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Nonlinear Differential Equations and Applications NoDEA

The bounded slope condition for parabolic equations with time-dependent integrands

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Abstract. In this paper, we study the Cauchy–Dirichlet problem

$$\begin{cases} \partial_t u - \operatorname{div}\left(D_\xi f(t, Du)\right) = 0 & \text{in } \Omega_T, \\ u = u_o & \text{on } \partial_{\mathcal{P}}\Omega_T, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a convex and bounded domain, $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is L^1-integrable in time and convex in the second variable. Assuming that the initial and boundary datum $u_o : \overline{\Omega} \to \mathbb{R}$ satisfies the bounded slope condition, we prove the existence of a unique variational solution that is Lipschitz continuous in the space variable.

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1. Introduction and results

It follows from classical theory [15, 17, 18, 26, 29] (see also [14, Chapter 1]) that any variational functional $F : W^{1,\infty}(\Omega) \to \mathbb{R}$ of the form

$$F(v) := \int_\Omega f(Dv)\,\mathrm{d}x,$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is convex and $\Omega \subset \mathbb{R}^n$ is a convex domain, admits a unique Lipschitz continuous minimizer in the class $\{v \in W^{1,\infty}(\Omega) : v = v_o \text{ on } \partial\Omega\}$ provided that the boundary datum v_o satisfies the bounded slope condition (see Definition 2.1). Modern elliptic results involving one-sided bounded slope conditions or more general integrands include for example [2–4, 10, 13, 22–24].

Surprisingly, while Hardt and Zhou [16, Chapter 4] used the bounded slope condition in a regularity argument in a time-dependent setting involving functionals with linear growth, an evolutionary analogue of the above stationary theorem was established only rather recently by Bögelein, Duzaar, Marcellini and Signoriello [7]. They considered the Cauchy–Dirichlet problem

$$\begin{cases} \partial_t u - \operatorname{div}(Df(Du)) = 0 & \text{in} \quad \Omega_T, \\ u = u_o & \text{on} \quad \partial_{\mathcal{P}}\Omega_T, \end{cases}$$

where $\Omega_T := \Omega \times (0, T)$ with $\Omega \subset \mathbb{R}^n$ and $T \in (0, \infty]$ denotes a space-time cylinder and $\partial_{\mathcal{P}}\Omega_T := \partial\Omega \times (0, T) \cup (\overline{\Omega} \times \{0\})$ its parabolic boundary. Given a Lipschitz continuous initial and boundary datum u_o that satisfies the bounded slope condition, in [7] it was proven that the above problem admits a unique variational solution that is globally Lipschitz continuous with respect to the spatial variables. Moreover, if the integrand f fulfills an additional p-coercivity condition with some $p > 1$, Bögelein and Stanin [8] obtained the local Lipschitz continuity of variational solutions in space and time under the assumption that u_o is convex and Lipschitz continuous. Further, global continuity of u was proven in the case that Ω is uniformly convex.

For the same class of integrands and merely convex domains Ω, Stanin [30] showed that variational solutions are still globally Hölder continuous even if the convexity assumption on u_o is dropped. Equations with lower-order terms were considered by Rainer, Siltakoski and Stanin [27] who extended a stationary Haar-Rado type theorem by Mariconda and Treu [24] to the parabolic problem

$$\begin{cases} \partial_t u - \operatorname{div}(Df(Du)) + D_u g(x, u) = 0 & \text{in} \quad \Omega_T, \\ u = u_o & \text{on} \quad \partial_{\mathcal{P}}\Omega_T, \end{cases}$$

where f is convex and p-coercive with some $p > 1$ and the lower-order term g satisfies a technical condition, in particular convexity with respect to u. As a corollary, the authors in [27] obtained the global Lipschitz continuity with respect to the spatial variables of variational solutions under the classical two-sided bounded slope condition provided that $f \in C^2$ is uniformly convex in a suitable sense.

Existence and regularity of solutions under general growth conditions, such as the so called $p-q$-growth conditions, have been recently considered by many authors, see for example [21,25] and the references therein. We emphasize that in the present manuscript, because of the bounded slope condition, no special growth conditions are imposed on the elliptic part of the operator.

The objective of the present paper is to extend the result of [7] to include time-dependent integrands. In order to focus on the novelty and to include integrands f with linear growth, we consider the classical bounded slope condition and avoid lower-order terms. We are concerned with parabolic partial differential equations of the form

$$\partial_t u - \operatorname{div}(D_\xi f(t, Du)) = 0 \quad \text{in } \Omega_T, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a convex and bounded domain and $T \in (0, \infty]$. The integrand $f \colon [0, T] \times \mathbb{R}^n \to \mathbb{R}$ is assumed to be a Carathéodory function that satisfies the following assumptions:

$$\begin{cases} \xi \mapsto f(t, \xi) \text{ is convex in } \mathbb{R}^n & \text{for a.e. } t \in [0, T], \\ t \mapsto f(t, \xi) \in L^1(0, \tau) & \text{for all } \xi \in \mathbb{R}^n \text{ and } \tau \in (0, T] \cap \mathbb{R}. \end{cases} \tag{1.2}$$

In particular, for any $L > 0$ and $\tau \in (0, T] \cap \mathbb{R}$ the map $t \mapsto \max_{|\xi| \le L} |f(t, \xi)|$ belongs to $L^1(0, \tau)$ (see Sect. 2.3 below). Therefore, for any $\tau \in (0, T] \cap \mathbb{R}$ and $V \in L^\infty(\Omega_T, \mathbb{R}^n)$ we have that

$$\iint_{\Omega_T} |f(t,V)| \,\mathrm{d}x\mathrm{d}t < \infty.$$

We emphasize that $t \mapsto f(t,\xi)$ is neither assumed to be continuous nor weakly differentiable.

Examples of admissible integrands are functionals with linear growth such as the area integrand $f(\xi) = \sqrt{1+|\xi|^2}$, integrands with exponential growth like $f(\xi) = \exp(|\xi|^2)$, Orlicz type functionals such as $f(\xi) = |\xi|\log(1+|\xi|)$ and time-dependent combinations thereof like $f(t,\xi) = \chi_{[0,t_o]} f_1(\xi) + \chi_{(t_o,T]} f_2(\xi)$ or more general $f(t,\xi) = \sum_{i=1}^m a_i(t) f_i(\xi)$ for functions $a_i \in L^1(0,T)$, $i = 1,\dots,m$.

In the present paper, we define variational solutions in the same way as in [5]. This notion of solution, inspired by Lichnewsky and Temam [20], was introduced by Bousquet [2,3] in the time-independent setting. We consider the following class of functions that are Lipschitz continuous in space

$$K^\infty := \{v \in L^\infty(\Omega_T) \cap C^0([0,T]; L^2(\Omega)) : Dv \in L^\infty(\Omega_T, \mathbb{R}^n)\}.$$

Further, we denote the subclass related to time-independent boundary values $u_o \in W^{1,\infty}(\Omega)$ by

$$K^\infty_{u_o} := \{v \in K^\infty(\Omega_T) : v = u_o \text{ on the lateral boundary } \partial\Omega \times (0,T)\}.$$

Definition 1.1. (*Variational solutions*) Assume that $f\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}$ satisfies (1.2) and consider a boundary datum $u_o \in W^{1,\infty}(\Omega)$. In the case $T \in (0,\infty)$ a map $u \in K^\infty_{u_o}(\Omega_T)$ is called a *variational solution* to the Cauchy–Dirichlet problem associated with (1.1) and u_o in Ω_T if and only if the variational inequality

$$\iint_{\Omega_T} f(t,Du)\,\mathrm{d}x\mathrm{d}t \le \iint_{\Omega_T} \partial_t v(v-u) + f(t,Dv)\,\mathrm{d}x\mathrm{d}t + \tfrac{1}{2}\|v(0)-u_o\|^2_{L^2(\Omega)} - \tfrac{1}{2}\|(v-u)(T)\|^2_{L^2(\Omega)} \qquad (1.3)$$

holds true for any comparison map $v \in K^\infty_{u_o}(\Omega_T)$ with $\partial_t v \in L^2(\Omega_T)$. If $T = \infty$ and $u \in K^\infty_{u_o}(\Omega_\infty)$ is a variational solution in Ω_τ for any $\tau \in (0,\infty)$, u is called a *global variational solution* or variational solution in Ω_∞ to the Cauchy–Dirichlet problem associated with (1.1) and u_o.

Our main result concerning the existence of variational solutions which are Lipschitz continuous with respect to the spatial variables can be formulated as follows.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded and convex set and $T \in (0,\infty]$. Assume that $f\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}$ satisfies hypotheses (1.2). Further, let $u_o \in W^{1,\infty}(\Omega_T)$ denote a boundary datum such that the bounded slope condition with some positive constant Q (see Definition 2.1 below) is fulfilled for $U_o := u_o|_{\partial\Omega}$. Then, there exists a unique variational solution u to the Cauchy–Dirichlet problem associated with (1.1) and u_o in Ω_T. Moreover, u satisfies the gradient bound*

$$\|Du\|_{L^\infty(\Omega_T,\mathbb{R}^n)} \le \max\{Q, \|Du_o\|_{L^\infty(\Omega,\mathbb{R}^n)}\}. \qquad (1.4)$$

Furthermore, we show that variational solutions to (1.1) are weak solutions and consequently, they are 1/2-Hölder continuous in time provided that the map $\xi \mapsto f(t,\xi)$ is C^1 and uniformly locally Lipschitz in the following sense: For each $L > 0$, there exists a constant $M_L > 0$ such that

$$\sup_{t\in(0,T)} |D_\xi f(t,\xi)| < M_L \quad \text{for all} \quad \xi \in B_L(0). \tag{1.5}$$

Theorem 1.3. *Suppose that the assumptions of Theorem 1.2 hold. Moreover, assume that the mapping $\xi \mapsto f(t,\xi)$ is in $C^1(\mathbb{R}^n)$ for almost all $t \in (0,T)$ and satisfies (1.5). Then the unique variational solution u to the Cauchy–Dirichlet problem associated with (1.1) and u_o is a weak solution (see (7.1)). Further, it is contained in the space of Hölder continuous functions $C^{0;1,1/2}(\overline{\Omega}_T)$.*

To prove Theorem 1.2, we may assume without a loss of generality that $T < \infty$, see the beginning of Sect. 6. The proof is divided into three parts. We first assume that the integrand is suitably regular and in particular has a weak derivative with respect to the time variable. Then the method of minimizing movements yields a solution u to the so called *gradient constrained obstacle problem* to (1.1), where the L^∞-norms of the gradients of the solution and the comparison maps are bounded by a fixed constant $L \in (0,\infty)$. Moreover, the regularity assumption on f ensures that u has a weak time derivative in $L^2(\Omega_T)$.

Next, under the same regularity assumptions on f as in the first step, a standard argument exploiting the bounded slope condition and the maximum principle yields the uniform gradient bound (1.4) for u. Choosing L large enough, this in turn allows us to deduce that u is in fact already a solution to the unconstrained problem in the sense of Definition 1.1.

To deal with a general integrand f, we consider its Steklov average f_ε. Since f_ε admits a weak time derivative, by the results mentioned in the preceding paragraph there exists a solution u_ε to the Cauchy–Dirichlet problem associated with f_ε in the sense of Definition 1.1. Moreover, since for each $\varepsilon > 0$ the solution u_ε satisfies the gradient bound (1.4) and $u_\varepsilon = u_o$ on $\partial\Omega \times (0,T)$, there exists a limit map $u \in L^\infty(\Omega_T)$ such that $u_\varepsilon \to u$ uniformly and $Du_\varepsilon \overset{*}{\rightharpoondown} Du$ weakly* up to a subsequence as $\varepsilon \downarrow 0$. This allows us to conclude that u is a variational solution in the sense of Definition 1.1, finishing the proof of Theorem 1.2.

The proof of Theorem 1.3 is similar to the one found in [7, Chapter 8]. The C^1 assumption on the integrand ensures the validity of the weak Euler–Lagrange equation, which lets us apply the argument from [6, pp. 23–24] to prove a Poincaré inequality for variational solutions. The Hölder continuity then follows from the Campanato space characterization of Hölder continuity by Da Prato [9].

The paper is organized as follows. Section 2 contains preliminary definitions and basic observations about the integrand. In Sect. 3 we prove certain properties of variational solutions that are required in later sections, including the comparison and maximum principles. Under additional regularity assumptions on f we use the method of minimizing movements to prove the existence of variational solutions to the gradient constrained problem in Sect. 4 and in Sect. 5 we consider the unconstrained problem. Finally, in Sect. 6 we consider general integrands and finish the proof of Theorem 1.2 and Hölder continuity in time is proven in Sect. 7 under additional regularity assumptions.

2. Preliminaries

2.1. Notation

Throughout the paper, for $p \in [1,\infty]$ and $m \in \mathbb{N}$ the space $L^p(\Omega,\mathbb{R}^m)$ denotes the usual Lebesgue space (we omit $\mathbb{R}^m$ if $m=1$) and $W^{1,p}(\Omega)$ and $W^{1,p}_0(\Omega)$ denote the usual Sobolev spaces. If Ω is a bounded Lipschitz domain, $W^{1,\infty}(\Omega)$ can be identified with the space $C^{0,1}(\overline{\Omega})$ of functions $v\colon \Omega \to \mathbb{R}$ that are Lipschitz continuous (with Lipschitz constant $[v]_{0,1} = \|Dv\|_{L^\infty(\Omega,\mathbb{R}^n)}$) up to the boundary of Ω. Note that in particular any convex set has a Lipschitz continuous boundary, since convex functions are locally Lipschitz [11, Corollary 2.4]. Further, for a Banach space X and an integrability exponent $p \in [1,\infty]$ we write $L^p(0,T;X)$ for the space of Bochner measurable functions $v\colon [0,T] \to X$ with $t \mapsto \|v(t)\|_X \in L^p(0,T)$. Moreover, $C^0([0,T];X)$ is defined as the space of the continuous functions $v\colon [0,T] \to X$. For maps v defined in Ω_T we also use the short notation $v(t)$ for the partial map $x \mapsto v(x,t)$ defined in Ω. Finally, for a set $A \subset \mathbb{R}^m$, the characteristic function $\chi_A\colon \mathbb{R}^m \to \{0,1\}$ is given by $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ else.

2.2. Bounded slope condition

In the proof of the existence result in Sect. 5 it is crucial that there exist affine comparison functions below and above the initial/boundary datum u_o coinciding with u_o in a point $x_o \in \partial\Omega$. This is ensured by applying the following bounded slope condition to $u_o|_{\partial\Omega}$.

Definition 2.1. A function $U\colon \partial\Omega \to \mathbb{R}$ satisfies the *bounded slope condition* with constant $Q > 0$ if for any $x_o \in \partial\Omega$ there exist two affine functions $w^{\pm}_{x_o}\colon \mathbb{R}^n \to \mathbb{R}$ with Lipschitz constants $[w^{\pm}_{x_o}]_{0,1} \le Q$ such that

$$\begin{cases} w^-_{x_o}(x) \le U(x) \le w^+_{x_o}(x) \text{ for any } x \in \partial\Omega, \\ w^-_{x_o}(x_o) = U(x_o) = w^+_{x_o}(x_o). \end{cases}$$

Note that unless U itself is affine, the convexity of Ω is necessary for the bounded slope condition to hold. Even strict convexity of Ω is not sufficient for general U, since the boundary can become "too flat". However, we know that for a uniformly convex, bounded C^2-domain Ω and $v \in C^2(\mathbb{R}^n)$ the restriction $U = v|_{\partial\Omega}$ fulfills the bounded slope condition. For more details, we refer to [14,26]. On the other hand, in the parabolic setting the following example is relevant: Consider a convex domain Ω with flat parts (such as a rectangle) and a Lipschitz continuous function u_o that vanishes at the boundary of Ω; i.e. we prescribe zero lateral boundary values, but the initial datum is not necessarily identical to zero.

We need the following lemma from [7, Lemma 2.3]. It states that if u_o is Lipschitz and $u_o|_{\partial\Omega}$ satisfies the bounded slope condition, then u_o can be squeezed between two affine functions that touch u_o at a given boundary boundary point and the Lipschitz constant of these affine functions is bounded by either the Lipschitz constant of u_o or the constant in the bounded slope condition.

Lemma 2.2. *Let $u_o \in C^{0,1}(\overline{\Omega})$ with Lipschitz constant $[u_o]_{0,1} \le Q_1$ such that the restriction $U := u_o|_{\partial\Omega}$ satisfies the bounded slope condition with constant Q_2. Then for any $x_o \in \partial\Omega$ there exist two affine functions $w^{\pm}_{x_o}$ with $[w^{\pm}_{x_o}]_{0,1} \le \max\{Q_1, Q_2\}$ such that*

$$\begin{cases} w^-_{x_o}(x) \le u_o(x) \le w^+_{x_o}(x) & \text{for any } x \in \overline{\Omega}, \\ w^-_{x_o}(x_o) = u_o(x_o) = w^+_{x_o}(x_o). \end{cases}$$

2.3. Dominating functions for the integrand

Observe that for any $L > 0$ the map $t \mapsto \max_{|\xi|\le L} f(t,\xi)$ is measurable, since we have that $\max_{|\xi|\le L} f(t,\xi) = \max_{\xi\in B_L(0)\cap\mathbb{Q}^n} f(t,\xi)$ and the maximum of countably many measurable functions is measurable. The same holds true for $t \mapsto \min_{|\xi|\le L} f(t,\xi)$. In the following lemma, we show that they are contained in $L^1(0,T)$.

Lemma 2.3. *Let $T \in (0,\infty)$ and assume that $f\colon [0,T]\times\mathbb{R}^n \to \mathbb{R}$ satisfies (1.2). Then, for any $L > 0$ there exists a function $g_L \in L^1(0,T)$ such that*

$$|f(t,\xi)| \le g_L(t) \quad \text{for all } t \in (0,T) \text{ and } \xi \in B_L(0). \tag{2.1}$$

Proof. First, we show that for any $L > 0$, we have that

$$t \mapsto \max_{|\xi|\le L} f(t,\xi) \in L^1(0,T). \tag{2.2}$$

To this end, fix $\xi_1,\dots,\xi_{n+1} \in \mathbb{R}^n$ such that the closed ball $B_L(0)$ is a subset of the simplex

$$\Delta := \left\{ \xi \in \mathbb{R}^n : \xi = \sum_{i=1}^{n+1} \lambda_i \xi_i \text{ with } 0 \le \lambda_i \le 1, i = 1,\dots,n+1, \sum_{i=1}^{n+1} \lambda_i = 1 \right\}.$$

Note that for any $t \in [0,T]$ such that $\mathbb{R}^n \ni \xi \mapsto f(t,\xi)$ is convex, the mapping $\xi \mapsto f(t,\xi)$ attains its maximum in one of the points $\xi_1,\dots,\xi_{n+1}$. Hence, for a.e. t we obtain that

$$f(t,0) \le \max_{|\xi|\le L} f(t,\xi) \le \sum_{i=1}^{n+1} |f(t,\xi_i)|.$$

Since the maps $t \mapsto f(t,0)$ and $t \mapsto f(t,\xi_i)$, $i = 1,\dots,n+1$, belong to $L^1(0,T)$ by $(1.2)_2$, this implies (2.2).

Next, we fix $L > 0$ and prove

$$t \mapsto \min_{|\xi|\le L} f(t,\xi) \in L^1(0,T). \tag{2.3}$$

Consider $t \in [0,T]$ such that $\xi \mapsto f(t,\xi)$ is convex. Then, there exist $\xi_{min}, \xi_{max} \in B_L(0)$ such that $f(t,\xi_{min}) = \min_{|\xi|\le L} f(t,\xi)$ and $f(t,\xi_{max}) = \max_{|\xi|\le L} f(t,\xi)$. Assume that $\xi_{min} \ne \xi_{max}$ (otherwise, $\xi \mapsto f(t,\xi)$ is constant in $B_L(0)$ and thus $f(t,0) = \min_{|\xi|\le L)} f(t,\xi))$. First, note that for $C := \frac{1}{2L}(f(t,\xi_{max}) - f(t,\xi_{min})) \in (0,\infty)$, we find that

$$f(t,\xi_{min}) \le f(t,\xi_{max}) - C|\xi_{max}-\xi_{min}|.$$

Furthermore, since $\xi \mapsto f(t,\xi)$ is convex in $\mathbb{R}^n$, its subdifferential at ξ_{max} is non-empty [11, Proposition 5.2], i.e. there exists $\eta = \eta(\xi_{max}) \in \mathbb{R}^n$ such that

$$f(t,\xi) \ge f(t,\xi_{max}) + \eta\cdot(\xi-\xi_{max})$$

for any $\xi \in \mathbb{R}^n$. In particular, we have that

$$\begin{aligned} f(t,\xi_{min}) &\ge f(t,\xi_{max}) + \eta\cdot(\xi_{min}-\xi_{max}) \\ &= f(t,\xi_{max}) + \cos(\alpha)|\eta||\xi_{min}-\xi_{max}|, \end{aligned}$$

where α denotes the angle between η and $\xi_{min}-\xi_{max}$. Together, the preceding two inequalities imply that

$$\cos(\alpha)|\eta| \le -C.$$

Next, choose $s>1$ such that $\xi_o := \xi_{min} + s(\xi_{max}-\xi_{min}) \in \partial B_{L+1}(0)$. Note that the vector $\xi_o - \xi_{max} = (1-s)(\xi_{min}-\xi_{max})$ points in the opposite direction as $\xi_{min}-\xi_{max}$. Therefore, the angle between η and $\xi_o - \xi_{max}$ is $\pi-\alpha$. Using the facts that $\cos(\pi-\alpha) = -\cos(\alpha)$ and $|\xi_o - \xi_{max}| \ge 1$, the preceding inequality and the definition of C, we conclude that

$$\begin{aligned} \max_{|\xi|\le L+1} f(t,\xi) \ge f(t,\xi_o) &\ge f(t,\xi_{max}) + \eta\cdot(\xi_o-\xi_{max}) \\ &= f(t,\xi_{max}) - \cos(\alpha)|\eta||\xi_o-\xi_{max}| \\ &\ge f(t,\xi_{max}) + C \\ &= \max_{|\xi|\le L} f(t,\xi) + \tfrac{1}{2L}(\max_{|\xi|\le L} f(t,\xi)) - \min_{|\xi|\le L} f(t,\xi))). \end{aligned}$$

This is equivalent to

$$(2L+1)\max_{|\xi|\le L} f(t,\xi) - 2L \max_{|\xi|\le L+1} f(t,\xi) \le \min_{|\xi|\le L} f(t,\xi) \le \max_{|\xi|\le L} f(t,\xi),$$

which holds for almost every $t\in[0,T]$. Since we have already shown that $t\mapsto \max_{|\xi|\le L} f(t,\xi)$ and $t\mapsto \max_{|\xi|\le L+1} f(t,\xi)$ are contained in $L^1(0,T)$, the preceding inequality proves (2.3). The claim of Lemma 2.3 follows by combining (2.2) and (2.3). □

2.4. Lower semicontinuity

In the course of the paper we will need the following result on the lower semicontinuity of integrals involving f with respect to the weak* topology of $L^\infty(\Omega_T,\mathbb{R}^n)$.

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $0<T<\infty$. Assume that $f\colon [0,T]\times\mathbb{R}^n \to \mathbb{R}$ satisfies (1.2). Then, for any sequence $(V_i)_{i\in\mathbb{N}} \subset L^\infty(\Omega_T,\mathbb{R}^n)$ and $V \in L^\infty(\Omega_T,\mathbb{R}^n)$ such that $V_i \overset{*}{\rightharpoondown} V$ weakly* in $L^\infty(\Omega_T,\mathbb{R}^n)$ as $i\to\infty$ we have that*

$$\iint_{\Omega_T} f(t,V)\,\mathrm{d}x\mathrm{d}t \le \liminf_{i\to\infty} \iint_{\Omega_T} f(t,V_i)\,\mathrm{d}x\mathrm{d}t.$$

Proof. Consider an arbitrary sequence $(V_i)_{i\in\mathbb{N}} \subset L^\infty(\Omega_T,\mathbb{R}^n)$ and a limit map $V \in L^\infty(\Omega_T,\mathbb{R}^n)$ such that $V_i \overset{*}{\rightharpoondown} V$ weakly* in $L^\infty(\Omega_T,\mathbb{R}^n)$ as $i\to\infty$. First, note that $(V_i)_{i\in\mathbb{N}}$ is bounded in $L^\infty(\Omega_T,\mathbb{R}^n)$ and set $M := \sup_{i\in\mathbb{N}}\|V_i\|_{L^\infty(\Omega_T,\mathbb{R}^n)} \geq \|V\|_{L^\infty(\Omega_T,\mathbb{R}^n)}$. We find that

$$C := \{W \in L^2(\Omega_T,\mathbb{R}^n) : \|W\|_{L^\infty(\Omega_T,\mathbb{R}^n)} \leq M\}$$

is a convex subset of $L^2(\Omega_T,\mathbb{R}^n)$. Therefore, since $\xi \mapsto f(t,\xi)$ is convex for a.e. $t\in[0,T]$ and since $\iint_{\Omega_T} f(t,W)\,\mathrm{d}x\mathrm{d}t$ is finite for any $W\in C$ by (2.1), we obtain that the functional $F\colon L^2(\Omega_T,\mathbb{R}^n)\to(-\infty,\infty]$ given by

$$F[W] := \begin{cases} \iint_{\Omega_T} f(t,W)\,\mathrm{d}x\mathrm{d}t & \text{if } W\in C,\\ \infty & \text{else}\end{cases}$$

is proper and convex. Further, F is lower semicontinuous with respect to the norm topology in $L^2(\Omega_T,\mathbb{R}^n)$. Indeed, assume that the sequence $(W_i)_{i\in\mathbb{N}} \subset L^2(\Omega_T,\mathbb{R}^n)$ converges strongly in $L^2(\Omega_T,\mathbb{R}^n)$ to a limit map $W\in L^2(\Omega_T,\mathbb{R}^n)$ as $i\to\infty$. If $\liminf_{i\to\infty} F[W_i]=\infty$, the assertion $F[W]\leq \liminf_{i\to\infty} F[W_i]$ holds trivially. Otherwise, there exists a subsequence $\mathfrak{K}\subset\mathbb{N}$ such that $W_i\in C$ for any $i\in\mathfrak{K}$, $\liminf_{i\to\infty} F[W_i] = \lim_{\mathfrak{K}\ni i\to\infty} F[W_i]$ and $W_i\to W$ a.e. in Ω_T as $\mathfrak{K}\ni i\to\infty$. By (2.1) and the dominated convergence theorem, we conclude that $F[W]=\lim_{\mathfrak{K}\ni i\to\infty} F[W_i] = \liminf_{i\to\infty} F[W_i]$. Therefore, F is also lower semicontinuous with respect to the weak topology in $L^2(\Omega_T,\mathbb{R}^n)$, cf. [11, Corollary 2.2]. Since Ω_T is bounded, we have that $V_i \rightharpoondown V$ weakly in $L^2(\Omega_T,\mathbb{R}^n)$ as $i\to\infty$ and hence

$$\iint_{\Omega_T} f(t,V)\,\mathrm{d}x\mathrm{d}t = F[V] \leq \liminf_{i\to\infty} F[V_i] = \liminf_{i\to\infty}\iint_{\Omega_T} f(t,V_i)\,\mathrm{d}x\mathrm{d}t.$$

This concludes the proof of the lemma. □

2.5. Steklov averages of the integrand

For the final approximation argument in the proof of Theorem 1.2 we need to regularize the integrand f with respect to time. To this end, extend f to $[0,\infty]\times\mathbb{R}^n$ by zero if $T<\infty$. For $\varepsilon>0$ define the *Steklov average* $f_\varepsilon\colon[0,T]\times\mathbb{R}^n\to\mathbb{R}$ of the extended integrand by

$$f_\varepsilon(t,\xi) := \fint_t^{t+\varepsilon} f(s,\xi)\,\mathrm{d}s. \tag{2.4}$$

In order to investigate convergence of the Steklov averages as $\varepsilon\downarrow 0$, first note that specializing the proof of [11, Corollary 2.4] gives us the following result.

Lemma 2.5. *Let $L>0$ and assume that $f\colon\mathbb{R}^n\to\mathbb{R}$ is a convex function with $\|f\|_{L^\infty(B_{L+1}(0))}\leq C$. Then, f satisfies the local Lipschitz continuity condition*

$$|f(\xi_1)-f(\xi_2)| \leq 2C|\xi_1-\xi_2| \quad \textit{for all } \xi_1,\xi_2\in B_L(0).$$

We also need the following variant of the dominated convergence theorem that can be found for example in [12, Theorem 1.20].

Lemma 2.6. *Assume that $v, v_k \in L^1(\mathbb{R}^n)$ and $w, w_k \in L^1(\mathbb{R}^n)$ are measurable for all $k \in \mathbb{N}$. Suppose that $w_k \to w$ a.e. in $\mathbb{R}^n$ and $|w_k| \le v_k$ for all $k \in \mathbb{N}$. Suppose moreover that $v_k \to v$ a.e. in $\mathbb{R}^n$ and*

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} v_k \, \mathrm{d}x = \int_{\mathbb{R}^n} v \, \mathrm{d}x.$$

Then

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} |w_k - w| \, \mathrm{d}x = 0.$$

With the preceding lemmas at hand, we prove the following convergence result.

Lemma 2.7. *Let $T \in (0, \infty)$ and assume that $f\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}$ satisfies hypotheses (1.2). For $\varepsilon > 0$ let $f_\varepsilon\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}$ denote the Steklov average of f given by (2.4). Then, we have that*

$$\lim_{\varepsilon\downarrow 0} \int_0^T \sup_{|\xi|\le L} |f_\varepsilon(t,\xi) - f(t,\xi)| \, \mathrm{d}t = 0 \quad \textit{for any } L > 0.$$

Proof. Fix $L > 0$. First of all, we show that

$$\lim_{\varepsilon\downarrow 0} \sup_{|\xi|\le L} |f_\varepsilon(t,\xi) - f(t,\xi)| = 0 \quad \text{for a.e. } t \in [0,T]. \tag{2.5}$$

By $(1.2)_2$, for fixed $\xi \in \mathbb{R}^n$ we have that $f_\varepsilon(t,\xi) \to f(t,\xi)$ for a.e. $t \in [0,T]$ by Lebesgue's differentiation theorem. Thus, there exists a set N of $\mathcal{L}^1$-measure zero such that

$$f_\varepsilon(t,\xi) \to f(t,\xi) \quad \text{for any } t \in [0,T] \setminus N \text{ and } \xi \in \mathbb{Q}^n. \tag{2.6}$$

Without loss of generality assume that additionally for all $t \in [0,T] \setminus N$ the map $\xi \mapsto f(t,\xi)$ is convex, the function g_{L+1} from (2.1) fulfills $g_{L+1}(t) < \infty$ and there holds $\fint_t^{t+\varepsilon} g_{L+1}(s) \, \mathrm{d}s \to g_{L+1}(t)$. Now, fix $t \in [0,T] \setminus N$. By (2.1) and Lemma 2.5 we conclude that $\xi \mapsto f(t,\xi)$ is Lipschitz continuous in $B_L(0)$ with Lipschitz constant $2g_{L+1}(t)$. Using this together with the definition of the Steklov average, for any $\xi_1, \xi_2 \in B_L(0)$ we compute that

$$\begin{aligned} |f_\varepsilon(t,\xi_1) - f_\varepsilon(t,\xi_2)| &\le \fint_t^{t+\varepsilon} |f(s,\xi_1) - f(s,\xi_2)| \, \mathrm{d}s \\ &\le 2 \fint_t^{t+\varepsilon} g_{L+1}(s) \, \mathrm{d}s \, |\xi_1 - \xi_2|. \end{aligned}$$

Since $\fint_t^{t+\varepsilon} g_{L+1}(s) \, \mathrm{d}s \to g_{L+1}(t)$, there exists $\varepsilon_o > 0$ such that $\xi \mapsto f_\varepsilon(t,\xi)$ is Lipschitz continuous with Lipschitz constant $4g_{L+1}(t)$ for all $\varepsilon \in (0, \varepsilon_o]$. This shows that the sequence $(f_\varepsilon(t,\cdot))_{\varepsilon\in(0,\varepsilon_o]}$ is equicontinuous in $B_L(0)$. Moreover, $(f_\varepsilon(t,\cdot))_{\varepsilon\in(0,\varepsilon_o]}$ is equibounded in $B_L(0)$, since for any $\xi \in B_L(0)$ and $\varepsilon \in (0, \varepsilon_o]$, we find that

$$|f_\varepsilon(t,\xi)| \le \fint_t^{t+\varepsilon} |f(s,\xi)| \, \mathrm{d}s \le \fint_t^{t+\varepsilon} g_{L+1}(s) \, \mathrm{d}s \le 2g_{L+1}(t).$$

Therefore, we infer from the Arzèla–Ascoli theorem that $(f_\varepsilon(t,\cdot))_{\varepsilon\in(0,\varepsilon_o]}$ converges uniformly in $B_L(0)$ as $\varepsilon\downarrow 0$ and the limit $f(t,\cdot)$ is determined by (2.6). This concludes the proof of (2.5). Next, since

$$\begin{aligned}\sup_{|\xi|\le L}|f_\varepsilon(t,\xi)-f(t,\xi)| &\le \sup_{|\xi|\le L}|f_\varepsilon(t,\xi)|+\sup_{|\xi|\le L}|f(t,\xi)|\\ &\le \fint_t^{t+\varepsilon} g_L(s)\,\mathrm{d}s+g_L(t),\end{aligned}$$

where $\fint_t^{t+\varepsilon} g_L(s)\,\mathrm{d}s\to g_L(t)$ in $L^1(0,T)$, the claim now follows from Lemma 2.6. □

2.6. Mollification in time

In general, variational solutions are not admissible as comparison maps in the variational inequality (1.3), since they do not necessarily admit a derivative with respect to time. Therefore, we use the following mollification procedure with respect to time. More precisely, consider a separable Banach space X, an initial datum $v_o\in X$ and a map $v\in L^r(0,T;X)$ for some $r\in[1,\infty]$. For $h>0$ define the mollification

$$[v]_h(t):=e^{-\frac{t}{h}}v_o+\tfrac{1}{h}\int_0^t e^{\frac{s-t}{h}}v(s)\,\mathrm{d}s\quad\text{for any } t\in[0,T]. \tag{2.7}$$

Later on, we will mainly use $X=L^q(\Omega)$ or $X=W^{1,q}(\Omega)$ for some $q\in[1,\infty)$. A vital feature of this mollification procedure is that $[v]_h$ solves the ordinary differential equation

$$\partial_t[v]_h=\tfrac{1}{h}\big(v-[v]_h\big) \tag{2.8}$$

with initial condition $[v]_h(0)=v_o$. This shows in particular that if v and $[v]_h$ are contained in a function space, the same holds true for the time derivative of $[v]_h$. The basic properties of time mollifications are collected in the following lemma (cf. [19, Lemma 2.2] and [5, Appendix B] for the proofs).

Lemma 2.8. *Let X be a separable Banach space and $v_o\in X$. If $v\in L^r(0,T;X)$ for some $r\in[1,\infty]$, then also $[v]_h\in L^r(0,T;X)$ and if $r<\infty$, then $[v]_h\to v$ in $L^r(0,T;X)$ as $h\downarrow 0$. Further, for any $t_o\in(0,T]$ there holds the bound*

$$\big\|[v]_h\big\|_{L^r(0,t_o;X)}\le\|v\|_{L^r(0,t_o;X)}+\Big[\tfrac{h}{r}\Big(1-e^{-\frac{t_or}{h}}\Big)\Big]^{\frac{1}{r}}\|v_o\|_X,$$

where the bracket $[\dots]^{\frac{1}{r}}$ has to be interpreted as 1 if $r=\infty$. Moreover, if $v\in C^0([0,T];X)$, then also $[v]_h\in C^0([0,T];X)$ with $[v]_h(0)=v_o$ and there holds $[v]_h\to v$ in $L^\infty(0,T;X)$ as $h\downarrow 0$.

For maps $v\in L^r(0,T;X)$ with $\partial_t v\in L^r(0,T;X)$ we have the following assertion.

Lemma 2.9. *Let X be a separable Banach space and $r\ge 1$. Assume that $v\in L^r(0,T;X)$ with $\partial_t v\in L^r(0,T;X)$. Then, for the mollification in time defined by*

$$[v]_h(t):=e^{-\frac{t}{h}}v(0)+\tfrac{1}{h}\int_0^t e^{\frac{s-t}{h}}v(s)\,\mathrm{d}s$$

the time derivative can be computed by

$$\partial_t[v]_h(t) = \tfrac{1}{h}\int_0^t e^{\frac{s-t}{h}}\partial_s v(s)\,\mathrm{d}s.$$

3. Properties of variational solutions

As mentioned in the introduction, besides variational solutions in the sense of Definition 1.1, we consider variational solutions of the so-called gradient constrained obstacle problem to (1.1). They enjoy the same basic properties as variational solutions to the unconstrained Cauchy–Dirichlet problem to (1.1) and proofs will be given in a unified way in this section.

Let $L \in (0,\infty]$. We define the following class of functions that are L-Lipschitz in space

$$K^L(\Omega_T) := \{v \in K^\infty(\Omega_T) : \|Dv\|_{L^\infty(\Omega_T,\mathbb{R}^n)} \le L\}$$

and given time-independent boundary values $u_o \in W^{1,\infty}(\Omega)$ with $\|Du_o\|_{L^\infty(\Omega,\mathbb{R}^n)} \le L$, we denote the subclass

$$K^L_{u_o}(\Omega_T) := \{v \in K^L(\Omega_T) : v = u_o \text{ on the lateral boundary } \partial\Omega \times (0,T)\}.$$

Definition 3.1. Assume that $f\colon [0,T]\times\mathbb{R}^n \to \mathbb{R}$ satisfies (1.2), consider a boundary datum $u_o \in W^{1,\infty}(\Omega)$ and let $L \in (0,\infty)$ be such that $\|Du_o\|_{L^\infty(\Omega,\mathbb{R}^n)} \le L$. In the case $T<\infty$ a map $u \in K^L_{u_o}(\Omega_T)$ is called a *variational solution* to the gradient constrained Cauchy–Dirichlet problem associated with (1.1) and u_o in Ω_T if and only if the variational inequality

$$\begin{aligned}\iint_{\Omega_T} f(t,Du)\,\mathrm{d}x\mathrm{d}t \le &\iint_{\Omega_T} \partial_t v(v-u) + f(t,Dv)\,\mathrm{d}x\mathrm{d}t \\ &+ \tfrac{1}{2}\|v(0)-u_o\|^2_{L^2(\Omega)} - \tfrac{1}{2}\|(v-u)(T)\|^2_{L^2(\Omega)}\end{aligned} \tag{3.1}$$

holds true for any comparison map $v \in K^L_{u_o}(\Omega_T)$ with $\partial_t v \in L^2(\Omega_T)$. If $T=\infty$ and $u \in K^L_{u_o}(\Omega_\infty)$ is a variational solution in Ω_τ for any $\tau>0$, u is called a *global variational solution* or variational solution in Ω_∞ to the gradient constrained Cauchy–Dirichlet problem associated with (1.1) and u_o.

3.1. Continuity with respect to time

In Definitions 1.1 and 3.1 we require that variational solutions are contained in the space $C^0([0,T];L^2(\Omega))$. However, this is already implied if u satisfies a variational inequality for a.e. $\tau\in[0,T]$. More precisely, we have the following Lemma, which will be applied with $L=\infty$ in Sect. 6.

Lemma 3.2. *Let $\Omega\subset\mathbb{R}^n$ be open and bounded and $T\in(0,\infty)$ and assume that $f\colon[0,T]\times\mathbb{R}^n\to\mathbb{R}$ satisfies (1.2). Let $L\in(0,\infty]$ and consider $u_o\in W^{1,\infty}(\Omega)$ such that $\|Du_o\|_{L^\infty(\Omega,\mathbb{R}^n)}\le L$. Further, consider $u\in L^\infty(\Omega_T)$ with $u=u_o$ on $\partial_{\mathcal{P}}\Omega_T$ and $\|Du\|_{L^\infty(\Omega_T,\mathbb{R}^n)}\le L$ if $L\in(0,\infty)$ and $\|Du\|_{L^\infty(\Omega_T,\mathbb{R}^n)}<\infty$ if $L=\infty$, respectively. Suppose that u satisfies the variational inequality*

$$\iint_{\Omega_\tau} f(t, Du)\,\mathrm{d}x\mathrm{d}t \le \iint_{\Omega_\tau} \partial_t v(v-u)\,\mathrm{d}x\mathrm{d}t + \iint_{\Omega_\tau} f(t, Dv)\,\mathrm{d}x\mathrm{d}t + \tfrac{1}{2}\,\|v(0)-u_o\|^2_{L^2(\Omega)} - \tfrac{1}{2}\,\|v(\tau)-u(\tau)\|^2_{L^2(\Omega)} \tag{3.2}$$

for almost all $\tau \in (0,T)$ whenever $v \in K^L_{u_o}(\Omega_T)$ with $\partial_t v \in L^2(\Omega_T)$. Then, we have that $u \in C^0([0,T]; L^2(\Omega))$.

Proof. The proof is similar to that of Lemma 2.6 in [28] except for the estimate of the second integral in (3.3) below. Denote by $[u]_h$ the time mollification of u with initial values u_o as defined in (2.7). In particular, observe that $[u]_h \in C^0([0,T]; L^2(\Omega))$, since we know that $\partial_t[u]_h \in L^2(\Omega_T)$ and $[u]_h(0) = u_o \in L^2(\Omega)$. Using $[u]_h$ as a comparison function in (3.2), taking the essential supremum over $\tau \in (0,T)$ and recalling that $([u]_h - u) = -h\partial_t[u]_h$, we obtain that

$$\begin{aligned}\sup_{\tau\in(0,T)} \tfrac{1}{2}\,\|[u]_h(\tau)-u(\tau)\|^2_{L^2(\Omega)} &\le \sup_{\tau\in(0,T)} \iint_{\Omega_\tau} \partial_t[u]_h([u]_h-u)\,\mathrm{d}x\mathrm{d}t \\ &\quad + \iint_{\Omega_T} f(t, D[u]_h) - f(t, Du)\,\mathrm{d}x\mathrm{d}t \\ &\le \iint_{\Omega_T} |f(t, D[u]_h) - f(t, Du)|\,\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.3}$$

Furthermore, we have that $D[u]_h \to Du$ almost everywhere in Ω_T as $h \downarrow 0$ (up to a subsequence) and that $|D[u]_h| \le |Du_o| + \sup_{\Omega_T} |Du|$. Therefore, by (2.1) and the dominated convergence theorem we find that the second integral in (3.3) vanishes in the limit $h \downarrow 0$. Hence, we have shown that $[u]_h \to u$ in $L^\infty(0,T; L^2(\Omega))$. Combining this with the fact that $[u]_h \in C^0([0,T]; L^2(\Omega))$, it follows that also $u \in C^0([0,T]; L^2(\Omega))$. □

3.2. Localization in time

Here, we show that a variational solution in a space-time cylinder Ω_T is also a solution in any sub-cylinder Ω_τ, $\tau \in (0,T)$.

Lemma 3.3. (Localization in time) *Let $T \in (0,\infty)$, assume that $\Omega \subset \mathbb{R}^n$ is open and bounded, and that $f\colon [0,T]\times\mathbb{R}^n \to \mathbb{R}$ satisfies (1.2). Consider $u_o \in W^{1,\infty}(\Omega)$ and $L \in (0,\infty]$ such that $\|Du_o\|_{L^\infty(\Omega,\mathbb{R}^n)} \le L$. Suppose that u is a variational solution to (1.1) in $K^L_{u_o}(\Omega_T)$ (in the sense of Definition 3.1 if $L < \infty$, in the sense of Definition 1.1 if $L = \infty$). Then $u|_{\Omega_\tau}$ is a variational solution to (1.1) in $K^L_{u_o}(\Omega_\tau)$ for any $\tau \in (0,T]$.*

Proof. For $\theta \in (0,\tau)$, consider the cut-off function

$$\xi_\theta(t) := \chi_{[0,\tau-\theta]}(t) + \frac{\tau - t}{\theta}\chi_{(\tau-\theta,\tau]}(t).$$

For $v \in K^L_{u_o}(\Omega_\tau)$ satisfying $\partial_t v \in L^2(\Omega_\tau)$ we define a function $v_\theta\colon \Omega_T \to \mathbb{R}$ by

$$v_\theta := \xi_\theta v + (1-\xi_\theta)[u]_h,$$

where $\xi_\theta v$ has been extended to Ω_T by zero and $[u]_h$ is defined according to (2.7) with initial datum u_o. Then we have $v_\theta \in K^L_{u_o}(\Omega_T)$ with $\partial_t v_\theta \in L^2(\Omega_T)$, and therefore we may use v_θ as a comparison map for u in the variational inequality. This yields

$$\iint_{\Omega_T} f(t, Du)\,\mathrm{d}x\mathrm{d}t \le \iint_{\Omega_T} \partial_t v_\theta (v_\theta - u) + f(t, Dv_\theta)\,\mathrm{d}x\mathrm{d}t + \tfrac{1}{2}\left\| v(0) - u_o \right\|^2_{L^2(\Omega)} - \tfrac{1}{2}\left\| ([u]_h - u)(T) \right\|^2_{L^2(\Omega)}. \quad (3.4)$$

The first term on the right-hand side of (3.4) is identical to the one in [7, Equation (3.2)] and can be estimated in the same way to obtain

$$\begin{aligned}\limsup_{\theta\to 0} &\iint_{\Omega_T} \partial_t v_\theta (v_\theta - u)\,\mathrm{d}x\mathrm{d}t \\ &\le \iint_{\Omega_\tau} \partial_t v(v-u)\,\mathrm{d}x\mathrm{d}t + \iint_{\Omega\times(\tau,T)} \partial_t [u]_h ([u]_h - u)\,\mathrm{d}x\mathrm{d}t \\ &\quad - \tfrac{1}{2}\int_\Omega (v - [u]_h)^2(\tau)\,\mathrm{d}x + \int_\Omega ([u]_h - u)(v - [u]_h)(\tau)\,\mathrm{d}x.\end{aligned}$$

The second term on the right-hand side of (3.4) is given by

$$\begin{aligned}\iint_{\Omega_T} f(t, Dv_\theta)\,\mathrm{d}x\mathrm{d}t = &\iint_{\Omega\times(\tau-\theta,\tau)} f(t, \xi_\theta Dv + (1-\xi_\theta)D[u]_h)\,\mathrm{d}x\mathrm{d}t \\ &+ \iint_{\Omega\times(0,\tau-\theta)} f(t, Dv)\,\mathrm{d}x\mathrm{d}t \\ &+ \iint_{\Omega\times(\tau,T)} f(t, D[u]_h)\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Since we know that

$$\begin{aligned}\|\xi_\theta Dv + (1-\xi_\theta)D[u]_h\|_{L^\infty(\Omega_T,\mathbb{R}^n)} &\le \|Dv\|_{L^\infty(\Omega_T,\mathbb{R}^n)} + \|D[u]_h\|_{L^\infty(\Omega_T,\mathbb{R}^n)} \\ &\le \|Dv\|_{L^\infty(\Omega_T,\mathbb{R}^n)} + \|Du_o\|_{L^\infty(\Omega,\mathbb{R}^n)} + \|Du\|_{L^\infty(\Omega_T,\mathbb{R}^n)} =: M < \infty,\end{aligned}$$

by (2.1) we find that

$$\left| \iint_{\Omega\times(\tau-\theta,\tau)} f(t, \xi_\theta Dv + (1-\xi_\theta)D[u]_h)\,\mathrm{d}x\mathrm{d}t \right| \le |\Omega| \int_{\tau-\theta}^{\tau} g_M(t)\,\mathrm{d}t \to 0$$

in the limit $\theta \downarrow 0$. Combining the preceding estimates we arrive at

$$\begin{aligned}\iint_{\Omega_T} f(t, Du)\,\mathrm{d}x\mathrm{d}t \le &\iint_{\Omega\times(0,\tau)} f(t, Dv)\,\mathrm{d}x\mathrm{d}t + \iint_{\Omega\times(\tau,T)} f(t, D[u]_h)\,\mathrm{d}x\mathrm{d}t \\ &- \tfrac{1}{2}\int_\Omega (v - [u]_h)^2(\tau)\,\mathrm{d}x + \int_\Omega ([u]_h - u)(v - [u]_h)(\tau)\,\mathrm{d}x \\ &+ \iint_{\Omega_\tau} \partial_t v(v-u)\,\mathrm{d}x\mathrm{d}t + \iint_{\Omega\times(\tau,T)} \partial_t [u]_h ([u]_h - u)\,\mathrm{d}x\mathrm{d}t \\ &+ \tfrac{1}{2}\left\| v(0) - u_o \right\|^2_{L^2(\Omega)} - \tfrac{1}{2}\left\| ([u]_h - u)(T) \right\|^2_{L^2(\Omega)}. \quad (3.5)\end{aligned}$$

Note that $[u]_h \to u$ in $L^\infty(0,T;L^2(\Omega))$ as $h \downarrow 0$, since $u \in C^0([0,T];L^2(\Omega))$. Further, we have that $D[u]_h \to Du$ pointwise almost everywhere in Ω_T as $h \downarrow 0$ (up to a subsequence) and that

$$\|D[u]_h\|_{L^\infty(\Omega_T,\mathbb{R}^n)} \le \|Du_o\|_{L^\infty(\Omega,\mathbb{R}^n)} + \|Du\|_{L^\infty(\Omega_T,\mathbb{R}^n)} =: L' < \infty \quad \text{for any } h > 0.$$

Therefore, assumption (2.1), the fact that Ω is bounded and the dominated convergence theorem imply that

$$\lim_{h\downarrow 0}\iint_{\Omega\times(\tau,T)} f(t,D[u]_h)\,\mathrm{d}x\mathrm{d}t=\iint_{\Omega\times(\tau,T)} f(t,Du)\,\mathrm{d}x\mathrm{d}t.$$

Hence, using that $\partial_t[u]_h([u]_h-u)\le 0$ and letting $h\downarrow 0$ in (3.5), we obtain the desired inequality

$$\begin{aligned}\iint_{\Omega_\tau} f(t,Du)\,\mathrm{d}x\mathrm{d}t \le &\iint_{\Omega_\tau} \partial_t v(v-u)+f(t,Dv)\,\mathrm{d}x\mathrm{d}t\\ &+\tfrac{1}{2}\left\|v(0)-u_o\right\|^2_{L^2(\Omega)}-\tfrac{1}{2}\left\|(v-u)(\tau)\right\|^2_{L^2(\Omega)}.\end{aligned}$$

□

3.3. The initial condition

As a consequence of the localization in time principle, we find that variational solutions attain the initial datum u_o in the C^0-L^2-sense. The precise statement is as follows.

Lemma 3.4. *Let $T\in(0,\infty)$, assume that $\Omega\subset\mathbb{R}^n$ is bounded and open, and that $f\colon[0,T]\times\mathbb{R}^n\to\mathbb{R}$ satisfies (1.2). Consider $u_o\in W^{1,\infty}(\Omega)$ and $L\in(0,\infty]$ such that $\|Du_o\|_{L^\infty(\Omega,\mathbb{R}^n)}\le L$. Suppose that u is a variational solution to (1.1) in $K^L_{u_o}(\Omega_T)$ (in the sense of Definition 3.1 if $L<\infty$, in the sense of Definition 1.1 if $L=\infty$). Then, there holds*

$$\lim_{\tau\downarrow 0}\|u(\tau)-u_o\|^2_{L^2(\Omega)}=0.$$

Proof. By Lemma 3.3, the function u is a variational solution in any smaller cylinder Ω_τ, $\tau\in(0,T]$. Using $v\colon\Omega_\tau\to\mathbb{R}$, $v(x,t):=u_o(x)$ as a comparison function for u and taking (2.1) with $M:=\max\{\|Du\|_{L^\infty(\Omega_T,\mathbb{R}^n)},\|Du_o\|_{L^\infty(\Omega,\mathbb{R}^n)}\}$ into account, we obtain that

$$\tfrac{1}{2}\left\|u(\tau)-u_o\right\|_{L^2(\Omega)}\le\iint_{\Omega_\tau} f(t,Du_o)-f(t,Du)\,\mathrm{d}x\mathrm{d}t\le 2\,|\Omega|\int_0^\tau g_M(t)\,\mathrm{d}t.$$

Since $g_M\in L^1(0,T)$, this implies the claim. □

3.4. Comparison principle

The following comparison principle ensures in particular that variational solutions to the problems considered in the present paper are unique.

Theorem 3.5. *(Comparison principle) Let $T\in(0,\infty)$, assume that $\Omega\subset\mathbb{R}^n$ is bounded and open, and that $f\colon[0,T]\times\mathbb{R}^n\to\mathbb{R}$ satisfies (1.2). Let $L\in(0,\infty]$ and suppose that u and $\tilde u$ are variational solutions to (1.1) in $K^L(\Omega_T)$ (in the sense of Definition 3.1 if $L<\infty$ and in the sense of Definition 1.1 if $L=\infty$) such that $\|Du(0)\|_{L^\infty(\Omega,\mathbb{R}^n)}$ and $\|D\tilde u(0)\|_{L^\infty(\Omega,\mathbb{R}^n)}$ are bounded by L if $L\in(0,\infty)$ and finite if $L=\infty$, respectively. Then the assumption that*

$$u\le\tilde u\quad\text{on}\quad\partial_{\mathcal{P}}\Omega_T$$

implies

$$u\le\tilde u\quad\text{in}\quad\Omega_T.$$

Proof. Let $\tau \in (0, T]$. By Lemma 3.3, the functions u and $\tilde{u}$ are variational solutions in $K^L(\Omega_\tau)$. Consider the functions

$$v := \min([u]_h, [\tilde{u}]_h) \quad \text{and} \quad w := \max([u]_h, [\tilde{u}]_h),$$

where $[u]_h$ and $[\tilde{u}]_h$ denote the mollifications of u and $\tilde{u}$ according to (2.7) with initial values $u(0) \in W^{1,\infty}(\Omega)$ and $\tilde{u}(0) \in W^{1,\infty}(\Omega)$, respectively. Since the boundary values attained by u and $\tilde{u}$ are independent of time, we have that $v \in K^L_u(\Omega_\tau)$ and $w \in K^L_{\tilde{u}}(\Omega_\tau)$ with $\partial_t v, \partial_t w \in L^2(\Omega_\tau)$. Therefore we may use v and w as comparison functions in the variational inequalities of u and $\tilde{u}$, respectively. Adding the resulting inequalities and using that $[u]_h(0) = u(0) \le \tilde{u}(0) = [\tilde{u}]_h(0)$, we obtain

$$\begin{aligned} 0 \le & \iint_{\Omega_\tau} \partial_t v(v-u) + \partial_t w(w-\tilde{u})\,\mathrm{d}x\mathrm{d}t \\ & + \iint_{\Omega_\tau} f(t,Dv) - f(t,Du) + f(t,Dw) - f(t,D\tilde{u})\,\mathrm{d}x\mathrm{d}t \\ & - \tfrac{1}{2}\left\|(v-u)(\tau)\right\|^2_{L^2(\Omega)} - \tfrac{1}{2}\left\|(w-\tilde{u})(\tau)\right\|^2_{L^2(\Omega)}. \end{aligned} \tag{3.6}$$

Using the identities

$$\begin{cases} v - u = \min([u]_h, [\tilde{u}]_h) - [u]_h - (u - [u]_h) = -([u]_h - [\tilde{u}]_h)_+ - h\partial_t[u]_h, \\ w - \tilde{u} = ([u]_h - [\tilde{u}]_h)_+ - h\partial_t[\tilde{u}]_h, \end{cases}$$

we compute that

$$\begin{aligned} &\partial_t v(v-u) + \partial_t w(w-\tilde{u}) \\ &\quad = \big(\partial_t[u]_h \chi_{\{[u]_h \le [\tilde{u}]_h\}} + \partial_t[\tilde{u}]_h \chi_{\{[\tilde{u}]_h < [u]_h\}}\big)\big(-\big([u]_h - [\tilde{u}]_h\big)_+ - h\partial_t[u]_h\big) \\ &\qquad + \big(\partial_t[\tilde{u}]_h \chi_{\{[u]_h \le [\tilde{u}]_h\}} + \partial_t[u]_h \chi_{\{[\tilde{u}]_h < [u]_h\}}\big)\big(\big([u]_h - [\tilde{u}]_h\big)_+ - h\partial_t[\tilde{u}]_h\big) \\ &\quad = \big(\partial_t[\tilde{u}]_h\big([u]_h - [\tilde{u}]_h\big)_+ - \partial_t[u]_h([u]_h - [\tilde{u}]_h)_+ - h(\partial_t[u]_h)^2 - h(\partial_t[\tilde{u}]_h)^2\big) \\ &\qquad \cdot \chi_{\{[u]_h \le [\tilde{u}]_h\}} \\ &\qquad + \big(\partial_t[u]_h\big([u]_h - [\tilde{u}]_h\big)_+ - \partial_t[\tilde{u}]_h\big([u]_h - [\tilde{u}]_h\big)_+ - h\partial_t[\tilde{u}]_h\partial_t[u]_h - h\partial_t[u]_h\partial_t[\tilde{u}]_h\big) \\ &\qquad \cdot \chi_{\{[\tilde{u}]_h < [u]_h\}} \\ &\quad \le \big(\partial_t[u]_h\big([u]_h - [\tilde{u}]_h\big)_+ - \partial_t[\tilde{u}]_h\big([u]_h - [\tilde{u}]_h\big)_+ - h\partial_t[\tilde{u}]_h\partial_t[u]_h - h\partial_t[u]_h\partial_t[\tilde{u}]_h\big) \\ &\qquad \cdot \chi_{\{[\tilde{u}]_h < [u]_h\}} \\ &\quad = \partial_t\big([u]_h - [\tilde{u}]_h\big)\big([u]_h - [\tilde{u}]_h\big)_+ - 2h\partial_t[u]_h\partial_t[\tilde{u}]_h \chi_{\{[\tilde{u}]_h < [u]_h\}} \\ &\quad \le \tfrac{1}{2}\partial_t\big(\big([u]_h - [\tilde{u}]_h\big)_+^2\big) + h\big(\big(\partial_t[u]_h\big)^2 + \big(\partial_t[\tilde{u}]_h\big)^2\big). \end{aligned}$$

Therefore, taking into account that $[u]_h(0) = u(0) \le \tilde{u}(0) = [\tilde{u}]_h(0)$, we find that

$$\begin{aligned} &\iint_{\Omega_\tau} \partial_t v(v-u) + \partial_t w(w-\tilde{u})\,\mathrm{d}x\mathrm{d}t \\ &\quad \le \tfrac{1}{2}\big\|\big([u]_h - [\tilde{u}]_h\big)_+(\tau)\big\|^2_{L^2(\Omega)} + \iint_{\Omega_\tau} h\big(\big(\partial_t[u]_h\big)^2 + \big(\partial_t[\tilde{u}]_h\big)^2\big)\,\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.7}$$

Furthermore, using $[u]_h$ as a comparison function for u and omitting the boundary term at time τ on the right-hand side of the variational inequality, we obtain

$$\iint_{\Omega_\tau} h\big(\partial_t[u]_h\big)^2\,\mathrm{d}x\mathrm{d}t = -\iint_{\Omega_\tau} \partial_t[u]_h\big([u]_h - u\big)\,\mathrm{d}x\mathrm{d}t \\ \leq \iint_{\Omega_\tau} f\big(t, D[u]_h\big) - f(t, Du)\,\mathrm{d}x\mathrm{d}t \tag{3.8}$$

and a similar inequality holds for $\tilde{u}$. Observe also that

$$\begin{aligned} &f(t, Dv) - f(t, Du) + f(t, Dw) - f(t, D\tilde{u}) \\ &\quad= \chi_{\{[u]_h \leq [\tilde{u}]_h\}} f\big(t, D[u]_h\big) + \chi_{\{[\tilde{u}]_h < [u]_h\}} f\big(t, D[\tilde{u}]_h\big) - f(t, Du) \\ &\qquad+ \chi_{\{[u]_h \leq [\tilde{u}]_h\}} f\big(t, D[\tilde{u}]_h\big) + \chi_{\{[\tilde{u}]_h < [u]_h\}} f\big(t, D[u]_h\big) - f(t, D\tilde{u}) \\ &\quad= f\big(t, D[u]_h\big) - f(t, Du) + f\big(t, D[\tilde{u}]_h\big) - f(t, D\tilde{u}). \end{aligned} \tag{3.9}$$

Combining the estimates (3.7), (3.8) and (3.9) with (3.6) we arrive at

$$-\tfrac{1}{2}\big\|\big([u]_h - [\tilde{u}]_h\big)_+(\tau)\big\|^2_{L^2(\Omega)} + \tfrac{1}{2}\|(v-u)(\tau)\|^2_{L^2(\Omega)} + \tfrac{1}{2}\|(w-\tilde{u})(\tau)\|^2_{L^2(\Omega)} \\ \leq 2\iint_{\Omega_\tau} f\big(t, D[u]_h\big) - f(t, Du) + f\big(t, D[\tilde{u}]_h\big) - f(t, D\tilde{u})\,\mathrm{d}x\mathrm{d}t. \tag{3.10}$$

By the same argument as in the end of the proof of Lemma 3.3 involving the dominated convergence theorem, the integral on the right-hand side of (3.10) vanishes in the limit $h \downarrow 0$. Writing $v - u = -([u]_h - [\tilde{u}]_h)_+ + [u]_h - u$ and $w - \tilde{u} = ([u]_h - [\tilde{u}]_h)_+ + [\tilde{u}]_h - \tilde{u}$ and using that $[u]_h \to u$ and $[\tilde{u}]_h \to \tilde{u}$ in $L^\infty([0,\tau], L^2(\Omega))$ as $h \downarrow 0$ since $u, \tilde{u} \in C^0([0,T]; L^2(\Omega))$, we conclude that

$$\lim_{h\downarrow 0}\Big(-\tfrac{1}{2}\big\|\big([u]_h - [\tilde{u}]_h\big)_+(\tau)\big\|^2_{L^2(\Omega)} + \tfrac{1}{2}\|(v-u)(\tau)\|^2_{L^2(\Omega)} + \tfrac{1}{2}\|(w-\tilde{u})(\tau)\|^2_{L^2(\Omega)}\Big) \\ = \tfrac{1}{2}\big\|(u-\tilde{u})_+(\tau)\big\|^2_{L^2(\Omega)}.$$

Hence, taking the limit $h \downarrow 0$ in (3.10), we infer

$$\tfrac{1}{2}\big\|(u-\tilde{u})_+(\tau)\big\|^2_{L^2(\Omega)} \leq 0,$$

which implies that $u \leq \tilde{u}$ in Ω_τ. Since τ was arbitrary, the claim follows. □

3.5. Maximum principle and localization in space for regular solutions

In this section, we consider more regular variational solutions u satisfying $\partial_t u \in L^2(\Omega_T)$. As a consequence, u is directly admissible as comparison map in its variational inequality without regularization with respect to the time variable. Further, due to the requirements of the proof of the existence result in Sect. 5, we will take time-dependent boundary values $u|_{\Omega\times(0,T)}$ into account here. In particular, the proof of the comparison principle in Theorem 3.5 is easily adapted to allow time-dependent boundary values if $\partial_t u$ and $\partial_t \tilde{u}$ are contained in $L^2(\Omega_T)$ by using $\min(u, \tilde{u})$ and $\max(u, \tilde{u})$ as comparison maps in the variational inequalities satisfied by u and $\tilde{u}$, respectively, and proceeding in a similar way as above. However, most arguments can be simplified, since mollification with respect to time is not necessary in the present situation. This allows us to deduce the following maximum principle.

Lemma 3.6. (Maximum principle) *Let $T \in (0,\infty)$, assume that $\Omega \subset \mathbb{R}^n$ is open and bounded, and that $f\colon [0,T]\times\mathbb{R}^n \to \mathbb{R}$ satisfies (1.2). Consider $L \in (0,\infty]$ and functions $u, \tilde{u} \in K^L(\Omega_T)$ such that $\partial_t u, \partial_t \tilde{u} \in L^2(\Omega_T)$. Suppose moreover that $\|Du(0)\|_{L^\infty(\Omega,\mathbb{R}^n)}$ and $\|D\tilde{u}(0)\|_{L^\infty(\Omega,\mathbb{R}^n)}$ are bounded by L if $L \in (0,\infty)$ and finite if $L=\infty$. Finally, assume that for any $\tau \in (0,T]$ the function u satisfies the variational inequality*

$$\iint_{\Omega_\tau} f(t,Du)\,\mathrm{d}x\mathrm{d}t \le \iint_{\Omega_\tau} \partial_t v(v-u) + f(t,Dv)\,\mathrm{d}x\mathrm{d}t + \tfrac{1}{2}\|u(0)-v(0)\|^2_{L^2(\Omega)} - \tfrac{1}{2}\|u(\tau)-v(\tau)\|^2_{L^2(\Omega)} \tag{3.11}$$

whenever $v \in K^L(\Omega_\tau)$ with $\partial_t v \in L^2(\Omega_\tau)$ and $v = u$ on $\Omega \times (0,\tau)$, and that $\tilde{u}$ fulfills the analogical inequality. Then

$$\sup_{\Omega_T}(u-\tilde{u}) = \sup_{\partial_{\mathcal{P}}\Omega_T}(u-\tilde{u}).$$

Proof. Let $\tau \in (0,T]$. Define

$$\hat{u} := \tilde{u} + \sup_{\partial_{\mathcal{P}}\Omega_T}(u-\tilde{u}).$$

Then $\hat{u}$ satisfies the variational inequality (3.11) with its own boundary values, and

$$u \le \hat{u} \quad \text{on} \quad \partial_{\mathcal{P}}\Omega_T. \tag{3.12}$$

Consider the functions $v := \min(u,\hat{u})$ and $w := \max(u,\hat{u})$. Then $v, w \in K^L(\Omega_\tau)$ with $\partial_t v, \partial_t w \in L^2(\Omega_\tau)$ and $v = u$, $w = \hat{u}$ on $\partial\Omega \times (0,\tau)$. Observe also that $v - u = -(u-\hat{u})_+$ and $w - \hat{u} = (u-\hat{u})_+$. Using v and w as comparison functions for u and $\hat{u}$ in the variational inequality (3.11), we obtain

$$\begin{aligned}
0 &\le \iint_{\Omega_\tau} \partial_t v(v-u) + \partial_t w(w-\hat{u})\,\mathrm{d}x\mathrm{d}t \\
&\quad + \iint_{\Omega_\tau} f(t,Dv) - f(t,Du) + f(t,Dw) - f(t,D\hat{u})\,\mathrm{d}x\mathrm{d}t \\
&\quad + \tfrac{1}{2}\|(v-u)(0)\|^2_{L^2(\Omega)} + \tfrac{1}{2}\|(w-\hat{u})(0)\|^2_{L^2(\Omega)} \\
&\quad - \tfrac{1}{2}\|(v-u)(\tau)\|^2_{L^2(\Omega)} - \tfrac{1}{2}\|(w-\hat{u})(\tau)\|^2_{L^2(\Omega)} \\
&= \iint_{\Omega_\tau} \tfrac{1}{2}\partial_t((u-\hat{u})_+)^2\,\mathrm{d}x\mathrm{d}t - \|(u-\hat{u})_+(\tau)\|^2_{L^2(\Omega)} \\
&= -\tfrac{1}{2}\|(u-\hat{u})_+(\tau)\|^2_{L^2(\Omega)},
\end{aligned}$$

where we used that $(v-u)(0) = (w-\hat{u})(0) = 0$ and that the terms with f cancel one another. As τ was arbitrary, we obtain

$$u \le \hat{u} = \tilde{u} + \sup_{\partial_{\mathcal{P}}\Omega_T}(u-\hat{u}) \quad \text{in } \Omega_T$$

so that

$$\sup_{\Omega_T}(u-\tilde{u}) \le \sup_{\partial_{\mathcal{P}}\Omega_T}(u-\tilde{u}).$$

Since the reverse inequality holds by continuity, this proves the claim. □

Lemma 3.7. (Localization in space) *Let $T \in (0,\infty)$, assume that $\Omega \subset \mathbb{R}^n$ is open and bounded, and that $f\colon [0,T]\times\mathbb{R}^n \to \mathbb{R}$ satisfies* (1.2)*. Consider $u_o \in W^{1,\infty}(\Omega)$ and $L \in (0,\infty]$ such that $\|Du_o\|_{L^\infty(\Omega,\mathbb{R}^n)} \le L$. Suppose that u is a variational solution to* (1.1) *in $K^L_{u_o}(\Omega_T)$, $L \in (0,\infty]$ (in the sense of Definition* 3.1 *if $L<\infty$, in the sense of Definition* 1.1 *if $L=\infty$). Moreover, suppose that $\partial_t u \in L^2(\Omega_T)$. Then for any domain $\Omega' \subset \Omega$ and any $\tau \in (0,T]$, the variational inequality*

$$\iint_{\Omega'_\tau} f(t,Du)\,\mathrm{d}x\mathrm{d}t \le \iint_{\Omega'_\tau} \partial_t v(v-u) + f(t,Dv)\,\mathrm{d}x\mathrm{d}t + \tfrac12\|u(0)-v(0)\|^2_{L^2(\Omega')} - \tfrac12\|u(\tau)-v(\tau)\|^2_{L^2(\Omega')} \tag{3.13}$$

holds whenever $v \in K^L_{u_o}(\Omega'_\tau)$ with $\partial_t v \in L^2(\Omega_\tau)$ and $v=u$ on $\partial\Omega' \times (0,\tau)$.

Proof. By Lemma 3.3 the function $u|_{\Omega_\tau}$ is a variational solution to (1.1) in the function space $K^L_{u_o}(\Omega_\tau)$. Observe that

$$w := \begin{cases} v & \text{in } \Omega'_\tau, \\ u & \text{in } (\Omega\setminus\Omega')_\tau, \end{cases}$$

is an admissible comparison function for $u|_{\Omega_\tau}$ in the variational inequality. Inserting w into the variational inequality (3.1) if $L<\infty$ (or (1.3) if $L=\infty$) with T replaced by τ immediately yields (3.13). □

4. Existence for the gradient constrained problem for regular integrands

In this section, we are concerned with integrands that admit a time derivative. More precisely, we consider $f\colon [0,T]\times\mathbb{R}^n \to \mathbb{R}$ such that

$$\begin{cases} \xi \mapsto f(t,\xi) \text{ is convex for any } t \in [0,T], \\ t \mapsto f(t,\xi) \in W^{1,1}(0,T) \text{ for any } \xi \in \mathbb{R}^n, \\ \text{for any } L>0 \text{ there exists } \tilde g_L \in L^1(0,T) \text{ such that } |\partial_t f(t,\xi)| \le \tilde g_L(t) \\ \text{for a.e. } t \in [0,T] \text{ and all } \xi \in B_L(0). \end{cases} \tag{4.1}$$

The aim of this section is to prove the following existence result.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $T \in (0,\infty)$. Consider a boundary datum $u_o \in W^{1,\infty}(\Omega)$ such that $\|Du_o\|_{L^\infty(\Omega,\mathbb{R}^n)} \le L$ for a constant $L \in (0,\infty)$. Further, assume that the integrand $f\colon [0,T]\times\mathbb{R}^n \to \mathbb{R}$ satisfies hypothesis* (4.1)*. Then, there exists a variational solution $u \in K^L_{u_o}(\Omega_T)$ to the gradient constrained problem in the sense of Definition* 3.1*. Further, there holds $\partial_t u \in L^2(\Omega_T)$ with the quantitative bound*

$$\iint_{\Omega_T} |\partial_t u|^2\,\mathrm{d}x\mathrm{d}t \le 4|\Omega|\Big(\sup_{|\xi|\le L}|f(0,\xi)| + \|\tilde g_L\|_{L^1(0,T)}\Big).$$

We prove Theorem 4.1 via the method of minimizing movements. The proof is divided into five steps.

4.1. A sequence of minimizers to elliptic variational functionals

Fix a step size $h := \frac{T}{m}$ for some $m \in \mathbb{N}$ and consider times ih, $i = 0, \ldots, m$. For $i = 0$, set $u_0 := u_o \in W^{1,\infty}(\Omega)$ with $\|Du_o\|_{L^\infty(\Omega,\mathbb{R}^n)} \leq L$. Further, for $i = 1, \ldots, m$, u_i is defined as the minimizer of the elliptic variational functional

$$F_i[v] := \int_\Omega f(ih, Dv)\,\mathrm{d}x + \tfrac{1}{2h}\int_\Omega |v - u_{i-1}|^2\,\mathrm{d}x$$

in the class $\mathcal{A} := \{v \in W^{1,\infty}(\Omega) : v = u_o \text{ on}\partial\Omega \text{ and}\|Dv\|_{L^\infty(\Omega,\mathbb{R}^n)} \leq L\}$. The existence of a minimizer to F_i in this class is ensured by the direct method in the calculus of variations. More precisely, note that $\mathcal{A} \neq \emptyset$, since $u_o \in \mathcal{A}$, and consider a minimizing sequence to F_i in $\mathcal{A}$, i.e. a sequence $(u_{i,j})_{j\in\mathbb{N}} \subset \mathcal{A}$ such that

$$\lim_{j\to\infty} F_i[u_{i,j}] = \inf_{v\in\mathcal{A}} F_i[v].$$

Further, by definition of $\mathcal{A}$ and Rellich's theorem there exists a limit map $u_i \in \mathcal{A}$ and a (not relabelled) subsequence such that

$$\begin{cases} u_{i,j} \to u_i \text{ strongly in } L^2(\Omega) \text{ as } j \to \infty, \\ Du_{i,j} \rightharpoonup Du_i \text{ weakly in } L^2(\Omega,\mathbb{R}^n) \text{ as} j \to \infty. \end{cases}$$

Since the functional $\widetilde{F}_i \colon W^{1,2}(\Omega) \to (-\infty, \infty]$,

$$\widetilde{F}_i[v] := \begin{cases} F_i[v] & \text{if } v \in \mathcal{A}, \\ \infty & \text{else} \end{cases}$$

is proper, convex and lower semicontinuous with respect to strong convergence in $W^{1,2}(\Omega)$, it is also lower semicontinuous with respect to weak convergence in $W^{1,2}(\Omega)$, see [11, Corollary 2.2]. Therefore, we obtain that

$$F_i[u_i] = \widetilde{F}_i[u_i] \leq \liminf_{j\to\infty} \widetilde{F}_i[u_{i,j}] = \lim_{j\to\infty} F_i[u_{i,j}] = \inf_{v\in\mathcal{A}} F_i[v].$$

4.2. Energy estimates

Since $u_{i-1} \in \mathcal{A}$ is an admissible comparison map for the minimizer u_i and f fulfills $(4.1)_3$, we have that

$$\begin{aligned}
\int_\Omega f(ih, Du_i)\,\mathrm{d}x &+ \tfrac{1}{2h}\int_\Omega |u_i - u_{i-1}|^2\,\mathrm{d}x = F_i[u_i] \\
&\leq F_i[u_{i-1}] \\
&= \int_\Omega f((i-1)h, Du_{i-1})\,\mathrm{d}x + \int_\Omega f(ih, Du_{i-1}) - f((i-1)h, Du_{i-1})\,\mathrm{d}x \\
&\leq \int_\Omega f((i-1)h, Du_{i-1})\,\mathrm{d}x + \iint_{\Omega\times((i-1)h,ih)} |\partial_t f(t, Du_{i-1})|\,\mathrm{d}x\mathrm{d}t \\
&\leq \int_\Omega f((i-1)h, Du_{i-1})\,\mathrm{d}x + |\Omega| \int_{((i-1)h,ih)} |\tilde{g}_L(t)|\,\mathrm{d}t.
\end{aligned}$$

Summing up the preceding inequalities from $i = 1$ to $i = m$, we find that

$$\sum_{i=1}^{m} \int_{\Omega} f(ih, Du_i)\,\mathrm{d}x\mathrm{d}t + \tfrac{1}{2h} \sum_{i=1}^{m} \int_{\Omega} |u_i - u_{i-1}|^2\,\mathrm{d}x \leq \sum_{i=1}^{m} \int_{\Omega} f((i-1)h, Du_{i-1})\,\mathrm{d}x + |\Omega| \int_{(0,T)} |\tilde{g}_L(t)|\,\mathrm{d}t.$$

Subtracting the first term on the left-hand side, we conclude that

$$\tfrac{1}{2h} \sum_{i=1}^{m} \int_{\Omega} |u_i - u_{i-1}|^2\,\mathrm{d}x \leq \int_{\Omega} f(0, Du_o)\,\mathrm{d}x - \int_{\Omega} f(T, Du_m)\,\mathrm{d}x + |\Omega| \|\tilde{g}_L\|_{L^1(0,T)} \leq 2|\Omega| \big(\sup_{|\xi| \leq L} |f(0,\xi)| + \|\tilde{g}_L\|_{L^1(0,T)} \big). \tag{4.2}$$

4.3. The limit map

In the following we denote the step size by h_m in order to emphasize the dependence on m. First, we join the minimizers u_i to a map that is piecewise constant with respect to time. More precisely, we define $u^{(m)} \colon \Omega \times (-h_m, T] \to \mathbb{R}$ by

$$u^{(m)}(t) := u_i \quad \text{for } t \in ((i-1)h_m, ih_m],\ i = 0, \ldots, m.$$

Observe that the sequence $\big(u^{(m)}\big)_{m \in \mathbb{N}}$ is bounded in $L^\infty(\Omega_T)$, since $\|u^{(m)}\|_{L^\infty(\Omega_T)} = \max_{i=0,\ldots,m} \|u_i\|_{L^\infty(\Omega)}$, $u_i \in \mathcal{A}$ for all $i = 0, \ldots, m$ and $\mathcal{A}$ is equibounded. Further, we know that $\|Du^{(m)}\|_{L^\infty(\Omega_T, \mathbb{R}^n)} = \max_{i=0,\ldots,m} \|Du_i\|_{L^\infty(\Omega, \mathbb{R}^n)} \leq L$ for any $m \in \mathbb{N}$. Therefore, there exists a subsequence $\mathfrak{K} \subset \mathbb{N}$ and a limit map $u \in L^\infty(\Omega_T)$ such that $\|Du\|_{L^\infty(\Omega_T, \mathbb{R}^n)} \leq L$, $u = u_o$ on $\partial\Omega \times (0,T)$ and

$$\begin{cases} u^{(m)} \overset{*}{\rightharpoondown} u \text{ weakly}^* \text{ in} L^\infty(\Omega_T) \text{ as} \mathfrak{K} \ni m \to \infty, \\ u^{(m)}(t) \to u(t) \text{ uniformly as } \mathfrak{K} \ni m \to \infty \text{ for each} t \in [0,T], \\ Du^{(m)} \overset{*}{\rightharpoondown} Du \text{ weakly}^* \text{ in} L^\infty(\Omega_T, \mathbb{R}^n) \text{ as} \mathfrak{K} \ni m \to \infty. \end{cases} \tag{4.3}$$

In order to prove that u has a time derivative, we consider the linear interpolation of minimizers $\tilde{u}^{(m)} \colon \Omega \times (-h_m, T] \to \mathbb{R}$ given by $\tilde{u}^{(m)}(t) := u_o$ for $t \in (-h_m, 0]$ and

$$\tilde{u}^{(m)}(t) := \Big(i - \tfrac{t}{h_m}\Big) u_{i-1} + \Big(1 - i + \tfrac{t}{h_m}\Big) u_i \quad \text{for } t \in ((i-1)h_m, ih_m],\ i = 1, \ldots, m.$$

Similar arguments as above ensure that $\big(\tilde{u}^{(m)}\big)_{m \in \mathbb{N}}$ is bounded in $L^\infty(\Omega_T)$ and that $\|D\tilde{u}^{(m)}\|_{L^\infty(\Omega_T, \mathbb{R}^n)} \leq L$ for any $m \in \mathbb{N}$. Moreover, by the energy bound (4.2) we obtain that

$$\iint_{\Omega_T} |\partial_t \tilde{u}^{(m)}|^2\,\mathrm{d}x\mathrm{d}t = \sum_{i=1}^{m} \iint_{\Omega \times ((i-1)h_m, ih_m]} \tfrac{1}{h_m^2} |u_i - u_{i-1}|^2\,\mathrm{d}x\mathrm{d}t = \tfrac{1}{h_m} \sum_{i=1}^{m} \int_{\Omega} |u_i - u_{i-1}|^2\,\mathrm{d}x \leq 4|\Omega| \big(\sup_{|\xi| \leq L} |f(0,\xi)| + \|\tilde{g}_L\|_{L^1(0,T)} \big). \tag{4.4}$$

Hence, $\big(\tilde u^{(m)}\big)_{m\in\mathbb{N}}$ is bounded in $W^{1,2}(\Omega_T)$. By Rellich's theorem we conclude that there exists a subsequence still labelled $\mathfrak{K}$ and a limit map $\tilde u \in L^\infty(\Omega_T)$ with $\|D\tilde u\|_{L^\infty(\Omega_T,\mathbb{R}^n)} \le L$, $\tilde u = u_o$ on $\partial\Omega\times(0,T)$ and $\partial_t\tilde u \in L^2(\Omega_T)$ such that

$$\begin{cases} \tilde u^{(m)} \to u \text{ strongly in } L^2(\Omega_T) \text{ as } \mathfrak{K}\ni m\to\infty,\\ \partial_t\tilde u^{(m)} \rightharpoonup \partial_t\tilde u \text{ weakly in } L^2(\Omega_T) \text{ as } \mathfrak{K}\ni m\to\infty. \end{cases} \tag{4.5}$$

Note that $\partial_t\tilde u \in L^2(\Omega_T)$ in particular implies that $\tilde u \in C^{0;\frac12}([0,T];L^2(\Omega))$ and therefore $\tilde u$ is contained in the class of functions $K^L_{u_o}(\Omega_T)$. Next, since $\big|\big(u^{(m)}-\tilde u^{(m)}\big)(t)\big| \le |u_i-u_{i-1}|$ for $t\in((i-1)h_m, ih_m]$, $i=1,\dots,m$, we infer from (4.2) that

$$\begin{aligned}\iint_{\Omega_T}\big|u^{(m)}-\tilde u^{(m)}\big|^2\,\mathrm{d}x\mathrm{d}t &\le h_m\sum_{i=1}^m\int_\Omega |u_i-u_{i-1}|^2\,\mathrm{d}x\\ &\le 4|\Omega|\big(\sup_{|\xi|\le L}|f(0,\xi)|+\|\tilde g_L\|_{L^1(0,T)}\big)h_m^2.\end{aligned}$$

Together with $(4.5)_1$ this implies that $u^{(m)}\to\tilde u$ strongly in $L^2(\Omega_T)$ as $\mathfrak{K}\ni m\to\infty$ and thus in particular that $u=\tilde u\in K^L_{u_o}(\Omega_T)$ with $\partial_t u\in L^2(\Omega_T)$. Finally, by lower semicontinuity with respect to weak convergence, (4.4) gives us the claimed bound

$$\iint_{\Omega_T}|\partial_t u|^2\,\mathrm{d}x\mathrm{d}t \le 4|\Omega|\big(\sup_{|\xi|\le L}|f(0,\xi)|+\|\tilde g_L\|_{L^1(0,T)}\big).$$

4.4. Minimizing property of the approximations

First, define piecewise constant approximations of the integrand by

$$f^{(m)}(t,\xi) := f(ih,\xi) \quad \text{for } t\in((i-1)h_m, ih_m],\ i=0,\dots,m.$$

We claim that $u^{(m)}$ is a minimizer of the functional

$$F^{(m)}[v] := \iint_{\Omega_T} f^{(m)}(t,Dv)\,\mathrm{d}x\mathrm{d}t + \tfrac{1}{2h_m}\iint_{\Omega_T}|v(t)-u^{(m)}(t-h_m)|^2\,\mathrm{d}x\mathrm{d}t$$

in the class of functions

$$\mathcal{A}_T := \{v\in L^\infty(\Omega_T) : \|Du\|_{L^\infty(\Omega_T,\mathbb{R}^n)}\le L \text{ and } u=u_o \text{ on } \partial\Omega\times(0,T)\}.$$

Indeed, consider an arbitrary map $v\in\mathcal{A}_T$. Since $v(t)\in\mathcal{A}$ for a.e. $t\in[0,T]$, by the minimizing property of u_i with respect to F_i in the class $\mathcal{A}$ we find that

$$F^{(m)}\big[u^{(m)}\big] = \sum_{i=1}^m\int_{((i-1)h_m,ih_m]}F_i[u_i]\,\mathrm{d}t \le \sum_{i=1}^m\int_{((i-1)h_m,ih_m]}F_i[v(t)]\,\mathrm{d}t = F^{(m)}[v].$$

A straightforward computation shows that this is equivalent to

$$\begin{aligned}&\iint_{\Omega_T} f^{(m)}\big(t,Du^{(m)}\big)\,\mathrm{d}x\mathrm{d}t\\ &\qquad\le \iint_{\Omega_T} f^{(m)}(t,Dv)\,\mathrm{d}x\mathrm{d}t\\ &\qquad\quad+\tfrac{1}{h_m}\iint_{\Omega_T}\tfrac12\big|v-u^{(m)}\big|^2+\big(v-u^{(m)}\big)\big(u^{(m)}-u^{(m)}(t-h_m)\big)\,\mathrm{d}x\mathrm{d}t\end{aligned}$$

for any $v \in \mathcal{A}_T$. Choosing the convex combination $u^{(m)} + s\big(v - u^{(m)}\big) \in \mathcal{A}_T$ with $s \in (0,1)$ as comparison map and using the convexity of $\xi \mapsto f(t,\xi)$ for all $t \in [0,T]$, we obtain that

$$\begin{aligned}
&\iint_{\Omega_T} f^{(m)}\big(t, Du^{(m)}\big)\,\mathrm{d}x\mathrm{d}t \\
&\quad\le (1-s)\iint_{\Omega_T} f^{(m)}\big(t, Du^{(m)}\big)\,\mathrm{d}x\mathrm{d}t + s\iint_{\Omega_T} f^{(m)}(t, Dv)\,\mathrm{d}x\mathrm{d}t \\
&\qquad + \tfrac{1}{h_m}\iint_{\Omega_T} \tfrac{s^2}{2}\big|v - u^{(m)}\big|^2 + s\big(v - u^{(m)}\big)\big(u^{(m)} - u^{(m)}(t-h_m)\big)\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Reabsorbing the first term on the right-hand side into the left-hand side, dividing the resulting inequality by s and taking the limit $s \downarrow 0$ gives us that

$$\begin{aligned}
&\iint_{\Omega_T} f^{(m)}\big(t, Du^{(m)}\big)\,\mathrm{d}x\mathrm{d}t \\
&\quad\le \iint_{\Omega_T} f^{(m)}(t, Dv)\,\mathrm{d}x\mathrm{d}t + \tfrac{1}{h_m}\iint_{\Omega_T} \big(v - u^{(m)}\big)\big(u^{(m)} - u^{(m)}(t-h_m)\big)\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Next, assume without loss of generality that $v(0) \in L^\infty(\Omega)$, extend v to $(-h_m, 0]$ by $v(0)$ and note that

$$\begin{aligned}
&\big(v - u^{(m)}\big)\big(u^{(m)} - u^{(m)}(t-h_m)\big) \\
&\quad= \big(v - u^{(m)}\big)\big(v - v(t-h_m)\big) + \tfrac12\big(v(t-h_m) - u^{(m)}(t-h_m)\big)^2 - \tfrac12\big(v - u^{(m)}\big)^2 \\
&\qquad - \tfrac12\big(v - v(t-h_m) - u^{(m)} + u^{(m)}(t-h_m)\big)^2 \\
&\quad\le \big(v - u^{(m)}\big)\big(v - v(t-h_m)\big) + \tfrac12\big(v(t-h_m) - u^{(m)}(t-h_m)\big)^2 - \tfrac12\big(v - u^{(m)}\big)^2.
\end{aligned}$$

Inserting this into the preceding inequality and recalling that $v(t) = v(0)$ for $t \in (-h_m, 0]$, we infer

$$\begin{aligned}
&\iint_{\Omega_T} f^{(m)}\big(t, Du^{(m)}\big)\,\mathrm{d}x\mathrm{d}t \\
&\quad\le \iint_{\Omega_T} f^{(m)}(t, Dv)\,\mathrm{d}x\mathrm{d}t + \tfrac{1}{h_m}\iint_{\Omega_T} \big(v - u^{(m)}\big)\big(v - v(t-h_m)\big)\,\mathrm{d}x\mathrm{d}t \\
&\qquad + \tfrac{1}{2h_m}\iint_{\Omega_T} \big(v(t-h_m) - u^{(m)}(t-h_m)\big)^2 - \big(v - u^{(m)}\big)^2\,\mathrm{d}x\mathrm{d}t \\
&\quad= \iint_{\Omega_T} f^{(m)}(t, Dv)\,\mathrm{d}x\mathrm{d}t + \tfrac{1}{h_m}\iint_{\Omega_T} \big(v - u^{(m)}\big)\big(v - v(t-h_m)\big)\,\mathrm{d}x\mathrm{d}t \\
&\qquad + \tfrac12\int_\Omega (v - u_o)^2\,\mathrm{d}x - \tfrac{1}{2h_m}\iint_{\Omega\times(T-h_m,T]} \big|v - u^{(m)}(T)\big|^2\,\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{4.6}$$

4.5. Variational inequality for the limit map

We fix an arbitrary map $v \in K^L_{u_o}(\Omega_T)$ with $\partial_t v \in L^2(\Omega_T)$. Thus, in particular we have that $v \in \mathcal{A}_T$, so v is an admissible comparison map in (4.6). Our goal is to pass to the limit $\mathfrak{K} \ni m \to \infty$ in (4.6) in order to deduce the variational inequality (3.1) for u. To this end, we consider the terms separately. First, we write the first term on the left-hand side of (4.6) as

$$\begin{aligned}\iint_{\Omega_T} f^{(m)}\big(t, Du^{(m)}\big)\,\mathrm{d}x\mathrm{d}t \\ = \iint_{\Omega_T} f\big(t, Du^{(m)}\big)\,\mathrm{d}x\mathrm{d}t + \iint_{\Omega_T} f^{(m)}\big(t, Du^{(m)}\big) - f\big(t, Du^{(m)}\big)\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

By Lemma 2.4 and $(4.3)_3$, we obtain that

$$\iint_{\Omega_T} f(t, Du)\,\mathrm{d}x\mathrm{d}t \le \liminf_{\mathfrak{K}\ni m\to\infty} \iint_{\Omega_T} f\big(t, Du^{(m)}\big)\,\mathrm{d}x\mathrm{d}t.$$

Further, since $\big\|Du^{(m)}\big\|_{L^\infty(\Omega_T,\mathbb{R}^n)} \le L$ for all $m \in \mathbb{N}$ and f fulfills $(4.1)_3$, we estimate

$$\begin{aligned}&\left|\iint_{\Omega_T} f^{(m)}\big(t, Du^{(m)}\big) - f\big(t, Du^{(m)}\big)\,\mathrm{d}t\right| \\ &\quad\le \sum_{i=1}^{m} \iint_{\Omega\times((i-1)h_m, ih_m]} \big|f\big(ih_m, Du^{(m)}\big) - f\big(t, Du^{(m)}\big)\big|\mathrm{d}x\mathrm{d}t \\ &\quad\le \sum_{i=1}^{m} \iint_{\Omega\times((i-1)h_m, ih_m]} \int_{((i-1)h_m, ih_m]} \big|\partial_t f\big(s, Du^{(m)}(t)\big)\big|\,\mathrm{d}s\,\mathrm{d}x\mathrm{d}t \\ &\quad\le |\Omega| h_m \sum_{i=1}^{m} \int_{((i-1)h_m, ih_m]} \tilde{g}_L(s)\,\mathrm{d}s \\ &\quad= |\Omega| \|\tilde{g}_L\|_{L^1(0,T)} h_m.\end{aligned}$$

Therefore, this term vanishes in the limit $m \to \infty$. Joining the preceding estimates, we conclude that

$$\iint_{\Omega_T} f(t, Du)\,\mathrm{d}x\mathrm{d}t \le \liminf_{\mathfrak{K}\ni m\to\infty} \iint_{\Omega_T} f^{(m)}\big(t, Du^{(m)}\big)\,\mathrm{d}x\mathrm{d}t. \tag{4.7}$$

Repeating the estimates in the penultimate inequality with $u^{(m)}$ replaced by v, for the first term on the right-hand side of (4.6) we find that

$$\iint_{\Omega_T} f(t, Dv)\,\mathrm{d}x\mathrm{d}t = \lim_{m\to\infty} \iint_{\Omega_T} f^{(m)}(t, Dv)\,\mathrm{d}x\mathrm{d}t. \tag{4.8}$$

Next, since $\frac{1}{h_m}(v(t) - v(t - h_m)) \to \partial_t v$ strongly in $L^2(\Omega_T)$ and $u^{(m)} \rightharpoonup u$ weakly in $L^2(\Omega_T)$ as $\mathfrak{K} \ni m \to \infty$ by $(4.3)_1$, we have that

$$\iint_{\Omega_T} \partial_t v(v - u)\,\mathrm{d}x\mathrm{d}t = \lim_{\mathfrak{K}\ni m\to\infty} \tfrac{1}{h_m} \iint_{\Omega_T} \big(v - u^{(m)}\big)\big(v - v(t - h_m)\big)\,\mathrm{d}x\mathrm{d}t. \tag{4.9}$$

Finally, by the fact that $v \in C^0([0, T]; L^2(\Omega))$ and by $(4.3)_2$, we obtain that

$$\begin{aligned}\tfrac{1}{2}\|(v - u)(T)\|^2_{L^2(\Omega)} \\ = \lim_{\mathfrak{K}\ni m\to\infty} \tfrac{1}{2h_m} \iint_{\Omega\times(T-h_m,T]} \big|v - u^{(m)}(T)\big|^2\,\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{4.10}$$

Collecting the assertions (4.7)–(4.10) yields

$$\iint_{\Omega_T} f(t, Du)\,\mathrm{d}x\mathrm{d}t \le \iint_{\Omega_T} f(t, Dv)\,\mathrm{d}x\mathrm{d}t + \iint_{\Omega_T} \partial_t v(v-u)\,\mathrm{d}x\mathrm{d}t \\ + \tfrac{1}{2}\|v(0)-u_o\|^2_{L^2(\Omega)} - \tfrac{1}{2}\|(v-u)(T)\|^2_{L^2(\Omega)}.$$

Since $v \in K^L_{u_o}(\Omega_T)$ with $\partial_t v \in L^2(\Omega_T)$ was arbitrary, we have shown that $u \in K^L_{u_o}(\Omega_T)$ is the desired variational solution. □

5. Existence for the unconstrained problem for regular integrands

In this section we show the existence of variational solutions to the unconstrained problem under the regularity condition (4.1) provided that the initial and boundary datum satisfies the bounded slope condition. To this end, we need the following lemma, whose proof is similar to that of [7, Lemma 7.1]. It states that affine functions independent of time are variational solutions to (1.1) with respect to their own initial and lateral boundary values.

Lemma 5.1. *Let Ω be open and bounded. Assume that $f\colon [0,T]\times\mathbb{R}^n \to \mathbb{R}$ satisfies (1.2). Let $w(x,t) := a + \xi\cdot x$ with constants $a \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$ be an affine function independent of time. Then w is a variational solution in the sense of Definition 1.1 in $K^\infty_w(\Omega_T)$.*

With the preceding lemma at hand, we are able to prove the following.

Theorem 5.2. *Let $T \in (0,\infty)$, assume that $\Omega \subset \mathbb{R}^n$ is open, bounded and convex, and that the integrand $f\colon [0,T]\times\mathbb{R}^n \to \mathbb{R}$ satisfies (4.1). Consider $u_o \in W^{1,\infty}(\Omega)$ such that $\|Du_o\|_{L^\infty(\Omega,\mathbb{R}^n)} \le Q$ and suppose that $u_o|_{\partial\Omega}$ satisfies the bounded slope condition with the same parameter Q. Then there exists a variational solution $u \in K^\infty_{u_o}(\Omega_T)$ to (1.1) in the sense of Definition 1.1. Further, we have the quantitative bound*

$$\|Du\|_{L^\infty(\Omega_T,\mathbb{R}^n)} \le Q. \tag{5.1}$$

Proof. Let $L > Q$. By Theorem 4.1 there exists a variational solution $u \in K^L_{u_o}(\Omega_T)$ with $\partial_t u \in L^2(\Omega_T)$ to the gradient constrained problem in the sense of Definition 3.1. We begin by proving the Lipschitz bound (5.1) and then show that u is in fact already a solution to the unconstrained problem.

Fix $x_o \in \partial\Omega$ and denote by $w^\pm_{x_o}$ the affine functions from Lemma 2.2. In particular we have $w^-_{x_o} \le u_o \le w^+_{x_o}$. Since by Lemma 5.1 the functions $w^-_{x_o}$ and $w^+_{x_o}$ are variational solutions, it follows from the comparison principle in Theorem 3.5 that

$$w^+_{x_o}(x) \le u(x,t) \le w^-_{x_o}(x) \quad \text{for all } (x,t)\in\Omega_T.$$

Consequently, there holds

$$|u(x,t) - u_o(x_o)| \le Q\,|x - x_o| \quad \text{for all } (x,t)\in\Omega_T.$$

Since $x_o \in \partial\Omega$ was arbitrary, we obtain that

$$\frac{|u(x,t)-u_o(x_o)|}{|x-x_o|} \le Q \quad \text{for all } x_o \in \partial\Omega, (x,t) \in \Omega_T. \tag{5.2}$$

Consider $x_1, x_2 \in \Omega$, $x_1 \neq x_2$, $t \in (0,T)$ and set $y := x_2 - x_1$. Define the shifted set $\widetilde{\Omega}_T := \big\{(x-y,t) \in \mathbb{R}^{n+1} : (x,t) \in \Omega_T\big\}$ and the shifted function $u_y \colon \widetilde{\Omega}_T \to \mathbb{R}$ by

$$u_y(x,t) := u(x+y,t).$$

Then u_y is a variational solution in $K^L(\widetilde{\Omega}_T)$. Since $\partial_t u, \partial_t u_y \in L^2((\Omega \cap \widetilde{\Omega})_T)$ by the spatial localization principle in Lemma 3.7, the functions u and u_y both satisfy variational inequality (3.11) from Lemma 3.6 in $(\Omega \cap \widetilde{\Omega})_T$. Therefore by Lemma 3.6 there exists $(x_o,t_o) \in \partial_{\mathcal{P}}((\Omega \cap \widetilde{\Omega}))_T$ such that

$$|u(x_1,t) - u_y(x_1,t)| \le |u(x_o,t_o) - u_y(x_o,t_o)|\,.$$

By definition of y and u_y, this yields

$$|u(x_1,t) - u(x_2,t)| \le |u(x_o,t_o) - u(x_o+y,t_o)|\,.$$

Since either $t_o = 0$ or one of the points x_o or $x_o + y$ belongs to $\partial\Omega$, it follows from the assumption $\|Du_o\|_{L^\infty(\Omega,\mathbb{R}^n)} \le Q$ and (5.2) that

$$|u(x_o,t_o) - u(x_o+y,t_o)| \le Q\,|y| = Q\,|x_1 - x_2|\,.$$

Combining this with the preceding estimate, we obtain (5.1).

It remains to show that u is a variational solution to the unconstrained problem. Let $w \in K^\infty_{u_o}(\Omega_T)$ with $\partial_t w \in L^2(\Omega_T)$ and choose the comparison map $v := u + s(w-u)$ for $0 < s \ll 1$; in particular, since $Q < L$, for s small enough we have that

$$\begin{aligned}&\|Dv\|_{L^\infty(\Omega_T,\mathbb{R}^n)}\\ &\quad \le \|Du\|_{L^\infty(\Omega_T,\mathbb{R}^n)} + s(\|Dw\|_{L^\infty(\Omega_T,\mathbb{R}^n)} + \|Du\|_{L^\infty(\Omega_T,\mathbb{R}^n)}) \le L.\end{aligned}$$

Thus v is an admissible comparison function for the gradient constrained problem and we obtain that

$$\begin{aligned}\iint_{\Omega_T} f(t,Du)\,\mathrm{d}x\mathrm{d}t \le &\iint_{\Omega_T} s\partial_t u(w-u) + sf(t,Dw) + (1-s)f(t,Du)\,\mathrm{d}x\mathrm{d}t\\ &+ \tfrac{s}{2}\,\|w(0)-u_o\|^2_{L^2(\Omega_T)} - \tfrac{s}{2}\,\|w(T)-u(T)\|^2_{L^2(\Omega_T)}\end{aligned}$$

Reabsorbing the integral with $f(t,Du)$ to the left-hand side and dividing by s, we see that u satisfies the variational inequality (1.3). Thus u is a variational solution in the sense of Definition 1.1. □

6. Existence for the unconstrained problem for general integrands

In this section we finish the proof of Theorem 1.2. Note that we only need to consider the case $T < \infty$. Indeed, assume that for any $\tau \in (0,\infty)$ we have constructed a variational solution with initial and boundary datum u_o in the sense of Definition 1.1 such that the gradient bound (1.4) holds in Ω_τ. Let $0 < \tau_1 < \tau_2 < \infty$ and denote by u_1 and u_2 the variational solutions in Ω_{τ_1} and Ω_{τ_2}, respectively. By the localization principle with respect to time in Lemma 3.3, u_2 is also a variational solution in Ω_{τ_1}. Further, u_1 and u_2 coincide in Ω_{τ_1} by the comparison principle in Theorem 3.5. Therefore, a unique global variational solution in the sense of Definition 3.1 can be constructed by taking an increasing sequence of times $(\tau_i)_{i\in\mathbb{N}}$ with $\lim_{i\to\infty}\tau_i = \infty$.

Thus we suppose that $T < \infty$. For $\varepsilon > 0$ we define the Steklov average $f_\varepsilon\colon [0,T]\times\mathbb{R}^n \to \mathbb{R}$ of f by (2.4). A straightforward computation shows that $\xi \mapsto f_\varepsilon(t,\xi)$ is convex for any $t \in [0,T]$. Further, for any $\varepsilon > 0$ the derivative of f_ε with respect to the time variable is given by

$$\partial_t f(t,\xi) = \tfrac{1}{\varepsilon}(f(t+\varepsilon,\xi) - f(t,\xi)).$$

Combining this with (2.1), for any $L > 0$ we have that

$$|\partial_t f(t,\xi)| \le \tfrac{1}{\varepsilon}(g_L(t+\varepsilon) + g_L(t)) \quad \text{for all } t \in [0,T], \xi \in B_L(0).$$

Hence, for any $\varepsilon > 0$, the integrand f_ε fulfills assumption (4.1). By Theorem 5.2 we conclude that for any $\varepsilon > 0$ there exists a variational solution $u_\varepsilon \in K^\infty_{u_o}(\Omega_T)$ to the Cauchy–Dirichlet problem associated with f_ε in the sense of Definition 1.1 satisfying the bound

$$\|Du_\varepsilon\|_{L^\infty(\Omega_T,\mathbb{R}^n)} \le \max\{Q, \|Du_o\|_{L^\infty(\Omega,\mathbb{R}^n)}\}.$$

Together with the fact that $u_\varepsilon = u_o$ on $\partial\Omega\times(0,T)$, this implies in particular that the sequence $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $L^\infty(\Omega_T)$. Thus, there exists a (not relabelled) subsequence and a limit map $u \in L^\infty(\Omega_T)$ such that $u = u_o$ on $\partial\Omega\times(0,T)$,

$$\|Du\|_{L^\infty(\Omega_T,\mathbb{R}^n)} \le \max\{Q, \|Du_o\|_{L^\infty(\Omega,\mathbb{R}^n)}\}$$

and in the limit $\varepsilon \downarrow 0$ there holds

$$\begin{cases} u_\varepsilon \xrightharpoonup{*} u \text{ weakly}^* \text{ in } L^\infty(\Omega_T), \\ u_\varepsilon(t) \to u(t) \text{ uniformly for a.e. } t \in [0,T], \\ Du_\varepsilon \xrightharpoonup{*} Du \text{ weakly}^* \text{ in } L^\infty(\Omega_T,\mathbb{R}^n). \end{cases} \tag{6.1}$$

It remains to show that u is a variational solution to the Cauchy–Dirichlet problem associated with f in the sense of Definition 1.1. To this end, note that u_ε satisfies the variational inequality

$$\begin{aligned}\iint_{\Omega_\tau} f_\varepsilon(t,Du_\varepsilon)\,\mathrm{d}x\mathrm{d}t \le \iint_{\Omega_\tau} \partial_t v(v-u_\varepsilon)\,\mathrm{d}x\mathrm{d}t + \iint_{\Omega_\tau} f_\varepsilon(t,Dv)\,\mathrm{d}x\mathrm{d}t \\ + \tfrac{1}{2}\|v(0)-u_o\|^2_{L^2(\Omega)} - \tfrac{1}{2}\|(v-u_\varepsilon)(\tau)\|^2_{L^2(\Omega)}\end{aligned} \tag{6.2}$$

for any $\tau \in [0,T]\cap\mathbb{R}$ and any comparison map $v \in K^\infty_{u_o}(\Omega_\tau)$ with $\partial_t v \in L^2(\Omega_\tau)$. In the following, we pass to the limit $\varepsilon \downarrow 0$ in (6.2). In order to treat the left-hand side, we rewrite

$$\iint_{\Omega_\tau} f_\varepsilon(t,Du_\varepsilon)\,\mathrm{d}x\mathrm{d}t = \iint_{\Omega_\tau} f(t,Du_\varepsilon)\,\mathrm{d}x\mathrm{d}t + \iint_{\Omega_\tau} f_\varepsilon(t,Du_\varepsilon) - f(t,Du_\varepsilon)\,\mathrm{d}x\mathrm{d}t.$$

By $(6.1)_3$ and Lemma 2.4 we obtain that

$$\iint_{\Omega_\tau} f(t, Du)\,\mathrm{d}x\mathrm{d}t \le \liminf_{\varepsilon\downarrow 0} \iint_{\Omega_\tau} f(t, Du_\varepsilon)\,\mathrm{d}x\mathrm{d}t.$$

Further, for $M := \max\{Q, \|Du_o\|_{L^\infty(\Omega,\mathbb{R}^n)}\}$ we find that

$$\left|\iint_{\Omega_\tau} f_\varepsilon(t, Du_\varepsilon) - f(t, Du_\varepsilon)\,\mathrm{d}x\mathrm{d}t\right| \le |\Omega| \int_0^\tau \sup_{|\xi|\le M} |f_\varepsilon(t,\xi) - f(t,\xi)|\,\mathrm{d}t \to 0$$

as $\varepsilon \downarrow 0$ by means of Lemma 2.7. Joining the preceding two estimates yields

$$\iint_{\Omega_\tau} f(t, Du)\,\mathrm{d}x\mathrm{d}t \le \liminf_{\varepsilon\downarrow 0} \iint_{\Omega_\tau} f_\varepsilon(t, Du_\varepsilon)\,\mathrm{d}x\mathrm{d}t. \tag{6.3}$$

Next, by $(6.1)_1$ we have that

$$\iint_{\Omega_\tau} \partial_t v(v-u)\,\mathrm{d}x\mathrm{d}t = \liminf_{\varepsilon\downarrow 0} \iint_{\Omega_\tau} \partial_t v(v-u_\varepsilon)\,\mathrm{d}x\mathrm{d}t. \tag{6.4}$$

For the second term on the right-hand side of (6.2), by Lemma 2.7 we conclude that

$$\begin{aligned}\left|\iint_{\Omega_\tau} f_\varepsilon(t, Dv) - f(t, Dv)\,\mathrm{d}x\mathrm{d}t\right| \\ \le |\Omega| \int_0^\tau \sup_{|\xi|\le M} |f_\varepsilon(t,\xi) - f(t,\xi)|\,\mathrm{d}t \to 0\end{aligned} \tag{6.5}$$

as $\varepsilon \downarrow 0$. Finally, $(6.1)_2$ shows that

$$\|(v-u)(\tau)\|^2_{L^2(\Omega)} = \lim_{\varepsilon\downarrow 0} \|(v-u_\varepsilon)(\tau)\|^2_{L^2(\Omega)} \tag{6.6}$$

for a.e. $\tau \in [0, T]$. Collecting (6.3)–(6.6), we infer that

$$\begin{aligned}\iint_{\Omega_\tau} f(t, Du)\,\mathrm{d}x\mathrm{d}t \le \iint_{\Omega_\tau} \partial_t v(v-u)\,\mathrm{d}x\mathrm{d}t + \iint_{\Omega_\tau} f(t, Dv)\,\mathrm{d}x\mathrm{d}t \\ + \tfrac{1}{2}\|v(0) - u_o\|^2_{L^2(\Omega)} - \tfrac{1}{2}\|(v-u)(\tau)\|^2_{L^2(\Omega)}\end{aligned}$$

for a.e. $\tau \in [0, T]$ and any $v \in K^\infty_{u_o}(\Omega_\tau)$ with $\partial_t v \in L^2(\Omega_\tau)$. In particular, this implies that $u \in C^0([0, T]; L^2(\Omega))$, see Lemma 3.2. Therefore, we have that $u \in K^\infty_{u_o}(\Omega_T)$ is a variational solution associated with the integrand f in the sense of Definition 1.1. Finally, by the comparison principle in Theorem 3.5, u is unique. This concludes the proof of Theorem 1.2.

7. Continuity in time (Proof of Theorem 1.3)

To prove Theorem 1.3, we begin by verifying that the unique variational solution u to the Cauchy–Dirichlet problem associated with (1.1) and u_o in Ω_T is a weak solution to (1.1) in Ω_T. To this end, let $\varphi \in C^\infty_0(\Omega_T)$ be a test function. We want to show that

$$\iint_{\Omega_T} u\partial_t\varphi\,\mathrm{d}x\mathrm{d}t = \iint_{\Omega_T} D_\xi f(t, Du)\cdot D\varphi\,\mathrm{d}x\mathrm{d}t. \tag{7.1}$$

We set $v_h := [u]_h + s[\varphi]_h$, where in the convolution we use the starting values u_o and $\varphi(0) = 0$ for u and φ, respectively. Using v_h as a comparison function in (1.3) and omitting the boundary term at T, we obtain that

$$0 \le \iint_{\Omega_T} \partial_t v_h (v_h - u)\,\mathrm{d}x\mathrm{d}t + \iint_{\Omega_T} f(t, Dv_h) - f(t, Du)\,\mathrm{d}x\mathrm{d}t. \tag{7.2}$$

Since by (1.4) we have that

$$\begin{aligned}\|Dv_h\|_{L^\infty(\Omega_T,\mathbb{R}^n)} &\le \|Du_o\|_{L^\infty(\Omega,\mathbb{R}^n)} + \|Du\|_{L^\infty(\Omega_T,\mathbb{R}^n)} + \|D\varphi\|_{L^\infty(\Omega_T,\mathbb{R}^n)} \\ &\le 2\|Du_o\|_{L^\infty(\Omega,\mathbb{R}^n)} + Q + \|D\varphi\|_{L^\infty(\Omega_T,\mathbb{R}^n)},\end{aligned}$$

it follows from (2.1) that the sequence of mappings $(x,t) \mapsto f(t, Dv_h(x,t))$ has an integrable dominant independent of h. Therefore by the dominated convergence theorem, we conclude that

$$\lim_{h\downarrow 0} \iint_{\Omega_T} f(t, Dv_h)\,\mathrm{d}x\mathrm{d}t = \iint_{\Omega_T} f(t, Du + sD\varphi)\,\mathrm{d}x\mathrm{d}t.$$

Further, by integration by parts and the convergence assertions from Lemmas 2.8 and 2.9, we find that

$$\begin{aligned}&\iint_{\Omega_T} \partial_t v_h (v_h - u)\,\mathrm{d}x\mathrm{d}t \\ &\quad = \iint_{\Omega_T} \partial_t [u]_h([u]_h - u) + s\partial_t[u]_h[\varphi]_h + s\partial_t[\varphi]_h([u]_h + s[\varphi]_h - u)\,\mathrm{d}x\mathrm{d}t \\ &\quad = \iint_{\Omega_T} \tfrac{1}{h}(u - [u]_h)([u]_h - u) - s\partial_t[\varphi]_h u\,\mathrm{d}x\mathrm{d}t \\ &\qquad + \int_\Omega s[u]_h[\varphi]_h(T) + \tfrac{s^2}{2}[\varphi]_h^2(T)\,\mathrm{d}x \\ &\quad \le -\iint_{\Omega_T} s\partial_t[\varphi]_h u\,\mathrm{d}x\mathrm{d}t + \int_\Omega s[u]_h[\varphi]_h(T) + \tfrac{s^2}{2}[\varphi]_h^2(T)\,\mathrm{d}x \\ &\quad \to -\iint_{\Omega_T} s\partial_t\varphi u\,\mathrm{d}x\mathrm{d}t\end{aligned}$$

in the limit $h \downarrow 0$. Thus, letting $h \downarrow 0$ in (7.2) and dividing by s we deduce that

$$\begin{aligned}\iint_{\Omega_T} u\partial_t\varphi\,\mathrm{d}x\mathrm{d}t &\le \iint_{\Omega_T} \tfrac{1}{s}(f(t, Du + sD\varphi) - f(t, Du))\,\mathrm{d}x\mathrm{d}t \\ &= \iint_{\Omega_T} \int_0^1 D_\xi f(t, Du + s\sigma D\varphi)\cdot D\varphi\,\mathrm{d}\sigma\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Finally, observe that by the gradient bound (1.4) and the assumption (1.5), the integrand at the right-hand side of the above inequality is bounded. Thus we may let $s \to 0$ to obtain that

$$\iint_{\Omega_T} u\partial_t\varphi\,\mathrm{d}x\mathrm{d}t \le \iint_{\Omega_T} D_\xi f(t, Du)\cdot D\varphi\,\mathrm{d}x\mathrm{d}t.$$

The reverse inequality in (7.1) follows by replacing φ by $-\varphi$.

Consider cylinders of the form

$$Q_r := B_r(x_0) \times (t_0 - r^2, t_0 + r^2) \cap \Omega_T$$

where $(x_0, t_0) \in \overline{\Omega_T}$ and $r > 0$. We show that u satisfies the Poincaré inequality

$$\begin{aligned} &\fint\!\!\fint_{Q_r} |u - (u)_{Q_r}|^2 \,\mathrm{d}x\mathrm{d}t \\ &\qquad \le C(n,\Omega) r^2 \Big(\fint\!\!\fint_{Q_r} |Du|^2 \,\mathrm{d}x\mathrm{d}t + \sup_{(x,t)\in Q_r} |D_\xi f(t, Du(x,t))|^2 \Big) \end{aligned} \tag{7.3}$$

for all small $r > 0$, where the mean value of u over Q_r is denoted by

$$(u)_{Q_r} := \fint\!\!\fint_{Q_r} u \,\mathrm{d}x\mathrm{d}t.$$

Thus the gradient bound (1.4) together with condition (1.5) yields

$$\fint\!\!\fint_{Q_r} |u - (u)_{Q_r}|^2 \,\mathrm{d}x\mathrm{d}t \le C\big(n, \Omega, Q, \|Du_o\|_{L^\infty(\Omega,\mathbb{R}^n)}, f\big) r^2 \tag{7.4}$$

for all $r > 0$. The claim then follows from [9, Theorem 3.1].

To prove (7.3), we first note that since Ω is a convex domain, there exist positive constants $R(\Omega)$ and $C(\Omega)$ such that for any $r \in (0, R)$ and $x_0 \in \overline{\Omega}$, the set $\Omega \cap B_r(x_0)$ contains a ball of radius $r/C(\Omega)$. Then we assume that Q_r with $r < R$ is given and denote $B_r := B_r(x_0)$, $t_1 := \max(t_0 - r^2, 0)$, $t_2 := \min(t_0 + r^2, T)$ so that $Q_r = (B_r \cap \Omega) \times (t_1, t_2)$. We fix a non-negative weight function $\eta \in C_0^\infty(B_r \cap \Omega)$ such that

$$\fint_{B_r\cap\Omega} \eta \,\mathrm{d}x = 1 \quad \text{and} \quad \|\eta\|_{L^\infty(\Omega)} + r\|D\eta\|_{L^\infty(\Omega;\mathbb{R}^n)} \le c(n,\Omega).$$

For the second assertion, we have used that $B_r \cap \Omega$ contains a ball of size $r/C(\Omega)$. Since $B_r \cap \Omega$ is convex, the Poincaré inequality

$$\int_{B_r\cap\Omega} \left|v - (v)_{B_r\cap\Omega}\right|^2 \mathrm{d}x \le \frac{r^2}{\pi^2} \int_{B_r\cap\Omega} |Dv|^2 \,\mathrm{d}x$$

holds for any $v \in W^{1,2}(B_r \cap \Omega)$, see for example [1]. An application of Hölder's and Minkowski's inequalities on the above further yields

$$\fint_{B_r\cap\Omega} |v - (v\eta)_{B_r\cap\Omega}|^2 \,\mathrm{d}x \le c r^2 \fint_{B_r\cap\Omega} |Dv|^2 \,\mathrm{d}x \tag{7.5}$$

with a constant $c = c(n, \Omega)$. We denote the weighted mean of u at time t by

$$u_\eta(t) := \fint_{B_r\cap\Omega} u(x,t)\eta(x) \,\mathrm{d}x$$

and decompose the left-hand side of (7.3) as follows

$$\begin{aligned}
&\fint\!\!\fint_{Q_r} |u-(u)_{Q_r}|^2 \,\mathrm{d}x\mathrm{d}t \\
&\quad\le c\fint_{t_1}^{t_2}\fint_{B_r\cap\Omega} |u_\eta(t)-(u)_{Q_r}|^2 \,\mathrm{d}x\mathrm{d}t + c\fint_{t_1}^{t_2}\fint_{B_r\cap\Omega} |u(x,t)-u_\eta(t)|^2 \,\mathrm{d}x\mathrm{d}t \\
&\quad= c\fint_{t_1}^{t_2} \Big|\fint_{t_1}^{t_2} u_\eta(t)-u_\eta(s)\,\mathrm{d}s + \fint_{t_1}^{t_2} u_\eta(s)\,\mathrm{d}s - (u)_{Q_r}\Big|^2 \,\mathrm{d}t \\
&\qquad + c\fint_{t_1}^{t_2}\fint_{B_r\cap\Omega} |u(x,t)-u_\eta(t)|^2 \,\mathrm{d}x\mathrm{d}t \\
&\quad\le c\fint_{t_1}^{t_2}\fint_{t_1}^{t_2} |u_\eta(t)-u_\eta(s)|^2 \,\mathrm{d}s\mathrm{d}t + c\Big|\fint_{t_1}^{t_2} u_\eta(s)\,\mathrm{d}s - (u)_{Q_r}\Big|^2 \\
&\qquad + c\fint_{t_1}^{t_2}\fint_{B_r\cap\Omega} |u(x,t)-u_\eta(t)|^2 \,\mathrm{d}x\mathrm{d}t \\
&\quad=: c\,(I_1+I_2+I_3).
\end{aligned}$$

To estimate I_3, we apply (7.5) to obtain that

$$I_3 \le c(n,\Omega) r^2 \fint_{B_r\cap\Omega} |Du|^2 \,\mathrm{d}x.$$

The same estimate holds for I_2 since by Hölder's inequality we have that

$$I_2 = \Big|\fint_{t_1}^{t_2}\fint_{B_r\cap\Omega} u_\eta(s)-u(x,s)\,\mathrm{d}x\mathrm{d}s\Big|^2 \le I_3.$$

To estimate I_1, let $\tau_1,\tau_2 \in (t_1,t_2)$ with $\tau_1<\tau_2$. As shown above, u is a weak solution to (1.1). That is, abbreviating $F(x,t) := D_\xi f(t, Du(x,t))$, we find that

$$\int_0^T\int_\Omega u\partial_t\varphi - F\cdot D\varphi \,\mathrm{d}x\mathrm{d}t = 0 \quad \text{for all } \varphi\in W_0^{1,\infty}(\Omega_T).$$

Fix $\delta>0$ and consider

$$\psi_\delta(t) := \begin{cases} 0, & t\in(0,\tau_1-\delta], \\ \frac{1}{2\delta}(t-(\tau_1-\delta)), & t\in(\tau_1-\delta,\tau_1+\delta), \\ 1, & t\in[\tau_1+\delta,\tau_2-\delta], \\ 1-\frac{1}{2\delta}(t-(\tau_2-\delta)), & t\in(\tau_2-\delta,\tau_2+\delta), \\ 0, & t\in[\tau_2+\delta,T). \end{cases}$$

Using the test function $\varphi(x,t) := \eta\psi_\delta$ in the weak Euler–Lagrange equation yields

$$\begin{aligned}
0 &= \int_0^T\int_\Omega u\eta\partial_t\psi_\delta - \psi_\delta F\cdot D\varphi \,\mathrm{d}x\mathrm{d}t \\
&= \fint_{\tau_1-\delta}^{\tau_1+\delta}\int_{B_r\cap\Omega} u\eta \,\mathrm{d}x\mathrm{d}t - \fint_{\tau_2-\delta}^{\tau_2+\delta}\int_{B_r\cap\Omega} u\eta \,\mathrm{d}x\mathrm{d}t + \int_{\tau_1-\delta}^{\tau_2+\delta}\int_{B_r\cap\Omega} \psi_\delta F\cdot D\eta \,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Passing to the limit $\delta \downarrow 0$, the preceding inequality implies that

$$\begin{aligned}|u_\eta(\tau_1) - u_\eta(\tau_2)| &\le \int_{\tau_1}^{\tau_2} \fint_{B_r \cap \Omega} |F \cdot D\eta| \, \mathrm{d}x\mathrm{d}t \\ &\le (\tau_2 - \tau_1) \|D\eta\|_{L^\infty(\Omega, \mathbb{R}^n)} \sup_{(x,t) \in Q_r} |F(x,t)| \\ &= 2c(n, \Omega) r \sup_{(x,t) \in Q_r} |D_\xi f(t, Du(x,t))|\end{aligned}$$

holds true for almost every $\tau_1, \tau_2 \in (t_1, t_2)$. In the last inequality, we used that $\tau_2 - \tau_1 \le t_2 - t_1 \le 2r^2$. Thus

$$I_1 \le c(n, \Omega) r^2 \sup_{(x,t) \in Q_r} |D_\xi f(t, Du(x,t))|^2. \tag{7.6}$$

Inequality (7.3) now follows by combining the estimates of I_1, I_2 and I_3.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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