma\}, \quad \mathfrak{x}(\rho) = \mathrm{span}_{\mathbb{R}}\{X_\gamma(\rho) \mid \gamma \in \Gamma\},\\
\mathfrak{y}(\rho) = \mathrm{span}_{\mathbb{R}}\{Y_\gamma(\rho) \mid \gamma \in \Gamma\};\\
W_\Gamma = \sum_{\gamma \in \Gamma} W_\gamma, \quad X_\Gamma = \sum_{\gamma \in \Gamma} X_\gamma, \quad Y_\Gamma = \sum_{\gamma \in \Gamma} Y_\gamma, \quad E_{\pm\Gamma} = \sum_{\gamma \in \Gamma} E_{\pm\gamma}\\
sl_\Gamma(2,\mathbb{C}) = \mathrm{span}_{\mathbb{C}}\{W_\Gamma, X_\Gamma, Y_\Gamma\}, \quad su_\Gamma(2) = \mathrm{span}_{\mathbb{R}}\{W_\Gamma, X_\Gamma, Y_\Gamma\}.
\end{gathered}$$

The simple subalgebra $sl_\Gamma(2,\mathbb{C}) \subset \mathfrak{g}$ is generated by the semisimple element $H_\Gamma = -2\mathrm{i}W_\Gamma \in \mathfrak{h}$ and the nilpotent elements E_Γ, $E_{-\Gamma}$.

Remark 2.4. We have done our computations and theorems in the presence of ρ (see also Remark 2.6). In the formulas of this section we shall sometimes write X_γ instead of $X_\gamma(\rho)$ or $\mathfrak{x}$ instead of $\mathfrak{x}(\rho)$, etc. We hope that no confusion for the reader comes from this. In any case we remark that the subalgebras $sl_\gamma(2,\mathbb{C})$ and hence the subalgebras $su_\gamma(2) = sl_\gamma(2,\mathbb{C}) \cap \mathfrak{u}$ do not depend on ρ.

Obviously, for any ρ_γ with $|\rho_\gamma| = 1$, the elements $W_\gamma, X_\gamma(\rho), Y_\gamma(\rho)$ span $su_\gamma(2) \subset \mathfrak{f}_\mathfrak{u}$. By strong orthogonality of Γ we have three τ-invariant Cartan subalgebras of $\mathfrak{g}$ (the direct sums are orthogonal):

$$\mathfrak{h}_I = \mathfrak{w}^{\mathbb{C}} \oplus \mathfrak{o}, \quad \mathfrak{h}_K = \mathfrak{x}^{\mathbb{C}} \oplus \mathfrak{o}, \quad \mathfrak{h}_J = \mathfrak{y}^{\mathbb{C}} \oplus \mathfrak{o}. \tag{24}$$

Corollary 2.11 in Lie algebra language is the following:

Proposition 2.15. *If $\gamma \in \Gamma$, then the subspace $\mathcal{V}_\gamma$ is a representation of the stem subalgebra $\mathfrak{f}$ under ad. We denote it by $\mathbf{r}_\gamma : \mathfrak{f} \to sl(\mathcal{V}_\gamma)$. We denote by the same letter the corresponding representation $\mathbf{r}_\gamma : \mathfrak{f}_u \to su(\mathcal{V}_\gamma^u)$.*

(a) *If $\gamma, \delta \in \Gamma$, then the restriction of $\mathbf{r}_\gamma$ to $sl_\delta(2)$ may be nontrivial only if $\gamma \preceq \delta$. Moreover,*
(b) *If $\gamma \neq \delta$, then $\mathcal{V}_\gamma^+$ and $\mathcal{V}_\gamma^-$ are invariant under the ad representation of $sl_\delta(2)$.*
(c) *The action of $sl_\gamma(2)$ on $\mathcal{V}_\gamma$ decomposes into 2-dimensional irreducible components:* $\mathrm{span}_{\mathbb{C}}\{E_\alpha, E_{s_\gamma(\alpha)}\}$, $\alpha \in \Phi_\gamma^+$.

Proof. See Corollary 2.11. □

We shall need several formulas, describing the action of one-parameter subgroups of the stem subgroup $\mathbf{F}$:

Remark 2.5. We have $\tau(X_\gamma) = X_\gamma$ for $\gamma \in \Gamma$, hence $\exp(t\mathrm{ad}X_\gamma)$ preserves $\mathfrak{u}$. Strong orthogonality of Γ implies that if $\gamma, \delta \in \Gamma$, then

$$\exp(t\mathrm{ad}X_\gamma) \circ \exp(s\mathrm{ad}X_\delta) = \exp(s\mathrm{ad}X_\delta) \circ \exp(t\mathrm{ad}X_\gamma).$$

Proposition 2.16. *Let* $\gamma \in \Gamma$, $t \in \mathbb{R}$ *and* $H \in \mathfrak{h}$. *Then*

$$\begin{aligned}\exp(t\mathrm{ad}X_\gamma)(W_\gamma) &= \cos(t)W_\gamma - \sin(t)Y_\gamma,\\ \exp(t\mathrm{ad}X_\gamma)(Y_\gamma) &= \sin(t)W_\gamma + \cos(t)Y_\gamma,\\ \exp(t\mathrm{ad}X_\gamma)(H) &= H + \mathrm{i}\gamma(H)(\sin(t)Y_\gamma + (1-\cos(t))W_\gamma).\end{aligned}$$

Proof. The three formulas follow by induction from the following: $\mathrm{ad}X_\gamma(H) = \mathrm{i}\gamma(H)Y_\gamma$, $\mathrm{ad}X_\gamma(W_\gamma) = -Y_\gamma$, $\mathrm{ad}X_\gamma(Y_\gamma) = W_\gamma$. □

Corollary 2.17. *Let* $\gamma \in \Gamma$. *Then*

$$\exp(t\mathrm{ad}X_\gamma)(E_\gamma) = E_\gamma - \mathrm{i}\overline{\rho_\gamma}((\cos(t)-1)Y_\gamma + \sin(t)W_\gamma).$$

Proof. Follows from $\exp(t\mathrm{ad}X_\gamma)(X_\gamma) = X_\gamma$ and the formula for Y_γ in Proposition 2.16. □

Proposition 2.18. *Let* $\gamma \in \Gamma$, $\alpha \in \Phi^+_\gamma$. *Then*

$$\exp(t\mathrm{ad}X_\gamma)(E_\alpha) = \cos(t/2)E_\alpha + N_{\gamma,-\alpha}\overline{\rho_\gamma}\sin(t/2)E_{s_\gamma(\alpha)}.$$

Proof. By induction, using Proposition 2.14, for $n \geq 0$ we have

$$(\mathrm{ad}X_\gamma)^{2n+1}(E_\alpha) = \frac{(-1)^n}{2^{2n+1}}\overline{\rho_\gamma}N_{\alpha,-\gamma}E_{s_\gamma(\alpha)}, \quad (\mathrm{ad}X_\gamma)^{2n}(E_\alpha) = \frac{(-1)^n}{2^{2n}}E_\alpha,$$

whence the proposition follows by summation of the series. □

2.3. The opposition involution

Definition 2.8. Let $\Gamma = \{\gamma_i\}_{i=1}^d$ be the stem of Δ^+. For $\gamma \in \Gamma$ we denote

$$\phi_\gamma = \phi_\gamma[\rho] = \exp(\pi\mathrm{ad}X_\gamma(\rho)) \in \mathbf{Ad}(\mathfrak{g}).$$

For simplicity we write $\phi_k = \phi_{\gamma_k}[\rho]$ for $k = 1, \ldots, d$ and define

$$\phi = \phi[\rho] = \phi_1 \circ \cdots \circ \phi_d = \exp(\pi\mathrm{ad}X_\Gamma(\rho)).$$

Remark 2.6. It is well known (see, e.g., Tits [T]), that if $\gamma \in \Delta$, $\rho \in \mathbb{C} \setminus \{0\}$ and we define $X_\gamma = \frac{1}{2}(\rho E_\gamma - \frac{1}{\rho}E_{-\gamma})$, then the inner automorphism $\exp(\pi\mathrm{ad}X_\gamma)$ is an extension of s_γ (the reflection along H_γ in $\mathfrak{h}$) to an automorphism of $\mathfrak{g}$. We have $\tau(X_\gamma(\rho)) = X_\gamma(\overline{\rho}^{-1})$, whence $X_\gamma(\rho) \in \mathfrak{u} \iff |\rho| = 1$.

The reflections $\{s_\gamma \mid \gamma \in \Gamma\}$ generate an abelian subgroup $\mathbf{W}_\Gamma \subset \mathbf{W}$, which is isomorphic to $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (d factors).

If we stay in the root system Δ, the point of this subsection is the fact that for any choice of Δ^+, hence of Γ, the product $s_{\gamma_1} \circ \cdots \circ s_{\gamma_d}$ is the opposition element in the Weyl group of Δ. However, for our purposes we need to make an explicit choice of a representative of the coset $s_{\gamma_1} \circ \cdots \circ s_{\gamma_d}$ in the exact sequence (2).

We recall that we denote by the same letter an automorphism $\psi \in \mathbf{N}(\mathfrak{h}) \subset \mathbf{Aut}(\mathfrak{g})$, its action on $\mathfrak{h}$ as an element of the Weyl group, and the conjugate action on $\mathfrak{h}^*$ given by $\psi(\alpha)(H) = \alpha(\psi^{-1}(H))$. In particular, from the third formula of Proposition 2.16 and the last remark, we see that for each γ we have $\phi_\gamma[\rho] \in \mathbf{N}_u(\mathfrak{h})$.

Proposition 2.19. *The automorphism ϕ represents the "opposition involution" in the Weyl group, that is, $\phi(\Delta^+) = \Delta^-$. The opposition involution equals the product $s_{\gamma_1} \circ \cdots \circ s_{\gamma_d}$.*

Proof. From Corollary 2.17 with $t = \pi$ it follows that $\phi_\gamma(E_\gamma) = -\overline{\rho}_\gamma^2 E_{-\gamma}$ and from Proposition 2.18 we obtain

$$\phi_\gamma(E_\alpha) = \overline{\rho_\gamma} N_{\gamma,-\alpha} E_{s_\gamma(\alpha)}, \quad \gamma \in \Gamma,\, \alpha \in \Phi_\gamma^+, \tag{25}$$

hence $\phi_\gamma(\mathfrak{g}(\gamma)) = \mathfrak{g}(-\gamma)$, $\phi_\gamma(\mathcal{V}_\gamma^+) = \mathcal{V}_\gamma^-$. The properties of Γ from Corollaries 2.11 and 2.8 imply $\phi_\gamma(\mathcal{V}_\delta^+) = \mathcal{V}_\delta^+$, $\phi_\gamma(E_\delta) = E_\delta$, $s_\gamma(\Phi_\delta^+) = \Phi_\delta^+$, $s_\gamma(\delta) = \delta$ for $\gamma \neq \delta$. It follows that $\phi(\mathfrak{n}^+) = \mathfrak{n}^-$, $s_{\gamma_1} \circ \cdots \circ s_{\gamma_d}(\Delta^+) = \Delta^-$. The proposition is proved. □

Proposition 2.20. *We have*

$$\mathfrak{o} = \{H \in \mathfrak{h} \mid \phi(H) = H\}; \quad \mathfrak{w}^{\mathbb{C}} = \{H \in \mathfrak{h} \mid \phi(H) = -H\}.$$

Proof. From the third formula in Proposition 2.16 and strong orthogonality of Γ ($\gamma(H_\delta) = 0$ if $\gamma \neq \delta$) it follows that

$$\phi(H) = H - \sum_{\gamma \in \Gamma} \gamma(H) H_\gamma \qquad H \in \mathfrak{h}. \tag{26}$$

Recalling the definitions of $\mathfrak{o}$ (in Definition 2.5) and of $\mathfrak{w}$ (in Definition 2.7) we see that (26) implies the proposition. □

Definition 2.9. Let θ be the contragredience automorphism of $\mathfrak{g}$ with respect to $\mathfrak{h}$(see (5)), and let $\phi \in \mathbf{W}$ be the opposition automorphism of $\mathfrak{g}$ with respect to $\mathfrak{h}$ (see Definition 2.8). We denote

$$\star = \theta \circ \phi = \phi \circ \theta \in \mathbf{Aut}(\mathfrak{h}).$$

We denote by the same symbol the adjoint involution $\star \in \mathbf{Aut}(\mathfrak{h}^*)$.

It is well known that $\star \in \mathbf{Aut}_\Pi(\Delta)$ and that in a reduced irreducible Δ the involution $\star$ is nontrivial only when $\Delta = \mathfrak{A}_n, n > 1, \Delta = \mathfrak{D}_{2n+1}, n \geq 1, \Delta = \mathfrak{E}_6$. We show now (improving Proposition 2.3) how $\star$ determines the number of elements of the stem:

Proposition 2.21. *Let $\gamma \in \Gamma$. Then*

a) $\star\gamma = \gamma$, $\star(\Phi_\gamma)^+ = \Phi_\gamma^+$, $\star(\Phi_\gamma^+ \cap \Pi) = \Phi_\gamma^+ \cap \Pi$;
b) *we have a trichotomy:*
   i) $\gamma \in \Pi$ *and* $\Phi_\gamma^+ = \varnothing$;
   ii) $\Phi_\gamma^+ \cap \Pi$ *has exactly one element;*
   iii) $\Phi_\gamma^+ \cap \Pi$ *has exactly two elements.*
c) *If* $\alpha, \beta \in \Phi_\gamma^+ \cap \Pi$ *and* $\alpha \neq \beta$, *then* $\star\alpha = \beta$.

Proof. To check a) see the proof of Proposition 2.19 (after (25)).

The trichotomy b) is Proposition 2.3 in the case when γ is a maximal root of Δ^+. To prove it for any $\gamma \in \Gamma$ we apply Proposition 2.3 to the closed root subsystem Θ_γ (see Proposition 2.10).

In c) we will use the order from Definition 2.4 (see also (18)). Let $\Gamma = \{\gamma_1, \dots, \gamma_d\}$ and let $\gamma = \gamma_k \in \Gamma, \Phi_k^+ \cap \Pi = \{\alpha, \beta\}$, $\alpha \neq \beta$.

Any $\zeta \in \Delta$ decomposes as follows:

$$\zeta = \sum_{\lambda \in \Pi} n_\lambda(\zeta)\lambda. \tag{27}$$

Obviously $n_\alpha(\alpha) = 1$. In order to prove $\star\alpha = \beta$ it is sufficient to prove that $n_\alpha(\star\alpha) = 0$ (by a). We proceed to do this.

By (19) the reflection s_{γ_j} leaves Φ_k pointwise fixed for $j = 1, \dots, k-1$, therefore $\star\alpha = -s_{\gamma_d} \circ \cdots \circ s_{\gamma_1}(\alpha) = -s_{\gamma_d} \circ \cdots \circ s_{\gamma_k}(\alpha)$.

Denote $\zeta = s_{\gamma_k}(\alpha) \in \Phi_k$. Then $n_\alpha(\zeta) = n_\alpha(\alpha) - n_\alpha(\gamma_k) = 0$, since $n_\alpha(\gamma_k) = 1$ (see the proof of Proposition 2.3).

The proposition will be proved if we show that for any $\zeta \in \Phi_k$ and $j > k$ we have $n_\alpha(\zeta) = n_\alpha(s_{\gamma_j}\zeta)$. The last equation follows obviously from the fact that for $j > k$ we have $n_\alpha(\gamma_j) = 0$. Indeed, by definition (see Proposition 2.5) of γ_j as maximal root of Δ_j we know that $n_\lambda(\gamma_j) \neq 0$ only for $\lambda \in \Pi \cap \Delta_j$ (see (27)). By the definition of stem, we have $\alpha \notin \Delta_j$. $\square$

From Proposition 2.21 and the definition of stem we get

Corollary 2.22. *If* Π *is a basis of* Δ *and* Γ *is the stem, then*

$$\#(\Gamma) = \#(\Pi/\{\mathrm{id}, \star\}).$$

In particular, $\frac{1}{2}\mathrm{rank}(\mathfrak{g}) \leq \#(\Gamma) \leq \mathrm{rank}(\mathfrak{g})$ *for any semisimple* $\mathfrak{g}$.

Example 2. If $\Delta = \mathfrak{A}_n$, then $d = [(n+1)/2]$; if $\Delta = \mathfrak{C}_n$, then $d = n$.

Corollary 2.23. *Denote* $\widetilde{\Gamma} = \{\gamma \in \Gamma \mid \Phi_\gamma^+ \cap \Pi = \{\alpha_\gamma, \beta_\gamma\}, \alpha_\gamma \neq \beta_\gamma\}$. *Then*

$$\Gamma^\perp = \mathrm{span}\{\alpha_\gamma - \beta_\gamma \mid \gamma \in \widetilde{\Gamma}\}; \quad \star(\zeta) = -\zeta,\ \zeta \in \Gamma^\perp.$$

Proof. Obviously the set $\{\alpha_\gamma - \beta_\gamma; \gamma \in \widetilde{\Gamma}\}$ is linearly independent.

For any $\gamma \in \widetilde{\Gamma}$ and $\delta \in \Gamma$ by Proposition 2.21 we have $\star(\delta) = \delta$ and $\star(\alpha_\gamma) = \beta_\gamma$, hence, since $\star$ is isometry, $\langle \alpha_\gamma, \delta \rangle = \langle \beta_\gamma, \delta \rangle$. Therefore $\langle \alpha_\gamma - \beta_\gamma, \delta \rangle = 0$.

By Proposition 2.3 it follows that $\#(\Gamma) + \#(\widetilde{\Gamma}) = \#(\Pi)$ and the first formula follows. The rest follows from $\star(\alpha_\gamma) = \beta_\gamma, \gamma \in \widetilde{\Gamma}$. $\square$

2.4. The Cayley transform

We define an automorphism which is a square root of the opposition involution ϕ from the previous subsection; we use all notation introduced there.

Definition 2.10. For $p = 1, \ldots, d$ we denote $X_p = X_{\gamma_p}(\rho)$ and

$$\mathbf{c}_p = \mathbf{c}_p[\rho] = \exp\left(\frac{\pi}{2}\mathrm{ad}X_p(\rho)\right) \in \mathbf{Ad}(\mathfrak{g}).$$

We call the following automorphism

$$\mathbf{c} = \mathbf{c}[\rho] = \mathbf{c}_1 \circ \mathbf{c}_2 \circ \ldots \circ \mathbf{c}_d = \exp\left(\frac{\pi}{2}\mathrm{ad}X_\Gamma(\rho)\right)$$

the Cayley transform (with respect to the parameter ρ).

Remark 2.7. By Remark 2.5 we conclude that for $p = 1, \ldots, d$ we have $\mathbf{c}_p \circ \tau = \tau \circ \mathbf{c}_p$, so all $\mathbf{c}_p$ and $\mathbf{c}$ are automorphisms of $\mathfrak{u}$. Also from Remark 2.5, it follows that for $i, j = 1, \ldots, d$ we have $\mathbf{c}_i \circ \mathbf{c}_j = \mathbf{c}_j \circ \mathbf{c}_i$, whence the definition of the automorphism $\mathbf{c}$ does not depend on the order of the factors; and that is why we may define it as an exponent of one element, namely, $(\pi/2)\mathrm{ad}X_\Gamma$.

For each $k = 1, \ldots, d$ we have $\mathbf{c}_k^2 = \phi_k$ (see Definition 2.8), whence $\mathbf{c}^2 = \phi$, i.e., $\mathbf{c}$ is a square root of the opposition involution.

We shall need an explicit description of the action of $\mathbf{c}$ on $\mathfrak{f} \oplus \mathfrak{o}$ (Definition 2.5).

Proposition 2.24. *If $\gamma \in \Gamma$, then*

$$\mathbf{c}(X_\gamma) = X_\gamma, \quad \mathbf{c}(Y_\gamma) = W_\gamma, \quad \mathbf{c}(W_\gamma) = -Y_\gamma.$$

Proof. We use strong orthogonality of Γ and Proposition 2.16. □

Proposition 2.25. *We have*

$$\mathfrak{o} = \{H \in \mathfrak{h} \mid \mathbf{c}(H) = H\}. \tag{28}$$

Proof. From the third formula in Proposition 2.16 and strong orthogonality of Γ ($\gamma(W_\delta) = 0$ if $\gamma \neq \delta$) we compute for any $H \in \mathfrak{h}$:

$$\mathbf{c}^{\pm}(H) = H + \mathrm{i}\sum_{j=1}^{d} \gamma_j(H)(W_j \pm Y_j).$$

These formulas and Proposition 2.24 imply (28). □

Remark 2.8. We may define

$$\mathbf{c}_{\mathfrak{y}} = \exp\left(\frac{\pi}{2}\mathrm{ad}Y_\Gamma\right), \quad \mathbf{c}_{\mathfrak{w}} = \exp\left(\frac{\pi}{2}\mathrm{ad}W_\Gamma\right).$$

Writing for the sake of symmetry $\mathbf{c}_{\mathfrak{x}}$ for the Cayley transform defined at the beginning of this subsection, we have (Proposition 2.24):

$$\mathbf{c}_{\mathfrak{w}}(\mathfrak{w}) = \mathfrak{w}, \mathbf{c}_{\mathfrak{w}}(\mathfrak{x}) = \mathfrak{y}; \quad \mathbf{c}_{\mathfrak{x}}(\mathfrak{x}) = \mathfrak{x}, \mathbf{c}_{\mathfrak{x}}(\mathfrak{y}) = \mathfrak{w}; \quad \mathbf{c}_{\mathfrak{y}}(\mathfrak{y}) = \mathfrak{y}, \mathbf{c}_{\mathfrak{y}}(\mathfrak{w}) = \mathfrak{x}.$$

The elements $\mathbf{c}_{\mathfrak{x}}^2$ and $\mathbf{c}_{\mathfrak{y}}^2$ represent the opposition involution with respect to the Cartan subalgebra $\mathfrak{h}_I$; also $\mathbf{c}_{\mathfrak{w}}^2$ and $\mathbf{c}_{\mathfrak{y}}^2$ represent the opposition involution with respect to the Cartan subalgebra $\mathfrak{x}^{\mathbb{C}} \oplus \mathfrak{o}$, etc.

In order to prove the statement in this remark, there is no need for new computations. Actually we know that putting $\mathrm{i}\rho$ in the place of ρ we change X_γ to Y_γ and Y_γ goes to $-X_\gamma$ in all formulas of this section. The corresponding statements about $\mathbf{c}_{\mathfrak{w}}$ are easy to check.

3. Existence of a hypercomplex structure

Now we use the root combinatorics of the stem to find sufficient conditions for admissibility of a $\mathfrak{b}^+$ complex structure on $\mathfrak{u}$. We present our candidate for a match to I.

Definition 3.1. Let I be a $\mathfrak{b}^+$ complex structure on $\mathfrak{u}$. We denote

$$J = I_{\mathbf{c}} = \mathbf{c} \circ I \circ \mathbf{c}^{-1},$$

where $\mathbf{c}$ is the Cayley transform defined in Definition 2.10.

By definition, J is equivalent to I, so J is an integrable $\mathbf{c}(\mathfrak{b}^+)$ complex structure on $\mathfrak{u}$. We obviously have $\mathfrak{m}_J^+ = \mathbf{c}(\mathfrak{m}_I^+)$. Proposition 2.24 and formula (28) imply $\mathfrak{h}_J = \mathbf{c}(\mathfrak{h}_I) = \mathfrak{y}^{\mathbb{C}} \oplus \mathfrak{o}$. The notations $\mathfrak{o}$, $\mathfrak{f}$, etc., are explained in Definition 2.5. Recall also that $d = \#(\Gamma)$.

In this section we give a necessary and sufficient condition for $IJ = -JI$.

Remark 3.1. By the definition of Φ_γ^+ we see that $\dim(\mathcal{V}_\gamma^+)$ is even, whence $\dim(\mathcal{V}_\gamma)$ is divisible by 4, whence $\dim_{\mathbb{C}}(\mathcal{V})$ is divisible by 4. So $\dim_{\mathbb{R}}(\mathcal{V}^u)$ is divisible by 4.

From the decomposition $\mathfrak{u} = \mathcal{V}^u \oplus \mathfrak{f}_u \oplus \mathfrak{o}_u$ we see that $\dim(\mathfrak{u})$ is divisible by 4 if and only if $3d + \dim(\mathfrak{o}_u)$ is divisible by 4. So in the following we shall always assume (sometimes implicitly) that $\dim(\mathfrak{o}_u) = d + 2p$, where p is some even integer.

From Proposition 2.15 (or Corollary 2.11) it follows that $\mathcal{V}_\gamma$ is $\mathbf{c}$-stable (see also Proposiiton 2.18).

3.1. The structure J on $\mathcal{V}$

First we are going to prove that $J(\mathcal{V}_\gamma^+) = \mathcal{V}_\gamma^-$, whence $IJ = -JI$ holds on $\mathcal{V}$ without any further conditions. The notations X_γ, Y_γ, etc., are explained in Definition 2.7. We begin with

Proposition 3.1. *If $V \in \mathcal{V}_\gamma$, then $IV = 2\mathrm{ad}W_\gamma(V)$.*

Proof. If $\alpha \in \Phi_\gamma^+$, then by Corollary 2.7 we have

$$[2W_\gamma, E_\alpha] = \mathrm{i}\alpha(H_\gamma)E_\alpha = \mathrm{i}C(\alpha,\gamma)E_\alpha = \mathrm{i}E_\alpha,$$
$$[2W_\gamma, E_{-\alpha}] = -\mathrm{i}\alpha(H_\gamma)E_{-\alpha} = -\mathrm{i}E_{-\alpha}.$$

The proposition is proved. □

We use Proposition 3.1 to make the next step:

Proposition 3.2. *If $V \in \mathcal{V}_\gamma$, then $JV = -2\mathrm{ad}Y_\gamma(V) = -\phi_\gamma[\mathrm{i}\rho](V)$.*

Proof. If $\alpha \in \Phi_\gamma^+$, then using Propositions 3.1 we have

$$JE_\alpha = \mathbf{c}I\mathbf{c}^{-1}E_\alpha = 2\mathbf{c}[W_\gamma, \mathbf{c}^{-1}E_\alpha] = 2[\mathbf{c}W_\gamma, E_\alpha].$$

By Proposition 2.24 and formula (25) (using $Y_\gamma(\rho) = X_\gamma(\mathrm{i}\rho)$) we get

$$JE_\alpha = -[2Y_\gamma, E_\alpha] = \mathrm{i}\overline{\rho_\gamma}N_{\gamma,-\alpha}E_{s_\gamma(\alpha)} = -\phi_\gamma[\mathrm{i}\rho](E_\alpha). \tag{29}$$

Further $JE_{-\alpha} = \tau(J\tau E_{-\alpha}) = \tau([-2Y_\gamma, -E_\alpha]) = [-2Y_\gamma, E_{-\alpha}]$, whence the proposition follows. □

From (29) (recall that $s_\gamma(\alpha) = \alpha - \gamma \in \Phi_\gamma^-$ for $\alpha \in \Phi_\gamma^+$) it follows:

Corollary 3.3. *For each $\gamma \in \Gamma$ we have $J(\mathcal{V}_\gamma^+) = \mathcal{V}_\gamma^-$.*

At the end, from Corollary 3.3 (as in Lemma 1.3) we obtain

Corollary 3.4. *For each $V \in \mathcal{V}$ we have $IJV = -JIV$.*

Remark 3.2. In particular, we have proved that for each $V \in \mathcal{V}_\gamma$ we have $\mathrm{ad}W_\gamma \mathrm{ad}Y_\gamma(V) = -\mathrm{ad}Y_\gamma \mathrm{ad}W_\gamma(V)$, whence

$$\mathrm{ad}W_\gamma \mathrm{ad}Y_\gamma(V) = \tfrac{1}{2}[\mathrm{ad}W_\gamma, \mathrm{ad}Y_\gamma](V) = \tfrac{1}{2}\mathrm{ad}[W_\gamma, Y_\gamma](V) = -\tfrac{1}{2}\mathrm{ad}X_\gamma(V).$$

Denoting as usual $K = IJ$, for each $V \in \mathcal{V}_\gamma$ we have

$$KV = IJV = 2\mathrm{ad}X_\gamma(V) = \phi_\gamma(V). \tag{30}$$

3.2. The complex structure J on $\mathfrak{f} \oplus \mathfrak{o}$

As explained in Subsection 1.2 we have some freedom in defining a $\mathfrak{b}^+$ complex structure I on the Cartan subalgebra $\mathfrak{h}$. While $IX_\gamma = Y_\gamma$ is fixed by the convention that $IX = \mathrm{i}X \quad (IX = -\mathrm{i}X)$ on $\mathfrak{n}^+ \quad (\mathfrak{n}^-)$, we have substantial freedom choosing the elements $IW_\gamma \in \mathfrak{h}_u$. At the end, it turns out that the necessary and sufficient condition for admissibility of I is a condition on $I(\mathfrak{w})$.

Definition 3.2. Let I be a $\mathfrak{b}^+$ complex structure on $\mathfrak{u}$. For each $\gamma \in \Gamma$ we denote:

$$Z_\gamma = IW_\gamma, \quad \mathfrak{z} = \mathfrak{z}_I = I(\mathfrak{w}) = \mathrm{span}_\mathbb{R}\{Z_\gamma \mid \gamma \in \Gamma\} \subset \mathfrak{h}_u.$$

We call the subalgebra $\mathfrak{e} = \mathfrak{e}_I = \mathfrak{f}_u + \mathfrak{z}_I$ the *extended stem subalgebra.*

First we compute the operator $J = I_\mathbf{c}$ on $\mathfrak{w}$.

Proposition 3.5. *For each $\gamma \in \Gamma$ we have $JW_\gamma = -X_\gamma$.*

Proof. By Proposition 2.24 we compute $JX_\gamma = \mathbf{c} \circ I \circ \mathbf{c}^{-1} X_\gamma = \mathbf{c}Y_\gamma = W_\gamma$. □

Proposition 3.6. *Let I be a $\mathfrak{b}^+$ complex structure on $\mathfrak{u}$ and let $J = I_\mathbf{c}$. The following three conditions are equivalent:*

a) $\mathfrak{z} \subset \mathfrak{o}$.[10]
b) *For each $\gamma \in \Gamma$ we have $JZ_\gamma = Y_\gamma$.*
c) *For $X \in \mathfrak{f}_u + \mathfrak{z}$ we have $IJX = -JIX$.*

Proof. a) $\Rightarrow$ b). By (28) and Proposition 2.24 we have

$$JZ_\gamma = \mathbf{c} \circ I \circ \mathbf{c}^{-1} Z_\gamma = \mathbf{c} \circ IZ_\gamma = -\mathbf{c}W_\gamma = Y_\gamma.$$

b) $\Rightarrow$ c) We use the definition of I and Proposition 3.5. to compute

$$\begin{aligned} IJW_\gamma &= -IX_\gamma = -Y_\gamma = -JZ_\gamma = -JIW_\gamma; \\ IJZ_\gamma &= IY_\gamma = -X_\gamma = JW_\gamma = -JIZ_\gamma; \\ IJX_\gamma &= IW_\gamma = Z_\gamma = -JY_\gamma = -JIX_\gamma; \\ IJY_\gamma &= -IZ_\gamma = W_\gamma = JX_\gamma = -JIY_\gamma. \end{aligned}$$

c) $\Rightarrow$ a). From Propositions 3.5 and 2.24 we get

$$Z_\gamma = IW_\gamma = IJX_\gamma = -JIX_\gamma = -JY_\gamma = \mathbf{c}(IW_\gamma) = \mathbf{c}(Z_\gamma).$$

So by (28) we have $Z_\gamma \in \mathfrak{o}$. □

We collect the above results in the following:

[10] Recall that by definition $\mathfrak{o} = \cap_{\gamma \in \Gamma} \mathfrak{ker}(\gamma)$.

Corollary 3.7. *Let I be a $\mathfrak{b}^+$ complex structure on $\mathfrak{u}$ and let $J = I_{\mathbf{c}}$. Then $\mathfrak{z} \subset \mathfrak{o}$ if and only if*

$$IX_\gamma = Y_\gamma,\ IW_\gamma = Z_\gamma, \quad JX_\gamma = W_\gamma,\ JZ_\gamma = Y_\gamma,\ \gamma \in \Gamma.$$

Proof. The first three formulas obviously follow from the definition of I, Z_γ and $J = I_{\mathbf{c}}$ (see Propositions 3.5). The last formula above was proved in Proposition 3.6 to be equivalent to $\mathfrak{z} \subset \mathfrak{o}$. □

We also have obviously:

Corollary 3.8. *If $\mathfrak{z} \subset \mathfrak{o}$, then the extended stem subalgebra $\mathfrak{e} = \mathfrak{f}_u \oplus \mathfrak{z}$ is invariant under I, J.*

We are now ready to prove:

Proposition 3.9. *If $I(\mathfrak{w}) = \mathfrak{o}_u$, then $J = I_{\mathbf{c}}$ matches I.*

Proof. In any case J is integrable, because it is equivalent to I.

We have $\mathfrak{g} = \mathcal{V} \oplus \mathfrak{f} \oplus \mathfrak{o}$ and under our assumption we have $\mathfrak{f} \oplus \mathfrak{o} = \mathfrak{e}^{\mathbb{C}}$. From Corollaries 3.3 and 3.7 respectively we get

$$J(\mathcal{V}^u) = \mathcal{V}^u, \quad J(\mathfrak{e}) = \mathfrak{e}.$$

Now from Corollaries 3.4 and 3.6 we have $IJ = -JI$, on both direct summands. The theorem is proved. □

Remark 3.3. It is easy to see that the condition $I(\mathfrak{w}) = \mathfrak{o}_u$ is equivalent to $2d = \operatorname{rank}(\mathfrak{u})$ and $I(\mathfrak{w}) \subset \mathfrak{o}_u$, which implies that $\dim(\mathfrak{u})$ is divisible by 4 (see Remark 3.1). When $2d = \operatorname{rank}(\mathfrak{g})$, any complex structure I on $\mathfrak{h}_u$ with $I(\mathfrak{w}) = \mathfrak{o}_u$ extends in an obvious way to a $\mathfrak{b}^+$ complex structure on $\mathfrak{u}$. Thus by Proposition 3.9, if $\mathbf{U}$ is a compact Lie group such that $2d = \operatorname{rank}(\mathfrak{u})$, then $\mathbf{U}$ carries a left invariant hypercomplex structure.

In order to state the sufficient condition in the general case we need some more notation.

Definition 3.3. Let Γ be the stem of Δ^+ and let $\mathfrak{z} \subset \mathfrak{o}$. We denote

$$P_\gamma = W_\gamma - \mathrm{i}Z_\gamma, \quad Q_\gamma = W_\gamma + \mathrm{i}Z_\gamma = \tau P_\gamma, \qquad \gamma \in \Gamma;$$
$$\mathfrak{v} = (\mathfrak{w} \oplus \mathfrak{z})^{\mathbb{C}}, \quad \mathfrak{v}^+ = \mathfrak{v} \cap \mathfrak{h}^+, \quad \mathfrak{v}^- = \mathfrak{v} \cap \mathfrak{h}^-, \quad \mathfrak{v}_u = \mathfrak{w} \oplus \mathfrak{z};$$
$$\mathfrak{j}^+ = \mathfrak{o} \cap \mathfrak{h}^+, \quad \mathfrak{j}^- = \mathfrak{o} \cap \mathfrak{h}^-, \quad \mathfrak{j} = \mathfrak{j}^+ \oplus \mathfrak{j}^-, \quad \mathfrak{j}_u = \mathfrak{j} \cap \mathfrak{u}.$$

Proposition 3.10. *Let $I(\mathfrak{w}) \subset \mathfrak{o}$. Then*

a) $I(\mathfrak{j}) = \mathfrak{j}$, $I(\mathfrak{j}_u) = \mathfrak{j}_u$.
b) *If $H \in \mathfrak{j}$, then $I_{\mathbf{c}}H = IH$.*

Proof. If $H \in \mathfrak{j}$, then $H = A + B, \quad A \in \mathfrak{j}^+,\ B \in \mathfrak{j}^-$. Thus $IH = \mathrm{i}A - \mathrm{i}B \in \mathfrak{j}$. For the second equality, note that $\mathfrak{u}$ is also invariant under I. So a) is proved.

Because $\mathfrak{j} \subset \mathfrak{o}$, for $H \in \mathfrak{j}$ (28) implies $\mathbf{c}^{-1}H = H$, then by a) of this proposition we have $I\mathbf{c}^{-1}H = IH \in \mathfrak{j}$ and again by (28) we have $\mathbf{c}IH = IH$. Thus, item b) is proved. □

Proposition 3.11. *Let $I(\mathfrak{w}) \subset \mathfrak{o}$. Then*

$$\mathfrak{v}^+ = \mathrm{span}_{\mathbb{C}}\{P_\gamma \mid \gamma \in \Gamma\}, \quad \mathfrak{v}^- = \mathrm{span}_{\mathbb{C}}\{Q_\gamma \mid \gamma \in \Gamma\}.$$

We have

$$\mathfrak{h} = \mathfrak{v} \oplus \mathfrak{j}, \quad \mathfrak{h}^+ = \mathfrak{v}^+ \oplus \mathfrak{j}^+, \quad \mathfrak{h}^- = \mathfrak{v}^- \oplus \mathfrak{j}^-, \quad \mathfrak{o}_u = \mathfrak{z} \oplus \mathfrak{j}_u. \tag{31}$$

In particular, $\mathrm{rank}(\mathfrak{g}) = 2d + 2p$*, where p is some even nonnegative integer and $dim_{\mathbb{R}}(\mathfrak{j}_u) = 2p$.*

Proof. The condition $I(\mathfrak{w}) \subset \mathfrak{o}$ implies $\gamma_k(P_j) = \mathrm{i}\delta_{k,j}$, so $P_1, \dots, P_d$ is a basis of $\mathfrak{v}^+ \subset \mathfrak{h}^+$. On the other hand, $\mathfrak{j}^+ = \{H \in \mathfrak{h}^+ \mid \gamma_1(H) = \cdots = \gamma_d(H) = 0\}$, thus $\mathfrak{h}^+ = \mathfrak{v}^+ \oplus \mathfrak{j}^+$. In the same way $\mathfrak{h}^- = \mathfrak{v}^- \oplus \mathfrak{j}^-$.

Now, in order to prove $\mathfrak{h} = \mathfrak{v} \oplus \mathfrak{j}$, we have to show only that $\mathfrak{v} \cap \mathfrak{j} = \{0\}$. Let $X \in \mathfrak{v} \cap \mathfrak{j}$. We may decompose $X = X^+ + X^-$, where $X^+ \in \mathfrak{v}^+, X^- \in \mathfrak{v}^-$. We have $I(\mathfrak{j}) = \mathfrak{j}$ (Proposition 3.10a)), hence $I(X) = \mathrm{i}X^+ - \mathrm{i}X^- \in \mathfrak{j}$. The inclusions $X^+ + X^- \in \mathfrak{j}, \mathrm{i}X^+ - \mathrm{i}X^- \in \mathfrak{j}$ imply $X^+ \in \mathfrak{j}$, $X^- \in \mathfrak{j}$, therefore $X^+ \in \mathfrak{j}^+ \cap \mathfrak{v}^+$, $X^- \in \mathfrak{j}^- \cap \mathfrak{v}^-$. Now from $\mathfrak{v}^+ \cap \mathfrak{j}^+ = \mathfrak{v}^- \cap \mathfrak{j}^- = \{0\}$ we obtain $X^+ = X^- = 0$.

Now the last statement is clear (recall Remark 3.1). □

We have the following important

Remark 3.4. Note that, when $\mathfrak{z} \subset \mathfrak{o}$, the extended stem subalgebra $\mathfrak{e}$ (see Definition 3.2) is closed under the action of $I, I_{\mathbf{c}}$. The corresponding subgroup $\mathbf{E}_u$ may *not be a closed subgroup* of $\mathbf{U}$. If $\mathbf{E}_u$ is a closed subgroup, which is an arithmetic condition on the Z_γ (vacuously fulfilled when[11] $\mathrm{rank}(\mathfrak{u}) = 2d$), then $\mathbf{E}_u$ is a hypercomplex submanifold of $\mathbf{U}$.

Obviously also $\mathfrak{e}^{\mathbb{C}} = \mathfrak{f}^+ \oplus \mathfrak{f}^- \oplus \mathfrak{v}$ is always a subalgebra of $\mathfrak{g}$ invariant under the action of I, J (the complexified extended stem subalgebra).

The subspace $\mathfrak{n}^+ \oplus \mathfrak{n}^- \oplus \mathfrak{v}$ is also invariant under the action of I, J, but is *not* obliged to be *a subalgebra*. An example is $\mathfrak{g} = sl(3, \mathbb{C}) \oplus \mathfrak{c}$ where $\mathfrak{c} \cong \mathbb{C}^4$. Then $\Gamma = \{\gamma\}$, we may take $IW_\gamma \in \mathfrak{c}_u$ so $\mathfrak{v} = \mathrm{span}_{\mathbb{C}}\{P_\gamma, Q_\gamma\}$ does not contain $\mathfrak{h}_s$

Theorem 3.12. *Let $\mathfrak{u}$ be a compact Lie algebra, whose dimension is divisible by 4, and let I be a $\mathfrak{b}^+$ complex structure on $\mathfrak{u}$. If $\mathfrak{z} = I(\mathfrak{w}) \subset \mathfrak{o}_u$, then I is admissible.*

Proof. From Proposition 3.11 we have a decomposition (of real vector spaces) $\mathfrak{u} = \mathcal{V}^u \oplus \mathfrak{f}_u \oplus \mathfrak{z} \oplus \mathfrak{j}_u$ and (recall also that $\mathfrak{m}_I^+ = \mathfrak{h}^+ \oplus \mathfrak{n}^+$)

$$\mathfrak{m}_I^+ = \mathfrak{n}^+ \oplus \mathfrak{v}^+ \oplus \mathfrak{j}^+. \tag{32}$$

Let $S_1, \dots, S_p$ be a basis of $\mathfrak{j}^+$. Define $T_k = \tau(S_k)$, $k = 1, \dots, p$, then $T_1, \dots, T_p$ is a basis of $\mathfrak{j}^-$. Let $\mathbf{b}$ be any $p \times p$ complex matrix such that $\mathbf{b}\overline{\mathbf{b}} = -1$. Then we may define a complex structure B on $\mathfrak{j}_u$ by

$$BS_j = \sum_{k=1}^{p} b_{k,j} T_k, \quad BT_j = \sum_{k=1}^{p} \overline{b_{k,j}} S_k, \quad j = 1, \dots, p. \tag{33}$$

[11]We say in this case that $\mathfrak{u}$ is nearest to semisimple.

The pair (I, B) is a quaternionic structure on the vector space $\mathfrak{j}_u$, since $B(\mathfrak{j}^+) = \mathfrak{j}^-$(see also Proposition 3.10). We may decompose the $\mathfrak{j} = \mathfrak{j}_B^+ \oplus \mathfrak{j}_B^-$ into the i and $-$i eigenspaces of B, respectively.

Now we may define a matching complex structure J on $\mathfrak{u}$:

$$JX = \begin{cases} I_{\mathbf{c}}X & \text{if } X \in \mathcal{V}^u \oplus \mathfrak{f}_u \oplus \mathfrak{z}, \\ BX & \text{if } X \in \mathfrak{j}_u. \end{cases} \tag{34}$$

By Corollary 3.4 and Proposition 3.6 we have $IJ + JI = 0$. To show that J is integrable we note that J is a regular complex structure with respect to the Cartan subalgebra $\mathfrak{y}^{\mathbb{C}} \oplus \mathfrak{o}$ (see (28)). Indeed, from the definition of J and Corollary 3.7 we have

$$\mathfrak{m}_J^+ = \mathfrak{j}_B^+ \oplus \operatorname{span}_{\mathbb{C}}\{Z_\gamma - \mathrm{i}Y_\gamma \mid \gamma \in \Gamma\} \oplus \operatorname{span}_{\mathbb{C}}\{X_\gamma - \mathrm{i}W_\gamma \mid \gamma \in \Gamma\} \oplus \mathbf{c}(\mathcal{V}^+).$$

On the other hand, from Proposition 2.24 and (28) one computes $\mathfrak{j}_B^+ = \mathbf{c}(\mathfrak{j}_B^+)$, $\mathbf{c}(E_\gamma) = \overline{\rho}_\gamma(X_\gamma - \mathrm{i}W_\gamma)$, $\mathbf{c}(Z_\gamma + \mathrm{i}W_\gamma) = Z_\gamma - \mathrm{i}Y_\gamma$, therefore,

$$\mathfrak{m}_J^+ = \mathbf{c}\left(\mathfrak{j}_B^+ \oplus \operatorname{span}_{\mathbb{C}}\{Z_\gamma + \mathrm{i}W_\gamma \mid \gamma \in \Gamma\} \oplus \mathfrak{n}^+\right) = \mathbf{c}\left(\mathfrak{j}_B^+ \oplus \mathfrak{v}^+ \oplus \mathfrak{n}^+\right),$$

which is a subalgebra and the integrability of J follows. Certainly $\mathfrak{n}(\mathfrak{m}_J^+) = \mathfrak{b}_J^+ = \mathbf{c}(\mathfrak{b}_I^+)$. $\square$

Remark 3.5. In the classical description of quaternions we have a third complex structure $K = IJ$. In order to get it we should have used another Cayley transform (see Remark 2.8):

$$\mathbf{c}_{\mathfrak{y}} = \exp\left(\frac{\pi}{2}\mathrm{ad}Y_\Gamma\right), \quad K = -\mathbf{c}_{\mathfrak{y}} I \mathbf{c}_{\mathfrak{y}}^{-1} \text{ on } \mathcal{V}^u \oplus \mathfrak{e},$$

and define $K = IB$ on $\mathfrak{j}$. Obviously K is regular with respect to the Cartan subalgebra $\mathfrak{x}^{\mathbb{C}} \oplus \mathfrak{o}$.

If we perceive a hypercomplex structure on $\mathbf{U}$ as a representation of SU(2) on $\mathfrak{u}$, which splits into real 4-dimensionnal irreducible components, then the hypercomplex structures constructed in this section do not depend on ρ on each component.

4. The hypercomplex structures

In this section we prove that up to equivalence, the hypercomplex structures described in the preceding section (see Theorem 3.12 and formula (34) in its proof) are all the hypercomplex structures on $\mathfrak{u}$. Furthermore, we obtain a more precise description of the matching Cayley structure $I_{\mathbf{c}}$ than we achieved in Section 3.

So we assume that I is any admissible $\mathfrak{b}^+$ complex structure on $\mathfrak{u}$ and J is *any integrable* complex structure on $\mathfrak{u}$ matching I.

In this section we use freely the conventions and notations of sections 1.2 and 1.3. In particular, we use the direct decompositions

$$\mathfrak{g} = \mathfrak{m}_I^+ \oplus \mathfrak{m}_I^- = \mathfrak{n}^+ \oplus \mathfrak{h}^+ \oplus \mathfrak{n}^- \oplus \mathfrak{h}^- = \mathfrak{h} \bigoplus_{\alpha \in \Delta} \mathfrak{g}(\alpha). \tag{35}$$

When $\mathfrak{a}$ is a direct summand in one of these decompositions and we write $\mathrm{pr}_{\mathfrak{a}} : \mathfrak{g} \to \mathfrak{a}$ we always mean projection along the complementary component in the above formula. Obviously the basis of Definition 1.7 is well adapted to such practices.

We work with the “complexified” Nijenhuis tensor, i.e., we extend $N(X, Y)$ to $\mathfrak{g}$ by complex linearity.

4.1. The Nijenhuis tensor

Recall that that $\mathfrak{m}_I^{\pm}$ are subalgebras and that $J(\mathfrak{m}_I^+) = \mathfrak{m}_I^-$, $J(\mathfrak{m}_I^-) = \mathfrak{m}_I^+$. The coefficients $a_{\alpha\beta}$, $\eta_{\nu,q}$, etc., are introduced in Definition 1.7.

Proposition 4.1. *Let $\alpha, \beta \in \Delta^+$, $q = 1, \dots, m$. Then*

$$a_{\beta,\alpha}(\alpha+\beta)(U_q) = \mathrm{pr}_{\mathfrak{g}(-\beta)}\Big(\sum_{\nu\in\Delta^+} \eta_{\nu,q}[E_{-\nu}, E_\alpha]\Big). \tag{36}$$

Proof. We decompose the element $JN_J(U_q, E_\alpha) \in \mathfrak{g}$ in the basis of Definition 1.7 using formula (12). Our purpose is to compute the coefficient of $E_{-\beta}$. From the integrability of J we have

$$\begin{aligned}0 = JN_J(U_q, E_\alpha) &= J[JU_q, JE_\alpha] - J[U_q, E_\alpha] + [U_q, JE_\alpha] + [JU_q, E_\alpha]\\ &= A - \alpha(U_q)\sum_{\beta\in\Delta^+} a_{\beta,\alpha}E_{-\beta} - \sum_{\beta\in\Delta^+} a_{\beta,\alpha}\beta(U_q)E_{-\beta} + \sum_{\nu\in\Delta^+}\eta_{\nu,q}[E_{-\nu}, E_\alpha],\end{aligned}$$

where $A = J[JU_q, JE_\alpha] - \alpha(U_q)\mathrm{pr}_{\mathfrak{h}}(JE_\alpha) + \alpha(\mathrm{pr}_{\mathfrak{h}}(JU_q))E_\alpha \in \mathfrak{b}^+$. For $\beta \in \Delta^+$ the coefficient of $E_{-\beta}$ must vanish, whence the proposition. □

Corollary 4.2. *If $\alpha, \beta \in \Delta^+$, $\alpha+\beta \notin \Delta$, then $a_{\alpha,\beta} = a_{\beta,\alpha} = 0$.*

Proof. By the assumption, the right-hand side in formula (36) is 0. On the other hand, the functional $\alpha+\beta$ is real at $\mathfrak{h}_{\mathbb{R}}$ and nonzero, so by (6) we may choose such a q that $(\alpha+\beta)(U_q) \neq 0$. □

Corollary 4.3. *If γ is a maximal root, then $JE_\gamma \in \mathfrak{h}^-$.*

Proof. By Corollary 4.2, for all $\nu \in \Delta^+$ we have $a_{\nu,\gamma} = 0$. □

Further, (36) obviously implies

Corollary 4.4. *Let $\alpha, \beta \in \Delta^+$ and $q = 1, \dots, m$. If $\gamma = \alpha+\beta \in \Delta$, then*

$$a_{\beta,\alpha}\gamma(U_q) = -N_{\gamma,-\alpha}\eta_{\gamma,q}.$$

Corollary 4.5. *If $\alpha, \beta, \gamma \in \Delta^+$ and $\gamma = \alpha+\beta$, then $N_{\gamma,-\alpha}a_{\alpha,\beta} = N_{\gamma,-\beta}a_{\beta,\alpha}$.*

Proof. Under the condition $N_{\gamma,-\beta} \neq 0 \neq N_{\gamma,-\alpha}$, we choose a q so that $\gamma(U_q) \neq 0$ and apply twice the formula in Corollary 4.4. □

Corollary 4.6. *For any $\alpha, \beta \in \Delta^+$ we have $a_{\alpha,\beta} = 0 \iff a_{\beta,\alpha} = 0$.*

Proof. If $\alpha+\beta \notin \Delta^+$ then we use Corollary 4.2, otherwise, Corollary 4.5. □

Before going on with the Nijenhuis tensor we introduce some convenient notation. Assume that $\gamma \in \Delta^+$ is such that $JE_\gamma \in \mathfrak{h}$. We denote

$$V_\gamma = JE_\gamma \in \mathfrak{h}^-, \quad U_\gamma = JE_{-\gamma} = -\tau(JE_\gamma) = -\tau(V_\gamma) \in \mathfrak{h}^+, \tag{37}$$

From the above definition and $\alpha(\tau(H)) = -\overline{\alpha(H)}$ we get

$$\gamma(V_\gamma) = \overline{\gamma(U_\gamma)}. \tag{38}$$

Now we may compute

Proposition 4.7. *Let I be an admissible complex structure and let J match I. Let $\gamma \in \Delta^+$ be such that $JE_\gamma \in \mathfrak{h}$. Then*

$$|\gamma(U_\gamma)| = 1, \quad \gamma(IH_\gamma) = 0; \tag{39}$$

$$JE_\gamma = \tfrac{1}{2}\gamma(V_\gamma)\left(H_\gamma + \mathrm{i}IH_\gamma\right), \quad JE_{-\gamma} = \tfrac{1}{2}\gamma(U_\gamma)\left(H_\gamma - \mathrm{i}IH_\gamma\right). \tag{40}$$

Proof. Integrability gives

$$\begin{aligned} N_J(E_\gamma, E_{-\gamma}) &= [V_\gamma, U_\gamma] - [E_\gamma, E_{-\gamma}] - J[V_\gamma, E_{-\gamma}] - J[E_\gamma, U_\gamma] \\ &= -H_\gamma + \gamma(V_\gamma)U_\gamma + \gamma(U_\gamma)V_\gamma = 0. \end{aligned}$$

For the present computation we denote $a = \gamma(U_\gamma)$. Now we apply I on the last expression to get the second equation of the following system:

$$H_\gamma = \overline{a}U_\gamma + aV_\gamma, \quad IH_\gamma = \mathrm{i}\overline{a}U_\gamma - \mathrm{i}aV_\gamma. \tag{41}$$

First, we use (41) to compute

$$\begin{aligned} 0 = N_J(H_\gamma, IH_\gamma) &= -[aE_\gamma + \overline{a}E_{-\gamma}, \mathrm{i}aE_\gamma - \mathrm{i}\overline{a}E_{-\gamma}] \\ &\quad - [H_\gamma, IH_\gamma] - J[H_\gamma, \mathrm{i}aE_\gamma - \mathrm{i}\overline{a}E_{-\gamma}] + J[aE_\gamma + \overline{a}E_{-\gamma}, IH_\gamma] \\ &= 2\mathrm{i}(|a|^2 - 1)H_\gamma - \mathrm{i}\gamma(IH_\gamma)IH_\gamma. \end{aligned}$$

Because H_γ, IH_γ are linearly independent[12], integrability implies (39). Now using (39) we solve the system (41) to get (40). □

Remark 4.1. At first glance formula (40) contains something like a vicious circle — we determine $V_\gamma = JE_\gamma$ using a circle parameter $\gamma(V_\gamma)$ on the right-hand side.

As we shall prove further $JE_\gamma \in \mathfrak{h}$ if anf only if $\gamma \in \Gamma$ (see Theorem 4.12). Actually, formulas (40) are the same as

$$JE_\gamma = V_\gamma = -\mathrm{i}\gamma(V_\gamma)Q_\gamma, \quad JE_{-\gamma} = U_\gamma = -\mathrm{i}\gamma(U_\gamma)P_\gamma. \tag{42}$$

(See Definition 3.3 for P_γ, Q_γ). The important point here is that, due to Theorem 4.12, any matching complex structure J sends the stem nilpotent $E_\gamma \in \mathfrak{f}^+$ to $Q_\gamma \in \mathfrak{h}^+$ multiplied by a complex number of norm 1; thus we recover the parameters ρ_γ from Section 2. We use this further to identify the Cayley transform which produces J — see Definition 4.1 and further.

Proposition 4.8. *Let $\alpha, \beta, \gamma \in \Delta^+$, $JE_\gamma \in \mathfrak{h}$. Then*

$$a_{\beta,\alpha}(\beta + \alpha)(U_\gamma) = \mathrm{pr}_{\mathfrak{g}(-\beta)}([E_\alpha, E_{-\gamma}]).$$

Proof. By the integrability of J we have

$$\begin{aligned} 0 = N(E_{-\gamma}, E_\alpha) &= [U_\gamma, JE_\alpha] - [E_{-\gamma}, E_\alpha] - J[U_\gamma, E_\alpha] - J[E_{-\gamma}, JE_\alpha] \\ &= [E_\alpha, E_{-\gamma}] - \sum_{\beta \in \Delta^+} \beta(U_\gamma)a_{\beta,\alpha}E_{-\beta} - \alpha(U_\gamma) \sum_{\beta \in \Delta^+} a_{\beta,\alpha}E_{-\beta} + A, \end{aligned}$$

where $A \in \mathfrak{b}^+$. The statement of the proposition comes from equating to zero the coefficient of $E_{-\beta}$ in the last expression. □

[12] Otherwise, $H_\gamma \in \mathfrak{h}^+$ or $H_\gamma \in \mathfrak{h}^-$, which contradicts $\mathfrak{h}^\pm \cap \tau(\mathfrak{h}^\pm) = \{0\}$.

Corollary 4.9. *Let $\alpha, \beta, \gamma \in \Delta^+$, $\alpha + \beta = \gamma$, $JE_\gamma \in \mathfrak{h}$. Then*

$$a_{\beta,\alpha} = N_{\gamma,-\alpha}\gamma(V_\gamma) \neq 0.$$

Proof. This follows trivially from Proposition 4.8 by (39) and the fact that $N_{\gamma,-\alpha} \neq 0$. □

Lemma 4.10. *Let $\alpha, \beta \in \Delta^+, \gamma \in \Gamma, JE_\gamma \in \mathfrak{h}$. If $\alpha + \beta \neq \gamma$, then*

$$\sum_{\nu \in \Phi_\gamma^+} a_{\nu,\beta} a_{\alpha,\mu(\nu)} N_{\gamma,-\nu} = 0,$$

where μ is the involution of Δ^+ defined in (22).

Proof. In the follwing computation we keep explicit only terms with a component in $\mathfrak{n}^-$. We have

$$\begin{aligned} 0 = N(E_\gamma, E_\beta) &= [V_\gamma, JE_\beta] - J[V_\gamma, E_\beta] - J[E_\gamma, JE_\beta] + A \\ &- \sum_{\alpha \in \Delta^+} a_{\alpha,\beta}(\beta + \alpha)(V_\gamma) E_{-\alpha} - J\left[E_\gamma, \mathrm{pr}_{\mathfrak{h}}(JE_\beta) + \sum_{\nu \in \Delta^+} a_{\nu,\beta} E_{-\nu}\right] + A \\ &= - \sum_{\alpha \in \Delta^+} a_{\alpha,\beta}(\beta + \alpha)(V_\gamma) E_{-\alpha} - \sum_{\nu \in \Delta^+} a_{\nu,\beta} J[E_\gamma, E_{-\nu}] + B. \end{aligned}$$

Now from Corollary 4.6 and $J(E_\gamma) \in \mathfrak{h}$ it follows that $a_{\gamma,\beta} = 0$, hence the sum $\sum_{\nu \in \Delta^+} a_{\nu,\beta} J[E_\gamma, E_{-\nu}]$ reduces to $\sum_{\nu \in \Delta^+ \setminus \{\gamma\}} a_{\nu,\beta} J[E_\gamma, E_{-\nu}]$, and by the definition of Φ_γ^+ the $\mathfrak{n}^-$-component is $\sum_{\nu \in \Phi_\gamma^+} a_{\nu,\beta} J[E_\gamma, E_{-\nu}]$. Therefore we reduce the equation $0 = N(E_\gamma, E_\beta)$ to the following:

$$0 = - \sum_{\alpha \in \Delta^+} \left(a_{\alpha,\beta}(\alpha + \beta)(V_\gamma) + \sum_{\nu \in \Phi_\gamma^+} N_{\gamma,-\nu} a_{\nu,\beta} a_{\alpha,\mu(\nu)} \right) E_{-\alpha} + C.$$

We conclude that for each β, α, γ as assumed, we have

$$a_{\alpha,\beta}(\alpha + \beta)(V_\gamma) + \sum_{\nu \in \Phi_\gamma^+} N_{\gamma,-\nu} a_{\nu,\beta} a_{\alpha,\mu(\nu)} = 0.$$

When $\beta + \alpha \neq \gamma$, Proposition 4.8 gives[13] $a_{\alpha,\beta}(\alpha + \beta)(V_\gamma) = 0$, whence the lemma. □

4.2. The coefficients of a

We recall that Γ is the stem of Δ^+ and that μ is involution of Δ^+ defined in (22).

Proposition 4.11. *Let $\gamma \in \Gamma$, $J(E_\gamma) \in \mathfrak{h}$ and $\alpha \in \Phi_\gamma^+, \beta \in \Delta^+$. Then $a_{\alpha,\beta} \neq 0$[14] if and only if $\alpha + \beta = \gamma$.*

[13] Because $(\alpha + \beta)(V_\gamma) = \overline{(\alpha + \beta)(U_\gamma)}$.

[14] by Corollary 4.6 this is the same as $a_{\beta,\alpha} \neq 0$.

Proof. If $\beta \in \Phi^+_\gamma$, then from Corollary 2.6a), we know that $\alpha+\beta \in \Delta$ if anf only if $\alpha+\beta=\gamma$, so Corollaries 4.2 and 4.9 give

$$a_{\alpha,\beta} \neq 0 \iff \alpha+\beta=\gamma, \quad \alpha,\beta \in \Phi^+_\gamma. \tag{43}$$

Let $\beta \notin \Phi^+_\gamma$. If $\beta+\alpha \notin \Delta^+$, then $a_{\alpha,\beta}=0$ by Corollary 4.2. So we have to treat just the case $\gamma \neq \beta+\alpha \in \Delta^+$ as in Lemma 4.10. Now if $\alpha, \nu, \mu(\nu)=\gamma-\nu \in \Phi^+_\gamma$, then by (43) $a_{\alpha,\mu(\nu)} \neq 0$ if and only if $\nu=\alpha$. Thus, the equality from Lemma 4.10 reduces to $a_{\alpha,\beta}N_{\gamma,-\alpha}=0$. We have $\alpha \in \Phi^+_\gamma$, hence $N_{\gamma,-\alpha} \neq 0$, whence $a_{\alpha,\beta}=0$. The proposition is proved. □

Now we can prove

Theorem 4.12. *Let I be an admissible $\mathfrak{b}^+$ complex structure on $\mathfrak{u}$ and let J match I. Then*

(a) *If $\gamma \in \Gamma$, then $J(E_\gamma) \in \mathfrak{h}^-$.*
(b) *If $\alpha \in \Phi^+$, $\beta \in \Delta^+$, then $a_{\alpha,\beta} \neq 0$ if and only if $\beta=\mu(\alpha)$.*

Proof. Let $\Gamma=\{\gamma_1,\dots,\gamma_d\}$ be the stem of Δ^+. We know that γ_1 is a maximal root, so by Corollary 4.3 we have $JE_{\gamma_1} \in \mathfrak{h}$. Since $J(\mathfrak{m}^+_I)=\mathfrak{m}^-_I$, we have $JE_{\gamma_1} \in \mathfrak{h}^-$.

Now by Proposition 4.11 we conclude that for any $\alpha \in \Phi^+_{\gamma_1}$, $\beta \in \Delta^+$ we have

$$a_{\alpha,\beta} \neq 0 \iff \alpha+\beta=\gamma_1 \iff \beta=\mu(\alpha).$$

Now we assume that for some $k<d$ we have $J(E_{\gamma_i}) \in \mathfrak{h}, \quad i=1,\dots,k$ and

$$a_{\alpha,\beta} \neq 0 \iff \beta=\mu(\alpha), \quad \alpha \in \Phi^+_{\gamma_1} \cup \dots \cup \Phi^+_{\gamma_k}, \ \beta \in \Delta^+. \tag{44}$$

If $\alpha \in \Phi^+_{\gamma_1} \cup \dots \cup \Phi^+_{\gamma_k}$ then by the definition of the stem $\gamma_{k+1} \neq \mu(\alpha)$, hence by the induction assumption (44) and Corollary 4.6 we have $a_{\gamma_{k+1},\alpha}=a_{\alpha,\gamma_{k+1}}=0$.

If $\alpha \in \Phi^+_{\gamma_{k+1}} \cup \dots \cup \Phi^+_{\gamma_d} \cup \Gamma$, then by Corollary 2.11 we have $\alpha+\gamma_{k+1} \notin \Delta$, whence by Corollary 4.2 we conclude $a_{\gamma_{k+1},\alpha}=a_{\alpha,\gamma_{k+1}}=0$.

Thus for each $\alpha \in \Delta^+$ we have $a_{\gamma_{k+1},\alpha}=a_{\alpha,\gamma_{k+1}}=0$, which means

$$JE_{\gamma_{k+1}} \in \mathfrak{h}. \tag{45}$$

Now let $\alpha \in \Phi^+_{\gamma_{k+1}}$.

If $\beta=\mu(\alpha)$ then Corollary 4.9 and (45) give $a_{\alpha,\beta} \neq 0$.

If $\beta \in \Delta^+$ and $\beta \neq \mu(\alpha)$, we apply Proposition 4.11 so

$$a_{\alpha,\beta} \neq 0 \iff \alpha+\beta=\gamma_{k+1} \iff \beta=\mu(\alpha), \quad \alpha \in \Phi^+_{\gamma_{k+1}}, \ \beta \in \Delta^+,$$

which combined with the assumption (44) gives

$$a_{\alpha,\beta} \neq 0 \iff \mu(\alpha)=\beta, \quad \alpha \in \bigcup_{i=1}^{k+1} \Phi^+_{\gamma_i}, \ \beta \in \Delta^+.$$

Our induction is complete, and the theorem is proved. □

Corollary 4.13. *The matrix* $\mathbf{a}$ *is antisymmetric.*

Proof. From Theorem 4.12 we know that $a_{\alpha,\beta} \neq 0$ if anf only if $\alpha \in \Phi^+$ and $\beta = \mu(\alpha)$. The result follows from Proposition 2.14 and Corollary 4.5. $\square$

Corollary 4.14. $J(\mathfrak{f}^+) \subset \mathfrak{h}^-$.

Proof. Follows directly from Theorem 4.12 a). $\square$

Corollary 4.15. *If* $\operatorname{rank}(\mathfrak{u}) < 2d$*, then* $\mathbf{U}$ *carries no hypercomplex structure.*

Proof. The claim follows from Corollary 4.14, the fact that J is bijective, and $2\dim(\mathfrak{h}^-) = \operatorname{rank}(\mathfrak{g})$. $\square$

From here on, we assume (often implicitly) that $\operatorname{rank}(\mathfrak{g}) \geq 2d$.

Corollary 4.16. *A semisimple compact Lie group* $\mathbf{U}$ *carries a hypercomplex structure if and only if*

$$\mathbf{U} \cong \mathrm{SU}(2d_1+1) \times \cdots \times \mathrm{SU}(2d_n+1), \quad d_1, \ldots, d_n \in \mathbb{N}.$$

Proof. The only simple group with $\operatorname{rank}(\mathfrak{g}) = 2d$ is $\mathrm{SL}(2n+1,\mathbb{C})$; for the other simple groups we have $\operatorname{rank}(\mathfrak{g}) < 2d$ (this follows, e.g., from Corollary 2.22). On the other hand, existence of a hypercomplex structure for our $\mathbf{U}$ follows from Remark 3.3. $\square$

Now we are ready to determine the complex structure J on $\mathcal{V}$ (see Definition 2.5).

Proposition 4.17. *Let* I *be an admissible* $\mathfrak{b}^+$ *complex structure and let* J *match* I*. If* $\gamma \in \Gamma, \alpha \in \Phi_\gamma^+$*, then*

$$JE_\alpha = N_{\gamma,-\alpha}\gamma(V_\gamma)E_{s_\gamma(\alpha)}. \tag{46}$$

Proof. We denote $\beta = \mu(\alpha) = -s_\gamma(\alpha)$. From Theorem 4.12 we have

$$JE_\alpha = a_{\beta,\alpha}E_{-\beta} + H, \qquad H \in \mathfrak{h}^-.$$

In the following computation we keep explicit only terms which have nontrivial projection to $\mathfrak{h}$.

$$\begin{aligned} N_J(E_\gamma, E_\alpha) &= [V_\gamma, a_{\beta,\alpha}E_{-\beta} + H] - J[E_\gamma, a_{\beta,\alpha}E_{-\beta} + H] - J[V_\gamma, E_\alpha] \\ &= \gamma(H)V_\gamma - N_{\gamma,-\beta}a_{\beta,\alpha}JE_\alpha - \alpha(V_\gamma)JE_\alpha + A \\ &= \gamma(H)V_\gamma - (N_{\gamma,-\beta}a_{\beta,\alpha} + \alpha(V_\gamma))H + B, \end{aligned}$$

where $A, B \in \mathfrak{n}^-$. Now we use $a_{\beta,\alpha} = N_{\gamma,-\alpha}\gamma(V_\gamma)$ (see Corollary 4.9) and $N_{\gamma,-\alpha}N_{\gamma,-\beta} = -1$ (Proposition 2.14) to get

$$\gamma(H)V_\gamma + \beta(V_\gamma)H = 0. \tag{47}$$

We apply γ to this equation and obtain $\gamma(H)(\gamma+\beta)(V_\gamma) = 0$. By Proposition 4.7 and $\beta \in \Phi_\gamma^+$ we have

$$\beta(V_\gamma) = \gamma(V_\gamma)\left(\tfrac{1}{2} + \text{ imaginary number}\right) \neq 0.$$

Thus $(\gamma+\beta)(V_\gamma) \neq 0$, whence $\gamma(H) = 0$. Now by (47) and $\beta(V_\gamma) \neq 0$ we get $H = 0$. The Proposition follows. $\square$

Formula (46) obviously implies:

Corollary 4.18. *Let I be an admissible $\mathfrak{b}^+$ complex structure and let J_1, J_2 be two complex structures matching I. If $J_1E_\gamma = J_2E_\gamma$ for each $\gamma \in \Gamma$, then $J_1E_\alpha = J_2E_\alpha$ for each $\alpha \in \Delta$.*

Given an admissible $\mathfrak{b}^+$ complex structure I, Proposition 4.17 determines the action of a matching J on the invariant[15] subspace $\mathcal{V}$. The result is so clean that it gives us a more precise description of the matching Cayley structure $I_{\mathbf{c}}$ than we achieved in Subsection 3.

4.3. The action of J on the extended stem subalgebra

In Section 3 we used a conjugate $I_{\mathbf{c}}$ of a $\mathfrak{b}^+$ complex structure I (see Definition 3.1) to obtain a matching complex structure under the condition $I(\mathfrak{w}) \subset \mathfrak{o}_{\mathfrak{u}}$. We show here that $I(\mathfrak{w}) \subset \mathfrak{o}_{\mathfrak{u}}$ is also a necessary condition for the admissibility of I. We show also that for any admissible complex structure I on $\mathfrak{u}$ and any J matching I we have $J = I_{\mathbf{c}}$ on $\mathfrak{e} \oplus \mathcal{V}^{\mathfrak{u}}$ (see Definitions 3.2, 2.5) for a certain value of the torus parameter ρ, namely:

Definition 4.1. Let $\gamma \in \Gamma$. We denote:

$$\rho_\gamma = \mathrm{i}\gamma(JE_{-\gamma}), \quad \rho = \{\rho_\gamma \mid \gamma \in \Gamma\}.$$

The first equality in formula (39) gives $|\rho_\gamma| = 1$ whence we may use all the entities from Definition 2.7.

In particular, for any $\gamma \in \Gamma$ from (40) we get:

$$JE_\gamma = V_\gamma = \overline{\rho_\gamma}(W_\gamma + \mathrm{i}Z_\gamma), \quad JE_{-\gamma} = U_\gamma = -\rho_\gamma(W_\gamma - \mathrm{i}Z_\gamma). \tag{48}$$

Proposition 4.19. *Let I be an admissible $\mathfrak{b}^+$ complex structure and let J match I. Then $I(\mathfrak{w}) \subset \mathfrak{o}_{\mathfrak{u}}$ and for any $\gamma \in \Gamma$ we have*

$$IX_\gamma = Y_\gamma, \quad IW_\gamma = Z_\gamma; \qquad JX_\gamma = W_\gamma, \quad JZ_\gamma = Y_\gamma. \tag{49}$$

Proof. The first and second equality in (49) come from the definition of a $\mathfrak{b}^+$ complex structure I. The third and fourth equality come by solving the system (48) for W_γ, Z_γ. It remains to show $I(\mathfrak{w}) \subset \mathfrak{o}_{\mathfrak{u}}$, i.e., we must show that $\gamma(IH_\delta) = 0$ for any $\gamma, \delta \in \Gamma$.

From Proposition 4.7 and Theorem 4.12 it follows that $\gamma(IH_\gamma) = 0$ for any $\gamma \in \Gamma$.

Let γ, δ be non-equal elements of Γ. Using Theorem 4.12 and the strong orthogonality of Γ we compute $JN_J(E_\gamma, E_\delta)$ to get:

$$0 = [JE_\gamma, E_\delta] + [E_\gamma, JE_\delta] = \delta(V_\gamma)E_\delta - \gamma(V_\delta)E_\gamma,$$

hence $\gamma(JE_\delta) = \delta(JE_\gamma) = 0$. Now by (48) and the strong orthogonality of Γ we have $0 = \gamma(JE_\delta) = \overline{\rho_\delta}(\gamma(W_\delta) + \mathrm{i}\gamma(Z_\delta)) = \mathrm{i}\overline{\rho_\delta}\gamma(Z_\delta)$. The Proposition is proved. □

Thus we see that $\mathfrak{z} \subset \mathfrak{o}_u$ is a necessary condition for admissibility of I. Due to this fact, the coincidence of the formulas in Proposition 4.19 and Corollary 3.7 means that if J is any complex structure matching I, then $J = I_{\mathbf{c}}$ on the extended stem subalgebra $\mathfrak{e} = \mathfrak{f}_u \oplus \mathfrak{z}$ (with ρ as in Definition 4.1: see also Remark 2.6). Combining Corollary 4.18 with Proposition 4.19 we get:

[15] One point of Proposition 4.17 is proving the J invariance of $\mathcal{V}$.

Corollary 4.20. *Let I be an admissible $\mathfrak{b}^+$ complex structure and let J match I. Then each of the subspaces $\mathfrak{c}$, $\mathcal{V}^{\mathfrak{u}}$ is J-stable. Moreover, for each $X \in \mathfrak{c} \oplus \mathcal{V}^{\mathfrak{u}}$ we have $JX = \mathbf{c}I\mathbf{c}^{-1}X$, where the Cayley transform $\mathbf{c} = \mathbf{c}[\rho]$ is as in Definition 2.10 and $\rho_\gamma = \mathrm{i}\gamma(JE_{-\gamma})$ for each $\gamma \in \Gamma$.*

We have another useful consequence of $I(\mathfrak{w}) \subset \mathfrak{o}$:

Corollary 4.21. *If I is admissible and $\Gamma = \{\gamma_1, \dots, \gamma_d\}$, then $\gamma_j(P_k) = \gamma_j(Q_k) = \mathrm{i}\delta_{jk}$.*

Formula (48) and Proposition 3.11 give

Proposition 4.22. *Let J match I, then $J(\mathfrak{f}^+) = \mathfrak{v}^-$, $\quad J(\mathfrak{f}^-) = \mathfrak{v}^+$.*

If we add Theorem 3.12 to Proposition 4.19 we obtain our solution of Problem A:

Theorem 4.23. *Let $\mathfrak{u}$ be a compact Lie algebra and let I be a $\mathfrak{b}^+$ complex structure on $\mathfrak{u}$. Let Γ be the stem of Δ^+. Then I is admissible if and only if $\dim(\mathfrak{u})$ is divisible by 4 and for each $\gamma, \delta \in \Gamma$ we have $\gamma(IW_\delta) = 0$.*

We have chosen to express the necessary and sufficient condition for admissibility of I in the most classical terms, only using the notion of stem. Recalling Remark 3.1 we get:

Corollary 4.24. *A compact Lie group $\mathbf{U}$ carries a left-invariant hypercomplex structure if and only if $\mathrm{rank}(\mathfrak{u}) = 2d + 4k$, where d is the number of elements in the stem Γ and k is a nonnegative integer.*

4.4. The classification

Now we proceed to Problem B, that is, we assume that I is an admissible $\mathfrak{b}^+$ complex structure on $\mathfrak{u}$ and describe all complex structures J matching I.

We already described the matching complex structures on the extended stem subalgebra $\mathfrak{f}_u \oplus \mathfrak{z}$ and on $\mathcal{V}$. We showed also (Corollary 4.20) that any matching complex structure coincides on $\mathfrak{f}_u \oplus \mathfrak{z} \oplus \mathcal{V}$ with the structure $I_{\mathbf{c}}$ (see Definition 3.1) for some ρ.

So we go on to determine the remaining coefficients of the matrix of J (see Definition 1.7). We already showed that $\mathfrak{z} \subset \mathfrak{o}_u$, so we can use Propositions 3.11, 3.10 (see Definition 3.3 for the notations). Thus, we have decompositions:

$$\begin{gathered} \mathfrak{m}_I^+ = \mathfrak{v}^+ \oplus \mathfrak{j}^+ \oplus \mathfrak{f}^+ \oplus \mathcal{V}^+, \\ \dim(\mathfrak{v}^+) = \dim(\mathfrak{f}^+) = d,\ \dim(\mathfrak{j}^+) = p, \end{gathered} \tag{50}$$

where p is a nonnegative even integer and $\mathrm{rank}(\mathfrak{g}) = 2d + 2p$.

In the first place we have the vectors $P_\gamma = W_\gamma - \mathrm{i}Z_\gamma,\ Q_\gamma = W_\gamma + \mathrm{i}Z_\gamma$, which are a basis for the subspaces $\mathfrak{v}^+, \mathfrak{v}^-$ respectively.

Equality (48) and Proposition 4.17 describe completely the complex structures matching I on the component of $\mathfrak{g}$ which is complementary to $\mathfrak{j} = \mathfrak{j}^+ \oplus \mathfrak{j}^-$ with respect to the decomposition (50).

In the nearest to semi-simple case, i.e., when $2d = \mathrm{rank}(\mathfrak{g})$, we have $\mathfrak{j} = 0$ and then equality (48) and Proposition 4.17 describe completely the complex structures matching I; more precisely:

Theorem 4.25. *Let $2d = \mathrm{rank}(\mathfrak{g})$ and let I be an admissible $\mathfrak{b}^+$ complex structure. Then there is exactly one (up to choice of $\rho = \{\rho_\gamma \mid \gamma \in \Gamma\}$) matching complex structure J. For $\gamma \in \Gamma$ and $\alpha \in \Phi^+_\gamma$ the operator J is given by*

$$J(E_\gamma) = \overline{\rho_\gamma} Q_\gamma; \quad J(E_\alpha) = \mathrm{i}\bar{\rho}_\gamma N_{\gamma,-\alpha} E_{\mu(\alpha)}.$$

This theorem improves the constructive Proposition 3.9 in particular.

We give several equivalent forms of the admissibility condition, when $2d = \mathrm{rank}(\mathfrak{g})$. In Subsection 2.3 we introduced and studied a representative of the opposition involution $\phi = \exp(\pi \mathrm{ad} X_\Gamma)$.

Corollary 4.26. *Let $2d = \mathrm{rank}(\mathfrak{g})$, let I be a $\mathfrak{b}^+$ complex structure on $\mathfrak{u}$. Then I is admissible if and only if any of the following three equivalent conditions holds*

a) $\phi(\mathfrak{m}_I^+) = \mathfrak{m}_I^-$.
b) $\phi(\mathfrak{h}^+) = \mathfrak{h}^-$.
c) $\phi \circ I = -I \circ \phi$.

Proof. In any case $\phi(\mathfrak{n}^+) = \mathfrak{n}^-$, so a) is equivalent to b).

The proof of c) $\Leftrightarrow$ b) imitates the proof of Proposition 1.3.

Now assume that I is admissible. Then for any $W \in \mathfrak{w}$ we have $IW \in \mathfrak{o}$, whence by Corollary 2.20, $\phi(W - \mathrm{i}IW) = -W - \mathrm{i}IW = -\tau(W - \mathrm{i}IW)$ whence the condition b) holds.

Conversely let condition c) hold. Then for $W \in \mathfrak{w}$ we have $\phi(IW) = -I\phi(W) = IW$ and by Proposition 2.20 we have $I(\mathfrak{w}) \subset \mathfrak{o}$, whence I is admissible. $\square$

When $2d < \mathrm{rank}(\mathfrak{g})$ and I is an admissible $\mathfrak{b}^+$ complex structure we have to determine the action of a matching complex structure J on the subspace $\mathfrak{j} = \mathfrak{j}^+ \oplus \mathfrak{j}^- \subset \mathfrak{o}$ (see Definition 3.3).

Proposition 4.27. *If $\mathrm{rank}(\mathfrak{u}) = 2d + 2p, p \in 2\mathbb{N}$, then $J(\mathfrak{j}^+) = \mathfrak{j}^-$.*

Proof. Let $S_1, \dots, S_p$ be a basis of $\mathfrak{j}^+$; then $T_k = \tau(S_k),\ k = 1, \dots, p$ is a basis of $\mathfrak{j}^-$. Then by Proposition 3.11 $\{P_\gamma \mid \gamma \in \Gamma\} \cup \{S_1, \dots, S_p\}$ is a basis of $\mathfrak{h}^+$ and $\{Q_\gamma \mid \gamma \in \Gamma\} \cup \{T_1, \dots, T_p\}$ is a basis of $\mathfrak{h}^-$. Slightly changing notation for the elements of the matrix J (see (12), Proposition 1.4), for any $q = 1, \dots, p$ we have

$$J(S_q) = \sum_{\delta \in \Gamma} b_{\delta,q} Q_\delta + \sum_{t=1}^{p} b_{tq} T_t + \sum_{\beta \in \Delta^+} \eta_{\beta,q} E_{-\beta}. \tag{51}$$

From the integrability for $q = 1, \dots, p,\ \ \gamma \in \Gamma$ we have(in this computation we use Corollary 4.21 and $\mathfrak{j} \subset \mathfrak{o}$):

$$\begin{aligned} 0 = N_J(S_q, V_\gamma) &= -\Big[\sum_{\delta \in \Gamma} b_{\delta,q} Q_\delta + \sum_{t=1}^{p} b_{tq} T_t + \sum_{\beta \in \Delta^+} \eta_{\beta,q} E_{-\beta}, E_\gamma\Big] \\ &\quad + J[S_q, E_\gamma] - J\Big[\sum_{\delta \in \Gamma} b_{\delta,q} Q_\delta + \sum_{t=1}^{p} b_{tq} T_t + \sum_{\beta \in \Delta^+} \eta_{\beta,q} E_{-\beta}, V_\gamma\Big] \\ &= -b_{\gamma,q}\gamma(Q_\gamma) E_\gamma - \sum_{\beta \in \Delta^+} \eta_{\beta,q}[E_{-\beta}, E_\gamma] - \sum_{\beta \in \Delta^+} \eta_{\beta,q} \beta(V_\gamma) J E_{-\beta}. \end{aligned}$$

From Proposition 4.17 and formula (42) for $\beta \in \Phi_\gamma^+$ we have

$$JE_{-\beta} = -\tau(JE_\beta) = N_{\gamma,-\beta}\overline{\gamma(V_\gamma)}E_{\gamma-\beta}, \quad \beta(V_\gamma) = -\mathrm{i}\gamma(V_\gamma)\beta(Q_\gamma);$$

therefore we have (recall that $|\gamma(V_\gamma)| = 1$)

$$\begin{aligned} 0 = -\mathrm{i}b_{\gamma,q}E_\gamma + \sum_{\beta\in\Phi_\gamma^+} \eta_{\beta,q}N_{\gamma,-\beta}(1+\mathrm{i}\beta(Q_\gamma))E_{\gamma-\beta} \\ - \sum_{\beta\in\Delta^+\setminus\Phi_\gamma^+} \eta_{\beta q}([E_{-\beta},E_\gamma] + \beta(V_\gamma)JE_{-\beta}). \end{aligned} \tag{52}$$

From Proposition 4.17 it follows that

$$\sum_{\beta\in\Delta^+\setminus\Phi_\gamma^+} \eta_{\beta q}([E_{-\beta},E_\gamma] + \beta(V_\gamma)JE_{-\beta}) \in \mathfrak{h} + \sum_{\alpha\in\Delta\setminus\Phi_\gamma^+} \mathfrak{g}(\alpha).$$

Now from (52) it follows that for any $\beta \in \Phi_\gamma^+$ we have

$$\eta_{\beta,q}N_{\gamma,-\beta}(1+\mathrm{i}\beta(Q_\gamma)) = -\mathrm{i}N_{\gamma,-\beta}\eta_{\beta,q}\alpha(Q_\gamma) = 0,$$

where $\alpha = \mu(\beta) = \gamma - \beta$. But for $\alpha \in \Phi_\gamma^+$ we have $\alpha(Q_\gamma) \neq 0$ (see the end of the proof of Proposition 4.17), whence for $\beta \in \Phi^+,\ q = 1,\dots,p$ we have

$$\eta_{\beta,q} = 0. \tag{53}$$

Suppressing all terms containing $\eta_{\beta,q},\ \beta \in \Phi^+$, equation (52) reduces to (recall that $\delta(V_\gamma) = 0$ for $\gamma,\delta \in \Gamma$, $\gamma \neq \delta$):

$$\begin{aligned} 0 &= -\mathrm{i}b_{\gamma,q}E_\gamma - \sum_{\delta\in\Gamma} \eta_{\delta,q}([E_{-\delta},E_\gamma] + \delta(V_\gamma)JE_{-\delta}) \\ &= -\mathrm{i}b_{\gamma,q}E_\gamma - \eta_{\gamma,q}(H_\gamma + \gamma(V_\gamma)U_\gamma). \end{aligned}$$

By (40) and $I(H_\gamma) \in \mathfrak{o}$ it follows that $H_\gamma + \gamma(V_\gamma)U_\gamma \neq 0$ and we see that for $\gamma \in \Gamma, \quad q = 1,\dots,p$ we have

$$\eta_{\gamma,q} = b_{\gamma,q} = 0. \tag{54}$$

Thus we conclude (see (51), (53),(54)) that for any $q = 1,\dots,p$,

$$J(S_q) = \sum_{k=1}^{p} b_{kq}T_k. \tag{55}$$

The proposition is proved. $\square$

We present our solution of Problem B from the introduction.

Theorem 4.28. *Let* $\mathfrak{u}$ *be a compact Lie algebra with* $\operatorname{rank}(\mathfrak{u}) = 2d + 2p$*, where* d *is the number of roots in the stem* Γ *of* Δ^+ *and* p *is a nonnegative even integer. Let* I *be an admissible* $\mathfrak{b}^+$*-complex structure on* $\mathfrak{u}$*. Let* $S_1, \ldots, S_p$ *be a basis of* $\mathfrak{j}^+$*, and* $T_k = \tau(S_k)$*,* $k = 1, \ldots, p$*. Then all the complex structures matching* I *are determined by the following formulas (here* $\gamma \in \Gamma$ *and* $\alpha \in \Phi_\gamma$*):*

$$J(E_\gamma) = \overline{\rho_\gamma} Q_\gamma, \;\; J(E_\alpha) = \mathrm{i}\overline{\rho}_\gamma N_{\gamma,-\alpha} E_{\mu(\alpha)}, \;\; J(S_q) = \sum_{k=1}^{p} b_{kq} T_k, \tag{56}$$

where $\{\rho_\gamma\}_{\gamma\in\Gamma}$ *is any family of numbers on the unit circle and* $\mathbf{b}$ *is any* $p \times p$ *complex matrix satisfying* $\overline{\mathbf{b}}\mathbf{b} = -1$*.*

Proof. The first equality is in (48). The second is in Proposition 4.17. The third follows from Proposition 4.27. In the proof of Theorem 3.12 we show that all the complex structures defined by these formulas are integrable. □

4.5. Equivalences

From the explicit result in Theorem 4.23 we may obtain the parameter spaces of equivalence classes of admissible complex structures on a compact Lie group with respect to:

Definition 4.2. For any Lie group $\mathbf{G}$ we shall call two left invariant complex structures I and I' on $\mathbf{G}$ equivalent if there exists an analytic automorphism of the group $\mathbf{G}$, which is a biholomorphic transformation between the complex manifolds $(\mathbf{G}, I)$ and $(\mathbf{G}, I')$. We denote by $\mathcal{P}_{\mathbf{G}}$ the set of all equivalence classes of admissible left invariant complex structures on $\mathbf{G}$.

The table below gives for a nearest to simple compact Lie group $\mathbf{U}$[16] the parameter space $\mathcal{P}_{\mathbf{U}}$ for $\mathbf{U}$ as well as the space $\mathcal{P}_{\widetilde{\mathbf{U}}}$ for the covering simply connected group $\widetilde{\mathbf{U}}$,

$\mathbf{U}$	$\mathcal{P}_{\mathbf{U}}$	$\mathcal{P}_{\widetilde{\mathbf{U}}}$
$\mathbf{SU}(2d+1)$	$\mathbf{S}_2\backslash\mathbf{GL}_d(\mathbb{R})$	the same
$\mathbf{SU}(2d) \times \mathbf{T}^1$	$(\mathbf{S}_2 \times \mathbf{GL}_1(\mathbb{Z}))\backslash\mathbf{GL}_d(\mathbb{R})$	$\mathbb{Z}$ replaced by $\mathbb{R}$
$\mathbf{Spin}(2m) \times \mathbf{T}^{m-2}$ $m = 2q+1$	$(\mathbf{S}_2 \times \mathbf{GL}_{m-2}(\mathbb{Z}))\backslash\mathbf{GL}_{m-1}(\mathbb{R})$	$\mathbb{Z}$ replaced by $\mathbb{R}$
$\mathbf{E}_6 \times \mathbf{T}^2$	$(\mathbf{S}_2 \times \mathbf{GL}_2(\mathbb{Z}))\backslash\mathbf{GL}_4(\mathbb{R})$	$\mathbb{Z}$ replaced by $\mathbb{R}$
$\mathbf{Spin}(4q) \times \mathbf{T}^{2q}$ $q > 2$	$\mathbf{GL}_{2q}(\mathbb{Z})\backslash\mathbf{GL}_{2q}(\mathbb{R})/\mathbf{S}_2$	$\{1\}$
$\mathbf{Spin}(8) \times \mathbf{T}^4$	$\mathbf{GL}_4(\mathbb{Z})\backslash\mathbf{GL}_4(\mathbb{R})/\mathbf{S}_3$	$\{1\}$
$\mathbf{Spin}(2d+1) \times \mathbf{T}^d$	$\mathbf{GL}_d(\mathbb{Z})\backslash\mathbf{GL}_d(\mathbb{R})$	$\{1\}$
$\mathbf{Sp}(d) \times \mathbf{T}^d$	$\mathbf{GL}_d(\mathbb{Z})\backslash\mathbf{GL}_d(\mathbb{R})$	$\{1\}$
$\mathbf{E}_8 \times \mathbf{T}^8$	$\mathbf{GL}_8(\mathbb{Z})\backslash\mathbf{GL}_8(\mathbb{R})$	$\{1\}$
$\mathbf{E}_7 \times \mathbf{T}^7$	$\mathbf{GL}_7(\mathbb{Z})\backslash\mathbf{GL}_7(\mathbb{R})$	$\{1\}$
$\mathbf{F}_4 \times \mathbf{T}^4$	$\mathbf{GL}_4(\mathbb{Z})\backslash\mathbf{GL}_4(\mathbb{R})$	$\{1\}$
$\mathbf{G}_2 \times \mathbf{T}^2$	$\mathbf{GL}_2(\mathbb{Z})\backslash\mathbf{GL}_2(\mathbb{R})$	$\{1\}$

[16] $\mathbf{U} = \mathbf{U}_s \times T^{2d-\operatorname{rank}(\mathfrak{u}_s)}$, where $\mathbf{U}_s$ is any simple, simply connected, compact Lie group.

where we use the following notations:

To save space we write, e.g., $\mathbf{GL}_n(\mathbb{R})$ instead of $\mathbf{GL}(n,\mathbb{R})$.

By $\mathbf{S}_k$ we denote the symmetric group of k elements.

For any $d \geq 1$ and $0 < p < d$ we define action

$$(\mathbf{S}_2 \times \mathbf{GL}_p(\mathbb{Z})) \times \mathbf{GL}_d(\mathbb{R}) \to \mathbf{GL}_d(\mathbb{R}),$$
$$((+1,A),B) \mapsto \begin{bmatrix} \mathrm{I}_{d-p} & 0 \\ 0 & A \end{bmatrix} B; \quad ((-1,A),B) \mapsto \begin{bmatrix} -\mathrm{I}_{d-p} & 0 \\ 0 & A \end{bmatrix} B.$$

The set of orbits in $\mathbf{GL}_d(\mathbb{R})$ with respect to this action we denote by $(\mathbf{S}_2 \times \mathbf{GL}_p(\mathbb{Z}))\backslash\mathbf{GL}_d(\mathbb{R})$. In the case $p = 0$ we discard the factor $\mathbf{GL}_p(\mathbb{Z})$ in this definition and obtain $\mathbf{S}_2\backslash\mathbf{GL}_d(\mathbb{R})$ and in the case $p = d$ we discard the factor $\mathbf{S}_2$ and obtain $\mathbf{GL}_d(\mathbb{Z})\backslash\mathbf{GL}_d(\mathbb{R})$.

By $\mathbf{GL}_{2q}(\mathbb{Z})\backslash\mathbf{GL}_{2q}(\mathbb{R})/\mathbf{S}_2$ we denote the set of orbits in $\mathbf{GL}_{2q}(\mathbb{R})$ obtained with the action of $\mathbf{GL}_{2q}(\mathbb{Z})$ on $\mathbf{GL}_{2q}(\mathbb{R})$ by left multiplication of the matrices in $\mathbf{GL}_{2q}(\mathbb{R})$ with the matrices in $\mathbf{GL}_{2q}(\mathbb{Z})$ and the action of $\mathbf{S}_2$ by permutation of q-th and $2q$-th columns of the matrices in $\mathbf{GL}_{2q}(\mathbb{R})$.

By $\mathbf{GL}_4(\mathbb{Z})\backslash\mathbf{GL}_4(\mathbb{R})/\mathbf{S}_3$ we denote the set of orbits in $\mathbf{GL}_4(\mathbb{R})$ obtained with the action of $\mathbf{GL}_4(\mathbb{Z})$ on $\mathbf{GL}_4(\mathbb{R})$ by left multiplication and the action of $\mathbf{S}_3$ by permutation of the second, third, and fourth columns of the matrices in $\mathbf{GL}_4(\mathbb{R})$.

We omit here the proof of this result (the table), but give hints for the two most extreme cases (the first and the eighth raw of the table). The proof uses Theorem 4.23, the list of stems in [DT1, Subsect. 2.9], and the list of outer automorphisms of simple Lie algebras. We have already accounted for the inner automorphisms by fixing the Cartan and Borel subalgebras in the conditions of Theorem 4.23.

Example 3. Let $\mathbf{U} = \mathrm{Sp}(d) \times T^d$. The universal covering group is $\widetilde{\mathbf{U}} \cong \mathrm{Sp}(d) \times \mathbb{R}^d$. Now Γ is the set of long roots in Δ^+.

Up to equivalence, there is *exactly one* left invariant admissible complex structure on the (noncompact) universal cover group $\widetilde{\mathbf{U}}$. Indeed, all bases $Z_1, \ldots, Z_d$ of $\mathfrak{o}_u$ are equivalent under the action of the tangent maps to the automorphisms in $\mathbf{Aut}(\mathbb{R}^d) = \mathrm{GL}(d,\mathbb{R})$.[17]

The parameter space of equivalence classes of admissible complex structures on $\mathrm{Sp}(d) \times \mathbf{T}^d$ is $\mathbf{GL}_d(\mathbb{Z})\backslash\mathbf{GL}_d(\mathbb{R})$.[18]

Example 4. Let I and I' be two admissible $\mathfrak{b}^+$-complex structures on $\mathfrak{su}(2d+1)$. Then I is equivalent to I' if and only if either $I = I'$ or $IH_\gamma = -I'H_\gamma$ for each $\gamma \in \Gamma$. The only non-trivial equivalence comes from the Lie algebra automorphism corresponding to the Dynkin diagram symmetry.

The parameter space of equivalence classes of admissible complex structures on $\mathrm{SU}(2d+1)$ is $\mathbf{S}_2\backslash\mathbf{GL}(d,\mathbb{R})$.

Theorem 4.28 gives the corresponding hypercomplex structures.

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[17]By $\mathbf{Aut}(\mathbf{G})$ we denote the group of analytic automorphisms of $\mathbf{G}$.

[18]Recall that $\mathbf{Aut}(\mathbf{T}^d)$ is isomorphic to $\mathbf{GL}_d(\mathbb{Z})$.

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