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A symmetry result for fully nonlinear problems in exterior domains

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Abstract. We study an overdetermined fully nonlinear problem driven by one of the Pucci's Extremal Operators in an external domain. Under certain decay assumptions on the solution, we extend Serrin's symmetry result, i.e, every domain where the solution exists must be radial.

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1. Introduction

This paper aims to be a contribution on the field of symmetry of non negative solutions of fully nonlinear elliptic equations. Following the approach started by Serrin [14], employing the famous moving plane method, we show the radiality of positive solutions on an external domain setting.

We consider the problem:

$$-\mathcal{M}^{\pm}_{r,R}(D^2u) = f(u) \qquad \text{in } \mathbb{R}^N \setminus G,\ N \geq 2 \tag{1}$$

$$u > 0 \qquad \text{in } \mathbb{R}^N \setminus G. \tag{2}$$

$$\lim_{|x|\to\infty} u(x)|x|^{\gamma} = c_0 > 0 \tag{3}$$

where γ and c_0 are positive constants, $\mathcal{M}^{\pm}_{r,R}$ denotes either one of the extremal Pucci's operators whose definition will be recalled in Sect. 2. Throughout the text the subscript r, R will be omitted whenever confusion will not arise.

Here we assume that $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function satisfying

$$\frac{f(u) - f(v)}{u - v} \leq c(|u| + |v|)^{\alpha} \text{ for } |u|, |v| \text{ sufficiently small and } \alpha > \frac{2}{\gamma}, \tag{4}$$

This is a natural assumption which paired with the growth assumption (3) guarantees the validity of a maximum principle in unbounded domains.

We also assume that $\tilde{N} = \frac{r}{R}(N-1)+1$, is greater than 2.

In (1)–(2) G is a domain defined as:

$$G = \bigcup_{i=1}^{k} G_i \tag{5}$$

where $k \in \mathbb{N}^+$ and G_i are bounded $\mathcal{C}^2$ domains such that $\bar{G}_i \cap \bar{G}_j = \emptyset$ for $i \neq j$.

We furthermore impose the following boundary conditions. For every $1 \leq i \leq k$:

$$\frac{\partial u}{\partial \nu} = \alpha_i \leq 0 \qquad \text{on } \partial G_i. \tag{6}$$

$$u = a_i > 0 \qquad \text{on } \partial G_i. \tag{7}$$

where, α_i and a_i are constants and ν is the external normal with respect to the boundary of G.

In the particular case where the Pucci operator coincides with the Laplacian, i.e, $r = R = 1$, the problem has been extensively covered in the literature.

In the early 70's Serrin [14] proved that for Ω, a bounded set with $\mathcal{C}^2$ boundary and f a continuos differentiable function, every positive solution of

$$\begin{aligned} -\Delta u = f(u, |\nabla u|) \qquad & \text{in } \Omega \\ u > 0 \qquad & \text{in } \Omega \\ u = 0 \qquad & \text{on } \partial\Omega \\ \frac{\partial u}{\partial \nu} = \text{ constant} \qquad & \text{on } \partial\Omega \end{aligned}$$

is radial and furthermore Ω must be a ball.

Serrin's result has then been extended in several directions, and would be impossible to exhaustively list all the literature, however, we present a few references for the different extensions: for the p-Laplace operator (see [4] and [9]), in the nonlocal setting ([5] and [17]) and under partially overdetermined boundary conditions see [8].

Concerning overdetermined problems for the Laplacian in external domains, Reichel (see [12] and [13]), under strong assumptions, first considered domains with one cavity and then on a follow-up article extended the result to domains with multiple cavities. Under more general assumptions, Sirakov ([16]), also obtained the result.

Both approaches are based on a topological arguments and strongly rely on the classical Serrin's Corner Lemma (see [14]). This presents a great difficulty to study the analogous problem in the fully nonlinear setting, since Serrin's Corner Lemma is not true in general.

In the last few years, some advances have been made regarding extending Serrin's result for Pucci's Extremal Operators in a bounded domain Ω.

Two different kinds of assumptions have been suggested to treat the problem. In [2], I. Birindelli and F. Demengel have proved symmetry when $\mathcal{M}^{\pm}$

is a perturbation of the laplacian, by providing a variation of Serrin's Corner Lemma.

Instead of assuming a constraint on the elliptic constants, Silvestre and Sirakov also obtained the same result in [15], but under geometric assumptions on the domains.

Our approach follows [2] where a smallness regime of the ratio $\frac{R}{r}$ is assumed in order to circumvent the lack of Serrin's Corner Lemma.

Here we study the external domain problem (1)–(7) and prove:

Theorem 1.1. *Let u be a $\mathcal{C}^{2,\beta}$ solution of (1)–(7), with $\beta > 0$. There exists a positive t_1 which depends only on r and β such that if $1 \leq \frac{R}{r} < t_1$, then, G has only one connected component. Moreover G is a ball and the solution u is radial with respect to the center of this ball.*

The paper is organized as follows. In Sect. 2, we state some preliminary lemmas which are used to prove the main result and introduce the necessary notation. We reserve Sect. 3 to provide the proof of Theorem 1.1 which will be divided in 10 steps. The proof closely follows [16], although some arguments are modified to deal with the fully nonlinear case. For the reader's convenience we include full details also for the steps which are the same as in the semilinear case.

Remark 1.2. This work has been produced during the authors PhD studies at Sapienza Universita di Roma.

2. Notation and Preliminary Results

Here we introduce some notation needed to apply the Moving Plane Method (see [10]), in order to show that, for every direction $\gamma \in \mathcal{S}^{N-1}$, our solution is symmetric in that direction. Since our operator is invariant with respect to rotations we may, without loss of generality, set $\gamma = e_1$, the first vector of a canonical base in $\mathbb{R}^N$, $N \geq 2$.

Hence we define, for $\lambda \in \mathbb{R}$

$$
\begin{aligned}
T_\lambda &= \{x \in \mathbb{R}^N \,|\, x_1 = \lambda\} \\
D_\lambda &= \{x \in \mathbb{R}^N \,|\, x_1 > \lambda\}
\end{aligned}
$$

For every point $x = (x_1, x') \in \mathbb{R}^N$, with $x' \in \mathbb{R}^{N-1}$, we set $x^\lambda = (2\lambda - x_1, x')$, i.e, x^λ is the reflection of x with respect to T_λ.

Then for every $A \subset \mathbb{R}^N$, we define

$$A^\lambda = \text{ the reflection of A with respect to } T_\lambda,\ i.e, A^\lambda = \{x \in \mathbb{R}^N \,|\, x^\lambda \in A\}$$

In particular we set $\Sigma_\lambda = D_\lambda \backslash (\bar{G} \cup \bar{G}^\lambda)$.

We will also use the notation

$$\Gamma_t(A) = A - te_1, \text{ for } t \in \mathbb{R}$$

$$\Gamma(A) = \bigcup_{t\in\mathbb{R}} \Gamma_t(A)$$

and for $z \in \mathbb{R}^N \setminus G$ we define,

$$\Gamma_t(z) = z - te_1 \text{ for } t \in \mathbb{R}$$

For $i = 1, \ldots, k$ we define:

$$d_i = \inf\{\lambda \in \mathbb{R} \,|\, T_\mu \cap \bar{G}_i = \emptyset \text{ for all } \mu > \lambda\}$$

$$\lambda_i = \inf\{\lambda \in \mathbb{R} \,|\, (D_\mu \cap \bar{G}_i)^\mu \subset G_i \text{ and } \langle \nu(z), e_1\rangle > 0 \text{ for all } \mu > \lambda \text{ and all } z \in T_\mu \cap \partial G_i \}$$

$$d = \max d_i \qquad \lambda_* = \max \lambda_i$$

where, as before, $\nu(z)$ denotes the exterior normal with respect to ∂G_i.
Let $z \in \partial G_i^\lambda \cap D_\lambda$, for some $i \in \{1, \ldots, k\}$, be such that $\Gamma_t(z) \in G_i^\lambda$ for small positive values of t. Then we define:

$$\underline{t} = \underline{t}(z) = \min\{t > 0 \,|\, z - te_1 \in \partial G_i \cup \partial G_i^\lambda\}$$

Now we recall the definition of the Pucci's Operators and associated quantities.

For a twice differentiable real function u defined on an open set Ω we define for $x \in \Omega$

$$\mathcal{M}^+_{r,R}(D^2u)(x) = R\sum_{\mu_i>0} \mu_i(x) + r\sum_{\mu_i<0} \mu_i(x)$$

$$\mathcal{M}^-_{r,R}(D^2u(x)) = r\sum_{\mu_i>0} \mu_i(x) + R\sum_{\mu_i<0} \mu_i(x)$$

where $0 < r \le R$ are called ellipticity constants and $\mu_i = \mu_i(D^2u(x))$, $i = 1, \ldots, N$ represent the eigenvalues associated to the hessian matrix $D^2u(x)$. Furthermore we introduce (as in [6,7]) the dimension like quantities $\tilde{N}_+$ and $\tilde{N}_-$ for $\mathcal{M}^\pm_{r,R}$, defined as

$$\tilde{N}_+ = \frac{r}{R}(N-1) + 1 \qquad \text{and } \tilde{N}_- = \frac{R}{r}(N-1) + 1 \tag{8}$$

In the scope of this work, only $\tilde{N}_+$ plays a role, therefore throught the article we will refer to $\tilde{N}_+$ as $\tilde{N}$.

We will now recall some results obtained in previous papers which will be needed later on.

We start with a version of the Hopf Lemma for non-proper operators

Lemma 2.1. [1] *Let $\Omega \subset \mathbb{R}^N$ be a smooth domain and let $b, c \in L^\infty(\Omega)$. Suppose $u \in \mathcal{C}(\bar{\Omega})$ is a viscosity solution of*

$$\mathcal{M}^-(D^2u) - b(x)|Du| + c(x)u \le 0 \qquad \text{in } \Omega. \tag{9}$$

$$u \ge 0 \qquad \text{in } \Omega. \tag{10}$$

Then either $u \equiv 0$ or $u > 0$ in Ω. Furthermore, at any point $x_0 \in \partial\Omega$ where $u(x_0) = 0$, we have

$$\liminf_{t\to 0} \frac{u(x_0 + tv) - u(x_0)}{t} < 0 \tag{11}$$

for every $v \in \mathbb{R}^N$ such that $\langle v, \nu(x_0)\rangle > 0$, where $\nu(x_0)$ is the external normal at x_0 with respect to the boundary of Ω.

Lemma 2.2. [16] *Let $i \in \{1, \dots, k\}$, if $\lambda \geq \lambda_*$, then any $z \in D_\lambda \cap \partial {G_i}^\lambda$ has one and only one of the following properties,(see Fig. 1):*

I) $\Gamma_t(z) \in \Sigma_\lambda$ for small positive values of t or there exists a sequence $t_m \searrow 0$ such that $\Gamma_{t_m}(z) \in D_\lambda \cap \partial {G_i}^\lambda$

II) $\underline{t} \leq d(z, T_\lambda)$, the open segment $(\Gamma_{\underline{t}}(z), z)$ belongs to G_i^λ, and $\Gamma_{\underline{t}}(z) \in \partial G_i^\lambda$.

III) $\underline{t} < d(z, T_\lambda)$, the open segment $(\Gamma_{\underline{t}}(z), z)$ belongs to G_i^λ, and $\Gamma_{\underline{t}}(z) \in \partial G_i$.

IV) $\lambda = \lambda_*$ and $z \in \partial G_i^\lambda \cap \partial G_i$

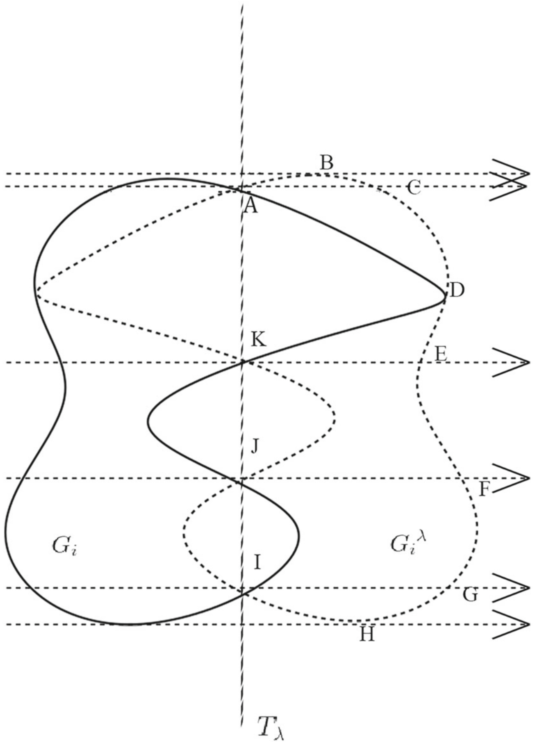

FIGURE 1. The four different types of points of $\partial G_i^\lambda \cap D_\lambda$: The arcs $(A, B], [H, I), (J, K)$ are of type (I). $(B, C], [E, F], [G, H)$ are of type (II), $(C, D), (D, E), (F, G)$ are of type (III) and D is of type (IV)

Lemma 2.3. [2] *Let f be a locally Lipschitz function, suppose that Ω is a bounded $\mathcal{C}^{2,\beta}$ domain and suppose that H_0 is an hyperplane such that*

- *there is $P \in H_0 \cap \partial\Omega$ such that $\nu(P) \in H_0$;*
- *Ω^- is the intersection of Ω with one of the half spaces bounded by H_0 and Ω^+, its reflection with respect to H_0 is contained in Ω.*

Let $u \geq 0$ be a viscosity solution of

$$\mathcal{M}^{\pm}_{r,R}(D^2u) + f(u) = 0 \qquad in\ \Omega$$

Let u_0 be the reflection of $u|\Omega^-$ in Ω^+. If $u_0 > u$ in a neighborhood of P in Ω, $u(P) = u_0(P)$ and $\nabla u_0(P) = \nabla u(P) \neq 0$, then there exists a $t_1 > 1$, such that if $\frac{R}{r} < t_1$ and for any direction v pointing inside Ω^+

$$\partial^2_v u_0(P) > \partial^2_v u(P) \tag{12}$$

Lemma 2.4. [2] *Let u be a $\mathcal{C}^1$ solution of (1)–(7), if ∂G is the graph of a $\mathcal{C}^2$ function ψ, then for every $x_0 \in \partial G$, $D^2u(x_0)$ only depends on $\psi(x_0), \nabla\psi(x_0)$ and $D^2\psi(x_0)$.*

Lemma 2.5. [11] *Let $\Omega \subset \mathbb{R}^N$ be an unbounded smooth domain. Suppose $u \in \mathcal{C}^2(\Omega)$ satisfies for $x \in \Omega$*

$$\mathcal{M}^{-}_{r,R}(D^2u)) + c(x)u \leq 0$$

where, $c(x)$ is a locally bounded real function such that

$$c(x) < \frac{-q(R(q+1) - r(N-1))}{|x|^2}$$

for some $q \in (0, \tilde{N} - 2)$. If, $\liminf_{|x|\to\infty} u(x)|x|^q \geq 0$ and $u \geq 0$ on $\partial\Omega$ then $u \geq 0$ in Ω.

3. Proof of Theorem 1

Using all the notations introduced in Sect. 2, we consider, for $x \in \Sigma_\lambda$, the function

$$w_\lambda(x) = u_\lambda(x) - u(x) = u(x^\lambda) - u(x).$$

where u is a solution of (1)–(7) for either one of the operators $\mathcal{M}^{\pm}_{r,R}$.

The proof of Theorem 1 will be obtained through several steps.

Step 1: $\exists\bar{\lambda} \in \mathbb{R}$ such that $\forall\lambda \geq \bar{\lambda}, w_\lambda \geq 0$,

Proof. Since $u \to 0$ as $|x| \to \infty$, we can take $\bar{\lambda}$ such that $\bar{\lambda} > d$, and, for $i \in \{1, \ldots, k\}$

$$u(x) \leq \frac{1}{2}\min_i a_i \qquad \text{for } |x| > \bar{\lambda}.$$

Hence

$$w_\lambda|\partial G_i^\lambda = u_\lambda|\partial G_i^\lambda - u|\partial G_i^\lambda > \frac{a_i}{2} > 0. \text{ for } \lambda > \bar{\lambda}.$$

since $z \in \partial G_i^\lambda$ implies $|z| > |\lambda|$.

The above proves that w_λ is positive on ∂G_i^λ. The proof on the rest of the domain is as follows:

Note that regardless of the operator, $\mathcal{M}^+_{r,R}$ or $\mathcal{M}^-_{r,R}$, w_λ satisfies

$$\mathcal{M}^-_{r,R}(D^2u) + \frac{f(u_\lambda) - f(u)}{w_\lambda} w_\lambda \leq 0 \qquad \text{in } \Sigma_\lambda. \tag{13}$$

$$w_\lambda \geq 0 \qquad \text{on } \partial\Sigma_\lambda. \tag{14}$$

Since by hypothesis $u = O(|x|^{-\gamma})$ and (4) holds, we have that, $\frac{f(u_\lambda)-f(u)}{w_\lambda} = O(|x|^{-2})$ and $\liminf_{|x|\to\infty} w_\lambda |x|^q = 0$, for some positive $q < \min\{\gamma, \tilde{N}-2\}$. Thus, Lemma 2.5 applies and we get $w_\lambda \geq 0$. □

Now we define

$$\lambda_0 = \inf\{\lambda \in \mathbb{R} \,|\, w_\mu \geq 0 \text{ in } \Sigma_\lambda, \ \forall \mu > \lambda\}.$$

The above set is nonempty due to the previous step and clearly λ_0 is finite. Also it follows from the continuity of u that $w_{\lambda_0} \geq 0$.

Step 2: $\boldsymbol{\frac{\partial u}{\partial x_1} < 0}$ **in** $\boldsymbol{\{x = (x_1, x') \in \mathbb{R}^N \,|\, x_1 > \max\{\lambda_0, d\}\}}$

Proof. Let $\mu > \max\{d, \lambda_0\}$. By Lemma 2.1 either $w_\mu \equiv 0$ or $w_\mu > 0$ in Σ_μ with $\frac{\partial w_\mu}{\partial \nu} < 0$ on the points of $\partial\Sigma_\mu$ such that $w_\mu = 0$.

If $w_\mu > 0$ in Σ_μ, we get on T_μ

$$w_\mu = 0, \qquad 0 > \frac{\partial w_\mu}{\partial \nu} = \frac{\partial w_\mu}{\partial(-x_1)} = -\frac{\partial w_\mu}{\partial x_1} = 2\frac{\partial u}{\partial x_1} \tag{15}$$

and so the assertion holds. Let us prove that $w_\mu \equiv 0$ in Σ_μ cannot occur.

Indeed, if $w_\mu \equiv 0$ we have two cases:

1. $w_\lambda > 0$ in Σ_λ for all $\lambda > \mu$.
2. $w_{\tilde{\mu}} \equiv 0$ in $\Sigma_{\tilde{\mu}}$ for some $\tilde{\mu} > \mu$.

Case 1: We just repeat the above argument for every $\lambda > \mu$ and obtain

$$\frac{\partial u}{\partial x_1} < 0 \text{ in } T_\lambda \text{ so that, } \frac{\partial u}{\partial x_1} < 0 \text{ in } \Sigma_\mu = \bigcup_{\lambda > \mu} T_\lambda. \tag{16}$$

Then take $x \in \Sigma_\mu$ such that $\{x + te_1\,,\, t \in \mathbb{R}\} \cap G = \emptyset$ (such x always exists since G is bounded) and notice that:

$$u(x) \leq u(x^{\lambda_0}) = u((x^{\lambda_0})^\mu) < u(x) \tag{17}$$

where the last inequality follows from $\frac{\partial u}{\partial x_1} < 0$ in Σ_μ. The contradiction in (17) shows that Case 1 does not hold.

Case 2: We take $y \in \Sigma_\mu$ such that $\{y + te_1\,,\, t \in \mathbb{R}\} \cap G = \emptyset$ and notice that

$$0 < u(y) = u(y^{\tilde{\mu}}) = u((y^{\tilde{\mu}})^\mu) \tag{18}$$

Since $\tilde{\mu} - \mu > 0$, continuing to reflect with respect to $T_{\tilde{\mu}}$ and T_μ alternatively, we can construct a sequence of points y_n such that $u(y_n) = u(y)$ and $|y_n| \to \infty$. This is clearly a contradiction with the hypothesis that $u(x) \to 0$ as $|x| \to \infty$. So also Case 2 cannot hold. □

Step 3: $\lambda_0 \leq d$

Proof. Assume not, then, $\lambda_0 > d$. Since u is a continuous function, $w_{\lambda_0} \geq 0$ in Σ_{λ_0}. Moreover $\frac{\partial u}{\partial x_1} < 0$ in $\{x_1 > \lambda_0\}$ by Step 2. By the strong maximum principle, either $w_{\lambda_0} > 0$ or $w_{\lambda_0} \equiv 0$. Assume $w_{\lambda_0} \equiv 0$. Take $y = (y_1, y'), z = (z_1, z') \in \partial G_1^{\lambda_0}$ such that $\lambda_0 < y_1 < z_1$ and $y' = z'$. Then $u(z) < u(y)$. On the other hand we also have

$$u(y) = u(y^{\lambda_0}) = a_1 = u(z^{\lambda_0}) = u(z) \tag{19}$$

Thus we get a contradiction which shows that $w_{\lambda_0} \not\equiv 0$.

Now consider the case $w_{\lambda_0} > 0$. By definition of λ_0 there is a sequence $\{\lambda_m\}$ such that $\lambda_m \nearrow \lambda_0$ and for every m there exists a minimizer $x_m \in \Sigma_{\lambda_m}$ such that $w_{\lambda_m}(x_m) < 0$ and $\nabla w_{\lambda_m}(x_m) = 0$.

Since $\lambda_0 > d$ we fix m_0 such that

$$dist(G^{\bar{\lambda}_m}, T_{\lambda_0}) \geq \frac{1}{2} dist(G^{\bar{\lambda}_0}, T_{\lambda_0}) > 0 \tag{20}$$

for $m \geq m_0$. We will now break in two cases

1. $x_m \in Int(\Sigma_{\lambda_m})$ for every $m \geq m_0$.
2. $x_m \in \partial\Sigma_{\lambda_m}$ for some $m \geq m_0$

Case 1: If $x_m \to x_0$, then passing to the limit in the definition of x_m we obtain that $w_{\lambda_0}(x_0) = 0$ and $\nabla u(x_0) = 0$. Therefore we must have, by the strong maximum principle, $x_0 \in \partial\Sigma_{\lambda_0}$, but that is a clear contradiction since Hopf Lemma implies $\nabla u(x_0) \neq 0$.

If x_m diverges, then we consider the sequence of points $\{y_m\}$ that minimize $w_{\lambda_m}|x|^q$ in Σ_{λ_m}, for $q > 0$ to be chosen. Clearly if x_m diverges so does y_m. The function $w_{\lambda_m}|x|^q$ satisfies in $\Sigma_{\lambda_m} \cap \{w_{\lambda_m} < 0\}$:

$$\mathcal{M}^-_{r,R}(D^2 w_{\lambda_m}|x|^q) - b(x)|\nabla(w_{\lambda_m}|x|^q)| + c(x) w_{\lambda_m}|x|^q \leq 0 \tag{21}$$

where,

$$b(x) = \frac{2\sqrt{N} R |\nabla(|x|^{-q}|)}{|x|^{-q}}$$

and,

$$c_m(x) = \frac{f(u_{\lambda_m}) - f(u)}{w_\lambda} + \frac{q(R(q+1)) - r(N-1))}{|x|^2} < 0 \tag{22}$$

To see the above take ϕ as a test function for $w_{\lambda_m}|x|^q$. Thus, $\phi|x|^{-q}$ is a test function for w_{λ_m}. Since w_{λ_m} is a subsolution, we obtain by using $\phi|x|^{-q}$ as a test function:

$$0 \geq \frac{f(u_{\lambda_m}) - f(u)}{w_\lambda} \phi|x|^{-q} + \mathcal{M}^-_{r,R}(\phi|x|^{-q}) \tag{23}$$

Using

$$D^2(\phi|x|^q) = |x|^q D^2\phi + D^2(|x|^q)\phi + 2\nabla(|x|^q) \otimes \nabla\phi$$

and that for every $A \in \mathcal{M}_{NxN}$, whose eigenvalues belong to (r, R),

$$Tr(A(p \otimes q)) \leq \sqrt{N} R |p||q| \tag{24}$$

we obtain (21).

If q belongs to $(0, \tilde{N} - 2)$ and $|x| > L$, for L sufficiently large, by (3) and (4) we get $c_m < 0$. Now we fix m such that $|y_m| > L + 2$ and consider the domain $\Sigma_{\lambda_m} \setminus B_{L+1}$. By the weak maximum principle, which we can apply since $c_m < 0$, inequality (21) implies that $w_{\lambda_m}|x|^q$ cannot achieve its minimum at y_m in $\Sigma_{\lambda_m} \setminus B_{L+1}$ unless it is constant. However, if q is chosen to be smaller than γ, by 3, we obtain $\lim_{|x| \to \infty} w_{\lambda_m}|x|^q = 0$. Hence a contradiction.

Case 2: Since $w_{\lambda_m}|T_{\lambda_m} = 0$ we have that $x_m \in \partial G^{\lambda_m}$.

Since $dist(G^{\lambda_m}, T_{\lambda_0}) > 0$, we get that $\{x_m + te_1\} \subset D_{\lambda_0}$ for small positive values of t.

Now we will use Lemma 2.2 to obtain a contradiction. Clearly x_m is not a point of type (III) or (IV). Let us prove it cannot be of types (I) and (II).

Type (I): $\Gamma_t(x) \in \Sigma_{\lambda_m}$. Then, by Step 2,

$$0 \leq \frac{\partial w_{\lambda_m}}{\partial(-x_1)}(x_m) = \frac{\partial u}{\partial(x_1)}(x_m) + \frac{\partial u}{\partial(x_1)}(x_m^\lambda) < \frac{\partial u}{\partial(x_1)}(x_m^\lambda) \tag{25}$$

We will now show that $\frac{\partial u}{\partial(x_1)}(x_m^\lambda) \leq 0$ to obtain a contradiction

Since $\Gamma_t(x_m) \in \Sigma_{\lambda_m}$, this implies that $\langle \nu(x_m^\lambda, e_1) \geq 0$. Since $u \equiv a_i$ on ∂G_i we obtain

$$\frac{\partial u}{\partial \xi} = 0 \qquad \forall \xi \text{ in the tangent space to } \partial G_i$$

Therefore

$$\frac{\partial u}{\partial x_1}(x_m^\lambda) = \frac{\partial u}{\partial \nu}(x_m^\lambda)\langle \nu(x_m^\lambda, e_1)\rangle \leq 0 \tag{26}$$

Let us now see that it is not possible to have a sequence $t_m \searrow 0$ such that $\Gamma_{t_m} \in \partial G_i^{\lambda_m}$. If that was the case we would have, by taking $y \in \Gamma_{t_m}(x_m)$, that

$$w_{\lambda_m}(x_m) \leq w_{\lambda_m}(y) = a_i - u(y) < a_i - u(x_m) = w_{\lambda_m}(x_m). \tag{27}$$

which is not possible. Type (II): Clearly $\Gamma_{\underline{t}} \in \partial G_i^{\lambda_m} \cap D_{\lambda_0}$ hence as before

$$w_{\lambda_m}(x_m) \leq w_{\lambda_m}(\Gamma_{\underline{t}}) = a_i - u(\Gamma_{\underline{t}}) < a_i - u(x_m) = w_{\lambda_m}(x_m). \tag{28}$$

□

Step 4: For any $z \in \partial G$ and any unit vector η for which $\langle \eta, \nu(z)\rangle > 0$, we can find a small enough ball $B_\delta(z)$ such that $\frac{\partial u}{\partial \eta} < 0$ in $B_\delta(z) \setminus \bar{G}$

The proof of Step 4 is done by induction and will use some of the following steps. For clarity's sake we will delay the proof until all the remaining steps have been proved.

Step 5: $w_\lambda > 0$ for any $\lambda \in [\lambda_0, \infty) \cap (\lambda_*, \infty)$

Proof. Based on Steps 2 and 3, we may assume $\lambda \leq d$. By the strong maximum principle we just have to see that w_λ is not identically zero in a connected component Z of Σ_λ. We will proceed arguing by contradiction, i.e, we assume that such a connected component exists.

First we observe that for every Y connected component of Σ_λ we have

$$dist(\partial Y \cap D_\lambda, T_\lambda) = 0. \tag{29}$$

In fact, Y is connected, hence Y^λ is a connected component of $(\mathbb{R}^N \backslash \bar{G}) \backslash \bar{D_\lambda}$. From that it follows that either Y^λ contains a left neighborhood of T_λ or

$$dist(\partial Y^\lambda \setminus \bar{D_\lambda}, T_\lambda) = 0. \tag{30}$$

However, the former case is not possible since $\lambda \leq d$ means that G $\cap T_\lambda$ is non empty. Thus, (29) follows by reflecting (30). Therefore take $\{z_m\}$ a sequence in $\partial Z \cap D_\lambda$ such that $z_m \to z_0 \in T_\lambda \cap \partial Z \cap D_\lambda$.

Since $\lambda > \lambda_*$, $\langle \nu(z_m^\lambda), e_1 \rangle < 0$, for m sufficiently large, and the open segment (z_m, z_m^λ) is contained in the ball $B_\delta(z_0) \backslash \bar{G}$, where delta is the one given by Step 4. Therefore by Step 4, u strictly decreases in (z_m^λ, z_m), which means $w_\lambda(z_m) > 0$. This is a contradiction with the fact that w is identically zero in Z. □

Step 6: $\frac{\partial u}{\partial x_1} < 0$ in $D_{\tilde{\lambda}} \setminus \bar{G}$ for $\tilde{\lambda} = \max\{\lambda_0, \lambda_*\}$

Proof. The proof follows in the same way as in the proof of Step 2, using Step 5 and Hopf Lemma. □

Step 7: $\lambda_0 \leq \lambda_*$

Proof. We will argue by contradiction, suppose $\lambda_0 > \lambda_*$. By Step 5, we know that w_{λ_0} is positive in Σ_{λ_0}, therefore we may obtain as in Step 3, a sequence of $\lambda_m \nearrow \lambda_0$, such that $w_{\lambda_m}(x_m) < 0$, for some $x_m \in \Sigma_{\lambda_m}$. We will break into the following cases:

Case 1: There is a subsequence of $\{x_m\}$ such that $x_m \in \text{Int}\Sigma_{\lambda_m}$.

The cases where x_m diverges or converges to a point on the regular part of $\partial\Sigma_{\lambda_m}$ are analogous to what have been done in Step 3. Furthermore the proof of Step 5 shows that $w_{\lambda_m} > 0$ for m sufficiently large.

Case 2: There is a subsequence of $\{x_m\}$ such that $x_m \in \partial\Sigma_{\lambda_m}$.

Clearly x_m is not in T_{λ_m}, therefore $\{x_m\}$ is a bounded sequence and passing to a subsequence we may assume $x_m \in \partial G_i^{\lambda_m}$, for a fixed $i \in \{1, \ldots, k\}$. We will now proceed by using Lemma 2.2 to reach a contradiction.

First let us show that x_m is not contained in $\partial\Sigma_{\lambda_m} \cap \{x_1 \leq \lambda_0\}$. If that was the case then $x_m \to x_0 \in T_{\lambda_0} \cap \partial G$, therefore the same argument as in Step 5 would imply a contradiction. Henceforth we will assume that $x_m \in \partial G_{i_m^\lambda} \cap \{x_1 > \lambda_0\}$. It is clear that x_m is not of type (IV) since $\lambda_m > \lambda_*$. Arguing as in Step 3 we also can exclude the case where x_m is of type (I). We will now consider two cases in order to finish the proof. Let $y_m = \Gamma_{t_m}(x_m)$.

Case (a): $y_m \in D_{\lambda_0}$ for some $m \in \mathbb{N}$.

If x_m is of type (II), then $y_m \in \partial G_i^{\lambda_m} \cap D_{\lambda_0}$. Thus we may obtain a contradiction as in Step 3. If x_m is of type (III), then $y_m \in \partial G_i \cap D_{\lambda_0}$, by

Steps 4 and 6 we obtain that u is strictly decreasing from y_m to x_m. However this contradicts the fact $w_{m(x_m)} < 0$ since

$$w_{\lambda_m}(x_m) = a_i - u(x_m) = u(y_m) - u(x_m) > 0 \quad (31)$$

Case 2: $y_m \in \bar{D}_{\lambda_m} \setminus D_{\lambda_0}$ for every $m \in \mathbb{N}$.

Since $y_m \in \partial G \cup \partial G^{\lambda_m}$, $\{y_m\}$ is bounded. Thus we may obtain that up to a subsequence y_m converges to $y_0 \in \partial G_i^{\lambda_0} \cap T_{\lambda_0}$. Therefore for m sufficiently large, y_m and its projection on T_{λ_0} belong to $B_\delta(y_0)$, where δ is given by Step 4. Again combining Steps 4 and 6 we obtain that u decreases on the whole segment (y_m, x_m) and get a contradiction as in the previous case. □

Step 8: $w_{\lambda_*} \equiv 0$ in at least one connected component of Σ_{λ_*}

Proof. If there exists $z_0 \in \partial G \cap T_{\lambda_*}$ such that $\langle \nu(z_0), e_1 \rangle = 0$, we will show that w_{λ_*} is identically zero in the connected component that contains z_0 on its boundary. For that, assume $u > u_{\lambda_*}$ in a certain ball around z_0. We take t_1 as in the assumption of Theorem 1.1, then, since we assume $\frac{R}{r} < t_1$, Lemma 2.3 implies that for every direction v pointing inside Σ_{λ_*} either $\partial_v u_{\lambda_*}(z_0) > \partial_v u(z_0)$ or $\partial_v^2 u_{\lambda_*}(z_0) > \partial_v u^2 u(z_0)$.

The first inequality cannot happen, since on ∂G_i, $\frac{\partial u}{\partial_v}$ is constant. The second inequality also cannot happen since $D^2 u_{\lambda_*}(z_0) = D^2 u(z_0)$ because by Lemma 2.4 the hessian only depends on the shape of the boundary. Therefore we obtain $u \equiv u_{\lambda_*}$ in $B(z_0, R)$, by the strong maximum principle, and also $u \equiv u_{\lambda_*}$ in the whole connected component which contains z_0.

If there is not such z_0 then, by the definition of λ_* we can find a point $z_1 \in \partial G \cap D_{\lambda_*}$ such that $z_1^{\lambda_*} \in \partial G$ is a point of internal tangency. Due to the boundary conditions, $w_{\lambda_*}(z_1) = 0 = \frac{\partial w_{\lambda_*}}{\nu}(z_1)$. That contradicts Hopf Lemma. □

Step 9: Let Z be a connected component of Σ_{λ_*} such that $w_{\lambda_*} = 0$ in Z. Then $\partial Z \setminus T_{\lambda_*} \subset \partial G$

Proof. We shall use Lemma 2.2 in order to show that all points on $\partial Z \setminus T_{\lambda_*}$ are of symmetry type (IV). Suppose by contradiction that there is a point z that is not of symmetry type (IV). The point z cannot be of type (I), otherwise we could argue as in Step 3 and obtain a contradiction. Also it is not of type (II) or type(III). To see that just set $y = \Gamma_{\underline{t}}(z) \in \Sigma_{\lambda_*}$, then, by Step 6, u is strictly decreasing on the segment (y, z). If z was of type (II) we would have:

$$a_i = u(y^{\lambda_*}) \geq u(y) > u(z) = u(z^{\lambda_*}) = a_i. \quad (32)$$

which is a contradiction. Type(III) can be excluded in an analogousbreak fashion. □

Step 10: End of the proof of Theorem 1.1

Let us denote by G^C the complement of G in $\mathbb{R}^N$. Then, the set X,

$$X = Z \cup Z^{\lambda_*} \cup (\partial Z \cap G^C) \cup (\partial Z^{\lambda_*} \cap G^C) \quad (33)$$

is symmetric with respect to T_{λ_*}. Note that by Step 9:

$$\partial Z \cap G^C \subset T_{\lambda_*} \cap G^C \tag{34}$$

One may check that X is an open subset in G^C, therefore $G^C \setminus X = G^C \setminus \bar{X}$ which implies that $X = G^C$ since G^C is connected. This implies that u and G are symmetric in the x_1 direction Therefore applying the result for every direction we obtain that G is radially symmetric and the solution u is radial.

Proof of Step 4:

Now we give the proof of Step 4 that was previously omitted.

Proof. We will prove the result by induction on k. First assume that $k = 1$ in (5). If $\alpha_1 < 0$, the result is trivial by continuity, therefore lets assume $\alpha_1 = 0$, which implies $\nabla u \equiv 0$ and $|D^2 u| = |\partial_\nu^2 u|$ on ∂G.

Choose $z_0 \in T_d \cap \partial G$, then

$$\frac{\partial u}{\partial x_1}(z_0) = \frac{\partial u}{\partial \nu}(z_0) = 0 \tag{35}$$

Since $\lambda_0 \leq d$ by Step 3, then Steps 2 implies

$$\frac{\partial u}{\partial x_1}(\Gamma_t(z_0)) < 0$$

for negative t. This implies

$$0 \geq \frac{\partial^2 u}{\partial x_1^2}(z_0) = \frac{\partial^2 u}{\partial \nu^2}(z_0) \tag{36}$$

On the other hand, the eigenvalues of $D^2 u$ for $x \in \partial G$ are

- $\frac{\partial^2 u}{\partial \nu^2}(x)$ with multiplicity one, (associated to the normal direction at x).
- 0 with multiplicity $N - 1$, (associated to the tangent space to ∂G at x).

Therefore by (1) and (7), we obtain for x in ∂G

$$\mathcal{M}^+_{r,R}(D^2 u) = r \frac{\partial^2 u}{\partial \nu^2} | \partial G = -f(a_1) \tag{37}$$

Analogously one would obtain for $\mathcal{M}^-_{r,R}$:

$$\mathcal{M}^-_{r,R}(D^2 u) = R \frac{\partial^2 u}{\partial \nu^2} | \partial G = -f(a_1) \tag{38}$$

In either case, by (36) we get that $f(a_1)$ must be a non-negative value. If $f(a_1)$ is strictly positive, we are done since $u \in \mathcal{C}^2(\mathbb{R}^{\mathbb{N}} \setminus G)$ we would have

$$\frac{\partial^2 u}{\partial \eta^2} = \frac{\partial^2 u}{\partial \nu^2} \langle \eta, \nu \rangle^2 < 0 \text{ on } \partial G. \tag{39}$$

This together with (35) and (36) imply the assertion.

If $f(a_1) = 0$ then all the first and second derivatives of u vanish on ∂G, therefore extending u as a_1 on G would provide a solution on the whole $\mathbb{R}^N$. However that is not possible for such a solution in $\mathbb{R}^N$, either by known results of symmetry and monotonicity (see [11]) or by Steps 1–3, the solution cannot be flat in an open set. Suppose now that the result is valid for $k-1$. Therefore, by steps 5 to 10, Theorem 1.1 is proved for $k-1$ domains G_i.

Set

$$I = \{i \,|\, \alpha_i < 0 \text{ or } \frac{\partial^2 u}{\partial \nu^2} < 0 \text{ on } \partial G_i\} \tag{40}$$

and J being $\{1...k\} \setminus I$. We claim that J is empty. Define D as

$$D = \max_{j \in J} d_j \tag{41}$$

If $D < \lambda_*$, we could complete steps 5 to 10 and obtain a contradiction. If $D \geq \lambda_*$ the moving plane reaches at least one domain G_j with $j \in J$, therefore arguing as in the case $k = 1$ we obtain a contradiction. □

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