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ERRATUM TO: ON THE DIRAC COHOMOLOGY OF COMPLEX LIE GROUP REPRESENTATIONS

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Abstract. We correct an error in the paper [D]. For Proposition 1.3 and Theorem 1.4 there to hold, we need to assume that $\lambda_L + s\lambda_L$ is dominant for $\Delta^+(\mathfrak{g}_0, \mathfrak{h}_0)$.

Let G be a connected complex semisimple Lie group. Fix a Borel subgroup B of G containing H. Let us put $\Delta^+(\mathfrak{g}_0, \mathfrak{h}_0) = \Delta(\mathfrak{b}_0, \mathfrak{h}_0)$ and set

$$\Delta^+(\mathfrak{g}, \mathfrak{h}) = \Delta^+(\mathfrak{g}_0, \mathfrak{h}_0) \times \{0\} \cup \{0\} \times (-\Delta^+(\mathfrak{g}_0, \mathfrak{h}_0)),$$

which is θ-stable. As deduced by Barbasch and Pandžić [BP], to find all the irreducible unitary representations with non-zero Dirac cohomology, it suffices to consider $J(\lambda_L, -s\lambda_L)$ such that $2\lambda_L$ is dominant integral regular for $\Delta^+(\mathfrak{g}_0, \mathfrak{h}_0)$, where $s \in W$ is an involution. For Proposition 1.3 of [D] to hold, we need the additional requirement that $\lambda_L + s\lambda_L$ *is dominant for* $\Delta^+(\mathfrak{g}_0, \mathfrak{h}_0)$. Without this condition, that proposition can fail, as illustrated by the following example, told to me by Vogan.

Example. Take $G = Sp(4, \mathbb{C})$, and identify $\mathfrak{h}_0^*$ with $\mathbb{C}^2$. Let us put

$$\Delta^+(\mathfrak{g}_0, \mathfrak{h}_0) = \{(0, 2), (1, -1), (1, 1), (2, 0)\}.$$

Take $\lambda_L = (2, 1)$. Let s be the reflection in the root $(2, 0)$. Then $\lambda_R = -s\lambda_L = (2, -1)$. Let us consider the representation

$$J((2, 1), (2, -1)).$$

Now $\lambda_L - \lambda_R = (0, 2)$, which is not dominant for $\Delta^+(\mathfrak{g}_0, \mathfrak{h}_0)$. Let w be the reflection in the root $(1, -1)$. Then one can use the element $w \in W$ to conjugate $\lambda_L - \lambda_R$ to be the dominant one $\lambda := \{\lambda_L - \lambda_R\} = (2, 0)$, and use λ to define a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$. But the point is that we should conjugate the Zhelobenko parameters λ_L, λ_R *simultaneously* using the same $w \in W$, which then become

$$J((1, 2), (-1, 2)).$$

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Now $2w\lambda_L = (2,4)$, which has a negative pairing with the root $(1,-1)$ in $\mathfrak{u}$. Thus the above representation is not in the good range, and Proposition 1.3 of [D] fails: instead of being irreducible, the cohomologically induced module in the RHS of equation (8) of [D] now has five composition factors, one of which is $J((2,1),(2,-1))$. $\square$

Thus for Proposition 1.3 of [D] to hold, we need to assume that $\lambda_L + s\lambda_L$ *is dominant for* $\Delta^+(\mathfrak{g}_0, \mathfrak{h}_0)$. Accordingly, the correct version of Theorem 1.4 of [D] is stated as follows.

Theorem 1.4′. *Let $J(\lambda_L, -s\lambda_L)$ be an irreducible representation of G, where $2\lambda_L$ is dominant integral regular for $\Delta^+(\mathfrak{g}_0, \mathfrak{h}_0)$, and $s \in W$ is an involution such that $\lambda := \lambda_L + s\lambda_L$ is dominant for $\Delta^+(\mathfrak{g}_0, \mathfrak{h}_0)$. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be the θ-stable parabolic subalgebra of $\mathfrak{g}$ defined by λ. Then $J(\lambda_L, -s\lambda_L)$ is a unitary $(\mathfrak{g}, K)$-module if and only if $J_L(\lambda'_L, \lambda'_R)$ is a unitary $(\mathfrak{l}, L \cap K)$-module. In such a case, if*

$$H_D(J_L(\lambda'_L, \lambda'_R)) = m2^{[l_0/2]} F_{2\lambda_L - \rho(\mathfrak{u}) - \rho_c^L},$$

where $l_0 = \dim \mathfrak{a}$, m is a non-negative integer, and F_ν denotes the $\widetilde{K_L}$-type with highest weight ν, then

$$H_D(J(\lambda_L, -s\lambda_L)) = m2^{[l_0/2]} E_{2\lambda_L - \rho_c},$$

where E_μ denotes the $\widetilde{K}$-type with highest weight μ. In particular, $H_D(J(\lambda_L, -s\lambda_L))$ is non-zero if and only if $H_D(J_L(\lambda'_L, \lambda'_R))$ is non-zero.

Since $2\lambda_L$ is dominant integral regular, one sees easily that the additional assumption "$\lambda := \lambda_L + s\lambda_L$ is dominant for $\Delta^+(\mathfrak{g}_0, \mathfrak{h}_0)$" is equivalent to the requirement that $J(\lambda_L, -s\lambda_L)$ is in the good range. Therefore, Theorem 1.4′ handles exactly the irreducible unitary representations of G which are in the good range.

When $\lambda_L + s\lambda_L$ is *not* dominant, since we care the most about unitary representations, things can be remedied to a certain extent by adopting Parthasarathy's Dirac inequality, which requires that

$$\|2\lambda_L\| \leq \|\rho_c + \{\lambda_L + s\lambda_L\}\|.$$

For instance, the representation considered in the above example is not unitary. It is elementary to deduce the following: fix any λ_L such that $2\lambda_L$ is dominant integral regular, and any involution $s \in W$ such that $\lambda_L + s\lambda_L$ is not dominant; there exists a positive integer N depending on λ_L and s such that

$$\|2k\lambda_L\| > \|\rho_c + \{k\lambda_L + sk\lambda_L\}\|,$$

where $k \geq N$ is an arbitrary integer. Theorem 1.4′ *cannot* handle the possible unitary representations in the family $J(k\lambda_L, -sk\lambda_L)$, $1 \leq k < N$.

Finally, we take this opportunity to correct the typos in [D]:

Page 72, line -4: replace "by Λ^+" with "by a subset of Λ^+".

Subsection 7.3: all the four places of $J(\lambda_L, -\lambda_L)$ should be $J(\lambda_L, \lambda_L)$.

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References

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