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Intrinsic sub-Laplacian for hypersurface in a contact sub-Riemannian manifold

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Abstract. We construct and study the intrinsic sub-Laplacian, defined outside the set of characteristic points, for a smooth hypersurface embedded in a contact sub-Riemannian manifold. We prove that, away from characteristic points, the intrinsic sub-Laplacian arises as the limit of Laplace–Beltrami operators built by means of Riemannian approximations to the sub-Riemannian structure using the Reeb vector field. We carefully analyse three families of model cases for this setting obtained by considering canonical hypersurfaces embedded in model spaces for contact sub-Riemannian manifolds. In these model cases, we show that the intrinsic sub-Laplacian is stochastically complete and in particular, that the stochastic process induced by the intrinsic sub-Laplacian almost surely does not hit characteristic points.

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Contents

Birkhäuser

1. Introduction

Recent years have seen increased activity in the study of hypersurfaces embedded in contact sub-Riemannian manifolds, with notable subtleties as well as distinctions compared to the Riemannian setting arising in the presence of characteristic points. These are points on the hypersurface where the tangent space coincides with the contact hyperplane.

First works in this direction have concerned the study of geometry of hypersurfaces in Heisenberg groups, and more generally in Carnot groups, in particular related to a notion of horizontal mean curvature and isoperimetric inequalities. For a review of these topics, see [9, 12] and references therein.

For surfaces embedded in the Heisenberg group the horizontal mean curvature, which may blow up at characteristic points, is locally integrable with respect to the sub-Riemannian perimeter measure, as shown by Danielli, Garofalo and Nhieu. Their conjecture given in [13] that around isolated characteristic points the horizontal mean curvature is also locally integrable with respect to the Riemannian induced measure is verified by Rossi [23] for characteristic points which are isolated and mildly degenerate. Rizzi and Rossi [24] give an example for a domain in the Heisenberg group where a higher-order coefficient in the asymptotic expansion for the heat content of smooth non-characteristic domains blows up at an isolated characteristic point.

Diniz and Veloso [14], assuming absence of characteristic points, and Balogh, Tyson and Vecchi [7, 8], allowing for characteristic points, introduce a Gauss–Bonnet theorem for surfaces in the Heisenberg group, which is extended by Veloso [26] to surfaces without characteristic points in general three-dimensional contact sub-Riemannian manifolds. A Gauss–Bonnet theorem recovering topological information concentrated around the characteristic points is obtained by Grong, Hidalgo Calderón and Vega-Molino [16] for surfaces in three-dimensional contact sub-Riemannian manifolds.

The work [2] analyses the metric structure, particularly near characteristic points, induced on surfaces embedded in three-dimensional contact sub-Riemannian manifolds, and [3] introduces and studies properties of a canonical stochastic process on surfaces in three-dimensional contact sub-Riemannian manifolds which exhibits different behaviours near an elliptic characteristic point and a hyperbolic characteristic point.

The present article aims to initiate further studies of hypersurfaces embedded in higher-dimensional contact sub-Riemannian manifolds. We intrinsically construct a sub-Laplacian on hypersurfaces in contact sub-Riemannian manifolds, which for surfaces in three-dimensional contact sub-Riemannian manifolds gives rise to the generator of the stochastic process obtained in [3] by means of Riemannian approximations, and we use our analysis to propose model cases for this setting. Some notions such as horizontal connectivity, horizontal connection and horizontal mean curvature on hypersurfaces in sub-Riemannian manifolds are studied by Tan and Yang [25].

We close this literature overview by highlighting that whilst the powerful convex surface theory in three-dimensional contact topology, that is, without additionally equipping the contact structure with a fibre inner product, has been introduced by Giroux [17], the theory of convex hypersurfaces in higher-dimensional contact topology is still a relatively new endeavour, see Honda and Huang [20].

Let M be a smooth manifold of dimension $2n+1$ for $n \geq 1$, let $\mathcal{D}$ be a contact structure on M, and let g be a smooth fibre inner product on $\mathcal{D}$. Since this gives rise to a contact manifold $(M, \mathcal{D})$ and as $(\mathcal{D}, g)$ defines a sub-Riemannian structure on the manifold M, the triple $(M, \mathcal{D}, g)$ is called a contact sub-Riemannian manifold. Throughout, we shall assume that there exists a global one-form ω on M such that $\mathcal{D} = \ker \omega$ and $\omega \wedge (\mathrm{d}\omega)^n \neq 0$. Such a global one-form ω is called contact form for the contact structure $\mathcal{D}$. The existence of a contact form ω ensures that the manifold M is orientable as it can then be oriented by the volume form

$$\Omega = \omega \wedge (\mathrm{d}\omega)^n . \tag{1}$$

We shall further assume that the one-form ω is normalised such that

$$(\mathrm{d}\omega)^n|_{\mathcal{D}} = n! \, \mathrm{vol}_g, \tag{2}$$

with vol_g denoting the volume form on the distribution $\mathcal{D}$ induced by the fibre inner product g. The Reeb vector field X_0 on M associated with the contact form ω is uniquely characterised by requiring $\omega(X_0) = 1$ and $\mathrm{d}\omega(X_0, \cdot) = 0$. Subject to the normalisation condition (2), a fixed sub-Riemannian manifold $(M, \mathcal{D}, g)$ admits a unique Reeb vector field X_0 as there exists a unique one-form ω on M which both defines the contact structure $\mathcal{D}$ and satisfies (2).

Let S be an orientable hypersurface embedded in the contact manifold $(M, \mathcal{D})$. We denote by $C(S)$ the set of characteristic points of S, namely the set of points $x \in S$ such that $T_xS = \mathcal{D}_x$. Observe that $C(S)$ is a closed subset of S, which implies that $S \setminus C(S)$ is a well-defined hypersurface in M. Outside the set of characteristic points, that is, on $S \setminus C(S)$, we define the distribution $\mathcal{F} = \mathcal{D} \cap TS$ which by construction has corank one in the tangent bundle of the hypersurface $S \setminus C(S)$. Let ζ be the one-form on $S \setminus C(S)$ obtained by restricting the one-form ω defined on M to $S \setminus C(S)$, and note that the distribution $\mathcal{F}$ is given as $\mathcal{F} = \ker \zeta$.

A crucial observation to be made at this stage is that the case $n = 1$ needs to be treated differently from the case $n > 1$. Indeed, for $n = 1$, the manifold M has dimension three, the hypersurface S is a two-dimensional surface and the distribution $\mathcal{F}$ is a line field, which is always integrable. On the other hand, if $n > 1$, then $\mathcal{F}$ is a rank $2n-1$ distribution on a $2n$-dimensional hypersurface. As discussed in more details in Sect. 2, when $n > 1$, the distribution $\mathcal{F}$ is always bracket generating as a result of $\mathcal{F}$ defining a quasi-contact structure on $S \setminus C(S)$. In particular, the kernel $\ker \mathrm{d}\zeta|_{\mathcal{F}}$ has dimension one.

In this article, we construct an intrinsic sub-Laplacian on $S \setminus C(S)$ obtained by taking the divergence of the horizontal gradient on $S \setminus C(S)$ with respect to a volume form for $S \setminus C(S)$ which is the restriction of the volume form Ω on M. We show that the intrinsic sub-Laplacian arises as the limit of Laplace–Beltrami operators built by means of Riemannian approximations to the sub-Riemannian structure using the Reeb vector field. We further determine the radial part of the constructed intrinsic sub-Laplacian explicitly for canonical hypersurfaces in the sphere S^{2n+1} and the anti-de Sitter space H^{2n+1} both equipped with standard sub-Riemannian contact structures as well as in the higher-dimensional Heisenberg group $\mathbb{H}^n$, which together constitute model spaces for our setting.

The construction of the intrinsic sub-Laplacian presented below carries over for any alternative normalisation condition fixing the one-form ω in place of (2). The associated sub-Laplacian then still arises as the limit of Laplace–Beltrami operators, except that the Reeb vector field used to define the Riemannian approximations is uniquely characterised in terms of the new normalisation. As can be observed later, any choice of normalisation which only changes the one-form ω by a constant gives rise to the same sub-Laplacian as obtained subject to the condition (2).

1.1. Intrinsic sub-Laplacian on hypersurface

We start with the construction of a sub-Laplacian on the embedded hypersurface $S \setminus C(S)$ which is intrinsic to the contact sub-Riemannian manifold $(M, \mathcal{D}, g)$ and the normalisation condition (2). We then state the result that it emerges as the limit of Laplace–Beltrami operators. This particularly implies that the operator Δ_0 constructed in [3] on surfaces in three-dimensional contact sub-Riemannian manifolds coincides with the intrinsic sub-Laplacian defined in this article for the case $n = 1$.

The sub-Riemannian normal in $(M, \mathcal{D}, g)$ to the hypersurface S away from the set $C(S)$ of characteristic points is formed from directions contained in the contact structure $\mathcal{D}$ and orthogonal to the distribution $\mathcal{F}$. Once the orientations of S and M are fixed, we have a unique unit and normal vector field N compatible with the orientations, which is defined as follows.

Definition 1. The sub-Riemannian normal vector field N along the hypersurface $S \setminus C(S)$ in the contact sub-Riemannian manifold $(M, \mathcal{D}, g)$ is the unit-length vector field in the distribution $\mathcal{D}$, that is,

$$\omega(N) = 0 \quad \text{and} \quad g(N, N) = 1, \tag{3}$$

such that, for any vector field Y on $S \setminus C(S)$ and in the distribution $\mathcal{F}$,

$$g(N, Y) = 0, \tag{4}$$

and, for any positively oriented local orthonormal frame $(Z_1, \ldots, Z_{2n})$ for the hypersurface $S \setminus C(S)$, the frame $(N, Z_1, \ldots, Z_{2n})$ for M is positively oriented.

Using the volume form Ω on M given by (1) and the sub-Riemannian normal vector field N along $S \setminus C(S)$, we define a volume form μ on $S \setminus C(S)$ with respect to which we later take the divergence when constructing the intrinsic sub-Laplacian on $S \setminus C(S)$.

Definition 2. Let μ be the volume form defined on $S \setminus C(S)$ as

$$\mu = \iota_N \Omega,$$

that is, the contraction of the form Ω with the vector field N restricted to $S \setminus C(S)$.

From the compatibility of N with the orientations of M and S, it follows that $\mu = \iota_N \Omega$ is positive on $S \setminus C(S)$, meaning it has positive values when evaluated on positively oriented orthonormal frames.

The final ingredient needed before we can introduce the intrinsic sub-Laplacian of a smooth function $f\colon S \setminus C(S) \to \mathbb{R}$ is the horizontal gradient $\nabla_S f$.

Definition 3. Let $f\colon S \setminus C(S) \to \mathbb{R}$ be a smooth function. The horizontal gradient $\nabla_S f$ of the function f is the unique vector field in the distribution $\mathcal{F}$, that is,

$$\zeta(\nabla_S f) = 0,$$

such that, for any vector field Y in $\mathcal{F}$,

$$g(\nabla_S f, Y) = \mathrm{d}f(Y).$$

In particular, with a local orthonormal frame $(Y_1, \dots, Y_{2n-1})$ for $\mathcal{F}$, we can write

$$\nabla_S f = \sum_{i=1}^{2n-1} (Y_i f) Y_i, \tag{5}$$

which follows by noting that, for all $j \in \{1, \dots, 2n-1\}$,

$$g(\nabla_S f, Y_j) = \sum_{i=1}^{2n-1} (Y_i f) g(Y_i, Y_j) = Y_j f = \mathrm{d}f(Y_j).$$

The intrinsic sub-Laplacian Δ on $S \setminus C(S)$ is constructed as the divergence with respect to the volume form μ of the horizontal gradient ∇_S.

Definition 4. The intrinsic sub-Laplacian Δ for a hypersurface $S \setminus C(S)$ embedded in a contact sub-Riemannian manifold $(M, \mathcal{D}, g)$ is given by, for a smooth function $f\colon S \setminus C(S) \to \mathbb{R}$,

$$\Delta f = \operatorname{div}_\mu (\nabla_S f).$$

The sub-Laplacian Δ defined on $S \setminus C(S)$ arises as the limit of Laplace–Beltrami operators on S in the following way. For $\varepsilon > 0$, we consider the Riemannian metric $\overline{g}^\varepsilon$ on M obtained as

$$\overline{g}^\varepsilon = g \oplus \frac{1}{\varepsilon^2} (\omega \otimes \omega). \tag{6}$$

We use i for the inclusion map $i\colon S \to M$ and observe that $i^* \overline{g}^\varepsilon$ is the Riemannian metric on S induced by the Riemannian metric $\overline{g}^\varepsilon$ on M. The Laplace–Beltrami operator Δ^ε of the $2n$-dimensional Riemannian manifold $(S, i^* \overline{g}^\varepsilon)$ then converges to the intrinsic sub-Laplacian Δ uniformly on compacts as $\varepsilon \to 0$.

Theorem 5. *For any smooth function $f \in C_c^\infty(S \setminus C(S))$ compactly supported in $S \setminus C(S)$, the functions $\Delta^\varepsilon f$ converge uniformly on $S \setminus C(S)$ to Δf as $\varepsilon \to 0$.*

Since the operator Δ_0 introduced in [3] on surfaces in three-dimensional contact sub-Riemannian manifolds is constructed as the limit of Laplace–Beltrami operators and subject to the normalisation condition $\mathrm{d}\omega|_{\mathcal{D}} = -\mathrm{vol}_g$, it follows from Theorem 5 that, for $n = 1$, the intrinsic sub-Laplacian Δ from Definition 4 coincides with the operator Δ_0. Whilst the additional sign in the normalisation condition compared to (2) flips the direction of the Reeb vector field X_0, it does not affect the operator Δ because the divergence remains unchanged for measures differing by a non-zero constant factor.

1.2. Hypersurfaces in contact sub-Riemannian model spaces

In [3], the operator Δ_0 is explicitly determined for natural choices of surfaces in the three classes of model spaces for three-dimensional sub-Riemannian structures. We extend these considerations to higher dimensions by studying the intrinsic sub-Laplacian Δ from Definition 4 for canonical hypersurfaces in the three classes of model spaces for contact sub-Riemannian manifolds. Moreover, we analyse the radial part of the stochastic process with generator $\frac{1}{2}\Delta$ which is sufficient to deduce that in all these cases the sub-Laplacian Δ defined away from characteristic points is stochastically complete. At the same time, the geometry induced on each hypersurface minus characteristic points is not geodesically complete.

The model spaces for contact sub-Riemannian manifolds arise by equipping the Euclidean space $\mathbb{R}^{2n+1}$, the sphere S^{2n+1} and the hyperboloid H^{2n+1}, respectively, with a standard contact structure $\mathcal{D}$ and the following fibre inner product g on $\mathcal{D}$. For $\mathbb{R}^{2n+1}$, we choose g such that $(\mathbb{R}^{2n+1}, \mathcal{D}, g)$ gives rise to the higher-dimensional Heisenberg group $\mathbb{H}^n$. For the sphere S^{2n+1} embedded in $\mathbb{R}^{2n+2}$, we choose $k \in \mathbb{R}$ positive and set, with $\langle \cdot, \cdot \rangle$ denoting the Euclidean inner product on $\mathbb{R}^{2n+2}$,

$$g(\cdot,\cdot) = \frac{1}{k^2}\ \langle \cdot, \cdot \rangle|_{\mathcal{D}}\,.$$

This gives rise to a one-parameter family of model spaces with underlying manifold S^{2n+1} and parameter $k > 0$. Similarly, for the hyperboloid H^{2n+1} embedded in the Lorentzian space $\mathbb{R}^{2n,2}$ with signature $(2n, 2)$, we use the flat Lorentzian metric η on $\mathbb{R}^{2n,2}$ and $k \in \mathbb{R}$ positive to define

$$g(\cdot,\cdot) = \frac{1}{k^2}\ \eta(\cdot,\cdot)|_{\mathcal{D}}\,,$$

which yields a one-parameter family of model spaces with underlying manifold H^{2n+1} and parameter $k > 0$. The model spaces for contact sub-Riemannian manifolds are described in more details in Sect. 4.

The hypersurface which we consider embedded in $\mathbb{R}^{2n+1}$, in S^{2n+1} and in H^{2n+1}, respectively, serves as a model hypersurface in the corresponding model space and can be identified with $\mathbb{R}^{2n}$, S^{2n} and $\widetilde{H}^{2n}$, respectively, with a unique characteristic point in the first and third case, and with two antipodal characteristic points in the second case. We refer to Sects. 4.1, 4.2 and 4.3 for more details on the choice of the model hypersurface. In our analysis of the model cases, we first obtain expressions for the volume form Ω on M and for the sub-Riemannian normal vector field N to derive an expression for the volume form μ on the hypersurface away from characteristic points.

Proposition 6. *Let $(M,\mathcal{D},g)$ be a $(2n+1)$-dimensional contact sub-Riemannian model space. Set $I=(0,\frac{\pi}{k})$ if $M=S^{2n+1}$ associated with parameter $k>0$ and set $I=(0,\infty)$ otherwise. Define $h_k\colon I\to\mathbb{R}$ by, for $r\in I$,*

$$h_k(r)=\begin{cases} r & \text{if } M=\mathbb{R}^{2n+1} \\ k^{-1}\sin(kr) & \text{if } M=S^{2n+1} \\ k^{-1}\sinh(kr) & \text{if } M=H^{2n+1} \end{cases}.$$

For the model hypersurface S in the model space $(M,\mathcal{D},g)$ and in suitable coordinates $(r,\varphi_1,\ldots,\varphi_{2n-1})$ for $S\setminus C(S)$ with $r\in I$, $\varphi_1,\ldots,\varphi_{2n-2}\in[0,\pi]$ and $\varphi_{2n-1}\in[0,2\pi)$, the volume form μ defined on $S\setminus C(S)$ is given by

$$\mu=\frac{n!}{2}\left(h_k(r)\right)^{2n}\left(\prod_{i=1}^{2n-2}\left(\sin(\varphi_i)\right)^{2n-i-1}\right)\mathrm{d}r\wedge\mathrm{d}\varphi_1\wedge\cdots\wedge\mathrm{d}\varphi_{2n-1}.$$

We observe that, except for a leading constant, the volume forms induced on the Euclidean space $\mathbb{R}^{2n}$, the sphere S^{2n} and the hyperboloid $\widetilde{H}^{2n}$ differ from the standard Riemannian volume forms by a factor of h_k. This additional factor is the main reason why the radial part of the stochastic process with generator $\frac{1}{2}\Delta$ is of one order higher than in the model spaces for Riemannian manifolds of the same topological dimension. For a discussion on the radial process of Brownian motion on the model Riemannian manifolds, see e.g. Grigor'yan [18, Sect. 3.10] and Hsu [21, Sect. 3.3]. The radial part of sub-Riemannian Brownian motion in the setting of totally geodesic foliations is studied in [4].

Theorem 7. *Let $(M,\mathcal{D},g)$ be a $(2n+1)$-dimensional contact sub-Riemannian model space. For the model hypersurface S in the model space $(M,\mathcal{D},g)$, the radial part of the stochastic process with generator $\frac{1}{2}\Delta$ on $S\setminus C(S)$ is*

- *the Bessel process of order $2n+1$ if $M=\mathbb{R}^{2n+1}$,*
- *a Legendre process of order $2n+1$ if $M=S^{2n+1}$,*
- *a hyperbolic Bessel process of order $2n+1$ if $M=H^{2n+1}$.*

Since a Bessel process of order $2n+1$ and a hyperbolic Bessel process of order $2n+1$ for $n\geq 1$ almost surely neither hits the origin nor explodes in finite time, and as a Legendre process of order $2n+1$ for $n\geq 1$ almost surely hits neither endpoint of the interval $(0,\frac{\pi}{k})$, it is an immediate consequence of Theorem 7 that in all model cases considered the intrinsic sub-Laplacian Δ defined on $S\setminus C(S)$ is stochastically complete. On the other hand, the geometry induced on the hypersurface $S\setminus C(S)$ is not geodesically complete. This can be seen by noting that a radial ray, that is, a path along the radial direction emanating from one of the characteristic points, parameterised by arc length is a geodesic which cannot be extended indefinitely towards the characteristic point.

Organisation of the article

In Sect. 2, we first provide an overview of contact sub-Riemannian manifolds and of quasi-contact sub-Riemannian manifolds before showing that, for $n \geq 2$, a contact structure on a manifold M of dimension $2n+1$ induces a quasi-contact structure on a hypersurface embedded in M away from the set of characteristic points. We illustrate this phenomenon by considering a canonical hypersurface in the Heisenberg group $\mathbb{H}^2$. In Sect. 3, we describe the Laplace–Beltrami operators Δ^ε obtained by means of Riemannian approximations in a convenient way which allows us to subsequently prove Theorem 5. We proceed by explicitly determining the intrinsic sub-Laplacian for the considered hypersurface in $\mathbb{H}^2$. In Sect. 4, we analyse model cases for our setting, which results in proofs of Proposition 6 and Theorem 7.

2. Hypersurfaces in contact sub-Riemannian manifolds

We start by providing a concise overview of contact sub-Riemannian manifolds and of quasi-contact sub-Riemannian manifolds. For more exhaustive discussions, see e.g. [1], Boscain, Neel and Rizzi [6, Sect. 10], and Charlot [10]. For an in-depth account on contact geometry, one may consult Blair [5] and Geiges [15]. We then link contact sub-Riemannian manifolds and quasi-contact sub-Riemannian manifolds by showing that for a hypersurface S in a manifold M of dimension bigger than three, a contact structure on M induces a quasi-contact structure on the hypersurface S away from the set $C(S)$ of characteristic points.

A *contact sub-Riemannian manifold* is a triple $(M, \mathcal{D}, g)$ consisting of a smooth manifold M with $\dim M = 2n + 1$ for $n \geq 1$, a contact structure $\mathcal{D}$ on M and a smooth fibre inner product g defined on $\mathcal{D}$. The distribution $\mathcal{D}$ is called a contact structure on M if it is locally defined as the kernel $\mathcal{D} = \ker \omega$ of a one-form ω on M which satisfies the non-degeneracy condition $\omega \wedge (\mathrm{d}\omega)^n \neq 0$. The latter is equivalent to requiring that $\mathrm{d}\omega|_{\mathcal{D}}$ is non-degenerate and implies that the contact structure $\mathcal{D}$ is a corank one distribution in the tangent bundle TM. Recall we assume throughout that there exists a global one-form ω defining the contact structure $\mathcal{D}$, which also induces an orientation on M through the volume form $\omega \wedge (\mathrm{d}\omega)^n$.

We observe that for a smooth and positive function $f \colon M \to (0, \infty)$, we have

$$(f\omega) \wedge (\mathrm{d}(f\omega))^n = f^{n+1}\, \omega \wedge (\mathrm{d}\omega)^n$$

as well as $\ker f\omega = \ker \omega$. Thus, the one-forms ω and $f\omega$ define the same contact structure $\mathcal{D}$ on M and the associated sub-Riemannian structures are equivalent. Due to $\mathcal{D} = \ker \omega$, we further obtain that

$$\mathrm{d}(f\omega)|_{\mathcal{D}} = f\, \mathrm{d}\omega|_{\mathcal{D}}.$$

Hence, we can and do assume that the contact form ω satisfies the normalisation condition (2), that is,

$$(\mathrm{d}\omega)^n|_{\mathcal{D}} = n!\, \mathrm{vol}_g.$$

The Reeb vector field X_0 on M with respect to the one-form ω normalised according to (2) is uniquely characterised by requiring that $\omega(X_0) = 1$ and $\mathrm{d}\omega(X_0, \cdot) = 0$.

A *quasi-contact sub-Riemannian manifold* is a triple $(S, \mathcal{F}, g)$ which consists of a smooth even-dimensional manifold S where $\dim S = 2n$ for $n \geq 2$, a quasi-contact structure $\mathcal{F}$ on S and a smooth fibre inner product g defined on $\mathcal{F}$. A distribution $\mathcal{F}$ is called a quasi-contact structure on S if it has corank one in the tangent bundle TS and is locally given as $\mathcal{F} = \ker \zeta$ for a one-form ζ on S satisfying the non-degeneracy condition that $\mathrm{d}\zeta|_{\mathcal{F}}$ has one-dimensional kernel. Note that since the manifold S is of even dimension, the distribution $\mathcal{F}$ has odd rank and $\mathrm{d}\zeta|_{\mathcal{F}}$ necessarily possesses a non-trivial kernel. Therefore, the above non-degeneracy condition can be understood as a minimal degeneracy assumption. If there exists a global one-form ζ defining the quasi-contact structure $\mathcal{F}$, we call this one-form ζ a quasi-contact form.

The following property for quasi-contact structures is well-known but we include its proof for completeness as it implies that the triple $(S, \mathcal{F}, g)$ introduced above is indeed a sub-Riemannian manifold.

Lemma 8. *A quasi-contact structure $\mathcal{F}$ on a manifold S is a bracket generating distribution on S.*

Observe that we define quasi-contact structures only in dimension $2n$ for $n \geq 2$, which is an important condition here because a rank one distribution, that is, a line field, is always integrable.

Proof of Lemma 8. Let ζ be a one-form locally defining the distribution $\mathcal{F}$ through $\mathcal{F} = \ker \zeta$. Since the kernel $\ker \mathrm{d}\zeta|_{\mathcal{F}}$ has dimension one and as $\mathcal{F}$ is of rank at least three, we can locally choose two vector fields Y_1 and Y_2 in $\mathcal{F}$ such that $\mathrm{d}\zeta(Y_1, Y_2)$ is non-zero. Applying the Leibniz rule and the Cartan identity, we further obtain

$$\mathcal{L}_{Y_1}\left(\zeta(Y_2)\right) = \left(\mathcal{L}_{Y_1}\zeta\right)(Y_2) + \zeta\left([Y_1, Y_2]\right) = \mathrm{d}\zeta(Y_1, Y_2) + \iota_{Y_2}\,\mathrm{d}\left(\zeta(Y_1)\right) + \zeta\left([Y_1, Y_2]\right) . \tag{7}$$

For the vector fields Y_1 and Y_2, we have $\zeta(Y_1) = \zeta(Y_2) = 0$ with $\mathrm{d}\zeta(Y_1, Y_2)$ being non-zero. The above identity then implies that $\zeta\left([Y_1, Y_2]\right)$ is non-zero because (7) simplifies to

$$0 = \mathrm{d}\zeta(Y_1, Y_2) + \zeta\left([Y_1, Y_2]\right).$$

It follows that the Lie bracket $[Y_1, Y_2]$ is not a vector field in $\mathcal{F}$. As a quasi-contact structure is a distribution of rank $2n - 1$ on a manifold of dimension $2n$ for some $n \geq 2$, this concludes the proof. □

We now take a $(2n + 1)$-dimensional contact sub-Riemannian manifold $(M, \mathcal{D}, g)$ for $n \geq 1$ with contact form ω satisfying the normalisation condition (2), and we consider an orientable hypersurface S embedded in M, where $\dim S = 2n$. Recall that the set $C(S)$ of characteristic points of S is given by

$$C(S) = \{x \in S : T_x S = \mathcal{D}_x\}.$$

The distribution $\mathcal{F}$ defined on the hypersurface $S \setminus C(S)$ as $\mathcal{F} = \mathcal{D} \cap TS$ has, by construction, corank one in the tangent bundle of $S \setminus C(S)$. As previously remarked, in the case $n = 1$ studied in [2,3] the distribution $\mathcal{F}$ is a line field, which is always integrable. In contrast to this, the following result states that, for $n > 1$, the rank $2n-1$ distribution $\mathcal{F}$ is a quasi-contact structure on the $2n$-dimensional hypersurface $S \setminus C(S)$. Together with Lemma 8, this shows that the distribution $\mathcal{F}$ is bracket generating for $n > 1$.

Lemma 9. *For $n \geq 2$, the distribution $\mathcal{F}$ defined on $S \setminus C(S)$ as $\mathcal{F} = \mathcal{D} \cap TS$ is a quasi-contact structure on $S \setminus C(S)$.*

Proof. For any $x \in S \setminus C(S)$, we need to show that $\mathrm{d}\zeta_x|_{\mathcal{F}_x} : \mathcal{F}_x \times \mathcal{F}_x \to \mathbb{R}$, that is,

$$\mathrm{d}\omega_x|_{\mathcal{F}_x} : \mathcal{F}_x \times \mathcal{F}_x \to \mathbb{R}$$

has one-dimensional kernel, which is a consequence of the following linear algebra observation.

We recall the skew-symmetric bilinear form $\mathrm{d}\omega_x : \mathcal{D}_x \times \mathcal{D}_x \to \mathbb{R}$ is non-degenerate by assumption. This means that if $w \in \mathcal{D}_x$ satisfies

$$\mathrm{d}\omega_x(v, w) = 0 \text{ for all } v \in \mathcal{D}_x$$

then $w = 0$. Moreover, since $x \in S \setminus C(S)$, we know that $\mathcal{F}_x \subset \mathcal{D}_x$ is a subspace of codimension one. The non-degeneracy of $\mathrm{d}\omega_x$ then implies that the orthogonal complement $\mathcal{F}_x^\perp$ of $\mathcal{F}_x$ defined with respect to the bilinear form $\mathrm{d}\omega_x$, that is,

$$\mathcal{F}_x^\perp = \{w \in \mathcal{D}_x : \mathrm{d}\omega_x(v, w) = 0 \text{ for all } v \in \mathcal{F}_x\}$$

has

$$\dim \mathcal{F}_x^\perp = \dim \mathcal{D}_x - \dim \mathcal{F}_x = 1. \tag{8}$$

Let $\xi \in \mathcal{F}_x^\perp$ be non-zero. If $\xi \notin \mathcal{F}_x$, we would have $\mathcal{D}_x = \mathcal{F}_x \oplus \mathcal{F}_x^\perp$ and ξ would lie in the kernel of $\mathrm{d}\omega_x$. As this contradicts the non-degeneracy of $\mathrm{d}\omega_x$, it follows that $\xi \in \mathcal{F}_x$. Therefore, we obtain

$$\mathcal{F}_x^\perp = \ker \mathrm{d}\zeta_x|_{\mathcal{F}_x},$$

and the desired result follows from (8). □

As a direct consequence of Lemma 9, we recover the result [25, Theorem 1.1] concerning the horizontal connectivity of points on a hypersurface embedded in a contact sub-Riemannian manifold of dimension $2n+1$ for $n > 1$.

Moreover, we see that $\mathrm{d}\zeta|_{\mathcal{F}}$ induces a line field on $S \setminus C(S)$ which can be oriented and extended to the set $C(S)$ of characteristic points to yield an oriented singular line field on the hypersurface S, called the characteristic foliation of S and defined in [20, Definition 2.0.1].

By equipping the pair $(S \setminus C(S), \mathcal{F})$ with the restriction of the fibre inner product of the contact sub-Riemannian manifold $(M, \mathcal{D}, g)$ to the distribution $\mathcal{F}$, we obtain a quasi-contact sub-Riemannian manifold provided that $n \geq 2$.

Remark 10. Throughout the article, we consider the distribution $\mathcal{F}$ as defined on $S \setminus C(S)$, that is, away from the set $C(S)$ of points $x \in S$ where $T_xS = \mathcal{D}_x$. One may also regard $\mathcal{F}$ as a generalised distribution given at every point of S by setting $\mathcal{F}_x = T_xS$ for $x \in C(S)$.

In this viewpoint, $\mathcal{F}$ is not a rank-varying distribution in the sense of vector fields because there does not exist a family of globally defined vector fields $Y_1, \ldots, Y_m$ on S such that, for all $x \in S$,

$$\mathcal{F}_x = \operatorname{span}\{Y_1(x), \ldots, Y_m(x)\}.$$

Indeed, in such a case the map $x \mapsto \dim \mathcal{F}_x$ would be lower semicontinuous, which is not true in our situation. Instead, the dimension of $\mathcal{F}_x$ increases at singular points. This is typical of a distribution defined by Pfaffian equations, that is, the zero locus of a family of linear forms.

Examples illustrating the geometry and in particular the singular one-dimensional foliation induced on surfaces embedded in three-dimensional contact sub-Riemannian manifolds are discussed, among others, in [2,3,22]. For an example which demonstrates the geometry induced on hypersurfaces in higher-dimensional contact sub-Riemannian manifolds and which further highlights that for $n > 1$ the distribution $\mathcal{F}$ defined away from characteristic points becomes quasi-contact, we study a canonical hypersurface embedded in the Heisenberg group $\mathbb{H}^2$.

Example 11. Let (x_1, y_1, x_2, y_2, z) denote Cartesian coordinates on $\mathbb{R}^5$ and consider the contact form ω on $\mathbb{R}^5$ defined by

$$\omega = \frac{1}{2}\left(x_1 \, \mathrm{d}y_1 - y_1 \, \mathrm{d}x_1\right) + \frac{1}{2}\left(x_2 \, \mathrm{d}y_2 - y_2 \, \mathrm{d}x_2\right) - \mathrm{d}z. \tag{9}$$

Equipping the contact structure $\mathcal{D} = \ker \omega$ with the fibre inner product

$$g = \mathrm{d}x_1 \otimes \mathrm{d}x_1 + \mathrm{d}y_1 \otimes \mathrm{d}y_1 + \mathrm{d}x_2 \otimes \mathrm{d}x_2 + \mathrm{d}y_2 \otimes \mathrm{d}y_2, \tag{10}$$

we obtain the contact sub-Riemannian manifold $(\mathbb{R}^5, \mathcal{D}, g)$, which is the Heisenberg group $\mathbb{H}^2$. We observe that our choice of contact form ω satisfies the normalisation condition (2) since

$$\mathrm{d}\omega = \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \mathrm{d}x_2 \wedge \mathrm{d}y_2,$$

which implies that

$$(\mathrm{d}\omega)^2 = \mathrm{d}\omega \wedge \mathrm{d}\omega = 2 \, \mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}y_2,$$

and therefore, we have $(\mathrm{d}\omega)^2\big|_{\mathcal{D}} = 2 \operatorname{vol}_g$.

The hypersurface S which we study in the Heisenberg group $\mathbb{H}^2$ is the one defined by $\{z = 0\}$. It illustrates well the changes in properties of the distribution $\mathcal{F}$ for $n > 1$ compared to $n = 1$ whilst still allowing for explicit computations and constructions. From

$$\omega\left(\frac{\partial}{\partial x_1}\right) = -\frac{y_1}{2}, \quad \omega\left(\frac{\partial}{\partial y_1}\right) = \frac{x_1}{2}, \quad \omega\left(\frac{\partial}{\partial x_2}\right) = -\frac{y_2}{2}, \quad \omega\left(\frac{\partial}{\partial y_2}\right) = \frac{x_2}{2},$$

we see that the origin of $\mathbb{R}^5$ is the only characteristic point of this hypersurface S. The distribution $\mathcal{F}$ defined on $S \backslash \{0\}$ as $\mathcal{D} \cap TS$ is a subbundle of corank one in the tangent bundle of $S \setminus \{0\}$ and can be described as the kernel of the one-form

$$\zeta = \frac{1}{2}\left(x_1 \, \mathrm{d}y_1 - y_1 \, \mathrm{d}x_1\right) + \frac{1}{2}\left(x_2 \, \mathrm{d}y_2 - y_2 \, \mathrm{d}x_2\right), \tag{11}$$

which is obtained by restricting the contact form ω to the tangent bundle of $S \setminus \{0\}$.

To gain a better understanding of the distribution $\mathcal{F}$, we find an orthonormal frame for $\mathcal{F}$, which we later further work with to explicitly determine the intrinsic sub-Laplacian Δ on $S \setminus \{0\}$. Let U_1, U_2, U_3, U_4 be the vector fields on $S \setminus \{0\}$ defined by

$$\begin{aligned}
U_1 &= \frac{1}{\sqrt{x_1^2+y_1^2+x_2^2+y_2^2}}\left(x_1\frac{\partial}{\partial x_1}+y_1\frac{\partial}{\partial y_1}+x_2\frac{\partial}{\partial x_2}+y_2\frac{\partial}{\partial y_2}\right),\\
U_2 &= \frac{1}{\sqrt{x_1^2+y_1^2+x_2^2+y_2^2}}\left(y_2\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial y_1}-y_1\frac{\partial}{\partial x_2}-x_1\frac{\partial}{\partial y_2}\right),\\
U_3 &= \frac{1}{\sqrt{x_1^2+y_1^2+x_2^2+y_2^2}}\left(x_2\frac{\partial}{\partial x_1}-y_2\frac{\partial}{\partial y_1}-x_1\frac{\partial}{\partial x_2}+y_1\frac{\partial}{\partial y_2}\right),\\
U_4 &= \frac{1}{\sqrt{x_1^2+y_1^2+x_2^2+y_2^2}}\left(y_1\frac{\partial}{\partial x_1}-x_1\frac{\partial}{\partial y_1}+y_2\frac{\partial}{\partial x_2}-x_2\frac{\partial}{\partial y_2}\right).
\end{aligned}$$

Using (10) and (11), we verify that (U_1, U_2, U_3) is an orthonormal frame for $\mathcal{F}$, and we further note that (U_1, U_2, U_3, U_4) is a frame for the tangent bundle of $S \setminus \{0\}$. Due to

$$\mathrm{d}\zeta = \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \mathrm{d}x_2 \wedge \mathrm{d}y_2,$$

we obtain that

$$\mathrm{d}\zeta\,(U_1, U_2) = \mathrm{d}\zeta\,(U_1, U_3) = 0 \quad \text{and} \quad \mathrm{d}\zeta\,(U_2, U_3) = -1.$$

It follows that

$$\ker \mathrm{d}\zeta|_{\mathcal{F}} = \operatorname{span}\{U_1\} = \operatorname{span}\left\{x_1\frac{\partial}{\partial x_1}+y_1\frac{\partial}{\partial y_1}+x_2\frac{\partial}{\partial x_2}+y_2\frac{\partial}{\partial y_2}\right\}, \tag{12}$$

and thus, consistent with Lemma 9, the rank three distribution $\mathcal{F}$ is a quasi-contact structure on the four-dimensional hypersurface $S \setminus \{0\}$. According to Lemma 8, this implies that the distribution $\mathcal{F}$ is bracket generating on $S\setminus\{0\}$, which can be seen directly by noting that

$$\begin{aligned}
[U_2, U_3] &= \frac{2}{x_1^2+y_1^2+x_2^2+y_2^2}\left(-y_1\frac{\partial}{\partial x_1}+x_1\frac{\partial}{\partial y_1}-y_2\frac{\partial}{\partial x_2}+x_2\frac{\partial}{\partial y_2}\right)\\
&= -\frac{2U_4}{\sqrt{x_1^2+y_1^2+x_2^2+y_2^2}}\,.
\end{aligned}$$

We continue our analysis for this case by determining the intrinsic sub-Laplacian Δ in the forthcoming Example 16. Moreover, in Sect. 4.1, we discuss the radial part of the stochastic process with generator $\frac{1}{2}\Delta$, which as a result of (12) is exactly the stochastic process induced on the characteristic foliation of the hypersurface S.

3. Intrinsic sub-Laplacian as limit of Laplace–Beltrami operators

After discussing the construction of the Laplace–Beltrami operators Δ^ε on the hypersurface S using Riemannian approximations of the contact sub-Riemannian manifold $(M, \mathcal{D}, g)$, we proceed with proving Theorem 5.

The Riemannian approximation for $\varepsilon > 0$ to the contact sub-Riemannian manifold $(M, \mathcal{D}, g)$ with respect to the Reeb vector field X_0 equips the smooth manifold M with the Riemannian metric $\overline{g}^\varepsilon$ given by

$$\overline{g}^\varepsilon = g \oplus \frac{1}{\varepsilon^2}\left(\omega \otimes \omega\right).$$

In particular, if $(X_1, \ldots, X_{2n})$ is a positively oriented local orthonormal frame for the distribution $\mathcal{D}$ with respect to g, then $(X_1, \ldots, X_{2n}, \varepsilon X_0)$ is a positively oriented orthonormal frame for the tangent bundle TM with respect to the Riemannian metric $\overline{g}^\varepsilon$. Using this observation, we can establish the property for the volume form $\Omega^\varepsilon = \mathrm{vol}_{\overline{g}^\varepsilon}$ on M stated below.

Lemma 12. *For $\varepsilon > 0$, the volume forms Ω and Ω^ε on the manifold M are related by*

$$\varepsilon n!\, \Omega^\varepsilon = \Omega.$$

Proof. Let $(X_1, \ldots, X_{2n})$ be a positively oriented local orthonormal frame for the distribution $\mathcal{D}$. Then $\omega(X_i) = 0$ for all $i \in \{1, \ldots, 2n\}$ as well as $\omega(X_0) = 1$ and $\mathrm{d}\omega(X_0, \cdot) = 0$ together with (1) and the normalisation condition (2) yield

$$\Omega(X_1, \ldots, X_{2n}, \varepsilon X_0) = n!\, \omega(\varepsilon X_0)\mathrm{vol}_g(X_1, \ldots, X_{2n}) = \varepsilon n!\, .$$

On the other hand, we have, by construction,

$$\Omega^\varepsilon(X_1, \ldots, X_{2n}, \varepsilon X_0) = 1,$$

which implies the claimed result. □

Similarly to Definition 1 for the sub-Riemannian normal vector field N to $S \setminus C(S)$ in the contact sub-Riemannian manifold $(M, \mathcal{D}, g)$, we define the Riemannian normal vector field N^ε for $\varepsilon > 0$ to the hypersurface S embedded in the Riemannian manifold $(M, \overline{g}^\varepsilon)$ of dimension $2n + 1$.

Definition 13. The Riemannian normal vector field N^ε along the hypersurface S embedded in the Riemannian manifold $(M, \overline{g}^\varepsilon)$ is the unit-length vector field along S, that is,

$$\overline{g}^\varepsilon(N^\varepsilon, N^\varepsilon) = 1,$$

such that, for any vector field Z on S,

$$\overline{g}^\varepsilon(N^\varepsilon, Z) = 0,$$

and, for any positively oriented local orthonormal frame $(Z_1, \ldots, Z_{2n})$ for the hypersurface $S \setminus C(S)$, the frame $(N^\varepsilon, Z_1, \ldots, Z_{2n})$ for M is positively oriented.

The next result states that as $\varepsilon \to 0$ the Riemannian normal vector fields N^ε converge uniformly on compact subsets of $S \setminus C(S)$ to the sub-Riemannian normal vector field N.

Lemma 14. *Uniformly on compact subsets of $S \setminus C(S)$, we have*

$$N^\varepsilon \to N \quad as \quad \varepsilon \to 0.$$

Proof. We use that the hypersurface S is locally given as the zero set of some smooth function $u \in C^\infty(M)$ with $\mathrm{d}u \neq 0$ on S, and we fix a local orthonormal frame $(X_1, \dots, X_{2n})$ for the contact structure $\mathcal{D}$ with respect to the fibre inner product g.

Since $x \in S$ is a characteristic point of the hypersurface S if the tangent space $T_x S$ coincides with $\mathcal{D}_x$, that is, if

$$(X_i u)(x) = 0 \quad \text{for all } i \in \{1, 2, \dots, 2n\},$$

we have

$$C(S) = \left\{ x \in S : \sum_{i=1}^{2n} \left((X_i u)(x)\right)^2 = 0 \right\}. \tag{13}$$

In terms of the local orthonormal frame $(X_1, \dots, X_{2n})$ for the distribution $\mathcal{D}$ and with $\sigma = 0$ or $\sigma = 1$ depending on the orientation of S, the sub-Riemannian normal vector field N along $S \setminus C(S)$ can be written as

$$N = (-1)^\sigma \frac{\sum_{i=1}^{2n} (X_i u) X_i}{\sqrt{\sum_{i=1}^{2n} (X_i u)^2}}\ , \tag{14}$$

due to the following reasoning. The expression (14) is well-defined away from the set of characteristic points as a result of (13). Moreover, the conditions in Definition 1 are satisfied because of $(X_1, \dots, X_{2n})$ being an orthonormal frame for $\mathcal{D}$ and since, for any vector field Y in the distribution $\mathcal{F} = \mathcal{D} \cap TS$ on $S \setminus C(S)$,

$$g(N, Y) = (-1)^\sigma \frac{\sum_{i=1}^{2n} (X_i u) g(X_i, Y)}{\sqrt{\sum_{i=1}^{2n} (X_i u)^2}} = (-1)^\sigma \frac{Yu}{\sqrt{\sum_{i=1}^{2n} (X_i u)^2}} = 0.$$

Similarly, we verify that the Riemannian normal vector field N^ε for $\varepsilon > 0$ to the hypersurface S can be expressed as

$$N^\varepsilon = (-1)^\sigma \frac{\sum_{i=1}^{2n} (X_i u) X_i + \varepsilon^2 (X_0 u) X_0}{\sqrt{\sum_{i=1}^{2n} (X_i u)^2 + \varepsilon^2 (X_0 u)^2}}\ . \tag{15}$$

The claimed result then follows from (14), (15) and $u \in C^\infty(M)$. □

The Riemannian volume form μ^ε induced on the hypersurface S embedded in the Riemannian manifold $(M, \overline{g}^\varepsilon)$ is given on S by

$$\mu^\varepsilon = \iota_{N^\varepsilon} \Omega^\varepsilon, \tag{16}$$

and the Riemannian gradient $\nabla_S^\varepsilon f$ of a smooth function $f \colon S \to \mathbb{R}$ is uniquely characterised by requiring that, for any vector field Z on S,

$$\overline{g}^\varepsilon(\nabla_S^\varepsilon f, Z) = \mathrm{d}f(Z).$$

The Laplace–Beltrami operator Δ^ε on the Riemannian manifold $(S, i^*\overline{g}^\varepsilon)$, where $i\colon S \to M$ is the inclusion map, is then defined by, for a smooth function $f\colon S \to \mathbb{R}$,

$$\Delta^\varepsilon f = \operatorname{div}_{\mu^\varepsilon}\left(\nabla_S^\varepsilon f\right).$$

The following result is crucial in proving the convergence of the Laplace–Beltrami operators Δ^ε as $\varepsilon \to 0$ to the intrinsic sub-Laplacian Δ.

Lemma 15. *Uniformly on compact subsets of $S \setminus C(S)$, we have*

$$\varepsilon n!\, \mu^\varepsilon \to \mu \quad as \quad \varepsilon \to 0.$$

Proof. This is a direct consequence of Definition 2, the identity (16), Lemma 12 and Lemma 14. □

We are finally in a position to prove Theorem 5. Note that as a result of Lemma 8 and Lemma 9, the intrinsic sub-Laplacian Δ is indeed a hypoelliptic operator on $S \setminus C(S)$ as long as $n \geq 2$.

Proof of Theorem 5. Choose a local orthonormal frame $(Y_1, \ldots, Y_{2n-1})$ for $\mathcal{F}$. From the observation (5), it follows that the intrinsic sub-Laplacian Δ on $S \setminus C(S)$ can be written as

$$\Delta = \sum_{i=1}^{2n-1} \left(Y_i^2 + (\operatorname{div}_\mu Y_i)\, Y_i\right). \tag{17}$$

We now aim to extend the local orthonormal frame $(Y_1, \ldots, Y_{2n-1})$ for $\mathcal{F}$ to a local orthonormal frame for the tangent bundle of $S \setminus C(S)$ with respect to the Riemannian metric $\overline{g}^\varepsilon$. To this end, we again use that the hypersurface S is locally given as the zero set of some smooth function $u \in C^\infty(M)$ with $\mathrm{d}u \neq 0$ on S and we consider the vector field Z on $S \setminus C(S)$ given by

$$Z = X_0 - \frac{X_0 u}{N u} N.$$

This vector field Z can be seen as the projection of the Reeb vector field X_0 on M onto the hypersurface $S \setminus C(S)$. Using (3) and (6), we compute, for $\varepsilon > 0$,

$$\overline{g}^\varepsilon(Z, Z) = \frac{1}{\varepsilon^2} + \frac{(X_0 u)^2}{(N u)^2} > 0, \tag{18}$$

which implies that we can define a vector field Z^ε on $S \setminus C(S)$ by setting

$$Z^\varepsilon = \frac{Z}{\sqrt{\overline{g}^\varepsilon(Z, Z)}}.$$

Since both the Reeb vector field X_0 and the sub-Riemannian normal vector field N are orthogonal with respect to the Riemannian metric $\overline{g}^\varepsilon$ to any vector field in $\mathcal{F}$, it follows that $(Y_1, \ldots, Y_{2n-1}, Z^\varepsilon)$ is a local orthonormal frame for the tangent bundle of $S \setminus C(S)$ with respect to the Riemannian metric $\overline{g}^\varepsilon$. Similarly as above, we can then express the Laplace–Beltrami operator Δ^ε on $S \setminus C(S)$ as

$$\Delta^\varepsilon = \sum_{i=1}^{2n-1} \left(Y_i^2 + (\operatorname{div}_{\mu^\varepsilon} Y_i)\, Y_i\right) + (Z^\varepsilon)^2 + (\operatorname{div}_{\mu^\varepsilon} Z^\varepsilon)\, Z^\varepsilon. \tag{19}$$

From (18), we deduce

$$\frac{1}{\sqrt{\overline{g}^{\varepsilon}(Z,Z)}} \leq \varepsilon,$$

which shows that for any smooth function $f \in C_c^\infty(S\backslash C(S))$ compactly supported in $S \setminus C(S)$, we have, as $\varepsilon \to 0$ and uniformly on $S \setminus C(S)$,

$$(Z^\varepsilon)^2 f \to 0 \quad \text{and} \quad Z^\varepsilon f \to 0. \tag{20}$$

Therefore, it remains to analyse the divergence terms in the expression (19) for the Laplace–Beltrami operator Δ^ε. Working in a local coordinate chart $(x_1, \dots, x_{2n})$ for the hypersurface $S \setminus C(S)$, we let ρ and ρ^ε denote the local coefficient of μ and μ^ε, respectively, and we use Lemma 15 as well as the uniform convergence of the derivatives on compacts, which can be established similarly, to argue that, for all $i \in \{1, \dots, 2n\}$ and uniformly on compact subsets of $S \setminus C(S)$,

$$\begin{aligned}\lim_{\varepsilon\to 0} \operatorname{div}_{\mu^\varepsilon} Y_i &= \lim_{\varepsilon\to 0} \sum_{j=1}^{2n} \frac{1}{\varepsilon n!\, \rho^\varepsilon} \frac{\partial}{\partial x_j} \left(\varepsilon n!\, \rho^\varepsilon Y_{i,j}\right) \\ &= \sum_{j=1}^{2n} \frac{1}{\rho} \frac{\partial}{\partial x_j} \left(\rho Y_{i,j}\right) = \operatorname{div}_\mu Y_i.\end{aligned} \tag{21}$$

Similarly, we conclude that, as $\varepsilon \to 0$ and uniformly on compact subsets of $S \setminus C(S)$,

$$\operatorname{div}_{\mu^\varepsilon} Z \to \operatorname{div}_\mu Z.$$

Hence, as a consequence of

$$\operatorname{div}_{\mu^\varepsilon} Z^\varepsilon = \frac{\operatorname{div}_{\mu^\varepsilon} Z}{\sqrt{\overline{g}^{\varepsilon}(Z,Z)}} + \overline{g}^{\varepsilon}\left(\nabla_S^\varepsilon \left(\frac{1}{\sqrt{\overline{g}^{\varepsilon}(Z,Z)}}\right), Z\right)$$

and since, by definition of the gradient ∇_S^ε,

$$\overline{g}^{\varepsilon}\left(\nabla_S^\varepsilon \left(\frac{1}{\sqrt{\overline{g}^{\varepsilon}(Z,Z)}}\right), Z\right) = Z\left(\frac{1}{\sqrt{\overline{g}^{\varepsilon}(Z,Z)}}\right),$$

we obtain that, as $\varepsilon \to 0$ and uniformly on compact subsets of $S \setminus C(S)$,

$$\operatorname{div}_{\mu^\varepsilon} Z^\varepsilon \to 0.$$

Together with (20) and (21), the claimed result then follows from (17) and (19). □

As a first illustration of the general strategy laid out for constructing the intrinsic sub-Laplacian Δ, we return to our analysis for the hypersurface given by $\{z = 0\}$ in the Heisenberg group $\mathbb{H}^2$ started in Example 11 and we demonstrate how to derive an explicit expression for the intrinsic sub-Laplacian Δ on $\{z = 0\}$ away from the unique characteristic point at the origin.

Example 16. As discussed in Example 11, the quasi-contact structure $\mathcal{F}$ on $S\setminus\{0\}$ admits the orthonormal frame (U_1, U_2, U_3) with respect to g. It follows that the horizontal gradient $\nabla_S f$ of a smooth function $f\colon S\setminus\{0\}\to\mathbb{R}$ can be expressed as

$$\nabla_S f = (U_1 f)U_1 + (U_2 f)U_2 + (U_3 f)U_3.$$

It remains to determine the volume form μ on the hypersurface $S\setminus\{0\}$ and to compute the divergence of the vector fields U_1, U_2 and U_3 with respect to μ.

The volume form Ω on $\mathbb{R}^5$ defined by (1) in terms of the contact form ω in (9) is given by

$$\Omega = \omega\wedge(\mathrm{d}\omega)^2 = -2\,\mathrm{d}x_1\wedge\mathrm{d}y_1\wedge\mathrm{d}x_2\wedge\mathrm{d}y_2\wedge\mathrm{d}z,$$

and the sub-Riemannian normal vector field N along the hypersurface $S\setminus\{0\}$ in $\mathbb{H}^2$ characterised by (3) as well as (4) and compatible with the orientations on M and S can be written as

$$N = U_4 - \frac{\sqrt{x_1^2+y_1^2+x_2^2+y_2^2}}{2}\frac{\partial}{\partial z}.$$

It follows that defining the volume form μ on $S\setminus\{0\}$ as $\iota_N\Omega$ yields

$$\mu = \sqrt{x_1^2+y_1^2+x_2^2+y_2^2}\,\mathrm{d}x_1\wedge\mathrm{d}y_1\wedge\mathrm{d}x_2\wedge\mathrm{d}y_2.$$

This implies that

$$\operatorname{div}_\mu U_1 = \frac{4}{\sqrt{x_1^2+y_1^2+x_2^2+y_2^2}} \quad\text{and}\quad \operatorname{div}_\mu U_2 = \operatorname{div}_\mu U_3 = 0.$$

Thus, the intrinsic sub-Laplacian Δ on the hypersurface $S\setminus\{0\}$ in the Heisenberg group $\mathbb{H}^2$ can be expressed as

$$\Delta = U_1^2 + U_2^2 + U_3^2 + \frac{4U_1}{\sqrt{x_1^2+y_1^2+x_2^2+y_2^2}}\,. \tag{22}$$

Due to the quasi-contact structure $\mathcal{F}$ on $S\setminus\{0\}$ being bracket generating, the intrinsic sub-Laplacian Δ is hypoelliptic, see Hörmander [19]. This illustrates a crucial change in property of the intrinsic sub-Laplacian for $n>1$ compared to the case $n=1$. As seen in [3], the operator Δ on surfaces in three-dimensional contact sub-Riemannian manifolds is never hypoelliptic as a result of $\mathcal{F}$ being a line field in that setting.

We close by highlighting that, as discussed in more details in the forthcoming analysis in Sect. 4, thanks to the drift term in (22), the intrinsic sub-Laplacian Δ on $S\setminus\{0\}$ is stochastically complete, and in particular, the stochastic process with generator $\frac{1}{2}\Delta$ on $S\setminus\{0\}$ almost surely does not hit the unique characteristic point at the origin.

4. Canonical hypersurfaces in contact sub-Riemannian model spaces

We consider canonical hypersurfaces in contact sub-Riemannian model spaces which extend the family of model cases given in [3, Theorem 1.5]. Choosing suitable coordinates, we establish Proposition 6 by explicitly computing the volume form μ induced on the hypersurface away from characteristic points. This in turn allows us to prove Theorem 7, which characterises the radial part of the stochastic process with generator $\frac{1}{2}\Delta$ and which implies that in these model cases analysed the intrinsic sub-Laplacian Δ defined on the hypersurface away from characteristic points is stochastically complete, whilst the induced geometry is not geodesically complete.

We first study $\mathbb{R}^{2n}$ suitably embedded in the Heisenberg group $\mathbb{H}^n$ for $n \geq 1$, which pushes the analysis from Example 11 and Example 16 to all possible dimensions, with the exception we do not provide a full expression for the intrinsic sub-Laplacian. Instead, we restrict our attention to its radial contribution.

We then proceed by considering the sphere S^{2n} embedded in S^{2n+1} equipped with the standard sub-Riemannian contact structure subject to an additional parameter $k > 0$, and the hyperboloid $\widetilde{H}^{2n}$ embedded in H^{2n+1} equipped with the standard sub-Riemannian contact structure subject to an additional parameter $k > 0$.

4.1. $\mathbb{R}^{2n}$ embedded in $\mathbb{R}^{2n+1}$

Let $(x_1, \ldots, x_{2n}, x_{2n+1})$ be Cartesian coordinates on $\mathbb{R}^{2n+1}$. Use the contact form ω on $\mathbb{R}^{2n+1}$ given by

$$\omega = \frac{1}{2} \sum_{m=1}^{n} \left(x_{2m-1} \mathrm{d}x_{2m} - x_{2m} \mathrm{d}x_{2m-1} \right) - \mathrm{d}x_{2n+1} \tag{23}$$

to define the contact structure $\mathcal{D} = \ker \omega$ on $\mathbb{R}^{2n+1}$. As fibre inner product g on $\mathcal{D}$, we take

$$g = \sum_{i=1}^{2n} \mathrm{d}x_i \otimes \mathrm{d}x_i. \tag{24}$$

This is the unique fibre inner product on the distribution $\mathcal{D}$ such that the vector fields, for $m \in \{1, \ldots, n\}$,

$$X_{2m-1} = \frac{\partial}{\partial x_{2m-1}} - \frac{x_{2m}}{2} \frac{\partial}{\partial x_{2n+1}}, \quad X_{2m} = \frac{\partial}{\partial x_{2m}} + \frac{x_{2m-1}}{2} \frac{\partial}{\partial x_{2n+1}}$$

form an orthonormal frame $(X_1, \ldots, X_{2n})$ for $\mathcal{D}$.

We obtain the contact sub-Riemannian manifold $(\mathbb{R}^{2n+1}, \mathcal{D}, g)$, which is referred to as Heisenberg group $\mathbb{H}^n$. The contact form ω given in (23) satisfies the imposed normalisation condition (2) because

$$\mathrm{d}\omega = \sum_{m=1}^{n} \mathrm{d}x_{2m-1} \wedge \mathrm{d}x_{2m}$$

gives rise to

$$(\mathrm{d}\omega)^n = n! \bigwedge_{i=1}^{2n} \mathrm{d}x_i,$$

whilst (24) implies that

$$\mathrm{vol}_g = \bigwedge_{i=1}^{2n} \mathrm{d}x_i.$$

We further deduce that the volume form Ω on $\mathbb{R}^{2n+1}$ defined by (1) can be expressed as

$$\Omega = -n! \bigwedge_{i=1}^{2n+1} \mathrm{d}x_i. \tag{25}$$

The hypersurface S in $\mathbb{H}^n$ which we study closer is the one given by $\{x_{2n+1} = 0\}$. Since, for $m \in \{1, \ldots, n\}$, we have

$$\omega\left(\frac{\partial}{\partial x_{2m-1}}\right) = -\frac{x_{2m}}{2} \quad \text{and} \quad \omega\left(\frac{\partial}{\partial x_{2m}}\right) = \frac{x_{2m-1}}{2},$$

the set $C(S)$ of characteristic points contains only the origin of $\mathbb{R}^{2n+1}$. Moreover, the quasi-contact form ζ induced on $S \setminus C(S)$ by the contact form ω on $\mathbb{R}^{2n+1}$ is

$$\zeta = \frac{1}{2} \sum_{m=1}^{n} \left(x_{2m-1}\mathrm{d}x_{2m} - x_{2m}\mathrm{d}x_{2m-1}\right).$$

Due to the kernel $\ker \mathrm{d}\zeta|_{\mathcal{F}}$ with $\mathcal{F} = \ker \zeta$ being guaranteed to be one-dimensional by Lemma 9, we can verify directly that

$$\ker \mathrm{d}\zeta|_{\mathcal{F}} = \mathrm{span}\left\{\sum_{i=1}^{2n} x_i \frac{\partial}{\partial x_i}\right\}. \tag{26}$$

The lemma stated below provides an expression for the sub-Riemannian normal vector field N along $S \setminus C(S)$ in $\mathbb{H}^n$, which we prove in detail as a similar approach can be used to confirm the expressions for the sub-Riemannian normal vector fields in Sects. 4.2 and 4.3.

Lemma 17. *The sub-Riemannian normal vector field N along $S \setminus C(S)$ in $\mathbb{H}^n$ is given by*

$$N = \frac{1}{\sqrt{\sum_{i=1}^{2n} x_i^2}} \left(\sum_{m=1}^{n} \left(x_{2m}\frac{\partial}{\partial x_{2m-1}} - x_{2m-1}\frac{\partial}{\partial x_{2m}}\right) - \frac{1}{2}\sum_{i=1}^{2n} x_i^2 \frac{\partial}{\partial x_{2n+1}}\right).$$

Proof. Since the vector field N is well-defined along the hypersurface S away from the unique characteristic point at the origin of $\mathbb{R}^{2n+1}$, it remains to check that N satisfies the defining properties (3) as well as (4) and that it is compatible with the orientations.

From the expressions for the contact form ω in (23) and the fibre inner product g in (24), it follows that

$$\omega(N) = 0 \quad \text{and} \quad g(N, N) = 1.$$

Furthermore, using that any vector field Y in the distribution $\mathcal{F}$ satisfies $\zeta(Y)=0$, we deduce

$$g(N,Y) = -\frac{2\zeta(Y)}{\sqrt{\sum_{i=1}^{2n} x_i^2}} = 0.$$

Finally, we obtain from (25) that $\iota_N\Omega$ is positive on $S\setminus C(S)$, which shows that N is indeed the sub-Riemannian normal vector field along $S\setminus C(S)$ in $\mathbb{H}^n$ according to Definition 1. □

Using the expression (25) for the volume form Ω on $\mathbb{R}^{2n+1}$ as well as Lemma 17, we compute that the volume form μ defined on $S\setminus C(S)$ as $\iota_N\Omega$ is given by

$$\mu = \frac{n!}{2}\sqrt{\sum_{i=1}^{2n} x_i^2}\bigwedge_{i=1}^{2n} \mathrm{d}x_i\ . \tag{27}$$

At this point, it is convenient to change from Cartesian coordinates $(x_1,\ldots,x_{2n})$ for $S\setminus C(S)$ to spherical coordinates $(r,\varphi_1,\ldots,\varphi_{2n-1})$ with $r>0$, $\varphi_1,\ldots,\varphi_{2n-2}\in[0,\pi]$ and $\varphi_{2n-1}\in[0,2\pi)$, where

$$x_i = r\cos(\varphi_i)\prod_{l=1}^{i-1}\sin(\varphi_l)\quad \text{for } i\in\{1,\ldots,2n-1\},$$
$$x_{2n} = r\prod_{l=1}^{2n-1}\sin(\varphi_l).$$

By means of induction over $n\geq 1$, it can be shown explicitly that the determinant of the associated Jacobian matrix J_{2n} equals

$$\det J_{2n} = r^{2n-1}\prod_{i=1}^{2n-2}\left(\sin(\varphi_i)\right)^{2n-i-1}.$$

Since we further know that

$$\bigwedge_{i=1}^{2n}\mathrm{d}x_i = \det J_{2n}\,\mathrm{d}r\wedge\bigwedge_{i=1}^{2n-1}\mathrm{d}\varphi_i \quad\text{and}\quad \sqrt{\sum_{i=1}^{2n}x_i^2} = r,$$

the expression for the volume form μ on $S\setminus C(S)$ in $\mathbb{H}^n$ stated in Proposition 6 follows from (27).

We close by analysing the radial part of the stochastic process with generator $\frac{1}{2}\Delta$ on $S\setminus C(S)$. Using (24), (26) and

$$\frac{\partial}{\partial r} = \sum_{i=1}^{2n}\frac{\partial x_i}{\partial r}\frac{\partial}{\partial x_i} = \frac{1}{\sqrt{\sum_{i=1}^{2n}x_i^2}}\sum_{i=1}^{2n}x_i\frac{\partial}{\partial x_i},$$

we obtain that

$$\ker \mathrm{d}\zeta|_{\mathcal{F}} = \operatorname{span}\left\{\frac{\partial}{\partial r}\right\}\quad\text{as well as}\quad g\left(\frac{\partial}{\partial r},\frac{\partial}{\partial r}\right) = 1.$$

Thus, the vector field $R = \frac{\partial}{\partial r}$ defined on $S\backslash C(S)$ is a unit-length representative of the characteristic foliation induced on the hypersurface S by the contact structure $\mathcal{D}$. We compute

$$\mathrm{div}_\mu(R) = \frac{2n}{r},$$

which implies that the radial part of the stochastic process with generator $\frac{1}{2}\Delta$ on $S \setminus C(S)$ is the one-dimensional diffusion process on $(0,\infty)$ with generator

$$\frac{1}{2}\frac{\partial^2}{\partial r^2} + \frac{n}{r}\frac{\partial}{\partial r}.$$

This indeed gives rise to a Bessel process of order $2n+1$, which proves the first part of Theorem 7. Since a Bessel process of order $2n+1$ for all $n \geq 1$ almost surely neither hits the origin nor explodes in finite time, it follows that the intrinsic sub-Laplacian Δ on $S \setminus C(S)$ is stochastically complete. On the other hand, the geometry induced on the hypersurface $S \setminus C(S)$ is not geodesically complete because rays emanating from the characteristic point and parameterised by arc length are geodesics which cannot be extended indefinitely towards the characteristic point not included in the underlying space.

Remark 18. Taking $n = 1$, we recover the analysis for the plane $\{x_3 = 0\}$ in the Heisenberg group $\mathbb{H}^1$ which arises from [3, Sect. 4.1] by considering $a = 0$, with the contact forms differing by a sign as a result of the normalisation conditions differing by a sign.

4.2. S^{2n} embedded in S^{2n+1}

In terms of Cartesian coordinates $(x_1, \dots, x_{2n+2})$ for $\mathbb{R}^{2n+2}$, we take the sphere $S^{2n+1} \subset \mathbb{R}^{2n+2}$ to be

$$S^{2n+1} = \left\{ (x_1, \dots, x_{2n+2}) \in \mathbb{R}^{2n+2} : \sum_{i=1}^{2n+2} x_i^2 = 1 \right\}.$$

Fix $k \in \mathbb{R}$ positive and consider the contact form ω on the sphere S^{2n+1} given by

$$\omega = \frac{1}{2k^2} \sum_{m=1}^{n+1} \left(x_{2m-1} \mathrm{d}x_{2m} - x_{2m} \mathrm{d}x_{2m-1} \right). \tag{28}$$

We further equip the contact structure $\mathcal{D} = \ker \omega$ on S^{2n+1} with a smooth fibre inner product g obtained by restricting a positive constant multiple of the Euclidean inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{2n+2}$. More precisely, we set, for vector fields X_1 and X_2 in $\mathcal{D}$,

$$g(X_1, X_2) = \frac{1}{k^2} \langle X_1, X_2 \rangle.$$

This construction gives rise to the standard sub-Riemannian contact structure $(\mathcal{D}, g)$ on S^{2n+1} with an additional parameter $k > 0$ which mimics the introduction of an additional scalar in [3, Sect. 5.1] and which later allows us to recover all Legendre processes of order $2n + 1$.

It follows from the following considerations that the choice (28) of contact form ω is in line with the normalisation condition (2). We compute

$$\mathrm{d}\omega = \frac{1}{k^2}\sum_{m=1}^{n+1} \mathrm{d}x_{2m-1}\wedge \mathrm{d}x_{2m}$$

as well as

$$(\mathrm{d}\omega)^n = \frac{n!}{k^{2n}}\sum_{m=1}^{n+1} \bigwedge_{\substack{l=1\\ l\neq 2m-1,2m}}^{2n+2} \mathrm{d}x_l,$$

which implies that the volume form Ω on S^{2n+1} defined by (1) takes the form

$$\Omega = \frac{n!}{2k^{2n+2}}\sum_{i=1}^{2n+2}(-1)^{i-1}x_i \bigwedge_{\substack{l=1\\ l\neq i}}^{2n+2} \mathrm{d}x_l. \tag{29}$$

On the other hand, the volume form on Euclidean space $\mathbb{R}^{2n+2}$ with respect to the inner product $\frac{1}{k^2}\langle\cdot,\cdot\rangle$ can be expressed as

$$\frac{1}{k^{2n+2}}\bigwedge_{i=1}^{2n+2}\mathrm{d}x_i.$$

Since $(kx_1,\dots,kx_{2n+2})$ is the unit normal vector at $(x_1,\dots,x_{2n+2})\in S^{2n+1}$ for the inner product $\frac{1}{k^2}\langle\cdot,\cdot\rangle$, the above volume form on $\mathbb{R}^{2n+2}$ induces the volume form $\mathrm{vol}_k^{S^{2n+1}}$ on the sphere S^{2n+1} given by

$$\mathrm{vol}_k^{S^{2n+1}} = \frac{1}{k^{2n+1}}\sum_{i=1}^{2n+2}(-1)^{i-1}x_i \bigwedge_{\substack{l=1\\ l\neq i}}^{2n+2} \mathrm{d}x_l. \tag{30}$$

To restrict the volume form $\mathrm{vol}_k^{S^{2n+1}}$ to the contact structure $\mathcal{D}$, we use the vector field $\hat{X}_0$ defined by

$$\hat{X}_0 = k\sum_{m=1}^{n+1}\left(x_{2m-1}\frac{\partial}{\partial x_{2m}} - x_{2m}\frac{\partial}{\partial x_{2m-1}}\right), \tag{31}$$

which is the positive constant multiple of the Reeb vector field X_0 such that

$$\frac{1}{k^2}\langle\hat{X}_0,\hat{X}_0\rangle = 1.$$

To establish that the contact form ω indeed satisfies the normalisation condition (2), it remains to observe that (29), (30) and (31) imply

$$(\mathrm{d}\omega)^n|_{\mathcal{D}} = \frac{1}{\omega(\hat{X}_0)}\iota_{\hat{X}_0}\Omega = n!\,\iota_{\hat{X}_0}\mathrm{vol}_k^{S^{2n+1}} = n!\,\mathrm{vol}_g\,.$$

In the contact sub-Riemannian manifold $(S^{2n+1}, \mathcal{D}, g)$ with parameter $k > 0$, we study the hypersurface S given by $\{x_{2n+2} = 0\}$. Phrased differently, we choose

$$S = \left\{ (x_1, \ldots, x_{2n+2}) \in S^{2n+1} : \sum_{i=1}^{2n+1} x_i^2 = 1 \right\},$$

which shows that the hypersurface S can be identified with the sphere S^{2n}. The set $C(S)$ of characteristic points contains exactly the two poles given by

$$x_1 = x_2 = \cdots = x_{2n} = 0 \quad \text{and} \quad x_{2n+1} = \pm 1,$$

and the contact form ω on S^{2n+1} induces the quasi-contact form ζ on $S \setminus C(S)$ defined by

$$\zeta = \frac{1}{2k^2} \sum_{m=1}^{n} \left(x_{2m-1} \mathrm{d}x_{2m} - x_{2m} \mathrm{d}x_{2m-1} \right). \tag{32}$$

Using the approach demonstrated in the proof of Lemma 17, one verifies that the sub-Riemannian normal vector field N along $S \setminus C(S)$ in S^{2n+1} can be expressed as

$$N = \frac{k}{\sqrt{\sum_{i=1}^{2n} x_i^2}} \left(\sum_{m=1}^{n} \left(x_{2m} x_{2n+1} \frac{\partial}{\partial x_{2m-1}} - x_{2m-1} x_{2n+1} \frac{\partial}{\partial x_{2m}} \right) + \sum_{i=1}^{2n} x_i^2 \frac{\partial}{\partial x_{2n+2}} \right).$$

This allows us to prove the next result.

Lemma 19. *The volume form μ defined on $S \setminus C(S)$ as $\iota_N \Omega$ is given by*

$$\mu = \frac{n!}{2k^{2n+1}} \sqrt{\sum_{i=1}^{2n} x_i^2} \sum_{i=1}^{2n+1} (-1)^{i-1} x_i \bigwedge_{\substack{l=1 \\ l \neq i}}^{2n+1} \mathrm{d}x_l.$$

Proof. Since the hypersurface S is defined by $\{x_{2n+2} = 0\}$ in S^{2n+1}, the interior product $\iota_N \Omega$ on $S \setminus C(S)$ simplifies to

$$\begin{aligned} \iota_N \Omega &= \iota_N \left(\frac{n!}{2k^{2n+2}} \sum_{i=1}^{2n+1} (-1)^{i-1} x_i \bigwedge_{\substack{l=1 \\ l \neq i}}^{2n+2} \mathrm{d}x_l \right) \\ &= \frac{n!}{2k^{2n+2}} \sum_{i=1}^{2n+1} (-1)^{i-1} x_i \left(\iota_N \mathrm{d}x_{2n+2} \right) \bigwedge_{\substack{l=1 \\ l \neq i}}^{2n+1} \mathrm{d}x_l. \end{aligned}$$

Due to

$$\iota_N \mathrm{d}x_{2n+2} = k \sqrt{\sum_{i=1}^{2n} x_i^2}$$

the claimed result follows. □

To show that Lemma 19 gives rise to the expression for the volume form μ stated in Proposition 6, it remains to change to spherical coordinates $(r, \varphi_1, \ldots, \varphi_{2n-1})$ for $S \setminus C(S)$ where $r \in (0, \frac{\pi}{k})$, $\varphi_1, \ldots, \varphi_{2n-2} \in [0, \pi]$ and $\varphi_{2n-1} \in [0, 2\pi)$ are such that

$$\begin{aligned} x_i &= \sin(kr)\cos(\varphi_i)\prod_{l=1}^{i-1}\sin(\varphi_l) \quad \text{for } i \in \{1, \ldots, 2n-1\}, \\ x_{2n} &= \sin(kr)\prod_{l=1}^{2n-1}\sin(\varphi_l), \\ x_{2n+1} &= \cos(kr). \end{aligned}$$

Using $\sin(kr) > 0$ for $r \in (0, \frac{\pi}{k})$, we obtain

$$\sqrt{\sum_{i=1}^{2n} x_i^2} = \sin(kr).$$

Comparing the expressions in Cartesian coordinates and in spherical coordinates for the volume form of a $2n$-dimensional Euclidean sphere, we deduce that

$$\sum_{i=1}^{2n+1}(-1)^{i-1}x_i \bigwedge_{\substack{l=1\\ l\neq i}}^{2n+1} \mathrm{d}x_l$$

can be written as

$$k\left(\sin(kr)\right)^{2n-1}\left(\prod_{i=1}^{2n-2}\left(\sin(\varphi_i)\right)^{2n-i-1}\right)\mathrm{d}r \wedge \mathrm{d}\varphi_1 \wedge \cdots \wedge \mathrm{d}\varphi_{2n-1}.$$

The result claimed in Proposition 6 for $(S^{2n+1}, \mathcal{D}, g)$ with parameter $k > 0$ then follows from Lemma 19.

In the last part, we discuss the random dynamics induced by the operator $\frac{1}{2}\Delta$ on the characteristic foliation of S. We have, with $\mathcal{F} = \ker\zeta$,

$$\ker \mathrm{d}\zeta|_{\mathcal{F}} = \operatorname{span}\left\{\frac{\partial}{\partial r}\right\},$$

which is implied by (32) and

$$\frac{\partial}{\partial r} = \sum_{i=1}^{2n+1}\frac{\partial x_i}{\partial r}\frac{\partial}{\partial x_i} = \frac{k}{\sqrt{\sum_{i=1}^{2n} x_i^2}}\left(\sum_{i=1}^{2n} x_i x_{2n+1}\frac{\partial}{\partial x_i} - \sum_{i=1}^{2n} x_i^2\frac{\partial}{\partial x_{2n+1}}\right).$$

We further compute

$$g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = \frac{1}{\sum_{i=1}^{2n} x_i^2}\left(\sum_{i=1}^{2n} x_i^2 x_{2n+1}^2 + \left(\sum_{i=1}^{2n} x_i^2\right)^2\right) = x_{2n+1}^2 + \sum_{i=1}^{2n} x_i^2 = 1,$$

showing that the representative $R = \frac{\partial}{\partial r}$ on $S \setminus C(S)$ of the characteristic foliation of S is a unit-length vector field. From Proposition 6, we obtain

$$\operatorname{div}_\mu(R) = \frac{2nk\cos(kr)}{\sin(kr)} = 2nk\cot(kr).$$

This establishes the second part of Theorem 7 that the latitudinal process between the two characteristic points on S of the stochastic process with generator $\frac{1}{2}\Delta$ on $S \setminus C(S)$ follows the one-dimensional diffusion process on $(0, \frac{\pi}{k})$ with generator

$$\frac{1}{2}\frac{\partial^2}{\partial r^2} + nk\cot(kr)\frac{\partial}{\partial r},$$

that is, a Legendre process of order $2n+1$ and with parameter $k > 0$. Similarly to the observations made in Sect. 4.1, as a Legendre process of order $2n+1$ for $n \geq 1$ almost surely hits neither endpoint of the interval $(0, \frac{\pi}{k})$, we deduce that the intrinsic sub-Laplacian Δ on $S \setminus C(S)$ is stochastically complete, whilst the geometry induced on $S \setminus C(S)$ is not geodesically complete.

Remark 20. For $n = 1$, the analysis presented above is in line with the discussions in [3, Sect. 5.1] for the sphere S^2 embedded in $\mathrm{SU}(2)$, which is isomorphic to S^3, equipped with the standard sub-Riemannian contact structure.

4.3. $\widetilde{H}^{2n}$ embedded in H^{2n+1}

Our construction closely mimics the hyperboloid model for hyperbolic space. Let $(x_1, \ldots, x_{2n+2})$ denote Cartesian coordinates on $\mathbb{R}^{2n+2}$ and consider the $(2n+1)$-dimensional hyperboloid H^{2n+1} defined as

$$H^{2n+1} = \left\{ (x_1, \ldots, x_{2n+2}) \in \mathbb{R}^{2n+2} : \sum_{i=1}^{2n} x_i^2 - x_{2n+1}^2 - x_{2n+2}^2 = -1 \right\}.$$

Fix $k \in \mathbb{R}$ positive and let η be the Lorentzian metric on $\mathbb{R}^{2n,2}$ given by

$$\eta = \sum_{i=1}^{2n} \mathrm{d}x_i \otimes \mathrm{d}x_i - \mathrm{d}x_{2n+1} \otimes \mathrm{d}x_{2n+1} - \mathrm{d}x_{2n+2} \otimes \mathrm{d}x_{2n+2}.$$

Using the contact form ω on the anti-de Sitter space H^{2n+1} defined by

$$\omega = \frac{1}{2k^2} \sum_{m=1}^{n+1} \left(x_{2m-1}\mathrm{d}x_{2m} - x_{2m}\mathrm{d}x_{2m-1} \right), \tag{33}$$

we get the contact structure $\mathcal{D} = \ker\omega$ on H^{2n+1} which we equip with the smooth fibre inner product g obtained by setting, for vector fields X_1 and X_2 in $\mathcal{D}$,

$$g(X_1, X_2) = \frac{1}{k^2}\eta(X_1, X_2) \tag{34}$$

This yields the standard sub-Riemannian contact structure $(\mathcal{D}, g)$ on H^{2n+1} subject to an additional parameter $k > 0$. Note that (34) indeed defines a smooth fibre inner product g on the contact structure $\mathcal{D}$ because the Reeb vector field X_0 on H^{2n+1} given by

$$X_0 = 2k^2 \left(x_{2n+1} \frac{\partial}{\partial x_{2n+2}} - x_{2n+2} \frac{\partial}{\partial x_{2n+1}} - \sum_{m=1}^{n} \left(x_{2m-1} \frac{\partial}{\partial x_{2m}} - x_{2m} \frac{\partial}{\partial x_{2m-1}} \right) \right)$$

is timelike due to

$$\eta(X_0, X_0) = -4k^4,$$

which implies that the distribution $\mathcal{D}$ is spanned by spacelike vector fields.

As in Sect. 4.2, the volume form Ω on H^{2n+1} given by (1) can be expressed as

$$\Omega = \frac{n!}{2k^{2n+2}} \sum_{i=1}^{2n+2} (-1)^{i-1} x_i \bigwedge_{\substack{l=1 \\ l \neq i}}^{2n+2} \mathrm{d}x_l,$$

and one can show as before that the choice (33) for the contact form ω satisfies the normalisation condition (2).

Many of the subsequent computations are similar to the computations performed in Sect. 4.2, but because the sub-Riemannian metric in this section is obtained by restricting a Lorentzian metric we choose to treat these two families of model cases separately for clarity.

The hypersurface S in $(H^{2n+1}, \mathcal{D}, g)$ which we study below is the upper sheet of the hypersurface given by $\{x_{2n+2} = 0\}$ or, phrased differently,

$$S = \left\{ (x_1, \ldots, x_{2n+2}) \in H^{2n+1} : \sum_{i=1}^{2n} x_i^2 - x_{2n+1}^2 = -1, \;\; x_{2n+1} > 0 \right\},$$

that is, S can be identified with the upper sheet of a $2n$-dimensional two-sheeted hyperboloid. The hypersurface S has a unique characteristic point given by

$$x_1 = x_2 = \cdots = x_{2n} = 0 \quad \text{and} \quad x_{2n+1} = 1,$$

and inherits from the contact form ω on H^{2n+1} the quasi-contact form ζ on $S \setminus C(S)$ defined by

$$\zeta = \frac{1}{2k^2} \sum_{m=1}^{n} \left(x_{2m-1} \mathrm{d}x_{2m} - x_{2m} \mathrm{d}x_{2m-1} \right). \tag{35}$$

Analogous to Sect. 4.2, the sub-Riemannian normal vector field N along the hypersurface $S \setminus C(S)$ in H^{2n+1} can be written as

$$N = \frac{k}{\sqrt{\sum_{i=1}^{2n} x_i^2}} \left(\sum_{m=1}^{n} \left(x_{2m} x_{2n+1} \frac{\partial}{\partial x_{2m-1}} - x_{2m-1} x_{2n+1} \frac{\partial}{\partial x_{2m}} \right) + \sum_{i=1}^{2n} x_i^2 \frac{\partial}{\partial x_{2n+2}} \right)$$

and the volume form μ defined on $S \setminus C(S)$ as $\iota_N \Omega$ takes the form

$$\mu = \frac{n!}{2k^{2n+1}} \sqrt{\sum_{i=1}^{2n} x_i^2} \sum_{i=1}^{2n+1} (-1)^{i-1} x_i \bigwedge_{\substack{l=1 \\ l \neq i}}^{2n+1} \mathrm{d}x_l. \tag{36}$$

In terms of the spherical coordinates $(r, \varphi_1, \ldots, \varphi_{2n-1})$ for $S \setminus C(S)$ with $r > 0$, $\varphi_1, \ldots, \varphi_{2n-2} \in [0, \pi]$ and $\varphi_{2n-1} \in [0, 2\pi)$, where

$$\begin{aligned} x_i &= \sinh(kr) \cos(\varphi_i) \prod_{l=1}^{i-1} \sin(\varphi_l) \quad \text{for } i \in \{1, \ldots, 2n-1\}, \\ x_{2n} &= \sinh(kr) \prod_{l=1}^{2n-1} \sin(\varphi_l), \\ x_{2n+1} &= \cosh(kr), \end{aligned}$$

we have

$$\sqrt{\sum_{i=1}^{2n} x_i^2} = \sinh(kr).$$

By further observing that

$$\sum_{i=1}^{2n+1} (-1)^{i-1} x_i \bigwedge_{\substack{l=1 \\ l \neq i}}^{2n+1} \mathrm{d}x_l$$

and

$$k \left(\sinh(kr)\right)^{2n-1} \left(\prod_{i=1}^{2n-2} \left(\sin(\varphi_i)\right)^{2n-i-1} \right) \mathrm{d}r \wedge \mathrm{d}\varphi_1 \wedge \cdots \wedge \mathrm{d}\varphi_{2n-1}$$

are expressions for the volume form on a $2n$-dimensional hyperboloid in Cartesian coordinates and in spherical coordinates, respectively, we deduce from (36) that the volume form μ on $S \setminus C(S)$ embedded in $(H^{2n+1}, \mathcal{D}, g)$ with parameter $k > 0$ can be written as stated in Proposition 6. This concludes the proof of Proposition 6.

We further obtain

$$\frac{\partial}{\partial r} = \sum_{i=1}^{2n+1} \frac{\partial x_i}{\partial r} \frac{\partial}{\partial x_i} = \frac{k}{\sqrt{\sum_{i=1}^{2n} x_i^2}} \left(\sum_{i=1}^{2n} x_i x_{2n+1} \frac{\partial}{\partial x_i} + \sum_{i=1}^{2n} x_i^2 \frac{\partial}{\partial x_{2n+1}} \right),$$

which together with (35) implies that, for $\mathcal{F} = \ker \zeta$,

$$\ker \mathrm{d}\zeta|_{\mathcal{F}} = \operatorname{span}\left\{ \frac{\partial}{\partial r} \right\}.$$

Due to the fibre inner product g on the contact structure $\mathcal{D}$ arising by restricting a positive constant multiple of the Lorentzian metric η on $\mathbb{R}^{2n,2}$, we have

$$g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = \frac{1}{\sum_{i=1}^{2n} x_i^2}\left(\sum_{i=1}^{2n} x_i^2 x_{2n+1}^2 - \left(\sum_{i=1}^{2n} x_i^2\right)^2\right) = x_{2n+1}^2 - \sum_{i=1}^{2n} x_i^2 = 1.$$

It follows that the representative $R = \frac{\partial}{\partial r}$ on $S \setminus C(S)$ of the characteristic foliation of S is a vector field of unit length. Using Proposition 6, we compute

$$\operatorname{div}_\mu(R) = \frac{2nk\cosh(kr)}{\sinh(kr)} = 2nk\coth(kr),$$

which shows that the radial part of the stochastic process with generator $\frac{1}{2}\Delta$ on $S \setminus C(S)$ is the one-dimensional diffusion process on $(0, \infty)$ with generator

$$\frac{1}{2}\frac{\partial^2}{\partial r^2} + nk\coth(kr)\frac{\partial}{\partial r}.$$

Since this yields a hyperbolic Bessel process of order $2n + 1$ with parameter $k > 0$ that completes the proof of Theorem 7. Moreover, as in Sects. 4.1 and 4.2, we conclude that the intrinsic sub-Laplacian Δ on $S \setminus C(S)$ is stochastically complete despite the geometry induced on $S \setminus C(S)$ not being geodesically complete.

Remark 21. Since the Lie group $\mathrm{SL}(2, \mathbb{R})$ is isomorphic to the hyperboloid given by $x_1^2+x_2^2-x_3^2-x_4^2 = -1$, see Wang [27, Remark 2.1], we obtain the results from [3, Sect. 5.2] by taking $n = 1$ in the above analysis. Chang, Markina and Vasil'ev [11] study further properties of the sub-Riemannian structure on the anti-de Sitter space H^3 as well as of a sub-Lorentzian structure on H^3.

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Declarations

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