



Multiplicity results for generalized quasilinear critical Schrödinger equations in \mathbb{R}^N

Laura Baldelli and Roberta Filippucci

Abstract. Multiplicity results are proved for solutions both with positive and negative energy, as well as nonexistence results, of a generalized quasilinear Schrödinger potential free equation in the entire \mathbb{R}^N involving a nonlinearity which combines a power-type term at a critical level with a subcritical term, both with weights. The equation has been derived from models of several physical phenomena such as superfluid film in plasma physics as well as the self-channelling of a high-power ultra-short laser in matter. Proof techniques, also in the symmetric setting, are based on variational tools, including concentration compactness principles, to overcome lack of compactness, and the use of a change of variable in order to deal with a well defined functional.

Mathematics Subject Classification. Primary 35J62; Secondary 35J60, 35J20.

Keywords. Variational methods, Generalized quasilinear Schrödinger equations, Multiplicity results, Concentration compactness.

1. Introduction

In this paper, we are interested in multiplicity results for nontrivial weak solutions, both with negative and positive energy of the following generalized quasilinear Schrödinger equation involving a critical term

$$-\Delta_p u - \frac{\alpha}{2} \Delta_p(|u|^\alpha)|u|^{\alpha-2}u = \lambda V(x)|u|^{k-2}u + \beta K(x)|u|^{\alpha p^* - 2}u \quad \text{in } \mathbb{R}^N, \quad (1.1)$$

where $\Delta_m u = \operatorname{div}(|Du|^{m-2}Du)$ is the m -Laplacian of u , $\alpha > 1$, $\beta, \lambda > 0$, $1 < p < N$, $p^* = Np/(N-p)$ is the critical Sobolev's exponent, the exponent k is such that $\alpha < k < \alpha p^*$ and the weights are nontrivial and satisfy

$$0 \leq V \in L^r(\mathbb{R}^N) \cap C(\mathbb{R}^N), \quad r = \frac{\alpha p^*}{\alpha p^* - k}, \quad (1.2)$$

$$K \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N). \quad (1.3)$$

In particular, we emphasize that the weight K can change sign.

Actually, in the last section of the paper, we deal also with the singular case, given by $0 < \alpha < 1$, and, in particular, thanks to a technique used to attach the case $\alpha > 1$, we succeed in extending some previous results proved in [6]. Our multiplicity results are a first contribution in the study of critical generalized quasilinear Schrödinger equations involving p -Laplacian and general $\alpha > 1$. To the best of our knowledge, they are new even in the Laplacian case.

Solutions of problems of type (1.1) are related to the existence of standing wave solutions for Schrödinger equations of the form

$$i\partial_t\psi = -\Delta\psi + W(x)\psi - \varphi(|\psi|^2)\psi - \kappa\Delta\varrho(|\psi|^2)\varrho'(|\psi|^2)\psi, \quad (1.4)$$

where $\psi : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{C}$, W is a given potential, $\kappa \in \mathbb{R}$ and φ, ϱ real functions of mainly pure power form. By using the well known energy methods, the semilinear subcase $\kappa = 0$ of (1.4) was studied deeply in [20], see also and the references therein.

Equation (1.4), according to different types of ϱ , has been derived from models of several physical phenomena. For instance, when $\varrho(s) = s$, then (1.4) is interpreted as the superfluid film equation in plasma physics, while if $\varrho(s) = (1 + s)^{1/2}$, Eq. (1.4) models the self-channelling of a high-power ultra short laser in matter. We refer to [25], for applications, in the theory of Heisenberg ferromagnets and magnons and to [7], where models for binary mixtures of ultracold quantum gases are treated.

It is worth pointing out that there are many difficulties in treating this class of generalized critical quasilinear Schrödinger equations (1.1) in \mathbb{R}^N , such as lack of compactness, the presence of the term $\Delta_p(|u|^\alpha)|u|^{\alpha-2}u$ which prevents us from working directly in a classical working space, to manage weights and finally to deal with an “apparently” supercritical term $K(x)|u|^{\alpha p^* - 2}u$. In light of this, challenging tasks, tricky to be managed, appear.

Actually, when $\alpha > 1$, the exponent αp^* plays the role of the critical exponent, differently from the case $0 < \alpha < 1$ in which p^* behaves like the critical exponent, as it appears in the critical equation (7.1) below. In particular, Sect. 2 is aimed to justify the appearance of the new critical exponent αp^* in terms of the compactness of a suitable embedding and of the nonexistence of solutions beyond αp^* . In the spirit of Theorem 2.11 in [2], where the semilinear version ($p = 2$) of (1.1) is investigated. We refer also to [18, 19, 35].

In addition, the choice of the parameter α gives rise to a different nature of the equation under consideration since for $\alpha > 1$, the term $\Delta_p(|u|^\alpha)|u|^{\alpha-2}u$, $p > 1$, is degenerate at $u = 0$, while for $0 < \alpha < 1$ it becomes singular when $u = 0$.

In literature, critical Schrödinger equations in \mathbb{R}^N are mainly studied in their physical relevance, corresponding to the situation $p = 2$ and when a potential term is involved, starting from the pioneering paper by Brezis and Nirenberg [12]. We refer also to [13] for existence results for semilinear elliptic equations with singular potential. The degenerate case, $\alpha > 1$, is mostly studied when $\alpha = 2$, see [33], where existence results can be deduced

by applying the Mountain Pass Theorem in the superlinear subcritical case $4(= \alpha p) < k < 2 \cdot 2^*$ of (1.1). As for equations where the reaction combines the multiple effects generated by a singular term and a critical term even with nonhomogeneous operators and in bounded domains, we refer to the recent works [23, 29].

For existence results in the critical semilinear case when $\alpha > 1$ in (1.1) very few is known in the entire interval $\alpha < k < \alpha p^*$. Nonlinearities only subcritical are studied in [1] when $2^* < k < 2^* \alpha$. Among subcritical problems, we also mention [22], where multiplicity results for Schrödinger equations when $p = 2$ are proved. In the critical case, see also [18, 32] where a general nonlinearity is included. Furthermore, for $\alpha = 2$, concerning concentrating behavior of solutions, we quote [14] where a new phenomenon occurs passing from critical to supercritical nonlinearities in terms of the limit equation.

Passing to the general quasilinear case, $1 < p < N$, with $\alpha > 1$, it seems that there are no multiplicity results for quasilinear critical Schrödinger problems on \mathbb{R}^N when a potential term is not involved in the equation and the growth rate of the subcritical term is less than αp . Motivated by this observation and with the sake of completing the picture started in [6], where the singular case $0 < \alpha < 1$ in (1.1) is treated, in this paper we investigate the case $\alpha < k < p$ in the following theorem, where E_λ stands for the standard energy functional associated to Eq. (1.1), see Sect. 2.

Theorem 1. *Assume $N \geq 3$, $1 < \alpha < k < p$. Let V, K satisfying (1.2) and (1.3), respectively. Then,*

- (i) *For any $\lambda > 0$, there exists $\beta^* > 0$ such that for any $0 < \beta < \beta^*$, then Eq. (1.1) has infinitely many nontrivial solutions $(u_n)_n \subset D^{1,p}(\mathbb{R}^N)$ such that $E_\lambda(u_n) < 0$ and $\|u_n\| \rightarrow 0$ as $n \rightarrow \infty$.*
- (ii) *For any $\beta > 0$, there exists $\lambda^* > 0$ such that for any $0 < \lambda < \lambda^*$, then Eq. (1.1) has infinitely many nontrivial solutions $(u_n)_n \subset D^{1,p}(\mathbb{R}^N)$ such that $E_\lambda(u_n) < 0$ and $\|u_n\| \rightarrow 0$ as $n \rightarrow \infty$.*

The proof of the above multiplicity result relies on the concentration compactness principle, the truncation of the energy functional and the theory of Krasnosel'skii genus. In particular, we cannot manage directly the energy functional E_λ associated with (1) since it might be not well defined, but we need to perform a suitable change of variables, in the spirit of [17], to achieve a nice functional. This causes some further obstacles in recovering in some sense compactness beyond p^* , since, as noted above, $\alpha p^* > p^*$. While the behaviour of the norm of the solutions follows from the application of the symmetric Mountain Pass Theorem, see [3, 21, 34].

Moreover, we also need to restrict the entire sublinear interval $\alpha < k < \alpha p$ to $\alpha < k < p$ due to the shape of the transformed energy functional which yields tricky estimates, cfr. Remark 3. As far as we know, in [36] there is an attempt to cover the entire interval $\alpha < k < \alpha p$ when $K \equiv 1$, by using an inequality which seems difficult to be verified.

In the second part of our paper, we study (1.1) in a symmetric setting related to a subgroup T of $O(N)$, the group of orthogonal linear transformations

in \mathbb{R}^N . Here,

$$|T| := \inf_{x \in \mathbb{R}^N, x \neq 0} |T_x|,$$

where $|T_x|$ the cardinality of a T -orbit $T_x := \{\tau x | \tau \in T\}$ with $|T_\infty| := 1$ (so that $|T_0| = 1$).

We make use of T -symmetric functions $f : \Omega \rightarrow \mathbb{R}^N$, i.e. $f(\tau x) = f(x)$ for all $\tau \in T$ and $x \in \Omega$, with Ω open T -symmetric subset of \mathbb{R}^N (i.e. if $x \in \Omega$, then $\tau x \in \Omega$ for all $\tau \in T$). For example, even functions are T -symmetric functions with $T = \{id, -id\}$, thus $|T| = 2$, and radially symmetric functions are T -symmetric functions with $T = O(N)$, thus $|T| = \infty$. Then, we denote with $D_T^{1,p}(\mathbb{R}^N)$ the subspace of $D^{1,p}(\mathbb{R}^N)$ consisting of all T -symmetric functions, cf. Sect. 2.

A pioneering paper about critical symmetric problems in the entire \mathbb{R}^N is [9] by Bianchi, Chabrowskii and Szulkin, where existence and multiplicity results are obtained for $p = 2$ under appropriate assumptions on the single weight involved and on the group T , no subcritical terms are included. In this context, we mention also [16] where the symmetry of solutions is shown by applying the improved “moving plane” method.

Differently from Theorem 1, where no assumptions on the sign for the weight K are required, in the next result, nonnegativity for the weight K is needed. Furthermore, we consider solutions with positive energy, with no additional restrictions on α, p, k , indeed the range for α and k is the largest possible.

Theorem 2. *Assume $N \geq 3$, $1 < \alpha < k < \alpha p^*$. Let V, K be T -symmetric functions satisfying (1.2), (1.3), with K nonnegative (nontrivial) in \mathbb{R}^N . If*

$$K(0) = K(\infty) = 0 \quad \text{and} \quad |T| = \infty, \tag{1.5}$$

where $K(\infty) = \limsup_{|x| \rightarrow \infty} K(x)$. Then, for all $\lambda, \beta > 0$ Eq. (1.1) possesses infinitely many solutions $(u_n)_n \subset D_T^{1,p}(\mathbb{R}^N)$ with positive energy such that $E_\lambda(u_n) \rightarrow \infty$ as $n \rightarrow \infty$.

The main ingredient used in the proof of Theorem 2 is the Fountain Theorem, which requires the Palais Smale property for the functional at any positive level. This is in force by virtue of the crucial assumption (1.5).

The paper is structured as follows. The description of the functional setting, the reformulation of the problem by a suitable change of variable and motivations on the critical exponent αp^* are contained in Sect. 2. Section 3 encloses compactness properties thanks to which we overcome the lack of compactness, while in Sect. 4 we perform a deep analysis on the possible behaviours of the energy functional and of its truncated version, this latter introduced to restore the boundedness from below. The proof of Theorem 1 is disclosed in Sect. 5, whereas in Sect. 6 we deal with solutions with positive energy and we develop the proof Theorem 2 in the symmetric setting described before. Finally, Sect. 7, by using some ideas given in Sect. 3, is devoted to extending Theorem 1.1 in [6], relative to the singular case $0 < \alpha < 1$, covering the entire interval $1 < k < \alpha p$.

2. Preliminaries

In this section, we introduce the main notations and we present preliminary results useful for the proof of the main theorems of the paper, given in Sects. 5, 6.

Let $D^{1,p}(\mathbb{R}^N)$ be the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|u\|_{D^{1,p}} = \|Du\|_p$, where $\|\cdot\|_p$ is the L^p norm in \mathbb{R}^N . In particular, we can define the reflexive Banach space $D^{1,p}(\mathbb{R}^N) = \{u \in L^{p^*}(\mathbb{R}^N) : Du \in L^p(\mathbb{R}^N)\}$.

The Euler Lagrange functional associated with Eq. (1.1) is the following

$$E_\lambda(u) = \frac{1}{p} \int_{\mathbb{R}^N} g(u)^p |Du|^p dx - \frac{\lambda}{k} \int_{\mathbb{R}^N} V|u|^k dx - \frac{\beta}{\alpha p^*} \int_{\mathbb{R}^N} K|u|^{\alpha p^*} dx, \tag{2.1}$$

for $u \in D^{1,p}(\mathbb{R}^N)$, where

$$g(t) = \left[1 + \frac{\alpha^p}{2} |t|^{p(\alpha-1)} \right]^{1/p}, \quad t \in \mathbb{R}. \tag{2.2}$$

Due to the appearance of the coercive term g , indeed $g(t) \rightarrow \infty$ as $|t| \rightarrow \infty$, when $\alpha > 1$, the functional E_λ may be not well defined in $D^{1,p}(\mathbb{R}^N)$, so we cannot apply variational methods or nonsmooth critical point theory to deal directly with (2.1). For example, from [31], if we consider $u_e(x) = |x|^{-(N-p)/2p}$, $x \in B_1 \setminus \{0\}$ then $u_e \in D^{1,p}(\mathbb{R}^N)$ but $g(u_e)^p |Du_e|^p \notin L^1(\mathbb{R}^N)$ for any $\alpha \geq 2$. To overcome this difficulty, we make a change of variables developed in [17], following an idea in [25], precisely

$$v = G(u) = \int_0^u g(z) dz. \tag{2.3}$$

In particular, the function g defined in (2.2) is an even function in \mathbb{R} , $g(0) = 1$, g is increasing in \mathbb{R}^+ and decreasing in \mathbb{R}^- . For any $\alpha > 0$ and $t \in \mathbb{R}$, we have

$$|G(t)| \leq \int_0^{|t|} \left(1 + \frac{\alpha}{2^{1/p}} y^{\alpha-1} \right) dy = |t| + \frac{1}{2^{1/p}} |t|^\alpha \tag{2.4}$$

Thus, G is well defined and continuous in \mathbb{R} . Moreover, G is a strictly increasing function being $g \geq 1$, $G(0) = 0$ and $\lim_{|t| \rightarrow \infty} G(t) = \infty$. So, we can define G^{-1} , an invertible, odd and C^1 function such that $G^{-1}(s) \geq 0$ for any $s \geq 0$. Thanks to the change of variables described above, the energy functional E_λ can be written by the following functional

$$F_\lambda(v) := \frac{1}{p} \int_{\mathbb{R}^N} |Dv|^p dx - \frac{\lambda}{k} \int_{\mathbb{R}^N} V|G^{-1}(v)|^k dx - \frac{\beta}{\alpha p^*} \int_{\mathbb{R}^N} K|G^{-1}(v)|^{\alpha p^*} dx, \tag{2.5}$$

for $v \in D^{1,p}(\mathbb{R}^N)$. The proof of the regularity of F_λ takes the following steps, starting with the properties of g and G . Especially, the following lemma holds.

Lemma 1. *Let $\alpha > 1$. Then, it holds*

- (a) $\lim_{s \rightarrow 0} \frac{|G^{-1}(s)|}{|s|} = 1$ thus $\lim_{s \rightarrow 0} \frac{|G^{-1}(s)|^\alpha}{|s|} = 0$;
- (b) $\lim_{s \rightarrow \infty} \frac{|G^{-1}(s)|^\alpha}{|s|} = 2^{1/p}$ thus $\lim_{s \rightarrow \infty} \frac{|G^{-1}(s)|}{|s|} = 0$;
- (c) $|G^{-1}(s)| \leq |s|$, for every $s \in \mathbb{R}$;
- (d) $0 \leq \frac{g'(t)t}{g(t)} < \alpha - 1$, for every $t \in \mathbb{R}$;
- (e) $|G^{-1}(s)| \leq |G^{-1}(s)g(G^{-1}(s))| < \alpha|s|$, for every $s \in \mathbb{R}$;

(f) Take $\wp \geq 0$. Then the following hold for every $t \in \mathbb{R}$.

$$\text{If } \wp > \alpha \text{ then } \frac{\wp - \alpha}{g(t)} < \frac{(\wp - 1)g(t) - tg'(t)}{g^2(t)} \leq \wp - 1.$$

$$\text{If } 1 < \wp < \alpha \text{ then } \wp - \alpha < \frac{(\wp - 1)g(t) - tg'(t)}{g^2(t)} \leq \wp - 1.$$

(g) $|G^{-1}(s)|^\alpha \leq 2^{p-1}|s|$, for every $s \in \mathbb{R}$;

(h) $|G^{-1}(s)|^\alpha \geq |G^{-1}(1)|^\alpha |s|$, for every $s \in \mathbb{R}$ such that $|s| \geq 1$;

(i) $|G^{-1}(s)|^{\alpha-1}(G^{-1}(s))' \leq C$ with $C > 0$.

Proof. Since G^{-1} is odd, we only consider the case $s \geq 0$. Property (a) is trivial by Hospital's rule, being $t = G^{-1}(s)$ so that $G'(t) = g(t)$ with $g(0) = 1$. Also (b) follows immediately again from Hospital's rule since

$$\lim_{t \rightarrow \infty} \frac{t^\alpha}{G(t)} = \alpha \lim_{t \rightarrow \infty} \frac{t^{\alpha-1}}{g(t)} = 2^{1/p}, \quad \lim_{t \rightarrow \infty} \frac{t}{G(t)} = \lim_{t \rightarrow \infty} \frac{1}{g(t)} = 0,$$

where we have used also that

$$g(t) \sim \frac{\alpha}{2^{1/p}} t^{\alpha-1}, \quad \text{as } t \rightarrow \infty. \tag{2.6}$$

Condition (c) follows since $g(t) > 1$ for all $t > 0$, thus $G(t) = \int_0^t g(z) dz \geq t$ for every $t \geq 0$. To prove (d), since g is increasing in \mathbb{R}^+ and positive, we have

$$0 \leq \frac{g'(t)t}{g(t)} = (\alpha - 1) \cdot \frac{\alpha^p t^{p(\alpha-1)}}{2 + \alpha^p t^{p(\alpha-1)}} < \alpha - 1,$$

being $\alpha > 1$. To get (e), multiply (d) by $g(t) > 0$ and integrate so that

$$(\alpha - 1)G(t) > \int_0^t \{[g(z)z]' - g(z)\} dz = g(t)t - G(t), \quad t > 0.$$

In turn, inequality (e) follows taking $s = G(t)$. For the proof of (f), it is enough to multiply (d) by $-1/g(t)$ and then add $(\wp - 1)/g(t)$ so that

$$\frac{\wp - \alpha}{g(t)} < \frac{(\wp - 1)g(t) - tg'(t)}{g^2(t)} \leq \frac{\wp - 1}{g(t)},$$

yielding (f) depending on the case. In order to prove (g) and (h) take

$$\left(\frac{|G^{-1}(s)|^\alpha}{s} \right)' = \frac{G^{-1}(s)^{\alpha-1}[\alpha s - g(G^{-1}(s))G^{-1}(s)]}{s^2 g(G^{-1}(s))} > 0, \quad \text{for } s \geq 0$$

from (e). Consequently, $|G^{-1}(s)|^\alpha/s$ is strictly increasing and by (b) its limit at infinity is $2^{1/p}$, thus (g) follows immediately. In addition, we get $|G^{-1}(s)|^\alpha > |G^{-1}(1)|^\alpha s$ for $s \geq 1$ and (h) holds by virtue of symmetry. Finally, (i) follows from $G^{-1}(0) = 0$ and (2.6). \square

Remark 1. Note that, by definition, we have

$$[G^{-1}(s)g(G^{-1}(s))]' = 1 + \frac{G^{-1}(s)g'(G^{-1}(s))}{g(G^{-1}(s))} \in (1, \alpha) \tag{2.7}$$

thanks to Lemma 1-(d). Moreover, for any $v \in D^{1,p}(\mathbb{R}^N)$

$$D[G^{-1}(v)g(G^{-1}(v))] = \left[1 + \frac{G^{-1}(v)g'(G^{-1}(v))}{g(G^{-1}(v))} \right] Dv$$

so, by using (2.7), we obtain

$$|Dv| \leq |D[G^{-1}(v)g(G^{-1}(v))]| \leq \alpha|Dv|. \tag{2.8}$$

In addition, since $g \geq 1$, then we have $|DG^{-1}(v)| = |Dv|/g(G^{-1}(v)) \leq |Dv|$. Thus, for any $v \in D^{1,p}(\mathbb{R}^N)$ we have $G^{-1}(v)g(G^{-1}(v)), G^{-1}(v) \in D^{1,p}(\mathbb{R}^N)$. On the other hand, if we take $v \in D^{1,p}(\mathbb{R}^N)$, by using the definition of g in (2.2), we have

$$\begin{aligned} |D(|G^{-1}(v)|^\alpha)|^p &= \alpha^p |G^{-1}(v)|^{p(\alpha-1)} |DG^{-1}(v)|^p = \alpha^p |G^{-1}(v)|^{p(\alpha-1)} \left| \frac{Dv}{g(G^{-1}(v))} \right|^p \\ &= 2 \frac{\frac{\alpha^p}{2} |G^{-1}(v)|^{p(\alpha-1)}}{1 + \frac{\alpha^p}{2} |G^{-1}(v)|^{p(\alpha-1)}} |Dv|^p \leq 2|Dv|^p, \end{aligned} \tag{2.9}$$

so that for any $v \in D^{1,p}(\mathbb{R}^N)$ also $|G^{-1}(v)|^\alpha \in D^{1,p}(\mathbb{R}^N)$.

In what follows we make use of the next crucial lemma, for its proof we refer to Lemma 2.2 in [6] where we consider $|G^{-1}(\cdot)|^\alpha$ in place of $G^{-1}(\cdot)$ and, by Remark 1, $|G^{-1}(v)|^\alpha \in D^{1,p}(\mathbb{R}^N)$ for every $v \in D^{1,p}(\mathbb{R}^N)$.

Lemma 2. *Assume $v_n \rightharpoonup v$ in $D^{1,p}(\mathbb{R}^N)$, then $|G^{-1}(v_n)|^\alpha \rightharpoonup |G^{-1}(v)|^\alpha$ in $D^{1,p}(\mathbb{R}^N)$.*

Actually, as described in the Introduction, Eq. (1.1) is critical since the corresponding critical exponent in the nonlinearity is αp^* , as soon as $\alpha > 1$. For the Laplacian case $p = 2$, we refer to [2] where a detailed discussion in this direction is conducted, see also [18, 35]. In order to justify this assumption, we first prove nonexistence beyond αp^* , and then we recover continuity and local compactness until αp^* .

In the following theorem, by the celebrated variational identity by Pucci and Serrin [30], we immediately get that for $k \geq \alpha p^*$ nonexistence follows at all, generalizing the result in [26] where the authors studied the case $p = \alpha = 2$ provided nonexistence of solutions in $H^1(\mathbb{R}^N)$ with $|Du|^2|u|^2 \in L^1(\mathbb{R}^N)$.

Theorem 3. *Equation (1.1) does not admit any solutions if the following hold*

$$k \geq \alpha p^*, \quad x \cdot DV(x) \leq 0, \quad x \cdot DK(x) \leq 0 \tag{2.10}$$

Proof. By using the following Pucci and Serrin variational identity [30], in \mathbb{R}^N

$$\begin{aligned} 0 &= \int_{\mathbb{R}^N} \left\{ \mathcal{F}(x, u, Du) \operatorname{div} h + h_i \mathcal{F}_{x_i}(x, u, Du) - \left[\frac{\partial u}{\partial x_j} \frac{\partial h_j}{\partial x_i} + u \frac{\partial a}{\partial x_i} \right] \mathcal{F}_{p_i}(x, u, Du) \right. \\ &\quad \left. - a \left[\frac{\partial u}{\partial x_i} \mathcal{F}_{p_i}(x, u, Du) + u \mathcal{F}_u(x, u, Du) \right] \right\} dx, \end{aligned}$$

for

$$\mathcal{F}(x, u, Du) = \frac{1}{p} g(u)^p |Du|^p - \frac{\lambda}{k} V(x) |u|^k - \frac{\beta}{\alpha p^*} K(x) |u|^{\alpha p^*},$$

we get

$$\begin{aligned}
 0 &= \int_{\mathbb{R}^N} \left\{ \left[\frac{1}{p} g(u)^p |Du|^p - \frac{\lambda}{k} V |u|^k - \frac{\beta}{\alpha p^*} K |u|^{\alpha p^*} \right] \operatorname{div} h \right. \\
 &\quad + h_i \left(-\frac{\lambda}{k} \frac{\partial V}{\partial x_i} |u|^k - \frac{\beta}{\alpha p^*} \frac{\partial K}{\partial x_i} |u|^{\alpha p^*} \right) - g(u)^p |Du|^{p-2} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial h_j}{\partial x_i} \\
 &\quad - g(u)^p |Du|^{p-2} u \frac{\partial u}{\partial x_i} \frac{\partial a}{\partial x_i} - a g(u)^{p-1} |Du|^p [g(u) + u g'(u)] \\
 &\quad \left. + a [\lambda V |u|^k + \beta K |u|^{\alpha p^*}] \right\} dx.
 \end{aligned}$$

Choosing $h(x) = x$ and $a(x) = N/\alpha p^*$ we arrive to

$$\begin{aligned}
 0 &= \int_{\mathbb{R}^N} \left\{ \left[\frac{N}{p} g(u)^p |Du|^p - \frac{N\lambda}{k} V |u|^k - \frac{\beta N}{\alpha p^*} K |u|^{\alpha p^*} \right] \right. \\
 &\quad - \frac{\lambda}{k} \frac{\partial V}{\partial x_i} |u|^k - \frac{\beta}{\alpha p^*} \frac{\partial K}{\partial x_i} |u|^{\alpha p^*} - g(u)^p |Du|^{p-2} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \delta_{ij} \\
 &\quad \left. - \frac{N}{\alpha p^*} g(u)^{p-1} |Du|^p [g(u) + u g'(u)] + \frac{\lambda N}{\alpha p^*} V |u|^k + \frac{\beta N}{\alpha p^*} K |u|^{\alpha p^*} \right\} dx,
 \end{aligned}$$

namely

$$\begin{aligned}
 &\int_{\mathbb{R}^N} \left\{ \frac{N-p}{\alpha p} [(\alpha-1)g(u) - u g'(u)] g(u)^{p-1} |Du|^p - \lambda N \left(\frac{1}{k} - \frac{1}{\alpha p^*} \right) V |u|^k \right\} dx \\
 &= \int_{\mathbb{R}^N} \left\{ \frac{\lambda}{k} |u|^k x \cdot DV(x) + \frac{\beta}{\alpha p^*} |u|^{\alpha p^*} x \cdot DK(x) \right\} dx, \tag{2.11}
 \end{aligned}$$

which yields to a contradiction when (2.10) holds since the left hand side of (2.11) is positive being $k \geq \alpha p^*$, $\lambda > 0$ and thanks also to Lemma 1-(d), while the right hand side of (2.11) is non positive. \square

Moreover, in what follows we will prove the compactness of the inverse map of G defined in (2.3) in some sense until αp^* , taking into account similar results in [19, 25].

Theorem 4. *The map $|G^{-1}|^\alpha : D^{1,p}(\mathbb{R}^N) \rightarrow L^q(\mathbb{R}^N)$ is continuous for $p \leq q \leq p^*$ and $|G^{-1}|^\alpha : D^{1,p}(\mathbb{R}^N) \rightarrow L^q_{loc}(\mathbb{R}^N)$ is compact for $p \leq q < p^*$. Moreover, it holds the following*

$$G^{-1}(v_n(x)) \rightarrow G^{-1}(v(x)) \text{ a.e. } x \in \mathbb{R}^N. \tag{2.12}$$

Proof. Take $v \in D^{1,p}(\mathbb{R}^N)$ and, by using (2.9) and Sobolev’s inequality, we have

$$\begin{aligned}
 \|G^{-1}(v)\|_{\alpha p^*} &= \| |G^{-1}(v)|^\alpha \|_{p^*}^{1/\alpha} \leq S^{-1/\alpha p} \|D(|G^{-1}(v)|^\alpha)\|_p^{1/\alpha} \\
 &\leq (2/S)^{1/\alpha p} \|Dv\|_p^{1/\alpha}, \tag{2.13}
 \end{aligned}$$

where S is Sobolev’s constant, i.e. $S = \inf \{ \|Du\|_p^p \cdot \|u\|_p^{-p} : u \in D^{1,p}(\mathbb{R}^N) \setminus \{0\} \}$. So that (2.13) imply

$$\| |G^{-1}(v)|^\alpha \|_{p^*} = \|G^{-1}(v)\|_{\alpha p^*}^\alpha \leq C \|Dv\|_p, \tag{2.14}$$

so that we get continuity for $q = p^*$. To prove continuity for $q < p^*$, consider $x, y > 1$ with $1 < x < q$ and use Hölder's inequality with exponents y and $y' > 1$

$$\begin{aligned} \|G^{-1}(v)\|_{\alpha q} &= \left(\int_{\mathbb{R}^N} (|G^{-1}(v)|^\alpha)^{q-x} (|G^{-1}(v)|^\alpha)^x dx \right)^{1/\alpha q} \\ &\leq \left(\int_{\mathbb{R}^N} (|G^{-1}(v)|^\alpha)^{(q-x)y'} dx \right)^{1/\alpha q y'} \left(\int_{\mathbb{R}^N} (|G^{-1}(v)|^\alpha)^{xy} dx \right)^{1/\alpha q y} \end{aligned}$$

Now choose $y = p^*/x > 1$, $y' = p^*/(p^* - x)$ so that we get

$$\begin{aligned} \|G^{-1}(v)\|_{\alpha q} &\leq \left(\int_{\mathbb{R}^N} (|G^{-1}(v)|^\alpha)^{(q-x)p^*/(p^*-x)} dx \right)^{(p^*-x)/\alpha q p^*} \\ &\quad \left(\int_{\mathbb{R}^N} (|G^{-1}(v)|^\alpha)^{p^*} dx \right)^{x/\alpha q p^*} \\ &\leq \|v\|_{(q-x)p^*/(p^*-x)}^{(q-x)/\alpha q} \cdot \| |G^{-1}(v)|^\alpha \|_{p^*}^{x/\alpha q} \\ &\leq C \|Dv\|_p^{(q-x)/\alpha q} \cdot \|Dv\|_p^{x/\alpha q} = C \|Dv\|_p^{1/\alpha}, \end{aligned}$$

where we have used Lemma 1-(g), (2.14) and the continuity of embedding $D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{(q-x)p^*/(p^*-x)}(\mathbb{R}^N)$, since $q < p^*$. In turn $\| |G^{-1}(v)|^\alpha \|_q \leq C \|Dv\|_p$.

To prove the compactness of the map $|G^{-1}|^\alpha : D^{1,p}(\mathbb{R}^N) \hookrightarrow L^q_{loc}(\mathbb{R}^N)$ for $p \leq q < p^*$, we start from a bounded sequence $(v_n)_n$ in $D^{1,p}(\mathbb{R}^N)$, so that up to subsequences $v_n \rightharpoonup v$ in $D^{1,p}(\mathbb{R}^N)$ then, by Lemma 2 we have $|G^{-1}(v_n)|^\alpha \rightharpoonup |G^{-1}(v)|^\alpha$ in $D^{1,p}(\mathbb{R}^N)$ and by the compactness of the embedding of $D^{1,p}(\mathbb{R}^N)$ in $L^s_{loc}(\mathbb{R}^N)$ for any $1 < s < p^*$ we have

$$|G^{-1}(v_n)|^\alpha \rightarrow |G^{-1}(v)|^\alpha \text{ in } L^s(\omega), \quad \omega \subset \mathbb{R}^N, \quad 1 \leq s < p^*. \tag{2.15}$$

Finally, from (2.15), by using an increasing sequence of compact sets whose union is \mathbb{R}^N and a diagonal argument, we get (2.12). \square

In order to prove the regularity of F_λ , we need to analyze the regularity of

$$J(v) = \int_{\mathbb{R}^N} V |G^{-1}(v)|^k dx \quad \text{and} \quad H(v) = \int_{\mathbb{R}^N} K |G^{-1}(v)|^{\alpha p^*} dx$$

Lemma 3. *If $V \in L^r(\mathbb{R}^N)$ and $1 < \alpha < k < \alpha p^*$, then $J(v)$ is weakly continuous on $D^{1,p}(\mathbb{R}^N)$. Moreover, J is continuously differentiable and $J' : D^{1,p}(\mathbb{R}^N) \rightarrow [D^{1,p}(\mathbb{R}^N)]'$, for all $\psi \in D^{1,p}(\mathbb{R}^N)$, is given by*

$$J'(v)\psi = k \int_{\mathbb{R}^N} V \frac{|G^{-1}(v)|^{k-2} G^{-1}(v)}{g(G^{-1}(v))} \psi dx. \tag{2.16}$$

Proof. For any $v \in D^{1,p}(\mathbb{R}^N)$, by Theorem 4, then $|G^{-1}(v)|^\alpha \in L^{p^*}(\mathbb{R}^N)$, so that by Hölder inequality with exponents $r = \alpha p^*/(\alpha p^* - k)$ and $r' = \alpha p^*/k$ we have

$$\|V |G^{-1}(v)|^k\|_1 \leq \|V\|_r \| |G^{-1}(v)|^k \|_{\alpha p^*}^k \tag{2.17}$$

This implies that J is well defined. Let $(v_n)_n \in D^{1,p}(\mathbb{R}^N)$ such that $v_n \rightharpoonup v$ in $D^{1,p}(\mathbb{R}^N)$, thus, $(v_n)_n$ is bounded in $D^{1,p}(\mathbb{R}^N)$ and, by Theorem 4, also $(G^{-1}(v_n))_n$ is bounded in $L^{\alpha p^*}(\mathbb{R}^N)$ and $(|G^{-1}(v_n)|^k)_n$ in $L^{\alpha p^*/k}(\mathbb{R}^N)$.

Since $V \in L^r(\mathbb{R}^N)$, by (2.12) and (2.17), the latter applied with v_n instead of v , we get the weakly continuity of J , thanks to Lebesgue convergence Theorem, that is

$$J(v_n) = \int_{\mathbb{R}^N} V|G^{-1}(v_n)|^k dx \rightarrow \int_{\mathbb{R}^N} V|G^{-1}(v)|^k dx = J(v). \tag{2.18}$$

In order to prove the Fréchet differentiability, that is $J \in C^1$, it is enough to show that J is Gâteaux differentiable and has a continuous Gâteaux derivative on $D^{1,p}(\mathbb{R}^N)$. First, consider $v, \psi \in D^{1,p}(\mathbb{R}^N)$ and $0 < |t| < 1$, so that

$$\frac{J(v + t\psi) - J(v)}{t} = \int_{\mathbb{R}^N} V \frac{|G^{-1}(v + t\psi)|^k - |G^{-1}(v)|^k}{t} dx. \tag{2.19}$$

Using the mean value theorem, there exists $\delta \in (0, 1)$ such that

$$\begin{aligned} & \frac{||G^{-1}(v + t\psi)|^k - |G^{-1}(v)|^k|}{|t|} \\ &= k|G^{-1}(v + t\delta\psi)|^{k-1} |(G^{-1}(v + t\delta\psi))'| |\psi| \\ &\leq c|v + t\delta\psi|^{(k-\alpha)/\alpha} |\psi| \leq c \left(|v|^{(k-\alpha)/\alpha} |\psi| + |\psi|^{k/\alpha} \right), \end{aligned} \tag{2.20}$$

with $c > 0$, where we have used, for the first inequality, the following condition

$$\frac{|G^{-1}(s)|^{k-1} (G^{-1}(s))'}{s^{(k-\alpha)/\alpha}} \leq c \tag{2.21}$$

which holds for on bounded sets in \mathbb{R}_0^+ by using Lemma 1-(g), (i) . While, the last inequality in (2.20) follows from the elementary formula $(a + b)^r \leq C(a^r + b^r)$, $a, b, r > 0$ and $C > 0$.

Now, by applying Hölder’s inequality twice with exponents $r, \alpha p^*/(k-\alpha), p^*$ and $r, \alpha p^*/k$, we get

$$\int_{\mathbb{R}^N} V \left(|v|^{(k-\alpha)/\alpha} |\psi| + |\psi|^{k/\alpha} \right) dx \leq \|V\|_r \|\psi\|_{p^*} \left(\|v\|_{p^*}^{(k-\alpha)/\alpha} + \|\psi\|_{p^*}^{k/\alpha-1} \right).$$

The right hand side of the above inequality is finite thanks to the suitable summabilities of the functions V, ψ, v . Thus, by letting $t \rightarrow 0$ in (2.19) using (2.20), from the Lebesgue dominated convergence theorem, J is Gâteaux differentiable.

Before proving the continuity of the Gâteaux derivative, we claim that the functional $J'(v)$ is well defined for every $v \in D^{1,p}(\mathbb{R}^N)$, that is $J'(v) \in [D^{1,p}(\mathbb{R}^N)]'$. Indeed, as in Lemma 2.5 in [19], consider $\psi_n \rightarrow 0$ in $D^{1,p}(\mathbb{R}^N)$ so that, by (2.18), we get

$$\int_{\mathbb{R}^N} V|G^{-1}(\psi_n)|^k dx \rightarrow 0, \quad \text{as } n \rightarrow \infty \tag{2.22}$$

Now, by using that $g \geq 1$, (2.4) and Lemma 1-(i) we have

$$\begin{aligned} & \int_{\mathbb{R}^N} V|G^{-1}(v)|^{k-1}G^{-1}(v)'\psi_n dx \\ & \leq C_1 \left(\int_{\mathbb{R}^N} V|G^{-1}(v)|^k dx \right)^{1-1/k} \cdot \left(\int_{\mathbb{R}^N} V|G^{-1}(\psi_n)|^k dx \right)^{1/k} \\ & \quad + C_2 \left(\int_{\mathbb{R}^N} V|G^{-1}(v)|^k dx \right)^{1-\alpha/k} \cdot \left(\int_{\mathbb{R}^N} V|G^{-1}(\psi_n)|^k dx \right)^{\alpha/k} \end{aligned}$$

where we used Hölder’s inequality and the fact that $\alpha < k$. Now, applying (2.22), we get

$$\int_{\mathbb{R}^N} V|G^{-1}(v)|^{k-1}G^{-1}(v)'\psi_n dx \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

obtaining the continuity of J' and, consequently, the claim.

Now, to get (Fréchet) differentiability we check that the Gâteaux derivative $J' : D^{1,p}(\mathbb{R}^N) \rightarrow [D^{1,p}(\mathbb{R}^N)]'$ defined in (2.16) is continuous. To reach the claim, consider $v_n \rightarrow v$ in $D^{1,p}(\mathbb{R}^N)$, so that there exists $U \in L^{p^*}(\mathbb{R}^N)$ such that $|v_n| \leq U(x)$ a.e. in \mathbb{R}^N . Now, define $W(v) = V|G^{-1}(v)|^{k-2}G^{-1}(v)(G^{-1}(v))'$ hence, by (2.21) and Young’s inequality with exponents $\alpha(p^* - 1)/(\alpha p^* - k)$ and $\alpha(p^* - 1)/(k - \alpha)$, we get for $c > 0$

$$\begin{aligned} |W(v_n) - W(v)|^{(p^*)'} & \leq c(|W(v_n)|^{(p^*)'} + |W(v)|^{(p^*)'}) \\ & \leq c(|V|^{p^*/(p^*-1)}|v_n|^{p^*(k-\alpha)/\alpha(p^*-1)} \\ & \quad + |V|^{p^*/(p^*-1)}|v|^{p^*(k-\alpha)/\alpha(p^*-1)}) \\ & \leq c(|V|^r + |v_n|^{p^*} \\ & \quad + |v|^{p^*}) \leq c(|V|^r + U^{p^*} + |v|^{p^*}) \in L^1(\mathbb{R}^N). \end{aligned}$$

In turn, Lebesgue dominated convergence Theorem gives $\|W(v_n) - W(v)\|_{(p^*)'} \rightarrow 0$ as $n \rightarrow \infty$. Consequently, by Hölder’s inequality, for $\psi \in D^{1,p}(\mathbb{R}^N)$, we have

$$\begin{aligned} & |(J'(v_n) - J'(v))\psi| \\ & \leq k \int_{\mathbb{R}^N} |W(v_n) - W(v)||\psi| dx \leq k\|W(v_n) - W(v)\|_{(p^*)'} \|\psi\|_{p^*} \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$. Namely, $J \in C^1$. □

Lemma 4. *If $K \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$, then H is continuously differentiable in $D^{1,p}(\mathbb{R}^N)$ and its derivative $H' : D^{1,p}(\mathbb{R}^N) \rightarrow [D^{1,p}(\mathbb{R}^N)]'$, for all $v, \psi \in D^{1,p}(\mathbb{R}^N)$, is given by*

$$H'(v)\psi = \alpha p^* \int_{\mathbb{R}^N} K \frac{|G^{-1}(v)|^{\alpha p^* - 2} G^{-1}(v)}{g(G^{-1}(v))} \psi dx.$$

Proof. The proof relies on the one of Lemma 3 but with some adjustments. First, note that here there are no conditions on the exponent k . Trivially, the functional is well defined and weakly continuous on $D^{1,p}(\mathbb{R}^N)$. In order to prove

the Gâteaux differentiability of H , we use again the Mean Value Theorem but, instead of (2.20), we have

$$\begin{aligned} & \frac{||G^{-1}(v + t\psi)|^{\alpha p^*} - |G^{-1}(v)|^{\alpha p^*}|}{|t|} \\ &= \alpha p^* |G^{-1}(v + t\delta\psi)|^{\alpha p^* - 1} |(G^{-1}(v + t\delta\psi))'| |\psi| \\ & \leq c |G^{-1}(v + t\delta\psi)|^{\alpha(p^* - 1)} |\psi| \leq c |v + t\delta\psi|^{p^* - 1} |\psi| \leq c \left(|v|^{p^* - 1} |\psi| + |\psi|^{p^*} \right), \end{aligned}$$

where we have used Lemma 1-(g), (i) and the elementary formula $(a + b)^r \leq C(a^r + b^r)$, $a, b, r > 0$ and $C > 0$. Concerning the well definition of H on $[D^{1,p}(\mathbb{R}^N)]'$, we get

$$\begin{aligned} & \int_{\mathbb{R}^N} K |G^{-1}(v)|^{\alpha p^* - 1} G^{-1}(v)' \psi_n dx \\ & \leq C_1 \int_{\mathbb{R}^N} K |G^{-1}(v)|^{\alpha p^* - 1} G^{-1}(v)' G^{-1}(\psi_n) dx \\ & \quad + C_2 \int_{\mathbb{R}^N} K |G^{-1}(v)|^{\alpha p^* - 1} G^{-1}(v)' |G^{-1}(\psi_n)|^\alpha dx \\ & \leq C_1 \left(\int_{\mathbb{R}^N} |G^{-1}(v)|^{\alpha p^*} dx \right)^{1 - 1/\alpha p^*} \cdot \left(\int_{\mathbb{R}^N} |G^{-1}(\psi_n)|^{\alpha p^*} dx \right)^{1/\alpha p^*} \\ & \quad + C_2 \int_{\mathbb{R}^N} |G^{-1}(v)|^{\alpha(p^* - 1)} |G^{-1}(\psi_n)|^\alpha dx \\ & \leq C_1 \left(\int_{\mathbb{R}^N} |G^{-1}(v)|^{\alpha p^*} dx \right)^{1 - 1/\alpha p^*} \cdot \left(\int_{\mathbb{R}^N} |G^{-1}(\psi_n)|^{\alpha p^*} dx \right)^{1/\alpha p^*} \\ & \quad + C_2 \left(\int_{\mathbb{R}^N} |G^{-1}(v)|^{\alpha p^*} dx \right)^{1 - 1/p^*} \cdot \left(\int_{\mathbb{R}^N} |G^{-1}(\psi_n)|^{\alpha p^*} dx \right)^{1/p^*} \end{aligned}$$

where, for the integral near C_1 we use that $g \geq 1$ and Hölder's inequality with exponents αp^* and $\alpha p^*/(\alpha p^* - 1)$, while for the integral with C_2 we use in Lemma 1-(i) and Hölder's inequality with exponents p^* and $p^*/(p^* - 1)$.

Finally, the continuity of the Gâteaux derivative follows by defining \tilde{W} instead of W as $\tilde{W}(v) = |G^{-1}(v)|^{\alpha p^* - 2} G^{-1}(v) (G^{-1}(v))'$ and from the following

$$|W(v)|^{(p^*)'} \leq |G^{-1}(v)|^{\alpha p^*} \leq |v|^{p^*}$$

where Lemma 1-(g), (i) is used. □

Now, if we consider $v_n \rightarrow v$ in $D^{1,p}(\mathbb{R}^N)$, then by Lemma 2, also $|G^{-1}(v_n)|^\alpha \rightarrow |G^{-1}(v)|^\alpha$ in $D^{1,p}(\mathbb{R}^N)$. Since the first term of F_λ is a norm with exponent $p > 1$, and thanks to Lemmas 3 and 4 with $1 < \alpha < k$, then we get $F_\lambda \in C^1(D^{1,p}(\mathbb{R}^N))$, and $F'_\lambda : D^{1,p}(\mathbb{R}^N) \rightarrow (D^{1,p}(\mathbb{R}^N))'$, for all $v, \psi \in D^{1,p}(\mathbb{R}^N)$, is given by

$$\begin{aligned} F'_\lambda(v)\psi &= \int_{\mathbb{R}^N} |Dv|^{p-2} Dv D\psi dx - \lambda \int_{\mathbb{R}^N} V \frac{|G^{-1}(v)|^{k-2} G^{-1}(v)}{g(G^{-1}(v))} \psi dx \\ & \quad - \beta \int_{\mathbb{R}^N} K \frac{|G^{-1}(v)|^{\alpha p^* - 2} G^{-1}(v)}{g(G^{-1}(v))} \psi dx. \end{aligned} \tag{2.23}$$

We say that $v \in D^{1,p}(\mathbb{R}^N)$ is a (weak) *solution* of Eq. (1.1) if

$$F'_\lambda(v)\psi = 0 \quad \text{for all } \psi \in D^{1,p}(\mathbb{R}^N),$$

Clearly, (weak) *solutions* of (1.1) are exactly critical points of the Euler-Lagrange functional E_λ , or equivalently F_λ , associated with (1.1). Moreover, every critical point of F_λ correspond to a solution of the following equation

$$-\Delta_p v = \lambda V \frac{|G^{-1}(v)|^{k-2} G^{-1}(v)}{g(G^{-1}(v))} + \beta K \frac{|G^{-1}(v)|^{\alpha p^* - 2} G^{-1}(v)}{g(G^{-1}(v))}. \quad (2.24)$$

As a consequence, Theorems 7 and 2, whose statements are given in the Introduction in terms of u satisfying (1.1), can be stated in terms of $v = G(u)$ solutions of (2.24).

A key role in the proof of our results is the concentration compactness principles by Lions. For a detailed discussion on them, we refer to [4,5]. In particular, we are interested in the second concentration compactness principle, which regards a possible concentration only at finite points and where two different types (since we are in unbounded domains) of convergences are considered: the tight convergence of measures, whose symbol is $\overset{*}{\rightharpoonup}$, and the “weak” convergence, denoted with \rightharpoonup . Precisely,

Lemma 5. (Lemma I.1, [24]) *Assume $\Omega \subset \mathbb{R}^N$ a domain, $1 \leq p < N$. Let $(u_n)_n$ be a bounded sequence in $D^{1,p}(\Omega)$ converging weakly to some u and such that $|Du_n|^p dx \rightharpoonup \mu$ and either $|u_n|^{p^*} dx \rightharpoonup \nu$ if Ω is bounded or $|u_n|^{p^*} \overset{*}{\rightharpoonup} \nu$ if Ω is unbounded, where μ, ν are bounded nonnegative measures on Ω . Then there exists some at most countable set J such that*

$$\nu = |u|^{p^*} + \sum_{j \in J} \nu_j \delta_{x_j}, \quad \mu \geq |Du|^p + \sum_{j \in J} \mu_j \delta_{x_j}, \quad \nu_j \geq 0, \mu_j \geq 0$$

with $S \nu_j^{p/p^*} \leq \mu_j$ and $\sum_{j \in J} \nu_j^{p/p^*} < \infty$, where $(x_j)_{j \in J}$ are distinct points in Ω , δ_x is the Dirac-mass of mass 1 concentrated at $x \in \Omega$.

Since the tight convergence excludes a possible concentration at infinity, by using the lemma above, in order to get compactness, it remains only to show that concentration around points, described by ν_j , cannot occur. However, the proof of tightness by definition as well as using the first concentration compactness principle leads to rather cumbersome and tricky calculations. To overcome these difficulties, Chabrowskii presented a version at infinity of the second principle, cfr. Proposition 2 in [15], see also Bianchi et al. [9] for the Laplacian case. In this principle Chabrowskii manages to enclose the concentration at infinity in the parameter ν_∞ , in according to the ν_j s in Lemma 5 so that the non concentration at infinity occurs if one proves that $\nu_\infty = 0$. Later Ben-Naoum et al. in [8] obtain a version for the p -Laplacian, here reported for completeness.

Proposition 1. (Proposition 3.3, [8]) *Let $(u_n)_n$ be a bounded sequence in $D^{1,p}(\mathbb{R}^N)$ and define*

$$\nu_\infty = \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > R} |u_n|^{p^*} dx, \quad \mu_\infty = \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > R} |Du_n|^p dx.$$

Then, the quantities ν_∞ and μ_∞ exist and satisfy

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n|^{p^*} dx = \int_{\mathbb{R}^N} d\nu + \nu_\infty, \quad \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} |Du_n|^p dx = \int_{\mathbb{R}^N} d\mu + \mu_\infty,$$

with $S\nu_\infty^{p/p^} \leq \mu_\infty$, where ν and μ are as in (i) and (ii) in Lemma 5 and such that (iii) holds.*

Another key tool in proving the multiplicity result of solutions with positive energy, namely Theorem 2, is the Fountain Theorem. However, before its statement, we briefly report the setting needed. For some well-known basic definitions of actions, invariant functions, we refer to [5].

(A1) The compact group \mathcal{G} acts isometrically on the space $M = \overline{\bigoplus_{j \in \mathbb{N}} M_j}$, which is a Banach space, where the spaces M_j are \mathcal{G} -invariant and there exists a finite dimensional space W such that, for every $j \in \mathbb{N}$, $M_j \simeq W$ and the action of \mathcal{G} on W is admissible.

From the decomposition of the Banach space M in **(A1)**, we define Y_m and Z_m as follows

$$Y_m := \bigoplus_{j=0}^m M_j, \quad Z_m := \overline{\bigoplus_{j=m}^\infty M_j} \tag{2.25}$$

and set $B_m := \{u \in Y_m : \|u\| \leq \rho_m\}$, $N_m = \{u \in Z_m : \|u\| = r_m\}$ where $\rho_m > r_m > 0$.

Now we are ready to state the Fountain Theorem.

Theorem 5. (Theorem 3.6, [37]) *Under assumption **(A1)**. Let $\varphi \in C^1(M, \mathbb{R})$ be an invariant functional. If, for every $m \in \mathbb{N}$, there exists $\rho_m > r_m > 0$ such that*

- (A2)** $a_m = \max_{u \in Y_m, \|u\| = \rho_m} \varphi(u) \leq 0$,
- (A3)** $b_m = \inf_{u \in Z_m, \|u\| = r_m} \varphi(u) \rightarrow \infty, \quad m \rightarrow \infty$,
- (A4)** φ satisfies the $(PS)_c$ condition for every $c > 0$,

where Y_m and Z_m as in (2.25). Then φ has an unbounded sequence of critical values.

Remark 2. In our setting, as done in [5], we set $\mathcal{G} = \mathbb{Z}/2$, $M = D_T^{1,p}(\mathbb{R}^N)$ so that, since $D_T^{1,p}(\mathbb{R}^N)$ is a separable Banach space, there is a linearly independent sequence $(e_j)_j$ such that the decomposition in **(A1)** holds with $M_j = X_j := \text{span}\{e_j\}$. Note that X_j are trivially \mathcal{G} -invariant and isomorphic to \mathbb{R} . Thus, condition **(A1)** is satisfied with $W = \mathbb{R}$.

Finally, a crucial ingredient when a symmetric setting is involved is the following principle of symmetric criticality due to Palais, cfr. [27, 28], which states that any critical point of F_λ restricted on $D_T^{1,p}(\mathbb{R}^N)$, is a critical point of the same functional on $D^{1,p}(\mathbb{R}^N)$. For further details, we refer to [5].

Lemma 6. *Let $v \in D_T^{1,p}(\mathbb{R}^N)$. If $F_\lambda(v)\psi = 0$ for all $\psi \in D_T^{1,p}(\mathbb{R}^N)$, then $F_\lambda(v)\psi = 0$ for all $\psi \in D^{1,p}(\mathbb{R}^N)$.*

3. On Palais Smale sequences

We start this section by giving the definition of $(PS)_c$ sequence.

Definition 1. Let Y be a Banach space and $E : Y \rightarrow \mathbb{R}$ be a differentiable functional. A sequence $(u_n)_n \subset Y$ is called a $(PS)_c$ sequence for E if $E(u_k) \rightarrow c$ and $E'(u_k) \rightarrow 0$ as $k \rightarrow \infty$. Moreover, we say that E satisfies the $(PS)_c$ condition if every $(PS)_c$ sequence for E has a converging subsequence in Y .

As standard, we need to deal with bounded $(PS)_c$ sequences for the functional F_λ in (2.5). In addition, a useful inequality holds only if $\alpha < k < \alpha p$ is proved in the next lemma.

Lemma 7. *Assume $\alpha < k < \alpha p^*$ and the further condition $K \geq 0$ in \mathbb{R}^N if $\alpha p < k < \alpha p^*$. Let (1.2) and (1.3) be verified and consider $(v_n)_n \subset D^{1,p}(\mathbb{R}^N)$ a $(PS)_c$ sequence for F_λ for all $c \in \mathbb{R}$. Then $(v_n)_n$ is bounded in $D^{1,p}(\mathbb{R}^N)$.*

In particular, if $\alpha < k < \alpha p$ and $c < 0$, it holds

$$\|Dv_n\|_p < C_* \lambda^{\alpha/(\alpha p - k)}, \quad C_* = \left[\frac{p(\alpha p^* - k)}{\alpha k(p^* - p)} \cdot \left(\frac{2}{S} \right)^{k/\alpha p} \|V\|_r \right]^{\alpha/(\alpha p - k)} \tag{3.1}$$

where S is Sobolev’s constant.

Proof. We follow Lemma 4 in [4]. Let $(v_n)_n \subset D^{1,p}(\mathbb{R}^N)$ be a $(PS)_c$ sequence of F_λ for all $c \in \mathbb{R}$ that is, using Definition 1, $F_\lambda(v_n) = c + o(1)$, $F'_\lambda(v_n)\psi = o(1)\|\psi\|$ as $n \rightarrow \infty$, for every $\psi \in D^{1,p}(\mathbb{R}^N)$. Now take $\psi = G^{-1}(v_n)g(G^{-1}(v_n))$ as a test function, since $\psi \in D^{1,p}(\mathbb{R}^N)$ thanks to Remark 1, and using $|D[G^{-1}(v_n)g(G^{-1}(v_n))]| \leq \alpha|Dv_n|$ by (2.8), we have

$$\begin{aligned} o(1)\|\psi\| &= F'_\lambda(v_n)(\psi) \leq \alpha\|Dv_n\|_p^p \\ &\quad - \lambda \int_{\mathbb{R}^N} V|G^{-1}(v_n)|^k dx - \beta \int_{\mathbb{R}^N} K|G^{-1}(v_n)|^{\alpha p^*} dx \end{aligned} \tag{3.2}$$

Now we disjoint the proof in two cases.

Case $\alpha < k \leq \alpha p$: using (3.2), thanks to Lemma 1-(c), (e), (2.13) and Hölder’s inequalities with exponents r and r' we get

$$\begin{aligned} c + o(1) + o(1)\|Dv_n\|_p &\geq F_\lambda(v_n) - \frac{1}{\alpha p^*} F'_\lambda(v_n)(G^{-1}(v_n)g(G^{-1}(v_n))) \\ &\geq \left(\frac{1}{p} - \frac{1}{p^*} \right) \|Dv_n\|_p^p - \lambda \left(\frac{1}{k} - \frac{1}{\alpha p^*} \right) \|V\|_r \|G^{-1}(v_n)\|_{\alpha p^*}^k \\ &\geq \left(\frac{1}{p} - \frac{1}{p^*} \right) \|Dv_n\|_p^p - \lambda \left(\frac{1}{k} - \frac{1}{\alpha p^*} \right) (2/S)^{k/\alpha p} \|V\|_r \|Dv_n\|_p^{k/\alpha}. \end{aligned} \tag{3.3}$$

Thus, since $k < \alpha p < \alpha p^*$, we conclude that $\|Dv_n\|_p$ should be bounded.

Case $\alpha p < k < \alpha p^*$: arguing as in (3.3), with $1/\alpha p^*$ replaced by $1/k$, since $K(x) \geq 0$ in \mathbb{R}^N , we obtain

$$\begin{aligned} c + o(1) + o(1) \|Dv_n\|_p &\geq F_\lambda(v_n) - \frac{1}{k} F'_\lambda(v_n)(G^{-1}(v_n)g(G^{-1}(v_n))) \\ &\geq \left(\frac{1}{p} - \frac{\alpha}{k}\right) \|Dv_n\|_p^p - \beta \left(\frac{1}{\alpha p^*} - \frac{1}{k}\right) \int_{\mathbb{R}^N} K |G^{-1}(v_n)|^{\alpha p^*} dx \\ &\geq \left(\frac{1}{p} - \frac{\alpha}{k}\right) \|Dv_n\|_p^p. \end{aligned}$$

The conclusion follows immediately since $k > \alpha p$. For the proof of inequality (3.1), as in Lemma 4 in [4], it is enough to observe that if $c < 0$, then (3.3) gives

$$\|Dv_n\|_p^{(\alpha p - k)/\alpha} < \frac{\lambda p(\alpha p^* - k)}{\alpha k(p^* - p)} \|V\|_r (2/S)^{k/\alpha p},$$

from which (3.1) follows immediately. □

In the lemma below we prove the validity of $(PS)_c$ condition of F_λ , that is the point in which the lack of compactness becomes manifest.

Lemma 8. *Suppose (1.2), (1.3), $\alpha < k < \alpha p$, $c < 0$ and define c_1, c_2 as follows*

$$c_1 = \frac{1}{N} \left(\frac{S}{2}\right)^{p^*/(p^* - p)}, \quad c_2 = \left[\frac{2p(\alpha p^* - k)}{S\alpha(p^* - p)} \right]^{k/(\alpha p - k)} \frac{1}{k^{\alpha p/(\alpha p - k)}}. \quad (3.4)$$

Then

(I) *For any $\lambda > 0$, there exists $\beta_{PS}^* > 0$ defined as follows*

$$\beta_{PS}^* = \left(\frac{c_1}{c_2}\right)^{(p^* - p)/p} \cdot \frac{1}{(\lambda \|V\|_r)^{\alpha(p^* - p)/(\alpha p - k)}} \cdot \frac{1}{\|K\|_\infty}, \quad (3.5)$$

such that for every $\beta \in (0, \beta_{PS}^)$, then F_λ satisfies $(PS)_c$ condition.*

(II) *For any $\beta > 0$, there exists $\lambda_{PS}^* > 0$ defined as follows*

$$\lambda_{PS}^* = \left(\frac{c_1}{c_2}\right)^{(\alpha p - k)/\alpha p} \cdot \frac{1}{(\beta \|K\|_\infty)^{(\alpha p - k)/\alpha(p^* - p)}} \cdot \frac{1}{\|V\|_r}, \quad (3.6)$$

such that for every $\lambda \in (0, \lambda_{PS}^)$, then F_λ satisfies $(PS)_c$ condition.*

Proof. Let $(v_n)_n$ be a $(PS)_c$ sequence, by Lemma 7, then $(v_n)_n$ is bounded in $D^{1,p}(\mathbb{R}^N)$ and by Banach-Alaoglu's Theorem, there exists $v \in D^{1,p}(\mathbb{R}^N)$ such that, up to subsequences, we get $v_n \rightharpoonup v$ in $D^{1,p}(\mathbb{R}^N)$. On the other hand, By Lemma 2, follows that $|G^{-1}(v_n)|^\alpha \rightharpoonup |G^{-1}(v)|^\alpha$ in $D^{1,p}(\mathbb{R}^N)$, so that $(|G^{-1}(v_n)|^\alpha)_n$ is bounded in $D^{1,p}(\mathbb{R}^N)$. In addition, Theorem 4 gives (2.12). Applying in Proposition 1, there exist $\mu, \nu, \nu_\infty, \mu_\infty$ bounded nonnegative measures on \mathbb{R}^N such that

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} |G^{-1}(v_n)|^{\alpha p^*} dx = \int_{\mathbb{R}^N} d\nu + \nu_\infty, \quad (3.7)$$

and

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} |D(|G^{-1}(v_n)|^\alpha)|^p dx = \int_{\mathbb{R}^N} d\mu + \mu_\infty,$$

where

$$\begin{aligned} \nu_\infty &= \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > R} |G^{-1}(v_n)|^{\alpha p^*} dx, \\ \mu_\infty &= \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > R} |D(|G^{-1}(v_n)|^\alpha)|^p dx. \end{aligned}$$

Moreover, there exists at most countable set J , a family $(x_j)_{j \in J}$ of distinct points in \mathbb{R}^N and two families $(\nu_j)_{j \in J}, (\mu_j)_{j \in J} \in]0, \infty[$ so that

$$\begin{aligned} \nu &= |G^{-1}(v)|^{\alpha p^*} + \sum_{j \in J} \nu_j \delta_{x_j}, \quad \nu_j \geq 0, \quad \mu \geq |D(|G^{-1}(v_n)|^\alpha)|^p \\ &+ \sum_{j \in J} \mu_j \delta_{x_j}, \quad \mu_j \geq 0, \end{aligned}$$

satisfying

$$S\nu_j^{p/p^*} \leq \mu_j, \quad S\nu_\infty^{p/p^*} \leq \mu_\infty. \tag{3.8}$$

Following an idea in [6], where now we use (2.9) and that $V \in L^r(\mathbb{R}^N)$ with r given in (1.2), it holds

$$2\beta K(x_j)\nu_j \geq \mu_j. \tag{3.9}$$

Inequality (3.9) establishes that concentration of the measure μ cannot occur at points in which $K(x_j) \leq 0$ being the right hand side positive. Consequently, $K(x_j) > 0$ for all $j \in J$. We claim that $J = \emptyset$. Indeed, combining (3.8)₁ and (3.9), we arrive to

$$\nu_j \geq \left(\frac{S}{2\beta K(x_j)} \right)^{p^*/(p^*-p)} \geq \left(\frac{S}{2\beta \|K\|_\infty} \right)^{p^*/(p^*-p)}, \quad j \in J. \tag{3.10}$$

To reach the claim, we show that (3.10) cannot occur for λ or β belonging to a suitable interval. As in [4], assumption (3.10) forces that $|J| < \infty$. Now, being $(v_n)_n$ a $(PS)_c$ sequence, choose again $G^{-1}(v_n)g(G^{-1}(v_n))\psi_\varepsilon$ as a test function in (2.23), where $\psi_\varepsilon(x) = \psi((x - x_j)/\varepsilon)$ for $0 < \varepsilon < 1$ and $\psi \in C_c^\infty(\mathbb{R}^N)$ such that $0 \leq \psi \leq 1$ in \mathbb{R}^N , $\psi = 0$ for $|x| > 1$, $\psi = 1$ for $|x| \leq 1/2$. Then, using

(2.8) and $0 \leq \psi_\varepsilon \leq 1$, we have, for $n \rightarrow \infty$,

$$\begin{aligned}
 c + o(1)\|Dv_n\|_p &= F_\lambda(v_n) - \frac{1}{\alpha p^*} F'_\lambda(v_n)(G^{-1}(v_n)g(G^{-1}(v_n)))\psi_\varepsilon \\
 &= \frac{1}{p}\|Dv_n\|_p^p - \frac{\lambda}{k} \int_{\mathbb{R}^N} V|G^{-1}(v_n)|^k dx \\
 &\quad - \frac{1}{\alpha p^*} \int_{\mathbb{R}^N} |Dv_n|^{p-1} |D(G^{-1}(v_n)g(G^{-1}(v_n)))| \psi_\varepsilon dx \\
 &\quad - \frac{1}{\alpha p^*} \int_{\mathbb{R}^N} G^{-1}(v_n)g(G^{-1}(v_n)) |Dv_n|^{p-2} Dv_n \cdot D\psi_\varepsilon dx \\
 &\quad + \frac{\lambda}{\alpha p^*} \int_{\mathbb{R}^N} V|G^{-1}(v_n)|^k \psi_\varepsilon dx \\
 &\geq \frac{1}{N} \int_{\mathbb{R}^N} |Dv_n|^p \psi_\varepsilon dx \\
 &\quad - \frac{\lambda}{k} \int_{\mathbb{R}^N} V|G^{-1}(v_n)|^k dx \\
 &\quad - \frac{1}{\alpha p^*} \int_{B_\varepsilon(x_j) \setminus B_{\varepsilon/2}(x_j)} G^{-1}(v_n)g(G^{-1}(v_n)) |Dv_n|^{p-1} |D\psi_\varepsilon| dx.
 \end{aligned}$$

In particular, since $|D\psi_\varepsilon| \leq C/\varepsilon$ in the entire \mathbb{R}^N , by Lemma 1-(e), Lemma 7 and using Hölder’s inequality twice with exponents p, p' and $N/(N - p), N/p$ respectively, we obtain

$$\begin{aligned}
 &\left| \int_{B_\varepsilon(x_j) \setminus B_{\varepsilon/2}(x_j)} G^{-1}(v_n)g(G^{-1}(v_n)) |Dv_n|^{p-1} |D\psi_\varepsilon| dx \right| \\
 &\leq \alpha \int_{B_\varepsilon(x_j) \setminus B_{\varepsilon/2}(x_j)} |v_n| |Dv_n|^{p-1} |D\psi_\varepsilon| dx \\
 &\leq \alpha \left(\int_{\mathbb{R}^N} |Dv_n|^p \right)^{(p-1)/p} \cdot \left(\int_{B_\varepsilon(x_j) \setminus B_{\varepsilon/2}(x_j)} |v_n|^p |D\psi_\varepsilon|^p dx \right)^{1/p} \\
 &\leq C \left(\int_{B_\varepsilon(x_j) \setminus B_{\varepsilon/2}(x_j)} |v_n|^{p^*} dx \right)^{1/p^*} \left(\int_{B_\varepsilon(x_j) \setminus B_{\varepsilon/2}(x_j)} |D\psi_\varepsilon|^N dx \right)^{1/N} \\
 &\leq C \left(\int_{B_\varepsilon(x_j) \setminus B_{\varepsilon/2}(x_j)} |v_n|^{p^*} dx \right)^{1/p^*} \rightarrow 0,
 \end{aligned}$$

as $\varepsilon \rightarrow 0$, since $v_n \in L^{p^*}(\mathbb{R}^N)$. In turn, applying Hölder’s inequality with exponents r, r' and (2.13) we arrive to

$$c + o(1)\|Dv_n\|_p \geq \frac{1}{N} \int_{\mathbb{R}^N} |Dv_n|^p \psi_\varepsilon dx - \frac{\lambda}{k} \left(\frac{2}{S} \right)^{k/\alpha p} \|V\|_r \|Dv_n\|_p^{k/\alpha} + o(1) \tag{3.11}$$

as $n \rightarrow \infty$. Now, thanks to (2.9) and (3.1), from (3.11), we get

$$c + o(1)\|Dv_n\|_p > \frac{1}{2N} \int_{B_{\varepsilon/2}(x_j)} |D(|G^{-1}(v_n)|^\alpha)|^p dx - \frac{(C_*)^{k/\alpha}}{k}.$$

$$\left(\frac{2}{S}\right)^{k/\alpha p} \|V\|_r \lambda^{\alpha p/(\alpha p-k)} + o(1),$$

where C_* is given in (3.1), so that, letting $n \rightarrow \infty$, $\varepsilon \rightarrow 0$ and using (3.8) and (3.10), we arrive to

$$0 > c > c_1(\beta \|K\|_\infty)^{-p/(p^*-p)} - c_2(\|V\|_r \lambda)^{\alpha p/(\alpha p-k)},$$

where we have replaced the value of C_* and c_1, c_2 are defined in (3.4). To obtain the required contradiction we need to have

$$c_1 > c_2(\beta \|K\|_\infty)^{p/(p^*-p)} (\|V\|_r \lambda)^{\alpha p/(\alpha p-k)}. \tag{3.12}$$

Consequently, since $k < \alpha p$, if we choose any $\beta > 0$, then there exists λ_{PS}^* , defined in (3.6), such that for every $\lambda \in (0, \lambda_{PS}^*]$, inequality (3.12) is verified. Similarly, for any $\lambda > 0$ fixed, there exists β_{PS}^* , defined in (3.5), such that for every $\beta \in (0, \beta_{PS}^*]$, inequality (3.12) holds. Thus $J = \emptyset$, concluding the proof of the claim.

On the other hand, following the idea of Chabrowski [15] and Ben-Naoum et al. [8], also a possible concentration at infinity is refused.

Consequently, (3.7) gives

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |G^{-1}(v_n)|^{\alpha p^*} dx = \int_{\mathbb{R}^N} |G^{-1}(v)|^{\alpha p^*} dx.$$

Furthermore, since $G^{-1}(v_n)(x) \rightarrow G^{-1}(v)(x)$ a.e. in \mathbb{R}^N from (2.12), then Brezis Lieb Lemma in [11] implies

$$\|G^{-1}(v_n)\|_{\alpha p^*}^{\alpha p^*} - \|G^{-1}(v_n) - G^{-1}(v)\|_{\alpha p^*}^{\alpha p^*} = \|G^{-1}(v)\|_{\alpha p^*}^{\alpha p^*} + o(1). \tag{3.13}$$

in other words, $\lim_{n \rightarrow \infty} \|G^{-1}(v_n) - G^{-1}(v)\|_{\alpha p^*} = 0$. Using that by weak continuity of the functional $F_\lambda(v_n) \rightarrow F_\lambda(v) = c$, that is

$$\begin{aligned} & \int_{\mathbb{R}^N} |Dv_n|^p dx - \int_{\mathbb{R}^N} |Dv|^p dx + o(1) \\ &= \frac{\lambda p}{k} \int_{\mathbb{R}^N} V [|G^{-1}(v_n)|^k - |G^{-1}(v)|^k] dx \\ & \quad + \frac{\beta p}{\alpha p^*} \int_{\mathbb{R}^N} K [|G^{-1}(v_n)|^{\alpha p^*} - |G^{-1}(v)|^{\alpha p^*}] dx \end{aligned}$$

as $n \rightarrow \infty$. Thus, by (3.13), the right hand side tends to 0, so that $\|Dv_n\|_p \rightarrow \|Dv\|_p$ as $n \rightarrow \infty$. Consequently, since $D^{1,p}(\mathbb{R}^N)$ is uniformly convex and by $Dv_n \rightharpoonup Dv$ in \mathbb{R}^N , by Proposition 3.32 in [10], we immediately get

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |D(v_n - v)|^p dx = 0,$$

that is the strong convergence in $L^p(\mathbb{R}^N)$ of the sequence $(Dv_n)_n$. □

4. The truncated functional F_∞

In this section, we introduce F_∞ , the truncated functional of F_λ , which has the very useful property to be bounded from below, differently from F_λ .

Taking into account Theorem 4, in particular (2.14), Hölder’s inequality and by using (2.5), for all $v \in D^{1,p}(\mathbb{R}^N)$ we have

$$F_\lambda(v) \geq \frac{1}{p} \|v\|^p - \lambda c_V \|v\|^{k/\alpha} - \beta c_K \|v\|^{p^*}.$$

where $c_V = (2/S)^{k/\alpha p} \|V\|_r/k$ and $c_K = (2/S)^{p^*/p} \|K\|_\infty/\alpha p^*$ are positive constants.

For simplicity, just to describe some qualitative properties of F_λ , we define the auxiliary function $h(t) = t^p/p - \lambda c_V t^{k/\alpha} - \beta c_K t^{p^*}$ in \mathbb{R}_0^+ . Indeed, even if h is smaller than F_λ , it will appear in the proof of Theorem 1 that the behaviour of F_λ will be roughly the same of h near 0. Following the same argument of Sect. 4 in [6], we write

$$h(t) = t^{k/\alpha} \hat{h}(t), \quad \hat{h}(t) := -\lambda c_V + \frac{1}{p} t^{(\alpha p - k)/\alpha} - \beta c_K t^{(\alpha p^* - k)/\alpha},$$

with $\hat{h}(0) < 0$, $\hat{h}(t) \rightarrow -\infty$ as $t \rightarrow \infty$ and there is a unique point $T_M > 0$ such that

$$\hat{h}'(T_M) = 0, \quad T_M = \left[\frac{\alpha p - k}{\beta c_K p (\alpha p^* - k)} \right]^{1/(p^* - p)}, \quad \hat{h}'(t) > 0 \text{ for } 0 < t < T_M.$$

Hence, the maximum value of the function \hat{h} is given by

$$\begin{aligned} \hat{h}(T_M) &= \frac{\alpha(p^* - p)}{\alpha p - k} (\beta c_K)^{-(\alpha p - k)/\alpha(p^* - p)} \\ &\quad \left(\frac{\alpha p - k}{p(\alpha p^* - k)} \right)^{(\alpha p^* - k)/\alpha(p^* - p)} - \lambda c_V. \end{aligned} \tag{4.1}$$

In turn, since \hat{h} and h have the same zeros and the same sign in \mathbb{R}^+ , if $\hat{h}(T_M) \geq 0$, then there exists $T_0 \in (0, T_M)$ such that $\hat{h}(T_0) = h(T_0) = 0$ so that $\hat{h}(t) < 0$ in $(0, T_0)$ and consequently $h(t) < 0$ in $(0, T_0)$, cfr. Figure 1a and Figure 1b.

While, if $\hat{h}(T_M) < 0$ then $\hat{h}(t), h(t) < 0$ in \mathbb{R}^+ , so that h does not have zeros in all of \mathbb{R}^+ . In addition, in this latter case h is nonincreasing, indeed

$$h'(t) = t^{k/\alpha} \hat{h}(t) \left[\frac{k}{\alpha} \frac{1}{t} + \frac{\hat{h}'(t)}{\hat{h}(t)} \right] = t^{k/\alpha} \hat{h}(t) \left(\log \mathfrak{Z}(t) \right)', \quad \mathfrak{Z}(t) := t^{k/\alpha} |\hat{h}(t)|,$$

so that if $h'(t) \geq 0$, then it turns out that \mathfrak{Z} is non increasing, being $\hat{h}(t) < 0$ in \mathbb{R}^+ . Thus, $0 < \mathfrak{Z}(t) \leq \mathfrak{Z}(0) = 0$ in \mathbb{R}^+ , a contradiction. So, if $\hat{h}(T_M) < 0$, then $h'(t) < 0$ in \mathbb{R}^+ .

In conclusion, there is always a right neighborhood of 0, say $(0, P_0)$, with $P_0 \in (0, \infty)$ opportune, in which h is negative.

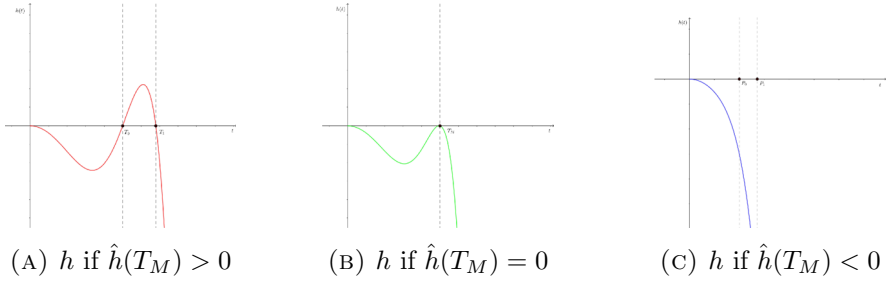


FIGURE 1. $h(t)$

Now, consider a cutoff function $\tau \in C^\infty(\mathbb{R}_0^+)$, nonincreasing and such that

$$\tau(t) = 1, \text{ if } 0 \leq t \leq P_0 \quad \text{and} \quad \tau(t) = 0 \text{ if } t \geq P_1,$$

for some real positive number $P_1 > P_0$ and P_0 given above. For instance, in the case in which $\hat{h}(T_M) > 0$, we can choose $P_0 = T_0$ and $P_1 = T_1$, respectively the first and the second zero of h , whose existence is guaranteed by the behavior of h , cfr. Figure 1a. Differently, if $\hat{h}(T_M) = 0$, then $h(T_M) = h'(T_M) = 0$, and we can choose $P_0 = T_M$, while P_1 any point in (T_M, ∞) , cfr. Figure 1b.

Finally, if $\hat{h}(T_M) < 0$ we can choose P_0 and P_1 in \mathbb{R}^+ without any restriction, cfr. Figure 1c. We point out that, by (4.1), this latter case occurs roughly either if $\lambda > 0$ arbitrary and β large or if $\beta > 0$ arbitrary and λ large. Precisely, define λ_T^* and β_T^* as follows

$$\lambda_T^* = k(p^* - p) \left(\frac{p^*(\alpha p - k)}{\beta \|K\|_\infty} \right)^{(\alpha p - k)/\alpha(p^* - p)} \left(\frac{S\alpha}{2p(\alpha p^* - k)} \right)^{(\alpha p^* - k)/\alpha(p^* - p)} \cdot \frac{1}{\|V\|_r} \tag{4.2}$$

$$\beta_T^* = p^*(\alpha p - k) \left(\frac{k(p^* - p)}{\lambda \|V\|_r} \right)^{\alpha(p^* - p)/(\alpha p - k)} \left(\frac{S\alpha}{2p(\alpha p^* - k)} \right)^{(\alpha p^* - k)/(\alpha p - k)} \cdot \frac{1}{\|K\|_\infty}. \tag{4.3}$$

Consequently, $\hat{h}(T_M) < 0$ holds either for all $\lambda > 0$ and $\beta > \beta_T^*$ or for all $\beta > 0$ and $\lambda > \lambda_T^*$. This property will appear crucial in the proof of Theorem 1.

By virtue of the cut-off function τ it is possible to define the truncated functional

$$F_\infty(v) = \frac{1}{p} \|Dv\|_p^p - \frac{\lambda}{k} \int_{\mathbb{R}^N} V |G^{-1}(v)|^k dx - \beta \frac{\tau(\|v\|_{D^{1,p}})}{\alpha p^*} \int_{\mathbb{R}^N} K |G^{-1}(v)|^{\alpha p^*} dx.$$

As above, we associate the real function $\bar{h}(t) = \frac{1}{p} t^p - \lambda c_V t^{k/\alpha} - \beta c_K t^{p^*} \tau(t)$, $t \in \mathbb{R}_0^+$, whose behaviour, can be represented for instance by cfr. Figure 2a

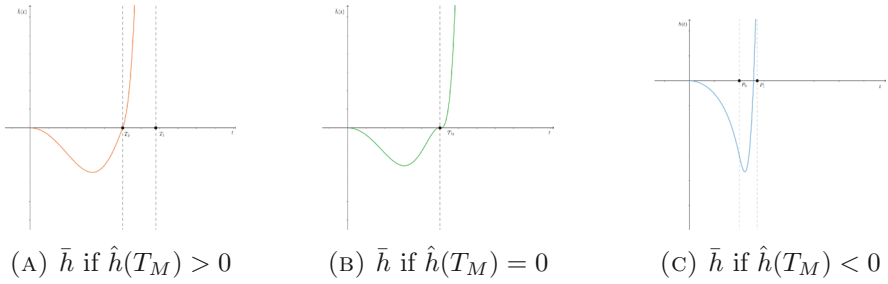


FIGURE 2. $\bar{h}(t)$

if $P_0 = T_0$, $P_1 = T_1$ and by Figures 2b, 2c for the corresponding cases to Figures 1b, 1c.

In particular, we have $F_\infty(v) \geq \bar{h}(\|v\|_{D^{1,p}})$ for all $v \in D^{1,p}(\mathbb{R}^N)$ and $F_\lambda(v) = F_\infty(v)$ if $0 \leq \|v\|_{D^{1,p}} \leq P_0$. Moreover, by the regularity both of the cut-off τ and of F_λ , we get $F_\infty \in C^1(D^{1,p}(\mathbb{R}^N), \mathbb{R})$ satisfying $(PS)_c$ for $\lambda < \lambda^*$ or $\beta < \beta^*$, as stated below, with

$$\lambda^* = \min\{\lambda_{PS}^*, \lambda_T^*\} = \begin{cases} \lambda_{PS}^* & \text{if } \left(\frac{\alpha p - k}{p}\right)^p > \left(\frac{\alpha p^* - k}{\alpha p^*}\right)^{p^*} \\ \lambda_T^* & \text{if } \left(\frac{\alpha p - k}{p}\right)^p < \left(\frac{\alpha p^* - k}{\alpha p^*}\right)^{p^*} \end{cases}$$

and

$$\beta^* = \min\{\beta_{PS}^*, \beta_T^*\} = \begin{cases} \beta_{PS}^* & \text{if } \left(\frac{\alpha p - k}{p}\right)^p > \left(\frac{\alpha p^* - k}{\alpha p^*}\right)^{p^*} \\ \beta_T^* & \text{if } \left(\frac{\alpha p - k}{p}\right)^p < \left(\frac{\alpha p^* - k}{\alpha p^*}\right)^{p^*} \end{cases}$$

with $\lambda_{PS}^*, \beta_{PS}^*$ given in (3.6), (3.5), while λ_T^*, β_T^* defined in (4.2), (4.3). Choosing either λ or β small is equivalent to fall in the cases described in Figures 1a-2a and 1b-2b.

Lemma 9. *Let F_∞ be the truncated functional of F_λ .*

- (a) *For all $\lambda > 0$ and $\beta \in (0, \beta^*)$, then F_∞ satisfies the $(PS)_c$ with $c < 0$.*
- (b) *For all $\beta > 0$ and $\lambda \in (0, \lambda^*)$, then F_∞ satisfies the $(PS)_c$ with $c < 0$.*

Proof. Assume either $\lambda > 0$ and $\beta \in (0, \beta^*)$ or $\beta > 0$ and $\lambda \in (0, \lambda^*)$, then it holds

- (h) *If $F_\infty(v) < 0$, then $\|v\|_{D^{1,p}} < P_0$ and $F_\lambda(u) = F_\infty(u)$ for all u in a small enough neighbourhood of v .*

Indeed, choosing λ, β as above, it immediately follows that $\hat{h}(T_M) \geq 0$, so that the existence of P_0 , with $h(P_0) = 0$, is guaranteed, cfr. Figures 1a, 1b. Consequently, we can apply the proof of Lemma 9 in [4], provided that the change of variables is taken into account. □

5. Proof of Theorem 1

This section is devoted to the proof of Theorem 1. Before stating the symmetric Mountain Pass Theorem, we need the notion of the genus.

Definition 2. Let $\Sigma = \{A \subset Y \setminus \{0\} \mid A \text{ closed and symmetric } u \in A \Rightarrow -u \in A\}$, with Y a real Banach space. Let $A \in \Sigma$, the genus $\gamma(A)$ of A is defined as the smallest integer N such that there exists $\Phi \in C(Y, \mathbb{R}^N \setminus \{0\})$ such that Φ is odd and $\Phi(x) \neq 0$ for all $x \in A$. We set $\gamma(\emptyset) = 0$ and $\gamma(A) = \infty$ if there are no integers with the above property.

For an even functional on a Banach space, the symmetric Mountain Pass Theorem due to Ambrosetti–Rabinowitz [3], Rabinowitz [34], as follows.

Theorem 6. Let $(Y, \|\cdot\|_Y)$ be an infinite dimensional Banach space and $E \in C^1(Y, \mathbb{R})$ satisfying the following assumptions.

- (C1) E is even, bounded from below, $E(0) = 0$ and E satisfies the Palais–Smale condition.
- (C2) For each $j \in \mathbb{N}$, there exists an $A_j \in \Sigma_j$ such that $\sup_{u \in A_j} E(u) < 0$.

Define $c_j = \inf_{A \in \Sigma_j} \sup_{u \in A} E(u)$, $\Sigma_j = \{A \subset Y \setminus \{0\} : A \text{ is closed in } Y, -A = A, \gamma(A) \geq j\}$.

Then each c_j is a critical value of E with $c_j \leq c_{j+1} < 0$ for $j \in \mathbb{N}$ and $(c_j)_j$ converges to zero. Moreover, if $c_j = c_{j+1} = \dots = c_{j+i} = c$, then $\gamma(K_c) \geq i + 1$, where K_c is defined by $K_c = \{u \in Y : E'(u) = 0, E(u) = c\}$.

Thus, for an even functional on an infinite dimensional Banach space, the symmetric Mountain Pass Theorem gives a sequence of critical values which converges to zero. Under the same assumptions on the functional, Kajikiya in [21] establishes the following critical point theorem which provides a sequence of critical points converging to zero.

Theorem 7. (Theorem 1, [21]) Let $(Y, \|\cdot\|_Y)$ be an infinite dimensional Banach space and $E \in C^1(Y, \mathbb{R})$ satisfying (C1) and (C2), then one of the following holds.

- There exists a sequence $(u_k)_k$ such that $E'(u_k) = 0$, $E(u_k) < 0$ and $\lim_{k \rightarrow \infty} \|u_k\|_Y = 0$.
- There exist two sequences $(u_k)_k$ and $(v_k)_k$ such that:
 - (i) $E'(u_k) = 0$, $E(u_k) = 0$, $u_k \neq 0$, $\lim_{k \rightarrow \infty} \|u_k\|_Y = 0$
 - (ii) $E'(v_k) = 0$, $E(v_k) < 0$, $\lim_{k \rightarrow \infty} E(v_k) = 0$ and $\lim_{k \rightarrow \infty} \|v_k\|_Y = c$, with $c \neq 0$.

Proof of Theorem 1. In order to get our multiplicity result, we need to prove conditions (C1) and (C2) of Theorem 6, as well as Theorem 7, for the functional F_∞ . In particular, F_∞ is even, bounded from below and satisfies the $(PS)_c$ condition for all $c < 0$, see Sect. 4, thus (C1) is proved. In order to prove (C2) consider $K_c = K_{c, F_\infty} = \{u \in D^{1,p}(\mathbb{R}^N) : F_\infty(u) = c, F'_\infty(u) = 0\}$ defined in

Theorem 6 and take $m \in \mathbb{N}^+$. For $1 \leq j \leq m$ let c_j and Σ_j as in Theorem 6. Our claim consists in proving that

$$c_j < 0 \quad \text{for all } j \geq 1. \tag{5.1}$$

To reach (5.1) it is enough to prove that for all $j \in \mathbb{N}$, there exists an $\varepsilon_j = \varepsilon(j) > 0$ s.t.

$$\gamma(F_\infty^{-\varepsilon_j}) \geq j, \quad F_\infty^a = \{v \in D^{1,p}(\mathbb{R}^N) : F_\infty(v) \leq a\} \quad \text{with } a \in \mathbb{R}. \tag{5.2}$$

Let $\Omega_V \subset \mathbb{R}^N$, $|\Omega_V| > 0$, be a bounded open set where $V > 0$ and extend functions in $D_0^{1,p}(\Omega_V)$ by 0 outside Ω_V , where $D_0^{1,p}(\Omega_V)$ is the closure of $C_0^\infty(\Omega_V)$ in the norm $\|v\|_{D_0^{1,p}(\Omega_V)}$. Take W_j a j -dimensional subspace of $D_0^{1,p}(\Omega_V)$, thus all the norms in W_j are equivalent. For every $v \in W_j \setminus \{0\}$, we write $v = r_j w$ with $w \in W_j$ and $\|w\|_{D_0^{1,p}(\Omega_V)} = 1$. In what follows we specify the choice of r_j . Note that there exists $d_j > 0$ such that

$$d_j \leq \int_{\Omega_V} V|w|^k dx \tag{5.3}$$

In particular, $V|w|^k \in L^1(\Omega_V)$ being Ω_V bounded, $V \in C(\mathbb{R}^N)$ and $k < p < p^*$.

By Lemma 1-(a), for $\varepsilon > 0$ there exists $\sigma = \sigma(\varepsilon)$ sufficiently small such that

$$|G^{-1}(t)| \geq \frac{1}{2^{1/k}}|t| \quad \text{if } |t| \leq \sigma. \tag{5.4}$$

Thus, since $\alpha < k < p$, by (5.3) and (5.4), we get

$$\begin{aligned} \int_{\Omega_V} V|G^{-1}(v)|^k dx &= \int_{\Omega_V} V\left(\frac{1}{2}|v|^k + |G^{-1}(v)|^k - \frac{1}{2}|v|^k\right) dx \\ &\geq \frac{1}{2} \int_{\Omega_V} V|v|^k dx + \int_{\{|v| \geq \sigma\} \cap \Omega_V} V\left(|G^{-1}(v)|^k - \frac{1}{2}|v|^k\right) dx \\ &\geq \frac{1}{2} d_j r_j^k - \frac{1}{2} \int_{\{|v| \geq \sigma\} \cap \Omega_V} V|v|^{k-p^*+p^*} dx \\ &\geq \frac{1}{2} d_j r_j^k - \frac{1}{2} \|V\|_{L^\infty(\Omega_V)} \sigma^{k-p^*} \|v\|_{p^*}^{p^*} = \frac{1}{2} d_j r_j^k - c r_j^{p^*}, \quad c > 0. \end{aligned}$$

Consequently, by Lemma 1-(g), the following holds

$$\begin{aligned} F_\lambda(v) &= \frac{1}{p} \int_{\Omega_V} |Dv|^p dx - \frac{\lambda}{k} \int_{\Omega_V} V|G^{-1}(v)|^k dx - \frac{\beta}{\alpha p^*} \int_{\Omega_V} K|G^{-1}(v)|^{\alpha p^*} dx \\ &\leq \frac{1}{p} r_j^p - \frac{\lambda}{k} \left(\frac{1}{2} d_j r_j^k - c r_j^{p^*}\right) + \frac{\beta 2^{(p-1)p^*}}{\alpha p^*} \|K\|_\infty \int_{\Omega_V} |v|^{p^*} dx \\ &\leq r_j^k \left[\frac{1}{p} r_j^{p-k} - \frac{\lambda}{2k} d_j + C r_j^{p^*-k}\right], \end{aligned}$$

where $C > 0$. Consequently, for every $v \in W_j$, $v \neq 0$, we can choose $r_j \in (0, P_0)$ sufficiently small such that $\|v\| = r_j$ and, since $\alpha < k < p$, we obtain $F_\infty(v) = F_\lambda(v) \leq -\varepsilon_j < 0$, by virtue of Lemma 9. Now, letting

$S_{r_j} = \{v \in D^{1,p}(\mathbb{R}^N) : \|v\|_{D_0^{1,p}(\Omega_V)} = r_j\}$, then $S_{r_j} \cap W_j \subset F_\infty^{-\varepsilon_j}$. By Definition 2, $\gamma(F_\infty^{-\varepsilon_j}) \geq \gamma(S_{r_j} \cap W_j) = \gamma(S^{j-1}) = j$, which proves claim (5.2). Thus, from $F_\infty^{-\varepsilon_j} \in \Sigma_j$, we obtain $c_j \leq \sup_{u \in F_\infty^{-\varepsilon_j}} F_\infty(v) \leq -\varepsilon_j < 0$.

In particular, applying Theorem 7, we get a sequence of solutions with negative energy of problem (1.1) such that, in both cases, either $\|u_n\| \rightarrow 0$ or $\|u_n\| \rightarrow c \neq 0$ and $E_\lambda(u_n) \rightarrow 0$ as $n \rightarrow \infty$. In addition, in the latter case, there is also another sequence of nontrivial solutions $(\bar{u}_n)_n$ such that $E_\lambda(\bar{u}_n) = 0$ and $\|\bar{u}_n\| \rightarrow 0$ as $n \rightarrow \infty$. This concludes the proof. \square

Remark 3. The case $p < k < \alpha p$ is not covered in Theorem 1 because of some considerable difficulties. Indeed, to prove (5.1) we need that F_∞ has to be negative near 0. Unfortunately, if $p < k$ it holds the opposite condition. Indeed, for any $v = r_j w \in W_j$ with $\|w\|_{D_0^{1,p}(\Omega_V)} = 1$ and r_j positive suitable, using Lemma 1-(c), (g) and Hölder’s inequality with exponents p^*/k and $p^*/(p^* - k)$, we have

$$\begin{aligned} F_\lambda(v) &\geq \frac{1}{p} r_j^p - \frac{\lambda}{k} \int_{\Omega_V} V|v|^k dx - \frac{\beta 2^{(p-1)p^*}}{\alpha p^*} \int_{\Omega_V} K|v|^{p^*} dx \\ &\geq \frac{1}{p} r_j^p - \frac{\lambda}{k} S^{-k/p} \|V\|_{L^\infty(\Omega_V)} |\Omega_V|^{(p^* - k)/p^*} r_j^k - \frac{\beta 2^{(p-1)p^*}}{\alpha p^*} S^{-p^*/p} \\ &\quad \|K\|_{L^\infty(\Omega_V)} r_j^{p^*}, \end{aligned}$$

since $V, K \in C(\mathbb{R}^N)$ and Ω_V is bounded, see the proof of Theorem 1. Consequently, if r_j is sufficiently small, say $r_j < P_0$, and $k > p$, then, by definition, $F_\infty(v) = F_\lambda(v) > 0$.

On the other hand, as it will be clear from the proof of Theorem 2 below, assuming $K \geq 0$ in \mathbb{R}^N , then one can obtain $F_\lambda(v) < 0$ for r_j sufficiently large, since $p, k/\alpha < p^*$. This choice of r_j fits with the behaviour of the functions h, \bar{h} given in Figure 1c and 2c, see Sect. 4 for details, namely when h is nonincreasing. However, this latter case cannot occur since $(PS)_c$ property for F_λ follows from Lemma 8 by requiring either λ fixed and β small or viceversa, in turn $\hat{h}(T_M) \geq 0$ by (4.1), that are situations described in Figures 1a-2a and 1b-2b.

6. Proof of Theorem 2

This section is devoted to the proof of the result under a symmetric setting. In particular, Theorem 2, whose statement is given in the Introduction, gives infinitely many solutions for problem (1) with no assumptions on the parameters λ and β . Indeed, thanks to the symmetric environment considered, by the corollary below, the $(PS)_c$ condition for the energy functional F_λ holds in a very general setting.

Corollary 1. *Let $\alpha < k < \alpha p^*$. If (1.5) holds, then the functional F_λ satisfies $(PS)_c$ condition in $D_T^{1,p}(\mathbb{R}^N)$ for every $c \in \mathbb{R}$.*

For the proof of the result above we refer to Corollary 1 in [5].

Remark 4. Unfortunately, we cannot improve Theorem 1 in a symmetric setting in view of the new property given by Corollary 1, since Lemma 9 seems difficult to be extended to any $\lambda, \beta > 0$. Indeed, in the proof of Lemma 9 we deduce the validity of $(PS)_c$ condition for the truncated energy functional E_∞ from property (h). This latter property is easily obtained for the cases 2a, 2b, valid roughly for either λ or β small, but, for the case in Figure 2c it fails since, if $F_\infty(v) < 0$, it could be possible that $\|v\|_{D^{1,p}} > P_0$, yielding $F_\infty(v) \neq F_\lambda(v)$.

Now we are ready to prove Theorem 2, by using the Fountain Theorem.

Proof of Theorem 2. The proof consists in applying Theorem 5 with $\mathcal{G} = \mathbb{Z}/2$ and $M = D_T^{1,p}(\mathbb{R}^N)$. Note that, by Remark 2, assumption (A1) is verified. The energy functional F_λ , from Corollary 1, satisfies the $(PS)_c$ condition for every $c \in \mathbb{R}$, so that also (A4) of Theorem 5 holds. Since $0 \neq K \geq 0$ in \mathbb{R}^N and $K \in C(\mathbb{R}^N)$, there exists an open bounded subset Ω_K of \mathbb{R}^N with $K > 0$ in Ω_K . By the T -symmetry of K , then also Ω_K is T -symmetric, in turn we can define $D_T^{1,p}(\Omega_K)$. Now, extending functions in $D_T^{1,p}(\Omega_K)$ by 0 outside Ω_K , we can consider $D_T^{1,p}(\Omega_K) \subset D_T^{1,p}(\mathbb{R}^N)$. Assume $(Y_m)_m$ be an increasing sequence of subspaces of $D_T^{1,p}(\Omega_K)$ with $\dim(Y_m) = m$. In particular, if $v \in Y_m, v \neq 0$, then we can write $v = \rho_m \omega$ with $\omega \in Y_m$ such that $\|\omega\| = 1$, so that $\rho_m = \|v\|$. Moreover, there exists a constant $\varepsilon_m > 0$ such that

$$\int_{\mathbb{R}^N} K|\omega|^{p^*} dx = \int_{\Omega_K} K|\omega|^{p^*} dx \geq \varepsilon_m. \tag{6.1}$$

By Lemma 1-b), for all $\varepsilon > 0$ there exists $M > 0$ large enough, such that $|G^{-1}(t)|^\alpha \geq p^{-1/p^*}|t|$ for $|t| \geq M$ choosing $\varepsilon = 2^{1/p} - p^{-1/p^*}$, so that, thanks to Lemma 1-g), (6.1) and since $K, V \geq 0$, the following holds

$$\begin{aligned} F_\lambda(v) &= \frac{1}{p}\rho_m^p - \frac{\lambda}{k} \int_{\Omega_K} V|G^{-1}(v)|^k dx - \frac{\beta}{\alpha p^*} \int_{\Omega_K} K \left[\frac{|v|^{p^*}}{p} + |G^{-1}(v)|^{\alpha p^*} - \frac{|v|^{p^*}}{p} \right] dx \\ &= \frac{1}{p}\rho_m^p - \frac{\lambda}{k} \int_{\Omega_K} V|G^{-1}(v)|^k dx - \frac{\beta}{\alpha p p^*} \int_{\Omega_K} K|v|^{p^*} dx \\ &\quad - \frac{\beta}{\alpha p^*} \int_{\{|v| \geq M\} \cap \Omega_K} K \left[|G^{-1}(v)|^{\alpha p^*} - \frac{|v|^{p^*}}{p} \right] dx \\ &\quad - \frac{\beta}{\alpha p^*} \int_{\{|v| < M\} \cap \Omega_K} K|G^{-1}(v)|^{\alpha p^*} dx + \frac{\beta}{\alpha p p^*} \int_{\{|v| < M\} \cap \Omega_K} K|v|^{p^*} dx \\ &\leq \frac{1}{p}\rho_m^p - \frac{\beta}{\alpha p p^*} \int_{\Omega_K} K|v|^{p^*} dx + \frac{\beta}{\alpha p p^*} \int_{\{|v| < M\} \cap \Omega_K} K|v|^{p^*} dx \\ &\leq \frac{1}{p}\rho_m^p - \frac{\beta}{\alpha p p^*} \int_{\Omega_K} K|v|^{p^*} dx + \frac{\beta}{\alpha p p^*} \|K\|_\infty M^{p^* - k/\alpha} \int_{\Omega_K} |v|^{k/\alpha} dx \\ &\leq \frac{1}{p}\rho_m^p - \frac{\beta \varepsilon_m}{\alpha p p^*} \rho_m^{p^*} + C \rho_m^{k/\alpha} < 0 \end{aligned}$$

for sufficiently large ρ_m , since $k < \alpha p^*$, where $C = \frac{\beta}{\alpha p p^*} \|K\|_\infty M^{p^* - k/\alpha} |\Omega_K|^{1 - k/\alpha p^*} S^{-k/\alpha p}$. This proves (A2) of Theorem 5. Condition (A3) follows exactly as in [5], taking into account the properties of v . Then applying Theorem 5, the energy functional F_λ has unbounded sequence of critical values in $D_T^{1,p}(\mathbb{R}^N)$, so that Theorem 2 is proved. \square

7. Further results in the singular case: $0 < \alpha < 1$

In this section we extend Theorem 1.1 in [6] where the so called singular version of Eq. (1.1), i.e. $0 < \alpha < 1$, is considered, and for which the critical exponent is p^* ,

$$-\Delta_p u - \frac{\alpha}{2} \Delta_p (|u|^\alpha) |u|^{\alpha-2} u = \lambda V(x) |u|^{k-2} u + \beta K(x) |u|^{p^*-2} u \quad \text{in } \mathbb{R}^N, \quad (7.1)$$

where $1 < k < p^*$. This case is known as “singular” since the function $g(t)$ in (2.2) is singular for $t = 0$, differently from the case $\alpha > 1$ where g is coercive at ∞ .

The natural energy functional associated with Eq. (7.1) is

$$\hat{E}_\lambda(v) = \frac{1}{p} \int_{\mathbb{R}^N} g(u)^p |Du|^p dx - \frac{\lambda}{k} \int_{\mathbb{R}^N} V |u|^k dx - \frac{\beta}{p^*} \int_{\mathbb{R}^N} K |u|^{p^*} dx,$$

while, the functional we have to deal with, after the same change of variable described in Sect. 2, is the following

$$\hat{F}_\lambda(v) = \frac{1}{p} \int_{\mathbb{R}^N} |Dv|^p dx - \frac{\lambda}{k} \int_{\mathbb{R}^N} V |G^{-1}(v)|^k dx - \frac{\beta}{p^*} \int_{\mathbb{R}^N} K |G^{-1}(v)|^{p^*} dx.$$

To manage the singular case, we have to take care of the different growth of G^{-1} at 0 and at ∞ respect to the case $\alpha > 1$. Multiplicity results for solutions of (7.1) with negative energy in the singular case are obtained using the same technique as for Theorem 1 but under the more restrictive condition $2 < k < \alpha p$. Here, we enlarge the interval for k, p up to consider $1 < k < \alpha p$ and we add properties on the behaviour of solutions, as follows.

Theorem 8. *Let $1 < p < N$ and $1 < k < \alpha p$. Assume that K satisfies (1.3) and $0 \leq V \in L^q(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ with $q = p^*/(p^* - k)$. Then,*

- (i) *For any $\lambda > 0$, there exists $\hat{\beta} > 0$ such that for any $0 < \beta < \hat{\beta}$, then Eq. (7.1) has infinitely many nontrivial solutions $(u_n)_n \subset D^{1,p}(\mathbb{R}^N)$ such that $\hat{E}_\lambda(u_n) < 0$ and $\|u_n\| \rightarrow 0$ as $n \rightarrow \infty$.*
- (ii) *For any $\beta > 0$, there exists $\hat{\lambda} > 0$ such that for any $0 < \lambda < \hat{\lambda}$, then Eq. (7.1) has infinitely many nontrivial solutions $(u_n)_n \subset D^{1,p}(\mathbb{R}^N)$ such that $\hat{E}_\lambda(u_n) < 0$ and $\|u_n\| \rightarrow 0$ as $n \rightarrow \infty$.*

The smaller interval $2 < k < \alpha p$ in [6] follows from the application of Lemma 3.3 in [6], which is crucial in order to obtain the validity of (PS)_c condition for \hat{F}_λ . We point out that the restriction $2 < k < \alpha p$ is not required in proving that concentration around points and at infinity cannot occur. On the other hand, Lemma 3.3 in [6] cannot be applied in the case $\alpha > 1$ since the critical exponent becomes $\alpha p^* > p^*$ so different arguments are required. Surprisingly, these last tools, applied to the singular case, allow us to cover the entire interval $1 < k < \alpha p$. In particular, it holds the following.

Lemma 10. *Suppose $0 \leq V \in L^q(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ with $q = p^*/(p^* - k)$ and K satisfy (1.3). Let $1 < k < p$ and $c < 0$. Then*

(I) For any $\lambda > 0$, there exists $\hat{\beta} > 0$ defined as follows

$$\hat{\beta} = \frac{\alpha}{\|K\|_\infty} \left(\frac{kp^*}{\lambda N \|V\|_r (p^* - k)} \right)^{p^2 / (N-p)(p-k)} S^{(p^* - k) / (p - k)}$$

such that for every $\beta \in (0, \hat{\beta}]$, then \hat{F}_λ satisfies $(PS)_c$ condition.

(II) For any $\beta > 0$, there exists $\hat{\lambda} > 0$ defined as follows

$$\hat{\lambda} = S^{(p^* - k) / (p^* - p)} \frac{kp^*}{N(p^* - k)} \cdot \frac{1}{\|V\|_r} \cdot \left(\frac{\alpha}{\beta \|K\|_\infty} \right)^{(p-k) / (p^* - p)},$$

such that for every $\lambda \in (0, \hat{\lambda}]$, then \hat{F}_λ satisfies $(PS)_c$ condition.

Proof. Let $(v_n)_n$ be a $(PS)_c$ sequences for the functional $\hat{F}_\lambda(v)$. We now repeat word by word up the proof of Lemma 3.4 in [6] until formula (3.28) obtained thanks to the application of the Brezis Lieb Lemma, precisely

$$\lim_{n \rightarrow \infty} \|G^{-1}(v_n) - G^{-1}(v)\|_{p^*} = 0. \tag{7.2}$$

At this point, using that $\hat{F}_\lambda(v_n) \rightarrow \hat{F}_\lambda(v) = c$ by weak continuity of the functional, we have

$$\begin{aligned} \int_{\mathbb{R}^N} |Dv_n|^p dx - \int_{\mathbb{R}^N} |Dv|^p dx + o(1) &= \frac{\lambda p}{k} \int_{\mathbb{R}^N} V [|G^{-1}(v_n)|^k - |G^{-1}(v)|^k] dx \\ &\quad + \frac{\beta p}{\alpha p^*} \int_{\mathbb{R}^N} K [|G^{-1}(v_n)|^{p^*} - |G^{-1}(v)|^{p^*}] dx \end{aligned}$$

as $n \rightarrow \infty$. Thus, by (7.2) and Hölder’s inequality, the right hand side of the above equation tends to 0 as $n \rightarrow \infty$, so that $\|Dv_n\|_p \rightarrow \|Dv\|_p$ as $n \rightarrow \infty$. Consequently, since $D^{1,p}(\mathbb{R}^N)$ is uniformly convex and by $Dv_n \rightharpoonup Dv$ in \mathbb{R}^N , by Proposition 3.32 in [10], we get the strong convergence in $L^p(\mathbb{R}^N)$ of the sequence $(Dv_n)_n$. \square

Proof of Theorem 8. It is enough to follow the proof Theorem 1 in [6] where, in place of Lemma 3.4, we use Lemma 10 above. In addition, to obtain the asymptotic behaviours we refer to Theorems 6, 7 given in Sect. 5. \square

As described in Remark 4 it seems difficult to extend the above theorem for every $\lambda, \beta > 0$ in the symmetric setting of Theorem 2, while for the corresponding multiplicity result for solutions with positive energy we refer to Theorem 1.3 in [6].

Acknowledgements

R. Filippucci and L. Baldelli are members of the *Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni* (GNAMPA) of the *Istituto Nazionale di Alta Matematica* (INdAM). R. Filippucci was partly supported by PRIN Project 2022 *Advanced theoretical aspects in PDEs and their applications* (2022BCFHN2) and by INdAM-GNAMPA Project 2023 titled *Equazioni differenziali alle derivate parziali nella modellizzazione di fenomeni reali* (E53C22001930001). L. Baldelli was partially supported by National Science Centre, Poland (Grant No. 2020/37/B/ST1/02742) and by INdAM-GNAMPA

Project 2023 titled *Problemi ellittici e parabolici con termini di reazione singolari e convettivi* (E53C22001930001).

Open Access. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

- [1] Adachi, S., Shibata, M., Watanabe, T.: Blow-up phenomena and asymptotic profiles of ground states of quasilinear elliptic equations with H^1 -supercritical nonlinearities. *J. Differ. Equ.* **256**, 1492–1514 (2014)
- [2] Adachi, S., Watanabe, T.: Uniqueness of the ground state solutions of quasilinear Schrödinger equations. *Nonlinear Anal.* **75**, 819–833 (2012)
- [3] Ambrosetti, A., Rabinowitz, P.H.: Dual variational methods in critical point theory and applications. *J. Funct. Anal.* **14**, 349–381 (1973)
- [4] Baldelli, L., Brizi, Y., Filippucci, R.: Multiplicity results for (p, q) -Laplacian equations with critical exponent in \mathbb{R}^N and negative energy. *Calc. Var. Partial Differ. Equ.* **60**, 30 (2021)
- [5] Baldelli, L., Brizi, Y., Filippucci, R.: On symmetric solutions for (p, q) -Laplacian equations in \mathbb{R}^N with critical terms. *J. Geom. Anal.* **32**, 25 (2022)
- [6] Baldelli, L., Filippucci, R.: Singular quasilinear critical Schrödinger equations in \mathbb{R}^N . *Commun. Pure Appl. Anal.* **21**, 2561–2586 (2022)
- [7] Bartsch, T., Jeanjean, L.: Normalized solutions for nonlinear Schrödinger systems. *Proc. R. Soc. Edinb. Sect. A* **148**, 225–242 (2018)
- [8] Ben-Naoum, A.K., Troestler, C., Willem, M.: Extrema problems with critical Sobolev exponents on unbounded domains. *Nonlinear Anal.* **26**, 823–833 (1996)
- [9] Bianchi, G., Chabrowski, J., Szulkin, A.: On symmetric solutions of an elliptic equation with a nonlinearity involving critical Sobolev exponent. *Nonlinear Anal.* **25**, 41–59 (1995)
- [10] Brezis, H.: *Functional Analysis. Sobolev Spaces and Partial Differential Equations*, vol. XIV, p. 599. Universitext, Springer, New York (2011)

- [11] Brezis, H., Lieb, E.: A relation between pointwise convergence of functions and convergence of functionals. *Proc. Am. Math. Soc.* **88**, 486–490 (1983)
- [12] Brezis, H., Nirenberg, L.: Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. *Commun. Pure Appl. Math.* **36**, 437–477 (1983)
- [13] Capuzzo Dolcetta, I., Tchou, N..A.: Semilinear elliptic equations with singular potential. *Comptes Rendus Acad. Sci. Paris Sér. I Math.* **309**, 1007–1012 (1989)
- [14] Cassani, D., Wang, Y.: Blow-up phenomena and asymptotic profiles passing from H^1 -critical to super-critical quasilinear Schrödinger equations. *Adv. Nonlinear Stud.* **21**, 855–874 (2021)
- [15] Chabrowski, J.: Concentration-compactness principle at infinity and semilinear elliptic equations involving critical and subcritical Sobolev exponents. *Calc. Var. Partial Differ. Equ.* **3**, 493–512 (1995)
- [16] Chen, K., Chen, K., Wang, H.: Symmetry of positive solutions of semilinear elliptic equations in infinite strip domains. *J. Differ. Equ.* **148**, 1–8 (1998)
- [17] Colin, M., Jeanjean, L.: Solutions for a quasilinear Schrödinger equations: a dual approach. *Nonlinear Anal.* **56**, 213–226 (2004)
- [18] Deng, Y.B., Peng, S.J., Yan, S.S.: Critical exponents and solitary wave solutions for generalized quasilinear Schrödinger equations. *J. Differ. Equ.* **260**, 1228–1262 (2016)
- [19] Do Ó, J..M..B., Severo, U.: Quasilinear Schrödinger equations involving concave and convex nonlinearities. *Commun. Pure Appl. Anal.* **8**, 621–644 (2009)
- [20] Jeanjean, L., Tanaka, K.: A positive solution for a nonlinear Schrödinger equation on \mathbb{R}^N . *Indiana Univ. Math. J.* **54**, 443–464 (2005)
- [21] Kajikiya, R.: A critical point theorem related to the symmetric mountain pass lemma and its applications to elliptic equations. *J. Funct. Anal.* **225**, 352–370 (2005)
- [22] Kristály, A.: Multiple solutions of a sublinear Schrödinger equation. *NoDEA Nonlinear Differ. Equ. Appl.* **14**, 291–301 (2007)
- [23] Kumar, D., Rădulescu, V.D., Sreenadh, K.: Singular elliptic problems with unbalanced growth and critical exponent. *Nonlinearity* **33**, 3336–3369 (2020)
- [24] Lions, P.L.: The concentration-compactness principle in the calculus of variations. The limit case. I. *Rev. Mat. Iberoam.* **1**, 145–201 (1985)
- [25] Liu, J., Wang, Y., Wang, Z.: Soliton solutions for quasilinear Schrödinger equations II. *J. Differ. Equ.* **187**, 473–493 (2003)
- [26] Liu, J., Wang, Y., Wang, Z.: Solutions for quasilinear Schrödinger equations via the Nehari method. *Commun. Partial Differ. Equ.* **29**, 879–901 (2004)
- [27] Palais, R.S.: The principle of symmetric criticality. *Commun. Math. Phys.* **69**, 19–30 (1979)

- [28] Papageorgiou, N.S., Rădulescu, V.D., Repovš, D.D.: *Nonlinear Analysis-Theory and Methods*. Springer Monographs in Mathematics, vol. XI, p. 577. Springer, Cham (2019)
- [29] Papageorgiou, N.S., Rădulescu, V.D., Repovš, D.D.: Nonlinear nonhomogeneous singular problems. *Calc. Var. Partial Differ. Equ.* **59**, 31 (2020)
- [30] Pucci, P., Serrin, J.: A general variational identity. *Indiana Univ. Math. J.* **35**, 681–730 (1986)
- [31] Severo, U.: Existence of weak solutions for quasilinear elliptic equations involving the p -Laplacian. *Electron. J. Differ. Equ.* **2008**, 1–16 (2008)
- [32] Shen, Y.T., Wang, Y.J.: A class of generalized quasilinear Schrödinger equations. *Commun. Pure Appl. Anal.* **15**, 853–870 (2016)
- [33] Silva, Elves A. de B., Vieira, G.F.: Quasilinear asymptotically periodic Schrödinger equations with critical growth. *Calc. Var. Partial Differ. Equ.* **39**, 1–33 (2010)
- [34] Rabinowitz, P.H.: *Minimax methods in critical point theory with applications to differential equations*. In: *CBMS Regional Conference Series in Mathematics*, vol. 65. Published for the Conference Board of the Mathematics Science, vol. VIII, p. 100. AMS, Washington (1986)
- [35] Wang, Y., Li, Z., Abdelgadir, A.A.: On singular quasilinear Schrödinger equations with critical exponents. *Math. Methods Appl. Sci.* **40**, 5095–5108 (2017)
- [36] Wang, L., Wang, J., Li, X.: Infinitely many solutions to quasilinear Schrödinger equations with critical exponent. *Electron. J. Qual. Theory Differ. Equ.* **5**, 16 (2019)
- [37] Willem, M.: *Minimax Theorems*. *Progress in Nonlinear Differential Equations and their Applications*, vol. 24, p. 162. Birkhäuser, Boston (1996)

Laura Baldelli
Department of Mathematics
University of Perugia
Via Vanvitelli 1
06123 Perugia
Italy
e-mail: lbaldelli@impan.pl

Roberta Filippucci
Institute of Mathematics
Polish Academy of Sciences
ul. Sniadeckich 8
00-656 Warsaw
Poland
e-mail: roberta.filippucci@unipg.it

Received: 2 May 2023.

Accepted: 13 October 2023.