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Boundary value problems associated with Hamiltonian systems coupled with positively-(p,q)-homogeneous systems

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Abstract. We study the multiplicity of solutions for a two-point boundary value problem of Neumann type associated with a Hamiltonian system which couples a system with periodic Hamiltonian in the space variable with a second one with positively-(p,q)-homogeneous Hamiltonian. The periodic problem is also treated.

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1. Introduction and statement of the main result

In the recent paper [7], a multiplicity result for a Neumann-type boundary value problem associated with a Hamiltonian system has been proved. It is the aim of this paper to extend this result to coupled systems, the first of which is of the type considered in [7], while the second one involves a positively-(p, q)-homogeneous and positive Hamiltonian function.

Denoting by J the standard symplectic matrix, our Hamiltonian system

$$J\dot{z} = \nabla_z H(t, z),$$

when writing $z = ((x, y), (u, v)) \in \mathbb{R}^{2M} \times \mathbb{R}^{2L}$, is driven by a Hamiltonian function of the type

$$H(t, z) = \mathcal{H}(t, x, y) + \mathscr{H}(u, v) + P(t, x, y, u, v).$$

To be more precise, we are dealing with the Hamiltonian system

$$\begin{cases} \dot{x} = \nabla_y \mathcal{H}(t,x,y) + \nabla_y P(t,x,y,u,v), \\ \dot{y} = -\nabla_x \mathcal{H}(t,x,y) - \nabla_x P(t,x,y,u,v), \\ \dot{u} = \nabla_v \mathscr{H}(u,v) + \nabla_v P(t,x,y,u,v), \\ \dot{v} = -\nabla_u \mathscr{H}(u,v) - \nabla_u P(t,x,y,u,v), \end{cases} \tag{1.1}$$

with Neumann-type boundary conditions

$$\begin{cases} y(a) = 0 = y(b), \\ v(a) = 0 = v(b). \end{cases} \tag{1.2}$$

We write

$$x = (x_1, \ldots, x_M) \in \mathbb{R}^M, \quad y = (y_1, \ldots, y_M) \in \mathbb{R}^M,$$
$$u = (u_1, \ldots, u_L) \in \mathbb{R}^L, \quad v = (v_1, \ldots, v_L) \in \mathbb{R}^L.$$

The functions $\mathcal{H} : [a,b] \times \mathbb{R}^{2M} \to \mathbb{R}$, $\mathscr{H} : \mathbb{R}^{2L} \to \mathbb{R}$ and $P : [a,b] \times \mathbb{R}^{2M+2L} \to \mathbb{R}$ are continuous, and continuously differentiable with respect to (x,y), (u,v) and (x,y,u,v), respectively.

In the sequel, the notation for the standard symplectic matrix

$$J = \left(\begin{array}{c|c} 0 & -I \\ \hline I & 0 \end{array} \right),$$

where I is the identity matrix, will also be used in any other dimensions.

Here are our hypotheses.

A1. For every $i \in \{1, \ldots, M\}$ there exists $\kappa_i > 0$ such that the functions $\mathcal{H}(t,x,y)$ and $P(t,x,y,u,v)$ are κ_i-periodic in the variable x_i.

The periodicity assumption $A1$ naturally leads us to consider the torus

$$\mathbb{T}^M = (\mathbb{R}/\kappa_1\mathbb{Z}) \times \cdots \times (\mathbb{R}/\kappa_M\mathbb{Z}).$$

Indeed, in view of this assumption, the x component of the solutions could sometimes be interpreted as belonging to $\mathbb{T}^M$.

A2. The function $P(t,x,y,u,v)$ has a bounded gradient with respect to (x,y,u,v).

Assumption $A2$ guarantees that the coupling term $P(t,x,y,u,v)$ can be seen as some kind of not so large perturbation term.

A3. All the solutions of system (1.1) satisfying $y(a) = v(a) = 0$ are defined on $[a,b]$.

In view of the results in [5,7], assumption $A3$ is surely satisfied if there exists a constant K_1 such that

$$|\nabla_x \mathcal{H}(t,x,y)| \leq K_1(1 + |y|), \quad \text{for every } (t,x,y) \in [a,b] \times \mathbb{T}^M \times \mathbb{R}^M.$$

A4. The function $\mathscr{H} : \mathbb{R}^{2L} \to \mathbb{R}$ is of the type

$$\mathscr{H}(u,v) = \sum_{j=1}^{L} \mathscr{H}_j(u_j, v_j),$$

for some functions $\mathscr{H}_j : \mathbb{R}^2 \to \mathbb{R}$ which are positively-(p_j, q_j)-homogeneous and positive, meaning that for some $p_j > 1$ and $q_j > 1$ with $(1/p_j) + (1/q_j) = 1$ we have

$$\mathscr{H}_j(\gamma^{q_j} r, \gamma^{p_j} s) = \gamma^{p_j + q_j} \mathscr{H}_j(r, s) > 0, \text{ for every } (r, s) \in \mathbb{R}^2 \setminus \{0\} \text{ and } \gamma > 0.$$

In this setting, the origin $(0, 0)$ is an isochronous center for the planar autonomous system

$$J\dot{\zeta} = \nabla \mathscr{H}_j(\zeta). \tag{1.3}$$

For every $j \in \{1, \ldots, L\}$, besides the origin all solutions of system (1.3) are periodic and have the same minimal period, which will be denoted by τ_j. Moreover, for all solutions $\zeta(t) = (\zeta_1(t), \zeta_2(t))$ of (1.3) starting with $\zeta_1(0) < 0$ and $\zeta_2(0) = 0$, there is a first time $\tau_{j_+} > 0$ for which $\zeta_2(\tau_{j_+}) = 0$, while $\zeta_2(t) > 0$ for all $t \in]0, \tau_{j_+}[$, and this time τ_{j_+} is independent of $\zeta_1(0) < 0$. Similarly, if $\zeta_1(0) > 0$ and and $\zeta_2(0) = 0$, there is a first time $\tau_{j_-} > 0$ for which $\zeta_2(\tau_{j_-}) = 0$, while $\zeta_2(t) < 0$ for all $t \in]0, \tau_{j_-}[$, and this time τ_{j_-} is independent of $\zeta_1(0) > 0$. Clearly enough, $\tau_j = \tau_{j_+} + \tau_{j_-}$.

Here is our main result.

Theorem 1.1. *Assume that A1 – A4 hold true. Let $\tau_{j_+} = \tau_{j_-}$ and*

$$\frac{b - a}{\tau_{j_+}} \notin \mathbb{N}, \quad \text{for every } j \in \{1, \ldots, L\}.$$

Then there are at least $M + 1$ geometrically distinct solutions of the boundary value problem (1.1)–(1.2).

Notice that, when a solution has been found, infinitely many others appear by just adding an integer multiple of κ_i to the x_i-th component. We say that two solutions are *geometrically distinct* if they cannot be obtained from each other in this way.

Let us remark here that a sufficient condition for having satisfied the assumption $\tau_{j_+} = \tau_{j_-}$ is that the function $\mathscr{H}_j$ is even in v. This is a frequent case in the applications, where, e.g., $\mathscr{H}_j$ is quadratic in v.

Theorem 1.1 generalizes the result in [7], where the case $P \equiv 0$ was treated, dealing only with the system in (x, y). In order to prove it, we first consider the case when, writing $w = (u, v)$, the second Hamiltonian functions is of the type $\mathscr{H}(w) = \frac{1}{2}\langle \mathbb{A}w, w \rangle$, where $\mathbb{A}$ is a particular diagonal matrix. Then, by a symplectic change of variables, we are able to transform the positively-(p, q)-homogeneous Hamiltonian in the quadratic one.

We also study the periodic problem for such kind of Hamiltonian systems, and obtain a similar multiplicity result when a suitable *twist condition* is assumed. This part of the paper is related to the Poincaré–Birkhoff Theorem [15], and we exploit some results obtained in [4], where any symmetric matrix $\mathbb{A}$ can be considered, provided that a nonresonance condition is also assumed. We thus generalize to this setting some results obtained in [3,8,9].

At the end of the paper we will analyze the possibility of dealing with any symmetric matrix $\mathbb{A}$, provided that a nonresonance condition is assumed,

also for the Neumann-type problem. However, we succeed doing this only in the case $L = 1$, while the case $L \geq 2$ remains an open problem.

Let us describe more in detail how the paper is organized.

In Sect. 2 we study the Neumann-type boundary value problem in the particular case when

$$\mathscr{H}(u, v) = \frac{1}{2} \sum_{j=1}^{L} \lambda_j (u_j^2 + v_j^2),$$

for some positive constants $\lambda_1, \ldots, \lambda_L$. The proof is variational, and it is modeled on the method developed in [7]. However, some delicate estimates are needed in order to prove the invertibility of the involved selfadjoint operator.

In Sect. 3 we provide the proof of Theorem 1.1. The idea is to construct a symplectic change of variables, so to reduce the problem to the one already treated in Sect. 2.

In Sect. 4 we study the periodic problem. Here we need to introduce a *twist condition*, which recalls the classical assumption in the Poincaré–Birkhoff Theorem. We obtain a similar multiplicity result as in Theorem 1.1 by applying a corollary of the main result in [4].

Some possible applications are given in Sect. 5. For example, we propose a system of the type

$$\begin{cases} \dot{x} = f(y) + E(t), \quad \dot{y} = -A \sin x - \partial_x P(t, x, u), \\ \dot{u} = |v|^{q-2} v, \quad \dot{v} = -\mu(u^+)^{p-1} + \nu(u^-)^{p-1} + \partial_u P(t, x, u), \end{cases}$$

where $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$. The first two equations can be seen as a generalization of the pendulum equation (obtained when $f(y) = y$), while the last two equations correspond to the scalar equation

$$\frac{d}{dt}\left(|\dot{u}|^{p-2}\dot{u}\right) + \mu(u^+)^{p-1} - \nu(u^-)^{p-1} = \partial_u P(t, x, u).$$

Notice that the particular case $p = 2$ leads to a classical asymmetric oscillator. Both Neumann-type and periodic problems are analyzed.

Finally, in Sect. 6 we end with some further remarks and proposing an open problem.

In all the rest of the paper we will denote by $\langle \cdot, \cdot \rangle$ and $|\cdot|$ the Euclidean scalar product and norm on $\mathbb{R}^k$, for any $k \in \mathbb{N}$.

2. Coupling with a linear system

In this section we consider a Hamiltonian system of the type

$$\begin{cases} \dot{x} = \nabla_y \mathcal{H}(t, x, y) + \nabla_y P(t, x, y, w), \\ \dot{y} = -\nabla_x \mathcal{H}(t, x, y) - \nabla_x P(t, x, y, w), \\ J\dot{w} = \mathbb{A}w + \nabla_w P(t, x, y, w). \end{cases} \tag{2.1}$$

Here, the functions $\mathcal{H} : [a, b] \times \mathbb{R}^{2M} \to \mathbb{R}$ and $P : [a, b] \times \mathbb{R}^{2M+2L} \to \mathbb{R}$ are continuous, and continuously differentiable with respect to (x, y) and (x, y, w),

respectively. The $2L \times 2L$ matrix $\mathbb{A}$ is of the type

$$\mathbb{A} = \left(\begin{array}{c|c} \mathbb{B}_L & 0 \\ \hline 0 & \mathbb{B}_L \end{array} \right), \tag{2.2}$$

where

$$\mathbb{B}_L = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & \lambda_L \end{bmatrix},$$

for some positive real numbers $\lambda_1, \ldots, \lambda_L$. Writing

$$x = (x_1, \ldots, x_M) \in \mathbb{R}^M, \quad y = (y_1, \ldots, y_M) \in \mathbb{R}^M,$$

and $w = (u, v) \in \mathbb{R}^{2L}$, with

$$u = (u_1, \ldots, u_L) \in \mathbb{R}^L, \quad v = (v_1, \ldots, v_L) \in \mathbb{R}^L,$$

we consider the Neumann-type boundary conditions

$$\begin{cases} y(a) = 0 = y(b), \\ v(a) = 0 = v(b). \end{cases} \tag{2.3}$$

Here is the main result of this section.

Theorem 2.1. *Assume that A1 – A3 hold true, and*

$$\frac{b-a}{\pi} \lambda_j \notin \mathbb{N}, \quad \text{for every } j \in \{1, \ldots, L\}.$$

Then, the boundary value problem (2.1)–(2.3) *has at least $M + 1$ geometrically distinct solutions.*

Proof. Without loss of generality, we may assume that $[a, b] = [0, \pi]$. By $A3$ and a standard compactness argument, there exists a constant $K_2 > 0$ such that, for any solution (x, y, w) of (2.1) satisfying $y(0) = v(0) = 0$, one has that

$$|y(t)| \leq K_2, \quad \text{for every } t \in [0, \pi]. \tag{2.4}$$

Notice indeed that, by the nonresonance assumption we have, one can find a C^1 a priori bound on w, which hence belongs to a compact subset of $C([0, \pi], \mathbb{R}^{2L})$. Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a C^∞-function such that

$$\sigma(s) = \begin{cases} 1, & \text{if } |s| \leq K_2, \\ 0, & \text{if } |s| \geq K_2 + 1, \end{cases}$$

set

$$\widehat{\mathcal{H}}(t, x, y) = \sigma(|y|) \mathcal{H}(t, x, y), \tag{2.5}$$

and consider the modified system

$$\begin{cases} \dot{x} = \nabla_y \widehat{\mathcal{H}}(t, x, y) + \nabla_y P(t, x, y, w), \\ \dot{y} = -\nabla_x \widehat{\mathcal{H}}(t, x, y) - \nabla_x P(t, x, y, w), \\ J\dot{w} = \mathbb{A}w + \nabla_w P(t, x, y, w). \end{cases} \tag{2.6}$$

The new Hamiltonian function is thus

$$\widetilde{H}(t, x, y, w) = \widehat{\mathcal{H}}(t, x, y) + \tfrac{1}{2}\langle \mathbb{A}w, w\rangle + P(t, x, y, w). \tag{2.7}$$

We will prove that the boundary value problem (2.6)–(2.3) has at least $M+1$ geometrically distinct solutions. By the above argument, these solutions will satisfy (2.4), hence they will be the solutions of (2.1)–(2.3) we are looking for.

The proof is variational, and it is based on a theorem by Szulkin recalled below. We will now introduce the function spaces and the needed functionals.

2.1. The function spaces

For any $\alpha \in \,]0, 1[$, we define X_α as the set of those real valued functions $\tilde{x} \in L^2(0, \pi)$ such that

$$\tilde{x}(t) \sim \sum_{m=1}^{\infty} \tilde{x}_m \cos(mt),$$

where $(\tilde{x}_m)_{m \geq 1}$ is a sequence in $\mathbb{R}$ satisfying

$$\sum_{m=1}^{\infty} m^{2\alpha} \tilde{x}_m^2 < \infty.$$

The space X_α is endowed with the inner product and norm

$$\langle \tilde{x}, \tilde{\phi}\rangle_{X_\alpha} = \sum_{m=1}^{\infty} m^{2\alpha} \tilde{x}_m \tilde{\phi}_m, \qquad \|\tilde{x}\|_{X_\alpha} = \sqrt{\sum_{m=1}^{\infty} m^{2\alpha} \tilde{x}_m^2}.$$

For any $\beta \in \,]0, 1[$, we define Y_β as the set of those real valued functions $y \in L^2(0, \pi)$ such that

$$y(t) \sim \sum_{m=1}^{\infty} y_m \sin(mt),$$

where $(y_m)_{m \geq 1}$ is a sequence in $\mathbb{R}$ satisfying

$$\sum_{m=1}^{\infty} m^{2\beta} y_m^2 < \infty.$$

The space Y_β is endowed with the inner product and norm

$$\langle y, \rho\rangle_{Y_\beta} = \sum_{m=1}^{\infty} m^{2\beta} y_m \rho_m, \qquad \|y\|_{Y_\beta} = \sqrt{\sum_{m=1}^{\infty} m^{2\beta} y_m^2}.$$

From now on, we will consider functions x, y, u, v which can be written as

$$x(t) = \bar{x} + \tilde{x}(t), \quad \bar{x} = \frac{1}{\pi}\int_0^\pi x(t)\,dt,$$

$$u(t) = \bar{u} + \tilde{u}(t), \quad \bar{u} = \frac{1}{\pi}\int_0^\pi u(t)\,dt,$$

where $\tilde{x}$ and y belong to the spaces X_α^M and Y_β^M respectively, while functions $\tilde{u}$ and v belong to the spaces X_α^L and Y_β^L respectively.

Choose two positive numbers α, β such that

$$\alpha < \tfrac{1}{2} < \beta \quad \text{and} \quad \alpha + \beta = 1.$$

Consider the space $E = X_\alpha^M \times Y_\beta^M \times (\mathbb{R}^L \times X_\alpha^L) \times Y_\beta^L$, and the torus $\mathbb{T}^M = (\mathbb{R}/\kappa_1\mathbb{Z}) \times \cdots \times (\mathbb{R}/\kappa_M\mathbb{Z})$. The space E is endowed with the scalar product

$$\langle (\tilde{x}, y, \bar{u}, \tilde{u}, v), (\widetilde{X}, Y, \overline{U}, \widetilde{U}, V) \rangle_E = \langle \tilde{x}, \widetilde{X} \rangle_{X_\alpha^M} + \langle y, Y \rangle_{Y_\beta^M} +$$
$$+ \langle \bar{u}, \overline{U} \rangle + \langle \tilde{u}, \widetilde{U} \rangle_{X_\alpha^L} + \langle v, V \rangle_{Y_\beta^L},$$

and the corresponding norm

$$\|(\tilde{x}, y, \bar{u}, \tilde{u}, v)\|_E = \sqrt{\|\tilde{x}\|_{X_\alpha^M}^2 + \|y\|_{Y_\beta^M}^2 + |\bar{u}|^2 + \|\tilde{u}\|_{X_\alpha^L}^2 + \|v\|_{Y_\beta^L}^2}.$$

Since X_α, Y_β and $\mathbb{R}$ are separable Hilbert spaces [7, Proposition 2.3 and 2.6], the same is true for E.

By $A1$, the Hamiltonian function $\widetilde{H}$ in (2.7) is κ_i-periodic in x_i for $i \in \{1, \ldots, M\}$, hence writing $x(t) = \bar{x} + \tilde{x}(t)$, with

$$\bar{x} = \frac{1}{\pi} \int_0^\pi x(t)\, dt,$$

we can assume that $\bar{x} \in \mathbb{T}^M$ and look for solutions $(z, \bar{x}) \in E \times \mathbb{T}^M$, where

$$z = (\tilde{x}, y, \bar{u}, \tilde{u}, v).$$

These solutions will be found as critical points of a suitable functional, by applying the following theorem of Szulkin [18] (see also [10,13]).

Theorem 2.2. ([18]) *If $\varphi : E \times \mathbb{T}^M \to \mathbb{R}$ is a continuously differentiable functional of the type*

$$\varphi(z, \bar{x}) = \frac{1}{2}\langle \mathscr{L}z, z \rangle_E + \psi(z, \bar{x}),$$

where $\mathscr{L} : E \to E$ is a bounded selfadjoint invertible operator and $d\psi(E \times \mathbb{T}^M)$ is relatively compact, then φ has at least $M + 1$ critical points.

2.2. The functional and the bilinear form

We define a functional $\psi : E \times \mathbb{T}^M \to \mathbb{R}$ as

$$\psi(z, \bar{x}) = \psi\big((\tilde{x}, y, \bar{u}, \tilde{u}, v), \bar{x}\big)$$
$$= \int_0^\pi \widetilde{H}\big(t, \bar{x} + \tilde{x}(t), y(t), \bar{u} + \tilde{u}(t), v(t)\big)\, dt.$$

In the following, we will treat $\mathbb{T}^M$ as being lifted to $\mathbb{R}^M$, so $E \times \mathbb{T}^M$ will often be identified with $E \times \mathbb{R}^M$. It has been shown in [7, Proposition 2.10] and [6, Proposition 19, Proposition 22] that ψ is continuously differentiable, and the gradient function $\nabla\psi$ has a relatively compact image. In what follows we introduce the operator $\mathscr{L}$.

We first consider the space

$$D = [\widetilde{C}^1([0, \pi])]^M \times [C_0^1([0, \pi])]^M \times F_L,$$

where
$$F_L = \big(\mathbb{R}^L \times [\widetilde{C}^1([0,\pi])]^L\big) \times [C_0^1([0,\pi])]^L,$$

and define a symmetric bilinear form $\mathcal{B} : D \times D \to \mathbb{R}$ as follows. For every $z = (\tilde{x}, y, \bar{u}, \tilde{u}, v)$ and $\mathcal{Z} = (\widetilde{X}, Y, \overline{U}, \widetilde{U}, V)$ in D,

$$\mathcal{B}(z, \mathcal{Z}) = \int_0^\pi \left[\langle y', \widetilde{X} \rangle - \langle \tilde{x}', Y \rangle - \langle J\dot{w}, W \rangle + \langle \mathbb{A}w, W \rangle \right] dt,$$

where $w = (\bar{u} + \tilde{u}, v), W = (\overline{U} + \widetilde{U}, V)$ are in F_L.

Proposition 2.3. *The set D is dense in E, and the bilinear form $\mathcal{B} : D \times D \to \mathbb{R}$ is continuous with respect to the topology of $E \times E$.*

Proof. We know by [7, Proposition 2.5 and 2.8] that D is a dense subspace of E. In order to prove the second part of the statement, let us write

$$\mathcal{B}(z, \mathcal{Z}) = \mathcal{B}_1\big((\tilde{x}, y), (\widetilde{X}, Y)\big) + \mathcal{B}_2(w, W),$$

where

$$\mathcal{B}_1\big((\tilde{x}, y), (\widetilde{X}, Y)\big) = \int_0^\pi \Big(\langle y', \widetilde{X} \rangle - \langle \tilde{x}', Y \rangle \Big) dt, \tag{2.8}$$

and

$$\mathcal{B}_2(w, W) = \int_0^\pi \Big(-\langle J\dot{w}, W \rangle + \langle \mathbb{A}w, W \rangle \Big) dt. \tag{2.9}$$

It has been proved in [6, Section 3.4] that $\mathcal{B}_1$ is continuous with respect to the topology of $X_\alpha^M \times Y_\beta^M$. We need to prove that $\mathcal{B}_2$ is continuous with respect to the topology of $\mathbb{R}^L \times X_\alpha^L \times Y_\beta^L$. For $w = (w_1, \ldots, w_L)$ and $W = (W_1, \ldots, W_L)$ in F_L we have

$$\int_0^\pi \langle J\dot{w}, W \rangle dt = \sum_{j=1}^L \int_0^\pi \langle J\dot{w}_j, W_j \rangle dt, \tag{2.10}$$

and, writing $w_j = (\bar{u}_j + \tilde{u}_j, v_j), W_j = (\overline{U}_j + \widetilde{U}_j, V_j)$,

$$\int_0^\pi \langle J\dot{w}_j, W_j \rangle dt = \int_0^\pi \dot{u}_j V_j dt - \int_0^\pi \dot{v}_j \overline{U}_j dt - \int_0^\pi \dot{v}_j \widetilde{U}_j dt. \tag{2.11}$$

We decompose the involved functions as

$$v_j = \sum_{m=1}^\infty v_m^j \sin(mt), \quad V_j = \sum_{m=1}^\infty V_m^j \sin(mt),$$

$$\tilde{u}_j = \sum_{m=1}^\infty \tilde{u}_m^j \cos(mt), \quad \widetilde{U}_j = \sum_{m=1}^\infty \widetilde{U}_m^j \cos(mt).$$

By the boundary condition $v(0) = 0 = v(\pi)$, we see that

$$\int_0^\pi \dot{v}_j \overline{U}_j dt = 0.$$

Recalling that $\alpha + \beta = 1$, we have

$$\left| \int_0^\pi \dot{u}_j V_j dt \right| = \frac{\pi}{2} \left| \sum_{m=1}^\infty -m\, \tilde{u}_m^j V_m^j \right|$$

$$\leq \frac{\pi}{2} \sum_{m=1}^\infty \left| m^\alpha \tilde{u}_m^j m^\beta V_m^j \right|$$

$$\leq \frac{\pi}{2} ||\tilde{u}_j||_{X_\alpha} ||V_j||_{Y_\beta},$$

and

$$\left| \int_0^\pi \dot{v}_j \widetilde{U}_j dt \right| = \frac{\pi}{2} \left| \sum_{m=1}^\infty m\, v_m^j \widetilde{U}_m^j \right|$$

$$\leq \frac{\pi}{2} \sum_{m=1}^\infty \left| m^\alpha \widetilde{U}_m^j m^\beta v_m^j \right|$$

$$\leq \frac{\pi}{2} ||\widetilde{U}_j||_{X_\alpha} ||v_j||_{Y_\beta}.$$

Going back to (2.11), for each $j \in \{1, \ldots, L\}$, we thus have

$$\left| \int_0^\pi \langle J\dot{w}_j, W_j \rangle dt \right| \leq \frac{\pi}{2} ||w_j||_{\mathbb{R} \times X_\alpha \times Y_\beta} ||W_j||_{\mathbb{R} \times X_\alpha \times Y_\beta}.$$

Hence, by (2.10),

$$\left| \int_0^\pi \langle J\dot{w}, W \rangle dt \right| \leq \frac{\pi}{2} ||w||_{\mathbb{R}^L \times X_\alpha^L \times Y_\beta^L} ||W||_{\mathbb{R}^L \times X_\alpha^L \times Y_\beta^L}.$$

We have thus proved the continuity of the first part of the bilinear form defined in (2.9).

For the second part, we can write

$$\int_0^\pi \langle \mathbb{A}w, W \rangle dt = \sum_{j=1}^L \lambda_j \int_0^\pi \langle w_j, W_j \rangle dt, \tag{2.12}$$

where

$$\int_0^\pi \langle w_j, W_j \rangle dt = \int_0^\pi \left\langle (\bar{u}_j + \tilde{u}_j, v_j), (\overline{U}_j + \widetilde{U}_j, V_j) \right\rangle dt$$

$$= \int_0^\pi (\bar{u}_j + \tilde{u}_j)(\overline{U}_j + \widetilde{U}_j) dt + \int_0^\pi v_j V_j \, dt.$$

Now for every $j = 1, \ldots, L$, we have

$$\left| \int_0^\pi (\bar{u}_j + \tilde{u}_j)(\overline{U}_j + \widetilde{U}_j) dt \right| \leq \left| \int_0^\pi \bar{u}_j \overline{U}_j \right| + \left| \int_0^\pi \tilde{u}_j \widetilde{U}_j \right|$$

$$\leq \pi |\bar{u}_j||\overline{U}_j| + \left| \frac{\pi}{2} \sum_{m=1}^\infty \tilde{u}_m^j \widetilde{U}_m^j \right|$$

$$\leq \pi |\bar{u}_j||\overline{U}_j| + \frac{\pi}{2} \sum_{m=1}^\infty \left| m^\alpha \tilde{u}_m^j m^\alpha \widetilde{U}_m^j \right|$$

$$\leq \pi |\bar{u}_j||\overline{U}_j| + \frac{\pi}{2} ||\tilde{u}_j||_{X_\alpha} ||\widetilde{U}_j||_{X_\alpha},$$

while

$$\left| \int_0^\pi v_j V_j dt \right| = \left| \frac{\pi}{2} \sum_{m=1}^\infty v_m^j V_{m_j} \right|$$

$$\leq \frac{\pi}{2} \sum_{m=1}^\infty \left| m^\beta v_m^j m^\beta V_m^j \right|$$

$$\leq \frac{\pi}{2} ||v_j||_{Y_\beta} ||V_j||_{Y_\beta}.$$

Thus we have

$$\left| \int_0^\pi \langle w_j, W_j \rangle dt \right| \leq \pi ||w_j||_{\mathbb{R} \times X_\alpha \times Y_\beta} ||W_j||_{\mathbb{R} \times X_\alpha \times Y_\beta},$$

and, going back to (2.12),

$$\left| \int_0^\pi \langle \mathbb{A}w, W \rangle dt \right| = \left| \sum_{j=1}^L \lambda_j \int_0^\pi \langle w_j, W_j \rangle dt \right|$$

$$\leq \sum_{j=1}^L \lambda_j \left| \int_0^\pi \langle w_j, W_j \rangle dt \right|$$

$$\leq \sum_{j=1}^L \pi \lambda_j ||w_j||_{\mathbb{R} \times X_\alpha \times Y_\beta} ||W_j||_{\mathbb{R} \times X_\alpha \times Y_\beta}$$

$$\leq \pi \lambda ||w||_{\mathbb{R}^L \times X_\alpha^L \times Y_\beta^L} ||W||_{\mathbb{R}^L \times X_\alpha^L \times Y_\beta^L},$$

where $\lambda = \max\{\lambda_1, \ldots, \lambda_L\}$. This shows that also the second part of the bilinear form $\mathcal{B}_2 : D \times D \to \mathbb{R}$ in (2.9) is continuous, and the proof is complete. $\square$

The bilinear form $\mathcal{B} : D \times D \to \mathbb{R}$ can thus be extended in a unique way to a continuous symmetric bilinear form $\mathcal{B} : E \times E \to \mathbb{R}$, for which we maintain the same notation. A bounded selfadjoint operator $\mathscr{L} : E \to E$ can thus be defined by

$$\langle \mathscr{L}z, \mathcal{Z} \rangle_E = \mathcal{B}(z, \mathcal{Z}),$$

for z and $\mathcal{Z}$ in E. Referring to (2.8) and (2.9), we can write

$$\mathscr{L}(\tilde{x}, y, \bar{u}, \tilde{u}, v) = (\mathscr{L}_1(\tilde{x}, y), \mathscr{L}_2(w)),$$

where

$$\langle \mathscr{L}_1(\tilde{x}, y), (\widetilde{X}, Y) \rangle_{X_\alpha^M \times Y_\beta^M} = \mathcal{B}_1\big((\tilde{x}, y), (\widetilde{X}, Y)\big),$$

and

$$\langle \mathscr{L}_2(w), W \rangle_{\mathbb{R}^L \times X_\alpha^L \times Y_\beta^L} = \mathcal{B}_2(w, W),$$

for every $z = (\tilde{x}, y, \bar{u}, \tilde{u}, v)$ and $\mathcal{Z} = (\widetilde{X}, Y, \overline{U}, \widetilde{U}, V)$ in E with $w = (\bar{u}, \tilde{u}, v)$, and $W = (\overline{U}, \widetilde{U}, V)$. It has been proved in [7, Proposition 2.14] that

$$||\mathscr{L}_1(\tilde{x}, y)||_{X_\alpha^M \times Y_\beta^M} = \frac{\pi}{2} ||(\tilde{x}, y)||_{X_\alpha^M \times Y_\beta^M}. \tag{2.13}$$

We now need the following.

Lemma 2.4. *There exist positive constants $\alpha, \beta, \widetilde{\delta}$ with $\alpha < \frac{1}{2} < \beta$, and $\alpha + \beta = 1$ such that*

$$||\mathscr{L}_2(w)||_{\mathbb{R}^L \times X_\alpha^L \times Y_\beta^L} \geq \widetilde{\delta}\, ||w||_{\mathbb{R}^L \times X_\alpha^L \times Y_\beta^L}, \tag{2.14}$$

for every $w \in \mathbb{R}^L \times X_\alpha^L \times Y_\beta^L$.

Proof. We first assume $L = 1$. Let $(\overline{\zeta}, \widetilde{\zeta}, \xi) \in \mathbb{R} \times X_\alpha \times Y_\beta$ be such that $\mathscr{L}_2(w) = (\overline{\zeta}, \widetilde{\zeta}, \xi)$, so that

$$\mathcal{B}_2(w, W) = \langle (\overline{\zeta}, \widetilde{\zeta}, \xi), W \rangle_{\mathbb{R} \times X_\alpha \times Y_\beta}, \tag{2.15}$$

for every $W = (U, V) \in \mathbb{R} \times X_\alpha \times Y_\beta$. Recalling that $w = (\overline{u}, \widetilde{u}, v)$, we decompose

$$\widetilde{u} = \sum_{m=1}^{\infty} u_m \cos(mt), \quad v = \sum_{m=1}^{\infty} v_m \sin(mt),$$

$$\widetilde{\zeta} = \sum_{m=1}^{\infty} \zeta_m \cos(mt), \quad \xi = \sum_{m=1}^{\infty} \xi_m \sin(mt).$$

By taking first $V = 0$ and then $U = 0$ in (2.15), and using (2.9), we obtain the following identities

$$\begin{cases} \overline{\zeta} = \lambda_1 \pi \overline{u}, \\ \zeta_m m^{2\alpha} = \frac{\pi}{2}[\lambda_1 u_m + m v_m], \\ \xi_m m^{2\beta} = \frac{\pi}{2}[m u_m + \lambda_1 v_m]. \end{cases} \tag{2.16}$$

Thus we have

$$\zeta_m m^\alpha = \frac{\pi}{2}[\lambda_1 m^{-\alpha} u_m + m^\beta v_m], \quad \xi_m m^\beta = \frac{\pi}{2}[m^\alpha u_m + \lambda_1 m^{-\beta} v_m],$$

and, by using the Young inequality,

$$\begin{aligned} \zeta_m^2 m^{2\alpha} + \xi_m^2 m^{2\beta} &= \frac{\pi^2}{4}\Big[\lambda_1^2 m^{-2\alpha} u_m^2 + m^{2\beta} v_m^2 + m^{2\alpha} u_m^2 \\ &\quad + \lambda_1^2 m^{-2\beta} v_m^2 + 2\lambda_1[m^{\alpha-\beta} + m^{\beta-\alpha}] u_m v_m\Big] \\ &\geq \frac{\pi^2}{4}\Big[\lambda_1^2 m^{-2\alpha} u_m^2 + m^{2\beta} v_m^2 + m^{2\alpha} u_m^2 \\ &\quad + \lambda_1^2 m^{-2\beta} v_m^2 - \lambda_1[m^{\alpha-\beta} + m^{\beta-\alpha}](u_m^2 + v_m^2)\Big] \\ &= \frac{\pi^2}{4} m^{-4\alpha}\Big[(\lambda_1 - m)(\lambda_1 - m^{4\alpha-1})\Big] m^{2\alpha} u_m^2 \\ &\quad + \frac{\pi^2}{4} m^{-4\beta}\Big[(\lambda_1 - m)(\lambda_1 - m^{4\beta-1})\Big] m^{2\beta} v_m^2. \end{aligned} \tag{2.17}$$

By hypothesis, we know that there exists a nonnegative integer n_1 such that

$$n_1 < \lambda_1 < n_1 + 1.$$

We now discuss separately the cases for $n_1 = 0$ and $n_1 \geq 1$.

Case 1. If $n_1 = 0$, then $0 < \lambda_1 < 1$, and so $\lambda_1 < m$ for all $m \geq 1$. Now for $m = 1$, (2.17) implies that

$$\zeta_1^2 + \xi_1^2 \geq \frac{\pi^2}{4}(\lambda_1 - 1)^2(u_1^2 + v_1^2). \tag{2.18}$$

For $m \geq 2$, we have

$$(\lambda_1 - m)(\lambda_1 - m^{4\alpha-1}) > (1 - m)(1 - m^{4\alpha-1}) = (m - 1)(m^{4\alpha-1} - 1).$$

By writing $m^{-4\alpha} = m^{-1}m^{-4\alpha+1}$, and choosing α such that

$$\frac{1}{4}\left(\frac{\log(4/3)}{\log 2} + 1\right) < \alpha < \frac{1}{2},$$

we have

$$m^{-4\alpha}(\lambda_1 - m)(\lambda_1 - m^{4\alpha-1}) > \left(1 - \frac{1}{m}\right)\left(1 - \frac{1}{m^{4\alpha-1}}\right)$$

$$\geq \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{2^{4\alpha-1}}\right) \geq \frac{1}{8} \geq \frac{\lambda_1^2}{8},$$

since $\lambda_1 < 1$. Similarly, since $\beta > \frac{1}{2} > \alpha$, we get

$$m^{-4\beta}(\lambda_1 - m)(\lambda_1 - m^{4\beta-1}) \geq \frac{\lambda_1^2}{8},$$

and thus (2.17) implies that

$$\zeta_m^2 m^{2\alpha} + \xi_m^2 m^{2\beta} \geq \frac{\pi^2}{4}\frac{\lambda_1^2}{8}[m^{2\alpha}u_m^2 + m^{2\beta}v_m^2]. \tag{2.19}$$

Combining (2.18), (2.19), and the first identity in (2.16), we have

$$\|\mathscr{L}_2(\bar{u}, \tilde{u}, v)\|_{\mathbb{R} \times X_\alpha \times Y_\beta}^2 = |\bar{\zeta}|^2 + \|\tilde{\zeta}\|_{X_\alpha}^2 + \|\tilde{\zeta}\|_{Y_\beta}^2$$

$$= \pi^2 \lambda_1^2 |\bar{u}|^2 + (\zeta_1^2 + \xi_1^2) + \sum_{m=2}^{\infty} (\zeta_m^2 m^{2\alpha} + \xi_m^2 m^{2\beta})$$

$$\geq \frac{\pi^2}{4}\frac{\lambda_1^2}{8}\left[|\bar{u}|^2 + \left(1 - \frac{1}{\lambda_1}\right)^2[u_1^2 + v_1^2] + \sum_{m=2}^{\infty} (u_m^2 m^{2\alpha} + v_m^2 m^{2\beta})\right]$$

$$\geq \tilde{\delta}^2\left[|\bar{u}|^2 + \sum_{m=1}^{\infty} (u_m^2 m^{2\alpha} + v_m^2 m^{2\beta})\right] = \tilde{\delta}^2\|(\bar{u}, \tilde{u}, v)\|_{\mathbb{R} \times X_\alpha \times Y_\beta}^2,$$

where

$$\tilde{\delta} = \frac{\pi}{8}\lambda_1 \min\left\{1, \left|1 - \frac{1}{\lambda_1}\right|\right\}.$$

This implies that (2.14) holds in this case, for $L = 1$.

Case 2. If $n_1 \geq 1$, then for $m \in \{1, \ldots, n_1\}$ we have $\lambda_1 - m \geq \lambda_1 - n_1 > 0$, and so

$$\lambda_1 - m^{4\alpha-1} \geq \lambda_1 - m \geq \lambda_1 - n_1 > 0.$$

This implies that

$$m^{-4\alpha}(\lambda_1 - m)(\lambda_1 - m^{4\alpha-1}) \geq n_1^{-4\alpha}(\lambda_1 - n_1)^2 \geq n_1^{-4\beta}(\lambda_1 - n_1)^2.$$

By choosing β such that

$$\frac{1}{2} < \beta < \frac{1}{4}\left(\frac{\log\left(\frac{1}{2}(\lambda_1 + n_1)\right)}{\log n_1} + 1\right), \tag{2.20}$$

we obtain that $\lambda_1 - m^{4\beta-1} \geq \lambda_1 - n_1^{4\beta-1} > \frac{1}{2}(\lambda_1 - n_1) > 0$, and so

$$m^{-4\beta}(\lambda_1 - m)(\lambda_1 - m^{4\beta-1}) \geq n_1^{-4\beta}\frac{1}{2}(\lambda_1 - n_1)^2.$$

Thus, for $m \in \{1, \ldots, n_1\}$, (2.17) and (2.20) imply that

$$\zeta_m^2 m^{2\alpha} + \xi_m^2 m^{2\beta} \geq \frac{\pi^2}{8}n_1^{-4\beta}(\lambda_1 - n_1)^2\left[m^{2\alpha}u_m^2 + m^{2\beta}v_m^2\right]. \tag{2.21}$$

For $m = n_1 + 1$, we have $\lambda_1 - m = \lambda_1 - (n_1 + 1) < 0$, and so

$$\lambda_1 - m^{4\beta-1} = \lambda_1 - (n_1 + 1)^{4\beta-1} < \lambda_1 - (n_1 + 1) < 0.$$

This implies that

$$m^{-4\beta}(\lambda_1 - m)(\lambda_1 - m^{4\beta-1}) \geq (n_1 + 1)^{-4\beta}(\lambda_1 - (n_1 + 1))^2.$$

By choosing α such that

$$\frac{1}{4}\left(\frac{\log\left(\frac{1}{2}(\lambda_1 + n_1 + 1)\right)}{\log(n_1 + 1)} + 1\right) \leq \alpha < \frac{1}{2}, \tag{2.22}$$

we obtain

$$\lambda_1 - m^{4\alpha-1} = \lambda_1 - (n_1 + 1)^{4\alpha-1} \leq \frac{1}{2}(\lambda_1 - (n_1 + 1)) < 0,$$

and so

$$m^{-4\alpha}(\lambda_1 - m)(\lambda_1 - m^{4\alpha-1}) \geq (n_1 + 1)^{-4\alpha}\frac{1}{2}(\lambda_1 - (n_1 + 1))^2$$

$$\geq (n_1 + 1)^{-4\beta}\frac{1}{2}(\lambda_1 - (n_1 + 1))^2.$$

Thus, for $m = n_1 + 1$, (2.17) and (2.22) imply that

$$\zeta_m^2 m^{2\alpha} + \xi_m^2 m^{2\beta} \geq \frac{\pi^2}{8}(n_1 + 1)^{-4\beta}(\lambda_1 - (n_1 + 1))^2\left[m^{2\alpha}u_m^2 + m^{2\beta}v_m^2\right]. \tag{2.23}$$

Lastly, for $m \geq n_1 + 2$, by choosing α such that

$$\frac{1}{4}\left(\frac{\log\left(\frac{2(n_1+1)(n_1+2)}{2n_1+3}\right)}{\log(n_1 + 2)} + 1\right) \leq \alpha < \frac{1}{2}, \tag{2.24}$$

we have

$$(\lambda_1 - m)(\lambda_1 - m^{4\alpha-1}) > (n_1 + 1 - m)(n_1 + 1 - m^{4\alpha-1})$$

$$= (m - (n_1 + 1))(m^{4\alpha-1} - (n_1 + 1)),$$

and, writing $m^{-4\alpha} = m^{-1}m^{-4\alpha+1}$,

$$m^{-4\alpha}(\lambda_1 - m)(\lambda_1 - m^{4\alpha-1}) > \left(1 - \frac{n_1 + 1}{m}\right)\left(1 - \frac{n_1 + 1}{m^{4\alpha-1}}\right)$$

$$\geq \left(1 - \frac{n_1 + 1}{n_1 + 2}\right)\left(1 - \frac{n_1 + 1}{(n_1 + 2)^{4\alpha-1}}\right)$$

$$\geq \frac{1}{2}\left(1 - \frac{n_1 + 1}{n_1 + 2}\right)^2 = \frac{1}{2}\frac{1}{(n_1 + 2)^2}.$$

Similarly, since $\beta > \frac{1}{2} > \alpha$, we obtain

$$m^{-4\beta}(\lambda_1 - m)(\lambda_1 - m^{4\beta-1}) \geq \frac{1}{2}\frac{1}{(n_1 + 2)^2}.$$

Hence for $m \geq n_1 + 2$, (2.17) and (2.24) imply that

$$\zeta_m^2 m^{2\alpha} + \xi_m^2 m^{2\beta} \geq \frac{\pi^2}{8}\frac{1}{(n_1 + 2)^2}\left[m^{2\alpha}u_m^2 + m^{2\beta}v_m^2\right]. \tag{2.25}$$

Combining (2.21), (2.23), (2.25), and the first identity in (2.16) we have

$$||\mathscr{L}_2(\bar{u}, \tilde{u}, v)||^2_{\mathbb{R} \times X_\alpha \times Y_\beta} = |\bar{\zeta}|^2 + ||\widetilde{\zeta}||^2_{X_\alpha} + ||\widetilde{\zeta}||^2_{Y_\beta}$$

$$= \pi^2 \lambda_1^2 |\bar{u}|^2 + \sum_{m=1}^{n_1}\left[\zeta_m^2 m^{2\alpha} + \xi_m^2 m^{2\beta}\right] +$$

$$+ \left[\zeta_{n_1+1}^2(n_1 + 1)^{2\alpha} + \xi_{n_1+1}^2(n_1 + 1)^{2\beta}\right] + \sum_{m=n_1+2}^{\infty}\left[\zeta_m^2 m^{2\alpha} + \xi_m^2 m^{2\beta}\right]$$

$$\geq \frac{\pi^2}{8}|\bar{u}|^2 + \frac{\pi^2}{8}(n_1 + 1)^{-4\beta}\left[\left(1 - \frac{n_1}{\lambda_1}\right)^2 \sum_{m=1}^{n_1}\left[m^{2\alpha}u_m^2 + m^{2\beta}v_m^2\right]\right.$$

$$+ \left(1 - \frac{n_1 + 1}{\lambda_1}\right)^2\left[(n_1 + 1)^{2\alpha}u_{n_1+1}^2 + (n_1 + 1)^{2\beta}v_{n_1+1}^2\right]\right] +$$

$$+ \frac{\pi^2}{8}\frac{1}{(n_1 + 2)^2}\sum_{m=n_1+2}^{\infty}\left[m^{2\alpha}u_m^2 + m^{2\beta}v_m^2\right]$$

$$\geq \widetilde{\delta}^2\left[|\bar{u}|^2 + \sum_{m=1}^{\infty}\left(u_m^2 m^{2\alpha} + v_m^2 m^{2\beta}\right)\right] = \widetilde{\delta}^2||(\bar{u}, \tilde{u}, v)||^2_{\mathbb{R} \times X_\alpha \times Y_\beta},$$

where

$$\widetilde{\delta} = \frac{\pi}{2\sqrt{2}}\min\left\{\frac{1}{n_1 + 2}, (n_1 + 1)^{-2\beta}\left|1 - \frac{n_1 + 1}{\lambda_1}\right|, (n_1 + 1)^{-2\beta}\left|1 - \frac{n_1}{\lambda_1}\right|\right\}.$$

This implies that (2.14) holds also in this case, for $L = 1$.

Finally, by using (2.10) and (2.12), we can easily see that (2.14) holds for any $L \geq 1$. $\qquad\square$

By combining (2.13) and (2.14) in Lemma 2.4, we can say that the selfadjoint operator $\mathscr{L} : E \to E$ is invertible, and the inverse operator $\mathscr{L}^{-1} : E \to E$ is continuous.

By Theorem 2.2, we conclude that the functional φ has at least $M + 1$ critical points. Arguing as in [6, Proposition 24], it can be seen that these

critical points correspond to the solutions of the boundary value problem (2.6)–(2.3) that we are looking for. The proof of Theorem 2.1 is thus completed.   □

3. Proof of Theorem 1.1

Without loss of generality, we may assume that $[a, b] = [0, \pi]$. We start assuming $L = 1$, and we first work on the planar system (1.3) so to transform it, by a symplectic change of variables, into a linear one. We will follow the approach developed in [1,8,11].

3.1. A symplectic change of variables

By using $A4$, we have that $\mathscr{H}(0,0) = 0$ and the generalized Euler Identity holds true, i.e.,

$$\left\langle \nabla \mathscr{H}(u, v), \left(\frac{u}{p}, \frac{v}{q}\right) \right\rangle = \mathscr{H}(u, v). \tag{3.1}$$

Choose the positive constant

$$\Upsilon = \min\left\{ \frac{1}{|w|^2} \mathscr{H}(w) : 1 \leq |w| \leq 2 \right\}, \tag{3.2}$$

and let $\eta : \mathbb{R} \to \mathbb{R}$ be a C^∞-function such that $\eta'(s) \leq 0$ for all $s \in \mathbb{R}$ and

$$\eta(s) = \begin{cases} 1, & \text{if } s \leq 1, \\ 0, & \text{if } s \geq 2. \end{cases}$$

For $w = (u, v)$, set

$$\widehat{\mathscr{H}}(w) = \eta(|w|)\Upsilon|w|^2 + (1 - \eta(|w|))\mathscr{H}(w), \tag{3.3}$$

and consider the new system

$$J\dot{w} = \nabla \widehat{\mathscr{H}}(w). \tag{3.4}$$

Notice that $\widehat{\mathscr{H}}(0) = 0$. For every $w \neq 0$, we have

$$\nabla \widehat{\mathscr{H}}(w) = \left(\Upsilon\eta'(|w|)|w| + 2\Upsilon\eta(|w|) - \frac{\eta'(|w|)}{|w|}\mathscr{H}(w)\right)w + (1 - \eta(|w|))\nabla\mathscr{H}(w).$$

Then, using (3.1) and (3.2), if $w = (u, v)$ is such that $1 \leq |w| \leq 2$, we have

$$\left\langle \nabla \widehat{\mathscr{H}}(w), \left(\frac{u}{p}, \frac{v}{q}\right) \right\rangle = \eta'(|w|)|w|\left(\frac{u^2}{p} + \frac{v^2}{q}\right)\left(\Upsilon - \frac{1}{|w|^2}\mathscr{H}(w)\right)$$

$$+ 2\eta(|w|)\Upsilon\left(\frac{u^2}{p} + \frac{v^2}{q}\right) + (1 - \eta(|w|))\mathscr{H}(w) > 0.$$

This implies that $\nabla \widehat{\mathscr{H}}(w) \neq 0$, for $1 \leq |w| \leq 2$. For $0 < |w| \leq 1$, the Hamiltonian function $\widehat{\mathscr{H}}$ is quadratic, so that $\nabla \widehat{\mathscr{H}}(w) \neq 0$. Lastly, for $|w| \geq 2$, we have $\nabla \widehat{\mathscr{H}}(w) = \nabla \mathscr{H}(w)$, and it is clear from (3.1) that $\nabla \mathscr{H}(w) \neq 0$. Hence $\nabla \widehat{\mathscr{H}}(w) \neq 0$ for every $w \neq 0$, and this shows that every non-zero solution of system (3.4) does not pass through the origin, and by Poincaré–Bendixson

theory, all the solutions of system (3.4) are periodic. Thus the origin is still a global center for the system (3.4).

Now for any $w_0 \in \mathbb{R}^2 \setminus \{0\}$, we denote by $\widehat{T}(w_0)$ the minimal period of the solution of (3.4) passing through w_0. We notice here that this solution is unique, even if we are not assuming $\nabla \mathscr{H}$ to be locally Lipschitz continuous, cf. [16]. The function $\widehat{T} : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ thus defined is continuously differentiable (see [1]).

Define

$$\delta^\star = [0, +\infty[\times \{0\},$$

and a function $\xi :\,]0, +\infty[\to]0, +\infty[$ as follows: for every $r > 0$, the level line $\{w \in \mathbb{R}^2 : \widehat{\mathscr{H}}(w) = r\}$ intersects $\delta^\star$ at the point $(\xi(r), 0)$. Such a point is unique, because for every $(\xi, 0) \in \delta^\star$ with $\xi \neq 0$ we have

$$\left\langle \nabla \widehat{\mathscr{H}}(\xi, 0), \left(\frac{\xi}{p}, 0 \right) \right\rangle > 0,$$

which implies that

$$\left\langle \nabla \widehat{\mathscr{H}}(\xi, 0), (\xi, 0) \right\rangle > 0.$$

Thus, if $w(t_0) = (u(t_0), v(t_0)) = (u(t_0), 0)$ is such that $u(t_0) > 0$, then $v'(t_0) < 0$, and so it is impossible for the level line $\{w \in \mathbb{R}^2 : \widehat{\mathscr{H}}(w) = r\}$ to intersect $\delta^\star$ at two different points.

Now define $\widehat{K} : \mathbb{R}^2 \to \mathbb{R}$ as

$$\widehat{K}(w) = \frac{1}{\tau} \int_0^{\widehat{\mathscr{H}}(w)} \widehat{T}(\xi(r), 0) \, dr.$$

This function is continuously differentiable, and

$$\nabla \widehat{K}(w) = \frac{\widehat{T}(w)}{\tau} \nabla \widehat{\mathscr{H}}(w).$$

Hence, the origin is an *isochronous* center for the system

$$J \dot{w} = \nabla \widehat{K}(w), \tag{3.5}$$

since all solutions except the equilibrium 0 are periodic with minimal period τ. Moreover,

$$\widehat{K}(w) = \frac{\pi}{\tau} |w|^2, \text{ if } |w| \leq 1.$$

Now, for every $w_0 \in \mathbb{R}^2 \setminus \{0\}$, let $\zeta(t; w_0)$ be the solution of system (3.5) satisfying $\zeta(0; w_0) = w_0$, and define $\theta(w_0) \in [0, 2\pi[$ as the minimum time for which

$$\zeta \left(-\frac{\tau}{2\pi} \theta(w_0); w_0 \right) \in \delta^\star.$$

As shown in [1], the restricted function $\theta : \mathbb{R}^2 \setminus \delta^\star \to\,]0, 2\pi[$ is continuously differentiable, and its gradient $\nabla \theta$ can be continuously extended to $\mathbb{R}^2 \setminus \{0\}$. We will still denote this extension by $\nabla \theta : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2$.

Hence, by [1, Proposition 2.2.], there exists a symplectic diffeomorphism $\Lambda : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\Lambda(w) = \begin{cases} \sqrt{\dfrac{\tau}{\pi}\widehat{K}(w)}\big(\cos\theta(w), -\sin\theta(w)\big), & \text{if } w \neq 0, \\ 0, & \text{if } w = 0, \end{cases}$$

such that, by the change of variable $z = \Lambda(w)$, system (3.5) is changed to the linear one

$$J\dot{z} = \frac{2\pi}{\tau}z.$$

3.2. The proof in the case $L = 1$

First, we can modify the function $\mathcal{H}$ as in (2.5) so to obtain the modified system

$$\begin{cases} \dot{x} = \nabla_y\widehat{\mathcal{H}}(t,x,y) + \nabla_y P(t,x,y,u,v)\,, \\ \dot{y} = -\nabla_x\widehat{\mathcal{H}}(t,x,y) - \nabla_x P(t,x,y,u,v), \\ \dot{u} = \nabla_v\mathscr{H}(u,v) + \nabla_v P(t,x,y,u,v), \\ \dot{v} = -\nabla_u\mathscr{H}(u,v) - \nabla_u P(t,x,y,u,v). \end{cases} \tag{3.6}$$

Using the argument in [5, Section 3], it can be seen that all the solutions of this system are globally defined. Moreover, those satisfying the boundary conditions

$$\begin{cases} y(0) = 0 = y(\pi), \\ v(0) = 0 = v(\pi) \end{cases} \tag{3.7}$$

are solutions of the original system (1.1).

Recalling the change of variables $\Lambda(w) = z$ in Sect. 3.1, we define a map

$$\widetilde{P}(t,x,y,z) = P(t,x,y,\Lambda^{-1}(z)).$$

Lemma 3.1. *The function $\widetilde{P}$ has a bounded gradient with respect to (x,y,z).*

Proof. Clearly, by $A2$ both

$$\partial_x\widetilde{P}(t,x,y,z) = \partial_x P(t,x,y,\Lambda^{-1}(z)), \quad \partial_y\widetilde{P}(t,x,y,z) = \partial_y P(t,x,y,\Lambda^{-1}(z))$$

are bounded and denoting by $\mathbb{M}^*$ the transpose of a matrix $\mathbb{M}$,

$$\nabla_z\widetilde{P}(t,x,y,z) = \big[(\Lambda^{-1}(z))'\big]^*\nabla_w P(t,x,y,\Lambda^{-1}(z))$$
$$= \big[(\Lambda'(\Lambda^{-1}(z)))^*\big]^{-1}\nabla_w P(t,x,y,\Lambda^{-1}(z)))\,.$$

Again by $A2$, $\nabla_w P(t,x,y,w)$ is bounded, so it is sufficient to show that $(\Lambda'(w))^{-1}$ is bounded. For $|w|$ large enough, we have that $\widehat{K}(w) = \mathscr{H}(w)$. By denoting $c(w) = \cos\theta(w)$ and $s(w) = \sin\theta(w)$, we have

$$\Lambda'(w) = \begin{bmatrix} a_{11}(w) & a_{12}(w) \\ a_{21}(w) & a_{22}(w) \end{bmatrix},$$

where

$$a_{11}(w) = \sqrt{\frac{\tau}{\pi}}\left(\frac{\partial_u\mathscr{H}(w)}{2\sqrt{\mathscr{H}(w)}}c(w) - \sqrt{\mathscr{H}(w)}\partial_u\theta(w)s(w)\right),$$

$$a_{12}(w) = \sqrt{\frac{\tau}{\pi}} \left(\frac{\partial_v \mathscr{H}(w)}{2\sqrt{\mathscr{H}(w)}} c(w) - \sqrt{\mathscr{H}(w)} \partial_v \theta(w) s(w) \right),$$

$$a_{21}(w) = \sqrt{\frac{\tau}{\pi}} \left(-\frac{\partial_u \mathscr{H}(w)}{2\sqrt{\mathscr{H}(w)}} s(w) - \sqrt{\mathscr{H}(w)} \partial_u \theta(w) c(w) \right),$$

$$a_{22}(w) = \sqrt{\frac{\tau}{\pi}} \left(-\frac{\partial_v \mathscr{H}(w)}{2\sqrt{\mathscr{H}(w)}} s(w) - \sqrt{\mathscr{H}(w)} \partial_v \theta(w) c(w) \right).$$

Recalling that Λ is symplectic, so $\det \Lambda'(w) = 1$, the inverse matrix is

$$(\Lambda'(w))^{-1} = \begin{bmatrix} a_{22}(w) & -a_{12}(w) \\ -a_{21}(w) & a_{11}(w) \end{bmatrix}.$$

From the definition of θ, for $w \neq 0$ and $\gamma > 0$ we see that $\theta(\gamma^q u, \gamma^p v) = \theta(u, v)$. Indeed, if $w(t) = (u(t), v(t))$ is a solution of system (3.5), then $w_\gamma = (\gamma^q u, \gamma^p v)$ is also a solution of system (3.5) with vertical component of the velocity equal to $\gamma^p \dot{v}(t)$. Hence, if $w(t)$ needs a time $\frac{\tau}{2\pi} \theta(u_0, v_0)$ to go from $\delta^\star$ to (u_0, v_0) (it has a vertical speed $\dot{v}(t)$), then the time for $w_\gamma(t)$ to go from $\delta^\star$ to $(\gamma^q u_0, \gamma^p v_0)$ must be the same, since its vertical speed is just γ^p times the vertical speed of $w(t)$. Thus we have

$$\partial_u \theta(\gamma^q u, \gamma^p v) \gamma^q = \partial_u \theta(u, v), \qquad \partial_v \theta(\gamma^q u, \gamma^p v) \gamma^p = \partial_v \theta(u, v),$$

for every $\gamma > 0$. For $w = (u, v)$ with $|w| \geq 2$, since $\mathscr{H}$ is positively-(p,q)-homogeneous, the following identities have been proved in [5]:

$$\frac{\partial \mathscr{H}}{\partial u}(\gamma^q u, \gamma^p v) = \gamma^{q(p-1)} \frac{\partial \mathscr{H}}{\partial u}(u, v) = \gamma^p \frac{\partial \mathscr{H}}{\partial u}(u, v),$$

$$\frac{\partial \mathscr{H}}{\partial v}(\gamma^q u, \gamma^p v) = \gamma^{p(q-1)} \frac{\partial \mathscr{H}}{\partial v}(u, v) = \gamma^q \frac{\partial \mathscr{H}}{\partial v}(u, v).$$

Thus we have

$$|a_{22}(w)| \leq \sqrt{\frac{\tau}{\pi}} \left(\frac{|\partial_v \mathscr{H}(w)|}{2\sqrt{\mathscr{H}(w)}} + \sqrt{\mathscr{H}(w)} |\partial_v \theta(w)| \right)$$

$$= \sqrt{\frac{\tau}{\pi}} \frac{|w|^q \left| \partial_v \mathscr{H}\left(\frac{u}{|w|^q}, \frac{v}{|w|^p} \right) \right|}{2|w|^{p+q} \sqrt{\mathscr{H}\left(\frac{u}{|w|^{2q}}, \frac{v}{|w|^{2p}} \right)}} +$$

$$+ \sqrt{\frac{\tau}{\pi}} \frac{|w|^{p+q}}{|w|^{p(p+q)}} \sqrt{\mathscr{H}\left(\frac{u}{|w|^{2q}}, \frac{v}{|w|^{2p}} \right)} \partial_v \theta\left(\frac{u}{|w|^{q(p+q)}}, \frac{v}{|w|^{p(p+q)}} \right)$$

$$\leq \sqrt{\frac{\tau}{\pi}} \frac{\left| \partial_v \mathscr{H}\left(\frac{u}{|w|^q}, \frac{v}{|w|^p} \right) \right|}{2\sqrt{\mathscr{H}\left(\frac{u}{|w|^{2q}}, \frac{v}{|w|^{2p}} \right)}} +$$

$$+ \sqrt{\frac{\tau}{\pi}} \sqrt{\mathscr{H}\left(\frac{u}{|w|^{2q}}, \frac{v}{|w|^{2p}} \right)} \partial_v \theta\left(\frac{u}{|w|^{q(p+q)}}, \frac{v}{|w|^{p(p+q)}} \right).$$

Define three types of sets as follow:

$$S = \left\{ \left(\frac{u}{|w|^q}, \frac{v}{|w|^p} \right) : w = (u, v), |w| \geq 1 \right\},$$

$$S' = \left\{ \left(\frac{u}{|w|^{2q}}, \frac{v}{|w|^{2p}} \right) : w = (u, v), |w| \geq 1 \right\},$$

and

$$S'' = \left\{ \left(\frac{u}{|w|^{q(p+q)}}, \frac{v}{|w|^{p(p+q)}} \right) : w = (u, v), |w| \geq 1 \right\}.$$

It is easy to see that the sets S, S', and S'' are subsets of the closed unit ball $\overline{B}(0, 1)$ of $\mathbb{R}^2$. This implies that $|a_{22}(w)|$ is bounded, since the functions $\mathscr{H}$ and θ are C^1. Similarly we can show that all of the other elements of the matrix $(\Lambda'(w))^{-1}$ are bounded, which thus proves that the map $\widetilde{P}$ has a bounded gradient with respect to z.   □

Now we consider the modified system

$$\begin{cases} \dot{x} = \nabla_y \widehat{\mathcal{H}}(t, x, y) + \nabla_y \widetilde{P}(t, x, y, \xi, \zeta), \\ \dot{y} = -\nabla_x \widehat{\mathcal{H}}(t, x, y) - \nabla_x \widetilde{P}(t, x, y, \xi, \zeta), \\ \dot{\xi} = \dfrac{2\pi}{\tau} \zeta + \partial_\zeta \widetilde{P}(t, x, y, \xi, \zeta), \\ \dot{\zeta} = -\dfrac{2\pi}{\tau} \xi - \partial_{\xi_j} \widetilde{P}(t, x, y, \xi, \zeta), \end{cases} \tag{3.8}$$

where $z = (\xi, \zeta)$. By the assumption $\tau_+ = \tau_-$, the boundary conditions become

$$\begin{cases} y(0) = 0 = y(\pi), \\ \zeta(0) = 0 = \zeta(\pi). \end{cases} \tag{3.9}$$

Thus, by taking $\lambda_1 = \frac{2\pi}{\tau}$, all the assumptions of Theorem 2.1 are satisfied, so that the boundary value problem (3.8)–(3.9) has at least $M + 1$ geometrically distinct solutions.

Recalling that Λ is a diffeomorphism, we can apply the inverse change of variables $w = \Lambda^{-1}(z)$, and obtain the $M + 1$ geometrically distinct solutions of system (3.6) satisfying the boundary conditions (3.7) we were looking for. This completes the proof of Theorem 1.1 in the case $L = 1$.   □

3.3. The proof in the higher dimensional case

We now consider the case $L \geq 2$, for which we will follow briefly the lines of the proof in the previous section. We can define $\widehat{\mathscr{H}_j}$ as in (3.3) and consider the new system

$$J\dot{\zeta} = \nabla \widehat{\mathscr{H}_j}(\zeta).$$

We define $\widehat{K}_j : \mathbb{R}^2 \to \mathbb{R}$ as

$$\widehat{K}_j(\zeta) = \frac{1}{\tau_j} \int_0^{\widehat{\mathscr{H}_j}(\zeta)} \widehat{T}_j(\xi_j(r), 0) \, dr,$$

so that the origin is an isochronous center for the system

$$J\dot{\zeta} = \nabla\widehat{K}_j(\zeta), \tag{3.10}$$

i.e., for every $j \in \{1, \ldots, L\}$, all solutions of system (3.10) except the origin are periodic and have the same minimal period τ_j. Now, for every $j \in \{1, \ldots, L\}$, there exists a symplectic diffeomorphism $\Lambda_j : \mathbb{R}^2 \to \mathbb{R}^2$ such that, by the change of variables $\rho = \Lambda_j(\zeta)$, system (3.10) becomes

$$J\dot{\rho} = \frac{2\pi}{\tau_j}\rho.$$

By the use of a cut-off function, we modify the Hamiltonian $\mathcal{H}$ like in (2.5), so that the new Hamiltonian $\widehat{\mathcal{H}}$ has a bounded gradient with respect to (x, y).

Defining $\Lambda : \mathbb{R}^{2L} \to \mathbb{R}^{2L}$ by

$$\Lambda(u, v) = (\Lambda_1(u_1, v_1), \ldots, \Lambda_L(u_L, v_L)),$$

we see that Λ is a symplectic diffeomorphism. By writing

$$\widetilde{P}(t, x, y, z) = P(t, x, y, \Lambda^{-1}(z)),$$

as in Lemma 3.1 we can show that the function $\widetilde{P}$ has a bounded gradient with respect to (x, y, z).

We apply the change of variables $z = \Lambda(w)$ and write $z = (\xi, \zeta)$ with

$$\xi = (\xi_1, \ldots, \xi_L), \quad \zeta = (\zeta_1, \ldots, \zeta_L),$$

so to obtain the modified system

$$\begin{cases} \dot{x} = \nabla_y\widehat{\mathcal{H}}(t, x, y) + \nabla_y\widetilde{P}(t, x, y, z), \\ \dot{y} = -\nabla_x\widehat{\mathcal{H}}(t, x, y) - \nabla_x\widetilde{P}(t, x, y, z), \\ \dot{\xi}_j = \dfrac{2\pi}{\tau_j}\zeta_j + \partial_{\zeta_j}\widetilde{P}(t, x, y, z), \quad j = 1, \ldots, L, \\ \dot{\zeta}_j = -\dfrac{2\pi}{\tau_j}\xi_j - \partial_{\xi_j}\widetilde{P}(t, x, y, z), \quad j = 1, \ldots, L. \end{cases} \tag{3.11}$$

Moreover, since $\tau_{j_+} = \tau_{j_-}$, the boundary conditions become the same as those in (3.9). Hence, by taking $\lambda_j = \frac{2\pi}{\tau_j}$, Theorem 2.1 implies that the modified system (3.11) has at least $M+1$ geometrically distinct solutions satisfying the boundary conditions (3.9).

Recalling that Λ is a diffeomorphism, we can apply the inverse change of variables $w = \Lambda^{-1}(z)$ and obtain the solutions of problem (1.1)–(1.2) we are looking for. □

4. The periodic problem

In this section, we consider the Hamiltonian system (1.1), where besides the regularity assumptions already made on the functions involved, we assume that all these functions are T-periodic in t. While maintaining assumptions $A1$, $A2$ and $A4$ we will reinforce assumption $A3$ by a *twist condition*, and for this we first recall some definitions.

By a convex body of $\mathbb{R}^M$, we mean a closed convex bounded subset $\mathcal{D}$ of $\mathbb{R}^M$ having nonempty interior. If in addition, $\mathcal{D}$ has a smooth boundary, then we denote the unit outward normal at $\zeta \in \partial\mathcal{D}$ by $\nu_{\mathcal{D}}(\zeta)$. Moreover, we say that $\mathcal{D}$ is strongly convex if for any $\bar{p} \in \partial\mathcal{D}$, the map $\mathcal{F} : \mathcal{D} \to \mathbb{R}$ defined by $\mathcal{F}(\xi) = \langle \xi - \bar{p}, \nu_{\mathcal{D}}(\bar{p}) \rangle$ has a unique maximum point at $\xi = \bar{p}$. Below is our *twist condition*.

$B3'$. There are a strongly convex body $\mathcal{D}$ of $\mathbb{R}^M$ having a smooth boundary and a symmetric regular $M \times M$ matrix $\mathbb{B}$ such that, for every C^1-function $\mathcal{W} : [0, T] \to \mathbb{R}^{2L}$, all the solutions (x, y) of system

$$\begin{cases} \dot{x} = \nabla_y \mathcal{H}(t, x, y) + \nabla_y P(t, x, y, \mathcal{W}(t)), \\ \dot{y} = -\nabla_x \mathcal{H}(t, x, y) - \nabla_x P(t, x, y, \mathcal{W}(t)), \end{cases} \tag{4.1}$$

starting with $y(0) \in \mathcal{D}$ are defined on $[0, T]$, and

$$y(0) \in \partial\mathcal{D} \quad \Rightarrow \quad \langle x(T) - x(0), \mathbb{B}\nu_{\mathcal{D}}(y(0)) \rangle > 0.$$

Here is our first result for the periodic problem.

Theorem 4.1. *Assume that A1, A2, B3' and A4 hold true, and let*

$$\frac{T}{\tau_j} \notin \mathbb{N}, \quad \text{for every } j \in \{1, \dots, L\}.$$

Then there are at least $M + 1$ geometrically distinct T-periodic solutions of system (1.1), *with $y(0) \in \mathring{\mathcal{D}}$.*

Proof. Following the lines of the proof of Theorem 1.1, we modify the problem so to have a coupling with a perturbed linear system. Then, [4, Corollary 2.4] applies (instead of Theorem 2.1), and the proof is readily completed. $\square$

We can state some variants of Theorem 4.1 replacing the twist assumption $B3'$ by $B3''$ or by $B3'''$ given below.

$B3''$. There exists a convex body $\mathcal{D}$ of $\mathbb{R}^M$, having a smooth boundary, such that for $\sigma \in \{-1, 1\}$ and for every C^1-function $\mathcal{W} : [0, T] \to \mathbb{R}^{2L}$, all the solutions (x, y) of system (4.1) starting with $y(0) \in \mathcal{D}$ are defined on $[0, T]$, and

$$y(0) \in \partial\mathcal{D} \quad \Rightarrow \quad x(T) - x(0) \notin \{\sigma\lambda\nu_{\mathcal{D}}(y(0)) : \lambda \geq 0\}.$$

$B3'''$. Let $\mathcal{D}$ be a rectangle in $\mathbb{R}^M$, i.e.

$$\mathcal{D} = [c_1, d_1] \times \cdots \times [c_M, d_M].$$

There exists an M-tuple $\sigma = (\sigma_1, \dots, \sigma_M) \in \{-1, 1\}^M$ such that, for every C^1-function $\mathcal{W} : [0, T] \to \mathbb{R}^{2L}$, all the solutions (x, y) of system (4.1) starting with $y(0) \in \mathcal{D}$ are defined on $[0, T]$, and, for every $i \in \{1, \dots, M\}$, we have

$$\begin{cases} y_i(0) = c_i \quad \Rightarrow \quad \sigma_i(x_i(T) - x_i(0)) < 0, \\ y_i(0) = d_i \quad \Rightarrow \quad \sigma_i(x_i(T) - x_i(0)) > 0. \end{cases}$$

The proofs of such results are similar to those of [8, Theorem 4.2, Theorem 4.3], so we avoid them for briefness.

5. Some possible applications

As an example of application of Theorem 1.1, we consider the following system for $L = M = 1$:

$$\begin{cases} \dot{x} = f(y) + E(t), \quad \dot{y} = -A\sin x - \partial_x P(t, x, u), \\ \dot{u} = |v|^{q-2}v, \quad \dot{v} = -\mu(u^+)^{p-1} + \nu(u^-)^{p-1} + \partial_u P(t, x, u), \end{cases} \tag{5.1}$$

with the Neumann-type boundary conditions

$$\begin{cases} y(a) = 0 = y(b), \\ v(a) = 0 = v(b). \end{cases} \tag{5.2}$$

Here we use the notation $u^+ = \max\{u, 0\}$, $u^- = \max\{-u, 0\}$. We assume that the constants A, μ, ν are positive, and the functions $f : \mathbb{R} \to \mathbb{R}$, $E : [a, b] \to \mathbb{R}$ and $P : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous. Assume further that $P(t, x, u)$ is 2π-periodic in x, continuously differentiable in (x, u), and that it has a bounded gradient with respect to (x, u). Since $\sin x$ and $\partial_x P(t, x, u)$ are bounded, assumption $A3$ clearly holds.

On the other hand, notice that the last two equations in system (5.1) correspond to the scalar equation

$$\frac{d}{dt}\big(|\dot{u}|^{p-2}\dot{u}\big) + \mu(u^+)^{p-1} - \nu(u^-)^{p-1} = \partial_u P(t, x, u).$$

If we define $\mathscr{H}$ by

$$\mathscr{H}(u, v) = \frac{|v|^q}{q} + \frac{1}{p}\big(\mu(u^+)^p + \nu(u^-)^p\big),$$

then $\mathscr{H}$ is positively-(p, q)-homogeneous and positive, and all the solutions of system $J\dot{w} = \nabla\mathscr{H}(w)$ with $w = (u, v)$ are periodic with the same minimal period

$$\tau = \pi_p(\mu^{-1/p} + \nu^{-1/p}), \tag{5.3}$$

(see [12, 17]), where

$$\pi_p = \frac{2(p-1)^{1/p}}{p\sin(\pi/p)}\pi.$$

We thus get the following immediate consequence of Theorem 1.1.

Corollary 5.1. *In the above setting, assume moreover that*

$$\frac{(\mu\nu)^{1/p}}{\mu^{1/p} + \nu^{1/p}} \neq \frac{n\pi_p}{2(b-a)}, \quad \text{for every } n \in \mathbb{N}.$$

Then problem (5.1)–(5.2) has at least two geometrically distinct solutions.

Remark 5.2. Surprisingly enough, besides continuity, in the above corollary no further assumption is needed on the function f.

Concerning the periodic problem, as a first example of application of Theorem 4.1 we consider the system

$$\begin{cases} \ddot{x} + A\sin x = e(t) + \partial_x P(t, x, u), \\ \frac{d}{dt}\left(|\dot{u}|^{p-2}\dot{u}\right) + \mu(u^+)^{p-1} - \nu(u^-)^{p-1} = \partial_u P(t, x, u), \end{cases} \tag{5.4}$$

where the constants A, μ, ν are positive. Assume that $P(t, x, u)$ is T-periodic in t and 2π-periodic in x, and that it has a bounded gradient with respect to (x, u). Setting $E(t) = \int_0^t e(s)\, ds$, system (5.4) is equivalent to

$$\begin{cases} \dot{x} = y + E(t), \quad \dot{y} = -A\sin x + \partial_x P(t, x, u), \\ \dot{u} = |v|^{q-2}v, \quad \dot{v} = -\mu(u^+)^{p-1} + \nu(u^-)^{p-1} + \partial_u P(t, x, u). \end{cases} \tag{5.5}$$

Assuming $e(t)$ to be T-periodic with

$$\int_0^T e(t)\, dt = 0,$$

the function $E(t)$ is T-periodic, as well.

Let us verify that the first two equations in (5.5) satisfy the twist condition $B3'''$, with $M = 1$. Notice that there exists $K_3 > 0$ such that, for every C^1-function $\mathcal{U} : [0, T] \to \mathbb{R}$, all the solutions (x, y) of the system

$$\dot{x} = y + E(t), \quad \dot{y} = -A\sin x + \partial_x P(t, x, \mathcal{U}(t))$$

are defined on $[0, T]$ and satisfy

$$|\dot{y}(t)| \le K_3, \quad \text{for every } t \in [0, T].$$

Define $d = K_3 T + \|E\|_\infty + 1$ and $c = -(K_3 T + \|E\|_\infty + 1)$. Then, if $y(0) = d$, we have

$$\dot{x}(t) = y(t) + E(t) = y(0) + \int_0^t \dot{y}(s)\, ds + E(t) \ge d - K_3 T - \|E\|_\infty > 0,$$

for every $t \in [0, T]$, and so $x(T) - x(0) > 0$. Similarly, if $y(0) = c$, then $x(T) - x(0) < 0$, which shows that the twist condition is satisfied.

As a consequence of Theorem 4.1 we then immediately have the following.

Corollary 5.3. *In the above setting, assume moreover that*

$$\frac{(\mu\nu)^{1/p}}{\mu^{1/p} + \nu^{1/p}} \ne \frac{n\pi_p}{T}, \quad \text{for every } n \in \mathbb{N}.$$

Then system (5.4) has at least two geometrically distinct T-periodic solutions.

A variant of the previous example is provided by the system

$$\begin{cases} \ddot{x} + A\sin x = e(t) + \partial_x P(t, x, u), \\ \dot{u} = \nu(v^-)^{q-1} - \mu(v^+)^{q-1}, \\ \dot{v} = \mu(u^+)^{p-1} - \nu(u^-)^{p-1} - \partial_u P(t, x, u). \end{cases} \tag{5.6}$$

where, being $w = (u, v)$, one has $w^+ = (u^+, v^+)$ and $w^- = (u^-, v^-)$. Assuming μ, ν to be positive, if we define $\mathscr{H}$ by

$$\mathscr{H}(u, v) = \frac{1}{q}\left(\mu(v^+)^q + \nu(v^-)^q\right) + \frac{1}{p}\left(\mu(u^+)^p + \nu(u^-)^p\right),$$

then $\mathscr{H}$ is positively-(p, q)-homogeneous and positive, and all the solutions of system $J\dot{w} = \nabla \mathscr{H}(w)$ with $w = (u, v)$ are periodic having the same minimal period τ, which can be computed as follows.

We first consider the dynamics in the first quadrant, i.e., when $u > 0$ and $v > 0$. In this case we can write $J\dot{w} = \nabla \mathscr{H}(w)$ as

$$\dot{u} = \mu v^{q-1}, \qquad \dot{v} = -\mu u^{p-1},$$

leading to the equation

$$\frac{d}{dt}\left(|\dot{u}|^{p-2}\dot{u}\right) + \mu^p u^{p-1} = 0.$$

Then, recalling (5.3), the time needed to pass from the positive v-axis to the positive u-axis is

$$\tau_1 = \frac{1}{4}\pi_p 2(\mu^p)^{-\frac{1}{p}} = \frac{\pi_p}{2\mu}.$$

Similarly, in the fourth quadrant, where $u > 0$ and $v < 0$, the system becomes

$$\dot{u} = -\nu|v|^{q-2}v, \qquad \dot{v} = \mu u^{p-1},$$

leading to the equation

$$\frac{d}{dt}\left(|\dot{u}|^{p-2}\dot{u}\right) + \mu\nu^{p-1}u^{p-1} = 0.$$

So, the time needed to pass from the positive u-axis to the negative v-axis is

$$\tau_2 = \frac{1}{4}\pi_p 2(\mu\nu^{p-1})^{-\frac{1}{p}} = \frac{\pi_p}{2\mu^{\frac{1}{p}}\nu^{\frac{1}{q}}}.$$

In a similar way, we obtain that the time needed to pass from the negative v-axis to the negative u-axis is

$$\tau_3 = \frac{\pi_p}{2\nu}$$

and the time needed to pass from the negative u-axis to the positive v-axis is

$$\tau_4 = \frac{\pi_p}{2\mu^{\frac{1}{q}}\nu^{\frac{1}{p}}}.$$

Hence,

$$\tau = \tau_1 + \tau_2 + \tau_3 + \tau_4 = \frac{\pi_p}{2}\left(\frac{1}{\mu} + \frac{1}{\nu} + \frac{1}{\mu^{\frac{1}{p}}\nu^{\frac{1}{q}}} + \frac{1}{\mu^{\frac{1}{q}}\nu^{\frac{1}{p}}}\right).$$

We thus get the following consequence of Theorem 4.1.

Corollary 5.4. *In the above setting, assume moreover that*

$$\frac{\pi_p}{2}\left(\frac{1}{\mu} + \frac{1}{\nu} + \frac{1}{\mu^{\frac{1}{p}}\nu^{\frac{1}{q}}} + \frac{1}{\mu^{\frac{1}{q}}\nu^{\frac{1}{p}}}\right) \neq \frac{T}{n}, \quad \text{for every } n \in \mathbb{N} \setminus \{0\}.$$

Then system (5.6) *has at least two geometrically distinct T-periodic solutions.*

Both Corollaries 5.3 and 5.4 generalize a classical theorem of Mawhin and Willem [14] on the multiplicity of periodic solutions for the pendulum equation.

6. Final remarks

In Theorem 2.1, dealing with the Neumann problem, we have only considered a diagonal matrix $\mathbb{A}$ like in (2.2). However, for the T-periodic problem, the first author with Gidoni in [4] were able to deal with *any* symmetric matrix $\mathbb{A}$, provided that the nonresonance condition $\sigma(J\mathbb{A}) \cap \frac{2\pi}{T} i\mathbb{Z} = \emptyset$ is assumed. We are confident that a similar result should also hold for the Neumann problem, but we have been able to prove it only when $L = 1$ and the matrix has a positive determinant. Here is our result.

Theorem 6.1. *Assume $L = 1$ and that A1 – A3 hold true. Let $\mathbb{A}$ be a symmetric 2×2 matrix such that $\det \mathbb{A} > 0$. If the non-resonance condition $\sigma(J\mathbb{A}) \cap \frac{\pi}{b-a} i\mathbb{Z} = \emptyset$ holds, then there are at least $M + 1$ geometrically distinct solutions of the boundary value problem (2.1)–(2.3).*

Proof. Consider the planar Hamiltonian system

$$J\dot{w} = \mathbb{A}w. \tag{6.1}$$

We can diagonalize $\mathbb{A}$ by a symplectic transformation. Indeed, there exist a matrix $\mathbb{U}$ with $\det \mathbb{U} = 1$ and a diagonal matrix $\mathbb{D}$ such that

$$\mathbb{A} = \mathbb{U}^{-1}\mathbb{D}\mathbb{U}.$$

Since $\det \mathbb{U} = 1$, and the dimension is 2, the change of variables $\varrho = \mathbb{U}w$ is symplectic. Hence, system (6.1) is transformed into the new Hamiltonian system

$$J\dot{\varrho} = \mathbb{D}\varrho, \tag{6.2}$$

with

$$\mathbb{D} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix},$$

for some α, β such that $\alpha\beta > 0$. Now, the symplectic change of variables $\varpi = \mathbb{M}\varrho$, with

$$\mathbb{M} = \begin{pmatrix} \sqrt[4]{\frac{\alpha}{\beta}} & 0 \\ 0 & \sqrt[4]{\frac{\beta}{\alpha}} \end{pmatrix}.$$

transforms system (6.2) into

$$J\dot{\varpi} = \lambda\varpi,$$

with $\lambda = \pm\sqrt{\alpha\beta}$, according to the signs of α and β. However, if $\lambda < 0$, a final change of variables $t \mapsto -t$ will lead to a positive λ. We can now apply Theorem 2.1 to conclude the proof. $\qquad\square$

The general case $L \geq 2$ remains an open problem. However, writing $w = (w_1, \ldots, w_L)$ with $w_j = (u_j, v_j)$ and applying componentwise the same procedure in the proof of Theorem 6.1 above, Theorem 2.1 can be generalized to the case when the last equation in system (2.1) is of the type

$$J\dot{w}_j = \mathbb{A}_j w_j + \nabla_{w_j} P(t, x, y, w), \quad j = 1, \ldots, L,$$

where $\mathbb{A}_j$ is a symmetric 2×2 matrix such that $\det \mathbb{A}_j > 0$. We omit the details, for briefness.

As a final remark, we recall that, for the periodic problem, Chen and Qian in [2] proved a multiplicity result, coupling *resonant* linear components with twisting components by using Ahmad-Lazer-Paul type resonance condition. In our case, a similar result can be expected for Neumann problem without any twist assumption. The problem remains open for further investigation.

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