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Behavior in time of solutions of a Keller–Segel system with flux limitation and source term

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Abstract. In this paper we consider radially symmetric solutions of the following parabolic–elliptic cross-diffusion system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u f(|\nabla v|^2)\nabla v) + g(u), \\ 0 = \Delta v - m(t) + u, \quad \int_\Omega v\, dx = 0, \\ u(x,0) = u_0(x), \end{cases}$$

in $\Omega \times (0,\infty)$, with Ω a ball in $\mathbb{R}^N$, $N \geq 3$, under homogeneous Neumann boundary conditions, where $g(u) = \lambda u - \mu u^k$, $\lambda > 0$, $\mu > 0$, and $k > 1$, $f(|\nabla v|^2) = k_f(1 + |\nabla v|^2)^{-\alpha}$, $\alpha > 0$, which describes gradient-dependent limitation of cross diffusion fluxes. The function $m(t)$ is the time dependent spatial mean of $u(x,t)$ i.e. $m(t) := \frac{1}{|\Omega|}\int_\Omega u(x,t)\, dx$. Under smallness conditions on α and k, we prove that the solution $u(x,t)$ blows up in L^∞-norm at finite time T_{max} and for some $p > 1$ it blows up also in L^p-norm. In addition a lower bound of blow-up time is derived. Finally, under largeness conditions on α or k, we prove that the solution is global and bounded in time.

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1. Introduction

Let us consider the chemotaxis system with flux limitation with source term,

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u f(|\nabla v|^2) \nabla v) + g(u), & x \in \Omega,\ t > 0, \\ 0 = \Delta v - m(t) + u, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

with Ω a ball in $\mathbb{R}^N$, $N \geq 3$, $m(t) = \frac{1}{|\Omega|} \int u(x,t)\, dx > 0$, $\int_\Omega v\, dx = 0$,

$$f(|\nabla v|^2) = k_f (1 + |\nabla v|^2)^{-\alpha} \tag{1.2}$$

with some $k_f > 0$ and $\alpha > 0$,

$$g(u) = \lambda u - \mu u^k \tag{1.3}$$

with $\lambda > 0,\ \mu > 0$, and $k > 1$, u_0 is a given nonnegative function.

The chemotaxis model (1.1) with $g(u) = 0$ and $f(|\nabla v|^2) = 1$ is just the classical Keller–Segel system (see [11]), which permits the concentration phenomena to result in the possible blowing up of solutions, and has been extensively studied since 1970 s, such as the existence of global bounded solutions and the detection of some solutions blowing up in either finite or infinite time, in a great number of literature (see [1,5,6,9,12,13,15–17] and the references therein).

We refer that in the case $f(|\nabla v|^2) = 1$, $\chi > 0$ with $g(u) = \lambda u - \mu u^k$, $\lambda \geq 0,\ \mu \geq 0$, and $1 < k < \frac{3}{2} + \frac{1}{2n-2}$, Ω a ball in $\mathbb{R}^N$, with $N \geq 5$, Winkler in [20] proved that there exist initial data such that the radially symmetric solution blows up in finite time. In [21], with Ω a ball in $\mathbb{R}^N, N \geq 3, \lambda \in \mathbb{R}, \mu > 0, k > 1$, and with $m(t)$ replaced by the function $v(x,t)$ in the second equation, under the assumption

$$k < \begin{cases} \frac{7}{6}, & \text{if } \ N \in \{3,4\}, \\ 1 + \frac{1}{2(N-1)}, & \text{if } \ N \geq 5, \end{cases}$$

the author derived a condition on the initial data sufficient to ensure the occurrence of blowing up solutions in finite time.

The range of k has been improved by Fuest in [8], where a nonnegative initial datum u_0 has been constructed such that the solution blows up in finite time when $\chi = 1$,

$$\begin{cases} 1 < k < \min\left\{2, \frac{N}{2}\right\}, & \mu > 0, \qquad \text{for } \ N \geq 3, \\ k = 2, & \mu \in \left(0, \frac{N-4}{N}\right), \quad \text{for } \ N \geq 5. \end{cases}$$

The value $k = 2$ is critical in the four and higher dimensions.

Recently the case f depending on the gradient of v (flux limitation term) received considerable attention in the biomathematical literature.

The most relevant results on flux limitation concern the case $g(u) = 0$.

In particular

◇ If $f(|\nabla v|^2) = |\nabla v|^{p-2}$, $\chi > 0$, $\Omega \subset \mathbb{R}^N$,

$$p \in (1, \infty) \quad \text{for } N = 1; \qquad p \in \left(1, \frac{N}{N-1}\right) \quad \text{for } N \geq 2,$$

Negreanu and Tello [17] obtained uniform bounds in $L^\infty(\Omega)$ and the existence of global in time solutions; for the one-dimensional case there exist infinitely many non-constant steady-states for $p \in (1, 2)$.

◇ If $f(|\nabla v|^2) = \frac{1}{\sqrt{1+|\nabla v|^2}}$ and Δu is replaced by $\nabla \cdot \left(\frac{u \nabla u}{\sqrt{u^2+|\nabla u|^2}}\right)$, Bellomo and Winkler [2] obtained the global existence of bounded classical solutions for arbitrary positive radial initial data $u_0 \in C^3(\overline{\Omega})$ when

$$\int_\Omega u_0 < \frac{1}{\sqrt{(\chi^2 - 1)_+}}, \quad \text{if } N = 1; \qquad \chi < 1, \quad N \geq 2.$$

In Bellomo and Winkler [3], the authors shows that the above conditions are essentially optimal in the sense that if $\chi > 1$ and

$$m > \frac{1}{\sqrt{\chi^2 - 1}}, \quad \text{if } N = 1; \qquad m > 0 \text{ arbitrary, if } N \geq 2$$

there exists $u_0 \in C^3(\overline{\Omega})$ with $\int_\Omega u_0 = m$, such that there exists a a unique blowing up classical solution.

◇ If $f(|\nabla v|^2) \geq K_f \left(1 + |\nabla v|^2\right)^{-\alpha}$, $K_f > 0$, $\chi = 1$, $0 < \alpha < \frac{N-2}{2(N-1)}$, Ω a ball in $\mathbb{R}^N$, with $N \geq 3$, for a considerably large set of radially symmetric initial data, the problem admits solutions blowing up in finite time in L^∞-norm for the first component. Otherwise, if $f(|\nabla v|^2) \leq K_f \left(1 + |\nabla v|^2\right)^{-\alpha}$, $\chi = 1$ and α satisfies

$$\begin{cases} \alpha > \frac{N-2}{2(N-1)}, & \text{for } N \geq 2, \\ \alpha \in \mathbb{R}, & \text{for } N = 1, \end{cases}$$

in general (not symmetric setting), a global bounded solution exists [22].

The case $\alpha = \frac{N-2}{2(N-1)}$ plays the role of a critical exponent and it is still an open problem.

◇ If $f(|\nabla v|^2) = K_f \left(1 + |\nabla v|^2\right)^{-\alpha}$, $K_f > 0$, $\chi = 1$, $0 < \alpha < \frac{N-2}{2(N-1)}$, $\Omega = B_R(0) \subset \mathbb{R}^N$, with $N \geq 3$, Marras, Vernier-Piro and Yokota [14], for suitable initial data, proved that a solution which blows up in L^∞-norm blows up also in L^p-norm for some $p > \frac{N}{2}$. Moreover, a safe time interval of existence of the solution $[0, T]$ is obtained, with T a lower bound of the blow-up time.

Less attention was payed to the case with f depending on the gradient of v in presence of a source term $g(u)$.

It is the purpose of the present paper to address the above question for a class of functions $g(u)$ modeling sources of logistic type: $g(u) = \lambda u - \mu u^k$, $\lambda > 0$, $\mu > 0$, and $k > 1$.

Main results The present work is addressed to study the behavior in time of the solutions of problem (1.1) with $\chi = 1$ in presence of the flux limitation term and the source term $g(u) = \lambda u - \mu u^k$ to varying $k \in (1, 2]$. In particular

in Sect. 3 we construct an initial data such that the solution of problem (1.1) blows up in L^∞-norm in the following sense.

Theorem 1.1. (Finite-time blow-up in L^∞-norm) Let $\Omega = B_R(0) \subset \mathbb{R}^N$, $R > 0$, $N \geq 3$. Suppose that

$$0 < \alpha < \frac{N-2}{2(N-1)}. \tag{1.4}$$

Then for all $m_0 > 0$ there exist radially symmetric as well as radially decreasing initial data

$$u_0 \in C^0(\bar{\Omega}), \quad u_0 \not\equiv 0 \tag{1.5}$$

with

$$\frac{1}{|\Omega|}\int_\Omega u_0\, dx = m_0$$

and some positive constant μ_0 such that if

$$\begin{aligned} N \geq 3, \qquad & k \in \left(1, \min\left\{2, 1+\frac{(N-2)^2}{4}\right\}\right) \quad \text{and } \mu > 0 \\ \text{or } N \geq 5, \qquad & k = 2 \quad \text{and } 0 < \mu \leq \mu_0, \end{aligned}$$

then (1.1) possesses a unique classical solution (u, v) in $\Omega \times (0, T_{max})$, for some $T_{max} \in (0, \infty)$, which blows up at T_{max} in the sense that

$$\limsup_{t \nearrow T_{max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \tag{1.6}$$

The second purpose of this paper is to prove that the solutions of (1.1) blow up at finite time in L^p-norm, for some $p > 1$, if they blow up in L^∞-norm (Sect. 4).

Theorem 1.2. (Finite-time blow-up in L^p-norm) *Let $\Omega = B_R(0) \subset \mathbb{R}^N$, $N \geq 3$ and $R > 0$. Then, a classical solution (u, v) of (1.1) for $t \in (0, T_{max})$, provided by Theorem 1.1, is such that for all $p > \frac{N}{2}$,*

$$\limsup_{t \nearrow T_{max}} \|u(\cdot, t)\|_{L^p(\Omega)} = \infty.$$

The investigation on blow-up solutions of system (1.1) goes on with the study of the behavior near the blow-up time T_{max} (Sect. 5). The goal is to obtain a safe time interval $(0, T)$, $(T < T_{max})$, of existence of the solutions of (1.1); to this end, we define, for all $p > 1$, the auxiliary function

$$\Psi(t) := \frac{1}{p}\|u(\cdot, t)\|^p_{L^p(\Omega)} \quad \text{with} \quad \Psi_0 := \Psi(0) = \frac{1}{p}\|u_0\|^p_{L^p(\Omega)}, \tag{1.7}$$

and we determine a lower estimate of the blow-up time T_{max}.

Theorem 1.3. (Lower bound of blow-up time) *Let $\Omega = B_R(0) \subset \mathbb{R}^N$, $N \geq 3$, $R > 0$ and let Ψ be defined in (1.7). Then, for all $p > \frac{N}{2}$ and some positive constants B_1, B_2, B_3, B_4, the blow-up time T_{max} for (1.1), provided by Theorem 1.1, satisfies the estimate*

$$T_{max} \geq T := \int_{\Psi_0}^\infty \frac{d\eta}{B_1\eta + B_2\eta^{\gamma_1} + B_3\eta^{\gamma_2} + B_4\eta^{\gamma}}, \tag{1.8}$$

with $\gamma_1 := \frac{p+1}{p}$, $\gamma_2 := \frac{2(p+1)-N}{2p-N}$, $\gamma := \frac{2(p+1)-\frac{N(p+1)(1+\epsilon)}{p+1+\epsilon}}{2p-\frac{N(1+\epsilon)(p+1)}{p+1+\epsilon}}$, $0 < \epsilon < \frac{2p}{N} - 1$.

Corollary 1.4. *Under the assumptions of Theorem 1.2, let* (u, v) *be a solution of (1.1) and* $\Psi(t)$ *and* Ψ_0 *defined in (1.7). Then there exists a safe interval of existence of* (u, v) *say* $[0, T]$ *with*

$$T := \frac{1}{\mathcal{A}(\gamma-1)\Psi_0^{\gamma-1}} \leq T_{max}.$$

We remark that $\frac{1}{\mathcal{A}(\gamma-1)\Psi_0^{\gamma-1}}$ *is explicitly computable.*

We observe that the blow-up phenomena can be avoided for different choises of the data. Moreover, we will prove that the results in Theorem 1.1 with $f(|\nabla v|^2) = k_f(1 + |\nabla v|^2)^{-\alpha}$ fulfilling $0 < \alpha < \frac{N-2}{2(N-1)}$ and $\kappa \leq 2$ cannot be improved. In fact if $\alpha > \frac{N-2}{2(N-1)}$ or $\kappa > 2$ we obtain that the global solution is bounded (Sect. 6).

Theorem 1.5. (Global existence and boundedness) *Let* $\Omega = B_R(0) \subset \mathbb{R}^N$, $N \geq 3$, $R > 0$. *Assume that either one of the following two conditions is satisfied:*

1. $\alpha > \dfrac{N-2}{2(N-1)}$ *and* $k > 1$,
2. $\alpha > 0$ *and* $k > 2$.

Then for all radially symmetric nonnegative initial data $u_0 \in C^0(\bar{\Omega})$, *system (1.1) possesses a unique global classical solution* (u, v) *in* $\Omega \times (0, \infty)$, *which is bounded in the sense that*

$$\sup_{t\in(0,\infty)} \|u(\cdot, t)\|_{L^\infty(\Omega)} < \infty.$$

2. Preliminaries

In this section, we present some preliminary lemmata which we shall use in the proof of our main results.

Lemma 2.1. *Let* $N \geq 1$, *and assume that* $\Omega = B_R(0) \subset \mathbb{R}^N$ *for some* $R > 0$, f, g *satisfy (1.2), (1.3) and that* $u_0 \in C^0(\bar{\Omega})$ *is nonnegative and radially symmetric with respect to* $x = 0$. *Then there exist* $T_{max} \in (0, \infty]$ *and a unique pair*

$$(u, v) \in \Big((C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max}))\Big)^2$$

which solves (1.1) in the classical sense in $\Omega \times (0, T_{max})$. *Moreover, we have* $u > 0$ *in* $\Omega \times (0, T_{max})$, *and both* $u(\cdot, t)$ *and* $v(\cdot, t)$ *are radially symmetric with respect to* $x = 0$ *for all* $t \geq 0$. *Finally,*

$$\text{if } T_{max} < \infty, \quad \text{then } \limsup_{t\nearrow T_{max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty.$$

We next give some properties of the Neumann heat semigroup which will be used later. For the proof, see [4, Lemma 2.1] and [19, Lemma 1.3].

Lemma 2.2. *Let $(e^{t\Delta})_{t\geq 0}$ be the Neumann heat semigroup in Ω, and let $\mu_1 > 0$ denote the first non zero eigenvalue of $-\Delta$ in Ω under Neumann boundary conditions. Then there exist $k_1, k_2 > 0$ which depend only on Ω and have the following properties:*

1. *if $1 \leq q \leq \mathrm{p} \leq \infty$, then*

$$\|e^{t\Delta} z\|_{L^{\mathrm{p}}(\Omega)} \leq k_1\big(1 + t^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{\mathrm{p}})}\big)e^{-\mu_1 t}\|z\|_{L^q(\Omega)}, \quad \forall\, t > 0 \tag{2.1}$$

   holds for all $z \in L^q(\Omega)$ satisfying $\int_\Omega z = 0$.
2. *If $1 < q \leq \mathrm{p} \leq \infty$, then*

$$\|e^{t\Delta}\nabla \cdot \boldsymbol{z}\|_{L^{\mathrm{p}}(\Omega)} \leq k_2\big(1 + t^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{q}-\frac{1}{\mathrm{p}})}\big)e^{-\mu_1 t}\|\boldsymbol{z}\|_{L^q(\Omega)}, \quad \forall\, t > 0 \tag{2.2}$$

   is valid for any $\boldsymbol{z} \in (L^q(\Omega))^N$, where $e^{t\Delta}\nabla\cdot$ is the extension of the operator $e^{t\Delta}\nabla\cdot$ on $(C_0^\infty(\Omega))^N$ to $(L^q(\Omega))^N$.

We observe that since constants are invariant under $e^{t\Delta}$ we can use (2.1) writing $\bar{z} = \frac{1}{|\Omega|}\int_\Omega z\,dx$ so that we have $\int_\Omega (z - \bar{z})\,dx = 0$ (see [19]).

We now derive an upper bound of the total mass functional $\int_\Omega u(x,t)dx$ in short time intervals.

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded and smooth domain, and $\lambda > 0$, $\mu > 0$, $k > 1$. Then for a solution (u, v) of (1.1) we have*

$$\int_\Omega u\,dx \leq \bar{m}, \quad \text{for all } t \in (0, \bar{T}), \tag{2.3}$$

with

$$\bar{m} = e^{\lambda}\int_\Omega u_0\,dx, \qquad \bar{T} = \min(1, T_{max}). \tag{2.4}$$

Proof. From the first equation in (1.1) we obtain

$$\frac{d}{dt}\int_\Omega u\,dx = \lambda\int_\Omega u\,dx - \mu\int_\Omega u^k dx \leq \lambda\int_\Omega u\,dx. \tag{2.5}$$

From (2.5) we infer that $z = \int_\Omega udx$, with $z(0) = z_0 = \int_\Omega u_0(x)dx$, satisfies

$$\begin{cases} z'(t) \leq \lambda z(t), & \text{for all } t \in [0, T_{max}), \\ z(0) = z_0. \end{cases}$$

From which we have

$$z(t) \leq e^{\lambda t} z_0, \quad \text{for all } t \in (0, T_{max}).$$

This clearly proves the lemma. □

In Sect. 5 we will use the Gagliardo–Nirenberg inequality in the following form.

Lemma 2.4. *Let Ω be a bounded and smooth domain of $\mathbb{R}^N$ with $N \geq 1$. Let $\mathsf{r} \geq 1$, $1 \leq \mathsf{q} < \mathsf{p} \leq \infty$, $\mathsf{s} > 0$. Then there exists a constant $C_{\mathrm{GN}} > 0$ such that*

$$\|f\|^{\mathsf{p}}_{L^{\mathsf{p}}(\Omega)} \leq C_{\mathrm{GN}}\Big(\|\nabla f\|^{\mathsf{p}a}_{L^{\mathsf{r}}(\Omega)}\|f\|^{\mathsf{p}(1-a)}_{L^{\mathsf{q}}(\Omega)} + \|f\|^{\mathsf{p}}_{L^{\mathsf{s}}(\Omega)}\Big) \tag{2.6}$$

for all $f \in L^{\mathsf{q}}(\Omega)$ with $\nabla f \in (L^{\mathsf{r}}(\Omega))^N$ and $a := \frac{\frac{1}{\mathsf{q}} - \frac{1}{\mathsf{p}}}{\frac{1}{\mathsf{q}} + \frac{1}{N} - \frac{1}{\mathsf{r}}} \in (0,1)$.

Proof. Following from the Gagliardo–Nirenberg inequality (see [18] for more details):

$$\|f\|^{\mathsf{p}}_{L^{\mathsf{p}}(\Omega)} \leq \Big[c_{\mathrm{GN}}\Big(\|\nabla f\|^{a}_{L^{\mathsf{r}}(\Omega)}\|f\|^{1-a}_{L^{\mathsf{q}}(\Omega)} + \|f\|_{L^{\mathsf{s}}(\Omega)}\Big)\Big]^{\mathsf{p}},$$

with some $c_{\mathrm{GN}} > 0$, and then from the inequality

$$(\mathsf{a}+\mathsf{b})^{\mathsf{p}} \leq 2^{\mathsf{p}}(\mathsf{a}^{\mathsf{p}} + \mathsf{b}^{\mathsf{p}}) \quad \text{for any } \mathsf{a}, \mathsf{b} \geq 0,\ \mathsf{p} > 0,$$

we arrive to (2.6) with $C_{\mathrm{GN}} = 2^{\mathsf{p}} c^{\mathsf{p}}_{\mathrm{GN}}$. □

Lemma 2.5. *Let $\beta > 0$, $\delta > 0$, $\gamma > 0$ and suppose that for some $T > 0$, $y \in C^0([0,T])$ is a nonnegative function satisfying*

$$y(t) \geq \beta + \delta \int_0^t y^{1+\gamma}(\tau)\, d\tau \quad \forall\, t \in (0,T).$$

Then $T \leq \frac{1}{\gamma\delta\beta^{\gamma}}$.

For the proof see [20, Lemma 2.4].

3. Blow-up in L^{∞}-norm

Transformation in nonlocal scalar parabolic equation:

Assume $\Omega = B_R(0)$, $R > 0$ and $u_0 \in C^0(\bar{\Omega})$ is radially symmetric with respect to $x = 0$. If (u, v) is the corresponding radial solution in $\Omega \times (0, T_{max})$ asserted by Lemma 2.1, we write $u = u(r,t)$ and $v = v(r,t)$ with $r = |x| \in [0, R]$.

Following Jäger–Luckhaus [10] we introduce the mass accumulation function

$$w(s,t) := \int_0^{s^{\frac{1}{N}}} \rho^{N-1} u(\rho, t)\, d\rho, \quad s = r^N \in [0, R^N], \quad t \in [0, T_{max}). \tag{3.1}$$

We have

$$w_s(s,t) = \frac{1}{N} u(s^{\frac{1}{N}}, t) \geq 0, \quad w_{ss}(s,t) = \frac{1}{N^2} s^{\frac{1}{N}-1} u_r(s^{\frac{1}{N}}, t).$$

From the second equation in (1.1) we deduce

$$\frac{1}{r^{N-1}}\big(r^{N-1} v_r(r,t)\big)_r = m(t) - u$$

and

$$r^{N-1} v_r(r,t) = m(t) \int_0^r \rho^{N-1}\, d\rho - \int_0^r \rho^{N-1} u(\rho, t)\, d\rho$$

$$= \frac{m(t)r^N}{N} - \int_0^r \rho^{N-1}u(\rho,t)\,d\rho.$$

Using (1.1) we obtain

$$\begin{aligned}
w_t(s,t) &= \int_0^{s^{\frac{1}{N}}} \rho^{N-1}u_t(\rho,t)\,d\rho \\
&= \int_0^{s^{\frac{1}{N}}} \left(\rho^{N-1}u_r\right)_r(\rho,t)\,d\rho - \int_0^{s^{\frac{1}{N}}} \left(\rho^{N-1}u(\rho,t)v_r f(v_r^2)\right)_r d\rho \\
&\quad + \lambda\int_0^{s^{\frac{1}{N}}} \rho^{N-1}u(\rho,t)\,d\rho - \mu\int_0^{s^{\frac{1}{N}}} \rho^{N-1}u^k(\rho,t)\,d\rho \\
&= s^{1-\frac{1}{N}}u_r(s^{\frac{1}{N}},t) - s^{1-\frac{1}{N}}uv_r f(v_r^2(s^{\frac{1}{N}},t)) \\
&\quad + \lambda\int_0^{s^{\frac{1}{N}}} \rho^{N-1}u(\rho,t)\,d\rho - \mu\int_0^{s^{\frac{1}{N}}} \rho^{N-1}u^k(\rho,t)\,d\rho \\
&= N^2 s^{2-\frac{2}{N}}w_{ss} + Nw_s\Big(w - \frac{m(t)}{N}s\Big)f\Big(s^{\frac{2}{N}-2}(w-\frac{m(t)}{N}s)^2\Big) \\
&\quad + \lambda w - \mu N^{k-1}\int_0^s w_s^k(\sigma,t)\,d\sigma
\end{aligned}$$

and

$$\begin{cases}
w_t = N^2 s^{2-\frac{2}{N}}w_{ss} + N(w-\frac{m(t)}{N}s)w_s f\big(s^{\frac{2}{N}-2}(w-\frac{m(t)}{N}s)^2\big) \\
\qquad +\lambda w - \mu N^{k-1}\int_0^s w_s^k(\sigma,t)\,d\sigma, \quad s\in(0,R^N),\ t\in(0,T_{max}), \\
w(0,t) = 0, \qquad w(R^N,t) = \frac{\mu R^N}{N}, \quad t\in(0,T_{max}), \\
w(s,0) = w_0(s), \quad s\in(0,R^N)
\end{cases} \tag{3.2}$$

with $w_0(s) = \int_0^{s^{\frac{1}{N}}} \rho^{N-1}u_0(\rho)d\rho$, $s\in[0,R^N]$.

Our aim is to prove that the functional $\int_0^{R^N} s^{-a}w^b(s,t)\,ds$, for suitable $a>0$ and $b\in(0,1)$ blows up in finite time. To this end, we use the estimate $w_s \le \frac{w}{s}$ proved by Fuest ([8, Lemma 3.3]):

Lemma 3.1. *Assume that u_0 satisfies (1.5).*
For all $s\in[0,R^N]$ and $t\in(0,T_{max})$,

$$w_s(s,t) \le \frac{w(s,t)}{s} \le w_s(0,t) \tag{3.3}$$

holds.

Proof. By a similar way as in [2, Lemma 2.3] where $\alpha = \frac{1}{2}$ and as in [7, Lemma 3.7], we can show that $u_r \le 0$ in $(0,R)\times(0,T_{max})$ and following the steps in [8] we arrive to (3.3). □

The next step is to prove that the functional $\int_0^{R^N} s^{-a}w^b(s,t)\,ds$ satisfies a differential inequality. First we obtain the following estimate.

Lemma 3.2. *Assume Lemma 2.3 and $\Omega = B_R(0) \subset \mathbb{R}^N$ with some $R > 0$ and $N \geq 2$. Let $u_0 \in C^0(\bar{\Omega})$ be radial, and let (u, v) denote the solution of (1.1) in $\Omega \times (0, T_{max})$. Then for all $a > 0$ and $b \in (0, 1)$, the function w defined in (3.1) satisfies*

$$\begin{aligned}\frac{1}{b}\int_0^{R^N} s^{-a} w^b(s,t)\,ds \geq\ & \frac{1}{b}\int_0^{R^N} s^{-a} w_0^b(s)\,ds \\ & - k_f \bar{m} |\Omega|^{-1} \int_0^t \int_0^{R^N} s^{1-a} w^{b-1} w_s\,ds d\tau \\ & + \frac{aNk_f}{2(b+1)} \bar{C} \int_0^t \int_0^{R^N} s^{-a-1} w^{b+1}\,ds d\tau \\ & + \frac{1}{2} N k_f \bar{C} \int_0^t \int_0^{R^N} s^{-a} w^b w_s\,ds d\tau \\ & + N^2(1-b) \int_0^t \int_0^{R^N} s^{2-\frac{2}{N}-a} w^{b-2} w_s^2\,ds d\tau \\ & - 2N(N-1) \int_0^t \int_0^{R^N} s^{1-\frac{2}{N}-a} w^{b-1} w_s\,ds d\tau \\ & - \mu N^{k-1} \int_0^t \int_0^{R^N} s^{-a} w^{b-1} \Big(\int_0^s w_s^k d\sigma \Big)\,ds d\tau, \end{aligned} \tag{3.4}$$

with

$$\bar{C} := \Big[\frac{N^2}{N^2 + 2|\Omega|^{-2} \bar{m}^2 R^2} \Big]^{\alpha}, \tag{3.5}$$

and $\bar{m}$ in (2.4).

Proof. Following the steps in [20, Lemma 2.1] we multiply the first equation in (3.2) by $(s+\epsilon)^{-a} w^{b-1}(s,\tau)$, $\epsilon > 0$, and integrate over $s \in (0, R^N)$. We obtain

$$\begin{aligned} & \frac{1}{b}\frac{d}{dt}\int_0^{R^N} (s+\epsilon)^{-a} w^b(s,t)\,ds \\ & \geq N^2 \int_0^{R^N} s^{2-\frac{2}{N}} (s+\epsilon)^{-a} w^{b-1} w_{ss}\,ds \\ & \quad + N \int_0^{R^N} (s+\epsilon)^{-a} w^{b-1} w_s \Big(w - \frac{m(t)}{N} s \Big) f\Big(s^{\frac{2}{N}-2} \Big(w - \frac{m(t)}{N} s \Big)^2 \Big)\,ds \\ & \quad - \mu N^{k-1} \int_0^{R^N} (s+\epsilon)^{-a} w^{b-1} \Big(\int_0^s w_s^k\,d\sigma \Big)\,ds = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3. \end{aligned} \tag{3.6}$$

Integrating by part we have

$$\begin{aligned} \mathcal{I}_1 &= N^2 \int_0^{R^N} s^{2-\frac{2}{N}} (s+\epsilon)^{-a} w^{b-1} w_{ss}\,ds \\ &= N^2 s^{2-\frac{2}{N}} (s+\epsilon)^{-a} w^{b-1} w_s \Big|_0^{R^N} \end{aligned}$$

$$
\begin{aligned}
&- N^2(b-1)\int_0^{R^N} s^{2-\frac{2}{N}}(s+\epsilon)^{-a}w^{b-2}w_s^2\,ds\\
&- N^2\int_0^{R^N}\frac{d}{ds}\big(s^{2-\frac{2}{N}}(s+\epsilon)^{-a}\big)w^{b-1}w_s\,ds\\
&\geq N^2(1-b)\int_0^{R^N} s^{2-\frac{2}{N}}(s+\epsilon)^{-a}w^{b-2}w_s^2\,ds\\
&- 2N(N-1)\int_0^{R^N} s^{1-\frac{2}{N}}(s+\epsilon)^{-a}w^{b-1}w_s\,ds \qquad (3.7)
\end{aligned}
$$

where in the last step we used $\frac{d}{ds}\big(s^{2-\frac{2}{N}}(s+\epsilon)^{-a}\big) = (2-\frac{2}{N})s^{1-\frac{2}{N}}(s+\epsilon)^{-a} - as^{2-\frac{2}{N}}(s+\epsilon)^{-a-1} \leq (2-\frac{2}{N})s^{1-\frac{2}{N}}(s+\epsilon)^{-a}$.

In $\mathcal{I}_2$ we have

$$
\begin{aligned}
\mathcal{I}_2 &= N\int_0^{R^N}(s+\epsilon)^{-a}w^{b-1}w_s\Big(w-\frac{m(t)}{N}s\Big)f\Big(s^{\frac{2}{N}-2}\Big(w-\frac{m(t)}{N}s\Big)^2\Big)\,ds\\
&= N\int_0^{R^N}(s+\epsilon)^{-a}w^{b}w_s f\big(s^{\frac{2}{N}-2}(w-\frac{m(t)}{N}s)^2\big)\,ds\\
&\quad - \int_0^{R^N} s(s+\epsilon)^{-a}w^{b-1}w_s m(t) f\Big(s^{\frac{2}{N}-2}\Big(w-\frac{m(t)}{N}s\Big)^2\Big)\,ds = \mathcal{I}_{21}+\mathcal{I}_{22}.
\end{aligned}
$$

Taking into account that $u \geq 0$ we have $w_s \geq 0$ in $(0,R^N)\times(0,T_{max})$ and from the boundary condition at $s = R^N$ we have $w(s,t) \leq \frac{m(t)R^N}{N}$ for all $s \in [0,R^N]$ and $t \in [0,T_{max})$.

By using $w \leq \frac{m(t)R^N}{N}$ and $s \leq R^N$, using (2.3) we arrive at

$$
\Big(\frac{m(t)}{N}s - w\Big)^2 \leq \frac{m^2(t)}{N^2}s^2 + w^2 \leq 2\frac{m^2(t)}{N^2}R^{2N} \leq 2\frac{|\Omega|^{-2}\bar{m}^2}{N^2}R^{2N}
$$

so that

$$
\begin{aligned}
f\Big(s^{\frac{2}{N}-2}\Big(w-\frac{m(t)}{N}s\Big)^2\Big) &= k_f\frac{1}{\big[1+s^{\frac{2}{N}-2}(\frac{m(t)}{N}s-w)^2\big]^{\alpha}}\\
&\geq k_f\frac{1}{\big[1+2\frac{|\Omega|^{-2}\bar{m}^2}{N^2}R^2\big]^{\alpha}} = k_f\bar{C},
\end{aligned}
$$

where $\bar{C}$ is a constant defined in (3.5).

We now split $\mathcal{I}_{21} = \frac{\mathcal{I}_{21}}{2} + \frac{\mathcal{I}_{21}}{2}$. Computing

$$
\begin{aligned}
\frac{\mathcal{I}_{21}}{2} &= \frac{1}{2}Nk_f\int_0^{R^N}(s+\epsilon)^{-a}w^b w_s f\Big(s^{\frac{2}{N}-2}\Big(w-\frac{m(t)}{N}s\Big)^2\Big)\,ds\\
&\geq \frac{1}{2}Nk_f\bar{C}\int_0^{R^N}(s+\epsilon)^{-a}w^b w_s\,ds
\end{aligned}
$$

and integrating by parts we get

$$\begin{aligned}
&\frac{1}{2}Nk_f\bar{C}\int_0^{R^N}(s+\epsilon)^{-a}w^b w_s\,ds\\
&=\frac{Nk_f}{2(b+1)}\bar{C}(s+\epsilon)^{-a}w^{b+1}\Big|_0^{R^N}\\
&\quad-\frac{Nk_f}{2(b+1)}\bar{C}\int_0^{R^N}\frac{d}{ds}\Big[(s+\epsilon)^{-a}\Big]w^{b+1}\,ds\\
&\geq-\frac{Nk_f}{2(b+1)}\bar{C}\int_0^{R^N}\frac{d}{ds}\Big[(s+\epsilon)^{-a}\Big]w^{b+1}\,ds\\
&=\frac{aNk_f}{2(b+1)}\bar{C}\int_0^{R^N}(s+\epsilon)^{-a-1}w^{b+1}\,ds.
\end{aligned}$$

This leads to

$$\frac{\mathcal{I}_{21}}{2}\geq\frac{aNk_f}{2(b+1)}\bar{C}\int_0^{R^N}(s+\epsilon)^{-a-1}w^{b+1}\,ds. \tag{3.8}$$

Now, since $\frac{1}{\left[1+s^{\frac{2}{N}-2}(\frac{m(t)}{N}s-w)^2\right]^{\alpha}}\leq 1$, we obtain

$$\begin{aligned}
\mathcal{I}_{22}&=-\int_0^{R^N}s(s+\epsilon)^{-a}w^{b-1}w_s m(t)f\Big(s^{\frac{2}{N}-2}\Big(w-\frac{m(t)}{N}s\Big)^2\Big)\,ds\\
&=-k_f\int_0^{R^N}s(s+\epsilon)^{-a}w^{b-1}w_s m(t)\frac{1}{\left[1+s^{\frac{2}{N}-2}(\frac{m(t)}{N}s-w)^2\right]^{\alpha}}\,ds\\
&\geq-k_f\int_0^{R^N}s(s+\epsilon)^{-a}w^{b-1}w_s m(t)\,ds\\
&\geq-k_f\bar{m}|\Omega|^{-1}\int_0^{R^N}s(s+\epsilon)^{-a}w^{b-1}w_s\,ds,
\end{aligned}\tag{3.9}$$

where in the last inequality we used (2.3).

Replacing (3.7), (3.8) and (3.9) in (3.6) and integrating from 0 to $t\in(0,T_{max})$ we arrive to

$$\begin{aligned}
\frac{1}{b}\int_0^{R^N}(s+\epsilon)^{-a}w^b(s,t)\,ds&\geq\frac{1}{b}\int_0^{R^N}(s+\epsilon)^{-a}w_0^b(s)\,ds\\
&\quad-k_f\bar{m}|\Omega|^{-1}\int_0^t\int_0^{R^N}s(s+\epsilon)^{-a}w^{b-1}w_s\,ds d\tau\\
&\quad+\frac{aNk_f}{2(b+1)}\bar{C}\int_0^t\int_0^{R^N}(s+\epsilon)^{-a-1}w^{b+1}\,ds d\tau\\
&\quad+\frac{1}{2}Nk_f\bar{C}\int_0^t\int_0^{R^N}(s+\epsilon)^{-a}w^b w_s\,ds d\tau
\end{aligned}$$

$$+ N^2(1-b)\int_0^t\int_0^{R^N} s^{2-\frac{2}{N}}(s+\epsilon)^{-a}w^{b-2}w_s^2\,dsd\tau$$
$$- 2N(N-1)\int_0^t\int_0^{R^N} s^{1-\frac{2}{N}}(s+\epsilon)^{-a}w^{b-1}w_s\,dsd\tau$$
$$- \mu N^{k-1}\int_0^t\int_0^{R^N}(s+\epsilon)^{-a}w^{b-1}\Big(\int_0^s w_s^k d\sigma\Big)\,dsd\tau. \tag{3.10}$$

Now, from the monotone convergence theorem, taking $\epsilon \searrow 0$ arrive at (3.4) □

Our aim is to construct an integral inequality for $y(t) = \int_0^{R^N} s^{-a}w^b(s,t)\,ds$, $t \in (0, T_{max})$ which ensure that $y(t)$ blows up in finite time inducing the chemotactic collapse of the solution of (1.1).

To this end, we estimate each term in (3.4).

In (3.4) we assume

$$c_1 := \min\left\{N^2(1-b),\ \frac{aNk_f}{2(b+1)}\bar{C}\right\} \tag{3.11}$$

to obtain

$$\frac{1}{b}\int_0^{R^N} s^{-a}w^b(s,t)\,ds \geq \frac{1}{b}\int_0^{R^N} s^{-a}w_0^b(s)\,ds$$
$$+ c_1\int_0^t\int_0^{R^N} s^{-a-1}w^{b+1}\,dsd\tau + \frac{1}{2}Nk_f\bar{C}\int_0^t\int_0^{R^N} s^{-a}w^b w_s\,dsd\tau$$
$$+ c_1\int_0^t\int_0^{R^N} s^{2-\frac{2}{N}-a}w^{b-2}w_s^2\,dsd\tau - k_f\bar{m}|\Omega|^{-1}\int_0^t\int_0^{R^N} s^{1-a}w^{b-1}w_s\,dsd\tau$$
$$- 2N(N-1)\int_0^t\int_0^{R^N} s^{1-\frac{2}{N}-a}w^{b-1}w_s\,dsd\tau$$
$$- \mu N^{k-1}\int_0^t\int_0^{R^N} s^{-a}w^{b-1}\Big(\int_0^s w_s^k\,d\sigma\Big)\,dsd\tau$$
$$= H_1 + H_2 + H_3 + H_4 - H_5 - H_6 - H_7, \qquad \text{for all } t \in (0, T_{max}). \tag{3.12}$$

Lemma 3.3. *Let H_5 and H_6 defined as in (3.12). If*

$$0 < a < \frac{N-2}{N}(b+1), \tag{3.13}$$

then

$$H_5 \leq \frac{1}{2}H_4 + \frac{1}{4}H_2 + c_4 t \tag{3.14}$$

$$H_6 \leq \frac{1}{2}H_4 + \frac{1}{4}H_2 + c_6 t, \ \textit{for all } t \in (0, T_{max}), \tag{3.15}$$

with $c_4,\ c_6 > 0$ and H_2, H_4 defined in (3.12).

Proof. Using Young's inequality we obtain

$$\begin{aligned} H_5 &= k_f \bar{m}|\Omega|^{-1} \int_0^t \int_0^{R^N} s^{1-a} w^{b-1} w_s \, ds d\tau \\ &\le \frac{c_1}{2} \int_0^t \int_0^{R^N} s^{2-\frac{2}{N}-a} w^{b-2} w_s^2 \, ds d\tau + c_2 \int_0^t \int_0^{R^N} s^{\frac{2}{N}-a} w^b \, ds d\tau \\ &\le \frac{c_1}{2} \int_0^t \int_0^{R^N} s^{2-\frac{2}{N}-a} w^{b-2} w_s^2 \, ds d\tau \\ &\quad + \frac{c_1}{4} \int_0^t \int_0^{R^N} s^{-a-1} w^{b+1} \, ds d\tau + c_3 \int_0^t \int_0^{R^N} s^{\frac{2}{N}-a+\frac{N+2}{N}b} \, ds d\tau. \end{aligned}$$

Since (3.13) holds we have $\frac{2}{N} - a + \frac{N+2}{N} b > -1$, and for some $c_4 > 0$ we obtain

$$H_5 \le \frac{1}{2} H_4 + \frac{1}{4} H_2 + c_4 t.$$

To estimate H_6 we apply Young's inequality:

$$\begin{aligned} H_6 &= 2N(N-1) \int_0^t \int_0^{R^N} s^{1-\frac{2}{N}-a} w^{b-1} w_s \, ds d\tau \\ &\le \frac{c_1}{2} \int_0^t \int_0^{R^N} s^{2-\frac{2}{N}-a} w^{b-2} w_s^2 \, ds d\tau + c_5 \int_0^t \int_0^{R^N} s^{-\frac{2}{N}-a} w^b \, ds d\tau \\ &\le \frac{c_1}{2} \int_0^t \int_0^{R^N} s^{2-\frac{2}{N}-a} w^{b-2} w_s^2 \, ds d\tau + \frac{c_1}{4} \int_0^t \int_0^{R^N} s^{-a-1} w^{b+1} \, ds d\tau \\ &\quad + \bar{c}_5 \int_0^t \int_0^{R^N} s^{-\frac{2}{N}-a+\frac{N-2}{N}b} \, ds d\tau \\ &\le \frac{1}{2} H_4 + \frac{1}{4} H_2 + c_6 t, \text{ for all } t \in (0, T_{max}), \end{aligned}$$

with $c_5,\ \bar{c}_5,\ c_6 > 0$ and by (3.13): $-\frac{2}{N} - a + \frac{N-2}{N} b > -1$. □

In order to estimate the term H_7 in (3.12) we prove the following lemma.

Lemma 3.4. *Let $N \ge 3$, $R > 0$ and H_7 be as in (3.12).*

◇ If $k = 2$ and u_0 satisfies (1.5), then there exists a constant $\mu_0 > 0$ such that for all $\mu \in (0, \mu_0]$ one can find $a > 1$ and $b \in (0,1)$ fulfilling (3.13) and

$$H_7 \le \frac{1}{4} H_2. \tag{3.16}$$

◇ If $k \in \left(1, \min\left\{2, 1 + \frac{(N-2)^2}{4}\right\}\right)$, then for all $\mu > 0$ one can find $a, b \in (0,1)$ fulfilling (3.13) and

$$H_7 \le \frac{1}{4} H_2 + \bar{c}_2 t, \ \ \bar{c}_2 > 0, \ \text{for all } t \in (0, T_{max}). \tag{3.17}$$

Proof. By Fubini's theorem we obtain

$$H_7 = \mu N^{k-1} \int_0^t \int_0^{R^N} s^{-a} w^{b-1} \Big(\int_0^s w_s^k d\sigma \Big)\, ds d\tau$$
$$= \mu N^{k-1} \int_0^t \int_0^{R^N} \Big(\int_\sigma^{R^N} s^{-a} w^{b-1} ds \Big) w_s^k(\sigma)\, d\sigma d\tau.$$

Since $b \in (0,1)$ and $w_s \geq 0$, then $w^{b-1}(s)$ decreases in s, we can write

$$H_7 \leq \mu N^{k-1} \int_0^t \int_0^{R^N} \Big(\int_\sigma^{R^N} s^{-a} ds \Big) w^{b-1}(\sigma) w_s^k(\sigma)\, d\sigma d\tau$$
$$= \frac{1}{1-a} \mu N^{k-1} \int_0^t \int_0^{R^N} \big(R^{N(1-a)} - \sigma^{1-a} \big) w^{b-1}(\sigma) w_s^k(\sigma)\, d\sigma d\tau.$$

In the case $k = 2$, $a > 1$ we neglect the negative term $-\frac{R^N}{a-1}$ and use (3.3) to obtain

$$H_7 \leq \frac{\mu N}{a-1} \int_0^t \int_0^{R^N} s^{1-a} w^{b-1}(s) w_s^2(s)\, ds d\tau$$
$$\leq \frac{\mu N}{a-1} \int_0^t \int_0^{R^N} s^{-a-1} w^{b+1}\, ds d\tau.$$

Now, if $0 < \mu \leq \mu_0$ with $\mu_0 \leq \frac{a-1}{4N} c_1$, and c_1 defined in (3.11), we note that one can find $a > 1$ and $b \in (0,1)$ fulfilling (3.13) such that (3.16) holds.

If $k \in \big(1, \min\{2, 1 + \frac{(N-2)^2}{4}\}\big)$, $a \in (0,1)$ we neglect the negative term $-\frac{1}{1-a}\sigma^{1-a}$ and arrive to

$$H_7 \leq \frac{\mu N^{k-1}}{1-a} R^{N(1-a)} \int_0^t \int_0^{R^N} w^{b-1}(s) w_s^k(s)\, ds d\tau.$$

We now fix $b = a \in \big(\sqrt{k-1}, \min\{1, \frac{N-2}{2}\}\big)$ fulfilling (3.13). This is possible in view of the choice of k, because (3.13) with $b = a$ is equivalent to $a < \frac{N-2}{2}$. Thus we see that $(a-1)\frac{a+1}{2-k} > -1$, and then (3.3) and Young's inequality lead to

$$H_7 \leq \frac{\mu N^{k-1}}{1-a} R^{N(1-a)} \int_0^t \int_0^{R^N} s^{-k} w^{k+a-1} ds d\tau$$
$$\leq \int_0^t \Big[\Big(\int_0^{R^N} s^{-a-1} w^{a+1} ds \Big)^{\frac{k+a-1}{a+1}} \Big(\int_0^{R^N} s^{(a-1)\frac{a+1}{2-k}}\, ds \Big)^{\frac{2-k}{a+1}} \Big] d\tau$$
$$\leq \frac{c_1}{4} \int_0^t \int_0^{R^N} s^{-a-1} w^{a+1}\, ds d\tau + \bar{c}_1 \int_0^t \int_0^{R^N} s^{(a-1)\frac{a+1}{2-k}}\, ds d\tau$$
$$= \frac{c_1}{4} \int_0^t \int_0^{R^N} s^{-a-1} w^{a+1}\, ds d\tau + \bar{c}_2 t, \ \ for\ all\ t \in (0, T_{max}),$$

with some $\bar{c}_2 > 0$. Thus we obtain (3.17) with $b = a$. □

Taking into account of Lemmata 3.2, 3.3 and 3.4, we derive an integral inequality for the functional $y(t) = \int_0^{R^N} s^{-a} w^b(s) ds$.

Lemma 3.5. *Suppose Lemma 3.3 and Lemma 3.4 hold. Let $N \geq 3$, $R > 0$, $m_0 > 0$, $\mu > 0$ and $k \in (1, 2]$. Then there exist $a > 0$, $b \in (0, 1)$, $\delta > 0$ and $C > 0$ such that if $u_0(r)$ is nonnegative in $B_R(0) \subset \mathbb{R}^N$ with $\frac{1}{|\Omega|}\int_\Omega u_0 = m_0$, for the corresponding solution (u, v) of (1.1) in $\Omega \times (0, T_{max})$ and w defined in (3.1), it holds*

$$\int_0^{R^N} s^{-a} w^b(s,t)\, ds \geq \int_0^{R^N} s^{-a} w_0^b(s)\, ds + \delta \int_0^t \Big(\int_0^{R^N} s^{-a} w^b(s,\tau)\, ds \Big)^{\frac{b+1}{b}} d\tau - Ct \tag{3.18}$$

for all $t \in (0, T_{max})$.

Proof. We analyse the following two cases separately.

Case (i) Assume $k = 2$, $1 < a < \frac{N-2}{N}(b+1)$, $N \geq 5$, $0 < \mu \leq \mu_0$. Thus $b \in (\frac{2}{N-2}, 1)$.

Substituting (3.14), (3.15) and (3.16) in (3.12) and neglecting the positive term H_3, we see that

$$\int_0^{R^N} s^{-a} w^b(s,t)\, ds \geq \int_0^{R^N} s^{-a} w_0^b(s)\, ds + \frac{bc_1}{4} \int_0^t \int_0^{R^N} s^{-a-1} w^{b+1}\, ds d\tau - Ct, \quad \forall\, t \in (0, T_{max}).$$

Case (ii) Assume $k \in \big(1, \min\{2, 1 + \frac{(N-2)^2}{4}\}\big)$, $b = a \in \big(\sqrt{k-1}, \min\{1, \frac{N-2}{2}\}\big)$, $N \geq 3$, $\mu > 0$.

Substituting (3.14), (3.15) and (3.17) in (3.12) we obtain (with $b = a$)

$$\begin{aligned}\int_0^{R^N} s^{-a} w^b(s,t)\, ds &\geq \int_0^{R^N} s^{-a} w_0^b(s)\, ds + \frac{bc_1}{4} \int_0^t \int_0^{R^N} s^{-a-1} w^{b+1}\, ds d\tau \\ &\quad + bc_1 \int_0^t \int_0^{R^N} s^{-a} w^b w_s\, ds d\tau - Ct \\ &\geq \int_0^{R^N} s^{-a} w_0^b(s)\, ds + \frac{bc_1}{4} \int_0^t \int_0^{R^N} s^{-a-1} w^{b+1}\, ds d\tau - Ct \quad \forall\, t \in (0, T_{max}).\end{aligned}$$

In both cases (i) and (ii) we arrive at the following type inequality:

$$\int_0^{R^N} s^{-a} w^b(s,t)\, ds \geq \int_0^{R^N} s^{-a} w_0^b(s)\, ds + \frac{bc_1}{4} \int_0^t \int_0^{R^N} s^{-a-1} w^{b+1}\, ds d\tau - Ct, \quad \forall\, t \in (0, T_{max}). \tag{3.19}$$

Now, by the Hölder inequality, we observe that

$$\int_0^{R^N} s^{-a}w^b\,ds = \int_0^{R^N} s^{-a+\frac{b(a+1)}{b+1}}\left(s^{-a-1}w^{b+1}\right)^{\frac{b}{b+1}}\,ds$$
$$\leq \left(\int_0^{R^N} s^{-a+b}\,ds\right)^{\frac{1}{b+1}}\left(\int_0^{R^N} s^{-a-1}w^{b+1}\,ds\right)^{\frac{b}{b+1}}$$

from which we have

$$\int_0^{R^N} s^{-a-1}w^{b+1}\,ds \geq \bar{c}_4\left(\int_0^{R^N} s^{-a}w^b\,ds\right)^{\frac{b+1}{b}} \tag{3.20}$$

with $\bar{c}_4 = \left(\frac{b+1-a}{R^{N(b+1-a)}}\right)^{\frac{1}{b}}$ and $-a+b > -1$.

Replacing (3.20) into (3.19) we arrive at (3.18) with $\delta = \frac{1}{4}bc_1\bar{c}_4$. □

Proof of theorem 1.1. By Lemma 3.5 with the aid of the Lemma 2.5 and following the steps in the proof of Theorem 0.1 in [20], we can conclude that $y(t) = \int_0^{R^N} s^{-a}w^b(s,t)ds$ blows up in finite time $T_{max} \leq \frac{b}{\delta\beta^{\frac{1}{b}}}$. □

4. Blow-up in L^p-norm

The aim of this section is to prove Theorem 1.2. To this end, first we prove the following lemma.

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$ be a bounded and smooth domain. Let (u,v) be a classical solution of system (1.1). If α satisfies (1.4) and if for some $p > \frac{N}{2}$ there exists $C > 0$ such that*

$$\|u(\cdot,t)\|_{L^p(\Omega)} \leq C, \quad \text{for any } t \in (0, T_{max}),$$

then, for some $\hat{C} > 0$,

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq \hat{C}, \quad \text{for any } t \in (0, T_{max}). \tag{4.1}$$

Proof. For any $t \in (0, T_{max})$, we set $t_0 := \max\{0, t-1\}$ and we consider the representation formula for u:

$$u(\cdot,t) = e^{(t-t_0)\Delta}u(\cdot,t_0) - k_f\int_{t_0}^t e^{(t-s)\Delta}\nabla\cdot\left(u(\cdot,s)\frac{\nabla v(\cdot,s)}{(1+|\nabla v(\cdot,t)|^2)^\alpha}\right)ds$$
$$+\int_{t_0}^t e^{(t-s)\Delta}\left(\lambda u(\cdot,s) - \mu u^k(\cdot,s)\right)ds =: u_1(\cdot,t) + u_2(\cdot,t) + u_3(\cdot,t)$$

and

$$0 \leq u(\cdot,t) \leq \|u_1(\cdot,t)\|_{L^\infty(\Omega)} + \|u_2(\cdot,t)\|_{L^\infty(\Omega)} + u_3(\cdot,t). \tag{4.2}$$

We have

$$\|u_1(\cdot,t)\|_{L^\infty(\Omega)} \leq \max\{\|u_0\|_{L^\infty(\Omega)}, 2e^{-\lambda}\bar{m}k_1\} =: \tilde{C}_1, \tag{4.3}$$

with $k_1 > 0$ introduced in (2.1) and $\bar{m}$ defined in (2.4). In fact, if $t \leq 1$, then $t_0 = 0$ and hence the maximum principle yields $u_1(\cdot,t) \leq \|u_0\|_{L^\infty(\Omega)}$. If $t > 1$, then $t - t_0 = 1$ and from (2.4) and (2.1) with p = ∞ and $q = 1$, we deduce that $\|u_1(\cdot,t)\|_{L^\infty(\Omega)} \leq k_1[1+(t-t_0)^{-\frac{N}{2}}]e^{-\mu_1(t-t_0)}\|u(\cdot,t_0)\|_{L^1(\Omega)} = 2e^{-\lambda}\bar{m}k_1$.

We next use (2.2) with $\mathrm{p} = \infty$, which leads to

$$\begin{aligned}
&\|u_2(\cdot,t)\|_{L^\infty(\Omega)} \\
&\quad \le k_2 k_f \int_{t_0}^t (1+(t-s)^{-\frac{1}{2}-\frac{N}{2q}}) e^{-\mu_1(t-s)} \left\| u(\cdot,s) \frac{\nabla v(\cdot,s)}{(1+|\nabla v|^2)^\alpha} \right\|_{L^q(\Omega)} ds \\
&\quad \le k_2 k_f \int_{t_0}^t (1+(t-s)^{-\frac{1}{2}-\frac{N}{2q}}) e^{-\mu_1(t-s)} \|u(\cdot,s)|\nabla v|^{1-2\alpha}\|_{L^q(\Omega)}\, ds, \qquad (4.4)
\end{aligned}$$

because $\frac{|\nabla v|}{(1+|\nabla v|^2)^\alpha} \le |\nabla v|^{1-2\alpha}$.

Here, we may assume that $\frac{N}{2} < p < N$, and then we can fix $N < q < \frac{Np}{N-p} = p^*$. Since $2\alpha < 1$, by Hölder's inequality, we can estimate the last term in (4.4) as

$$\begin{aligned}
&\|u(\cdot,s)|\nabla v(\cdot,s)|^{1-2\alpha}\|_{L^q(\Omega)} \\
&\quad \le \|u(\cdot,s)\|_{L^{\frac{q}{2\alpha}}(\Omega)} \|\nabla v(\cdot,s)\|^{1-2\alpha}_{L^q(\Omega)} \\
&\quad \le C_2 \|u(\cdot,s)\|_{L^{\frac{q}{2\alpha}}(\Omega)} \|\nabla v(\cdot,s)\|^{1-2\alpha}_{L^{p^*}(\Omega)} \quad \text{for all } s \in (0,T_{max}),
\end{aligned}$$

for some $C_2 > 0$. The Sobolev embedding theorem and elliptic regularity theory for the second equation in (1.1) tell us that $\|v(\cdot,s)\|_{W^{1,p^*}(\Omega)} \le C_3 \|v(\cdot,s)\|_{W^{2,p}(\Omega)} \le C_4$ with some $C_3, C_4 > 0$. Thus again by Hölder's inequality, the definition of $\bar{m}$ and interpolation's inequality, we obtain

$$\begin{aligned}
\|u(\cdot,s)|\nabla v(\cdot,s)|^{1-2\alpha}\|_{L^q(\Omega)} &\le C_5 \|u(\cdot,s)\|_{L^{\frac{q}{2\alpha}}(\Omega)} \\
&\le C_5 \|u(\cdot,s)\|^\theta_{L^\infty(\Omega)} \|u(\cdot,s)\|^{1-\theta}_{L^1(\Omega)} \\
&\le C_6 \|u(\cdot,s)\|^\theta_{L^\infty(\Omega)} \quad \text{for all } s \in (0,T_{max}),
\end{aligned}$$

with $\theta := 1 - \frac{2\alpha}{q} \in (0,1)$, $C_5 := C_2 C_4$ and $C_6 := C_5 \bar{m}^{1-\theta}$. Hence, combining this estimate and (4.4), we infer

$$\|u_2(\cdot,t)\|_{L^\infty(\Omega)} \le C_6 k_2 \int_{t_0}^t (1+(t-s)^{-\frac{1}{2}-\frac{N}{2q}}) e^{-\mu_1(t-s)} \|u(\cdot,s)\|^\theta_{L^\infty(\Omega)}\, ds.$$

Now fix any $T \in (0, T_{max})$. Then, since $t - t_0 \le 1$, we have

$$\begin{aligned}
\|u_2(\cdot,t)\|_{L^\infty(\Omega)} &\le C_6 k_2 \int_{t_0}^t (1+(t-s)^{-\frac{1}{2}-\frac{N}{2q}} e^{-\mu_1(t-s)})\, ds \sup_{t\in[0,T]} \|u(\cdot,t)\|^\theta_{L^\infty(\Omega)} \\
&\le C_7 \sup_{t\in[0,T]} \|u(\cdot,t)\|^\theta_{L^\infty(\Omega)}, \qquad (4.5)
\end{aligned}$$

where $C_7 := C_6 k_2 \big(1 + \mu_1^{\frac{N}{2q}-\frac{1}{2}} \int_0^\infty r^{-\frac{1}{2}-\frac{N}{2q}} e^{-r}\, dr\big) > 0$ is finite, because $\frac{1}{2} + \frac{N}{2q} < 1$ (i.e., $q > N$).

Now we prove that there exists a constant $c_8 \ge 0$ such that $u_3(\cdot,t) \le c_8$. In fact we observe that $g(u) = \lambda u - \mu u^k \le g(\tilde{u}) := c_8$, with $\tilde{u} = \big(\frac{\lambda}{\mu}\big)^{\frac{1}{k-1}}$

$$u_3(\cdot,t) = \int_{t_0}^t e^{(t-s)\Delta} \big[\lambda u(\cdot,s) - \mu u^k(\cdot,s)\big]\, ds \le c_8 \int_{t_0}^t ds \le c_8. \qquad (4.6)$$

Plugging (4.3), (4.5) and (4.6) into (4.2), we see that

$$0 \le u(\cdot,t) \le C_1 + C_7 \sup_{t\in[0,T]} \|u(\cdot,t)\|^{\theta}_{L^\infty(\Omega)}, \tag{4.7}$$

with $C_1 = \tilde{C}_1 + c_8$.

The inequality (4.7) implies

$$\sup_{t\in[0,T]} \|u(\cdot,t)\|_{L^\infty(\Omega)} \le C_1 + C_7\Big(\sup_{t\in[0,T]} \|u(\cdot,t)\|_{L^\infty(\Omega)}\Big)^{\theta} \text{ for all } T \in (0, T_{max}).$$

From this inequality with $\theta \in (0,1)$, we arrive at (4.1). □

Proof of Theorem 1.2. Since Theorem 1.1 holds, the unique local classical solution of (1.1) blows up at $t = T_{max}$ in the sense of (1.6), that is,

$$\limsup_{t\nearrow T_{max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty.$$

We prove that it blows up also in L^p-norm by contradiction.

In fact, if one supposes that there exist $p > \frac{N}{2}$ and $C > 0$ such that

$$\|u(\cdot,t)\|_{L^p(\Omega)} \le C, \quad \text{for all } t \in (0, T_{max}),$$

then, from Lemma 4.1, it would exist $\hat{C} > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \hat{C}, \quad \text{for all } t \in (0, T_{max}),$$

which contradics (1.6). Thus, if u blows up in L^∞-norm, then u blows up also in L^p-norm for all $p > \frac{N}{2}$. □

5. Lower bound of the blow-up time T_{max}

Throughout this section we assume that Theorem 1.2 holds.

We want to obtain a safe interval of existence of the solution of (1.1) $[0,T]$, with T a lower bound of the blow-up time T_{max}. To this end, first we construct a first order differential inequality for Ψ defined in (1.7) and by integration we get the lower bound.

Proof of Theorem 1.2. By differentiating (1.7) we have

$$\begin{aligned}\Psi'(t) &= \int_\Omega u^{p-1}\Delta u\,dx - \int_\Omega u^{p-1}\nabla\cdot(u\nabla v f(|\nabla v|^2)\,dx \\ &\quad + \lambda\int_\Omega u^p\,dx - \mu\int_\Omega u^{p+k-1}\,dx \\ &=: \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_4 \end{aligned} \tag{5.1}$$

with

$$\begin{aligned}\mathcal{J}_1 &= \int_\Omega u^{p-1}\Delta u\,dx \\ &= \int_\Omega \nabla\cdot\left(u^{p-1}\nabla u\right)dx - (p-1)\int_\Omega u^{p-2}|\nabla u|^2\,dx \\ &= -\frac{4(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2\,dx. \end{aligned} \tag{5.2}$$

In the second term of (5.1), integrating by parts and using the boundary conditions in (1.1), for all $t \in [0, T_{max})$ we obtain

$$\begin{aligned}
\mathcal{J}_2 &= -\int_\Omega u^{p-1}\nabla\cdot(u\nabla v f(|\nabla v|^2))\,dx \\
&= (p-1)\int_\Omega f(|\nabla v|^2)u^{p-1}\nabla u\cdot\nabla v\,dx \\
&= \frac{p-1}{p}\int_\Omega \nabla u^p\cdot\nabla v f(|\nabla v|^2)\,dx \\
&= -\frac{p-1}{p}\int_\Omega u^p\nabla\cdot[\nabla v f(|\nabla v|^2)]\,dx \\
&= -\frac{p-1}{p}\int_\Omega u^p[\Delta v f(|\nabla v|^2)]\,dx \\
&\quad -\frac{p-1}{p}\int_\Omega u^p f'(|\nabla v|^2)\nabla v\cdot\nabla(|\nabla v|^2)\,dx. \qquad (5.3)
\end{aligned}$$

Using the second equation of (1.1) and taking into account that $f(\xi) = k_f(1+\xi)^{-\alpha}$, $f'(\xi) = -\alpha k_f(1+\xi)^{-\alpha-1}$ in (5.3), we have

$$\begin{aligned}
\mathcal{J}_2 &= -k_f\frac{p-1}{p}\int_\Omega u^p\frac{m(t)-u}{(1+|\nabla v|^2)^\alpha}\,dx \\
&\quad +\alpha k_f\frac{p-1}{p}\int_\Omega u^p\frac{\nabla v\cdot\nabla(|\nabla v|^2)}{(1+|\nabla v|^2)^{\alpha+1}}\,dx \\
&\le k_f\frac{p-1}{p}\int_\Omega u^{p+1}\,dx + \alpha k_f\frac{p-1}{p}\int_\Omega u^p\frac{\nabla v\cdot\nabla(|\nabla v|^2)}{(1+|\nabla v|^2)^{\alpha+1}}\,dx, \qquad (5.4)
\end{aligned}$$

where we dropped the negative term $-k_f\frac{p-1}{p}\int_\Omega u^p\frac{m(t)}{(1+|\nabla v|^2)^\alpha}dx$ and used the inequality $\frac{1}{(1+|\nabla v|^2)^\alpha} \le 1$ as $\alpha > 0$.

In order to estimate the second term of (5.4) we recall the radially symmetric setting to obtain (with ω_N the surface area of the unit sphere in N dimension)

$$\begin{aligned}
\int_\Omega u^p\frac{\nabla v\cdot\nabla(|\nabla v|^2)}{(1+|\nabla v|^2)^{\alpha+1}}\,dx &= \omega_N\int_0^R u^p\frac{Nv_r(v_r^2)_r}{(1+v_r^2)^{\alpha+1}}r^{N-1}\,dr \\
&= 2N\omega_N\int_0^R u^p\frac{v_r^2 v_{rr}}{(1+v_r^2)^{\alpha+1}}r^{N-1}\,dr,
\end{aligned}$$

which together with $v_{rr} = \frac{m(t)}{N} - u + \frac{N-1}{r^N}\int_0^r \rho^{N-1}u\,d\rho$ implies

$$\begin{aligned}
&\int_\Omega u^p\frac{\nabla v\cdot\nabla(|\nabla v|^2)}{(1+|\nabla v|^2)^{\alpha+1}}\,dx \\
&= 2m(t)\omega_N\int_0^R u^p\frac{v_r^2}{(1+v_r^2)^{\alpha+1}}r^{N-1}\,dr \\
&\quad -2N\omega_N\int_0^R u^{p+1}\frac{v_r^2}{(1+v_r^2)^{\alpha+1}}r^{N-1}\,dr
\end{aligned}$$

$$
\begin{aligned}
&+ 2N(N-1)\omega_N \int_0^R u^p \frac{v_r^2}{(1+v_r^2)^{\alpha+1}} \frac{1}{r}\Big(\int_0^r \rho^{N-1} u \, d\rho \Big) \, dr \\
&\le 2\frac{\bar m}{|\Omega|}\omega_N \int_0^R u^p r^{N-1} \, dr \\
&\quad + 2N(N-1)\omega_N \int_0^R u^p \frac{1}{r}\Big(\int_0^r \rho^{N-1} u \, d\rho \Big) \, dr, \qquad (5.5)
\end{aligned}
$$

where we used (2.3), we dropped the negative term $-2N\omega_N \int_0^R u^{p+1} \frac{v_r^2}{(1+v_r^2)^{\alpha+1}} r^{N-1} \, dr$ and finally we used the inequality $\frac{v_r^2}{(1+v_r^2)^{\alpha+1}} \le 1$.

In the second term of (5.5), Hölder's inequality yelds that for all $\epsilon > 0$ there exists $c = c(\epsilon, N, p)$ such that

$$
\begin{aligned}
&\omega_N \int_0^R u^p \frac{1}{r}\Big(\int_0^r \rho^{N-1} u \, d\rho \Big) dr \\
&\le \omega_N \int_0^R u^p \frac{1}{r}\Big(\int_0^r \rho^{N-1} \, d\rho \Big)^{\frac{p}{p+1}} \Big(\int_0^r u^{p+1} \rho^{N-1} \, d\rho \Big)^{\frac{1}{p+1}} dr \\
&\le \Big(\frac{1}{N}\Big)^{\frac{p}{p+1}} \Big(\int_\Omega u^{p+1} \, dx \Big)^{\frac{1}{p+1}} \omega_N^{\frac{p}{p+1}} \int_0^R u^p r^{\frac{Np}{p+1}-1} \, dr \\
&\le \Big(\frac{1}{N}\Big)^{\frac{p}{p+1}} \Big(\int_\Omega u^{p+1} \, dx \Big)^{\frac{1}{p+1}} \omega_N^{\frac{p}{p+1}} \Big(\int_0^R u^{p+1+\epsilon} r^{N-1} \, dr \Big)^{\frac{p}{p+1+\epsilon}} \Big(\int_0^R r^{\frac{\epsilon N p}{p+1}-1} dr \Big)^{\frac{1+\epsilon}{p+1+\epsilon}} \\
&= c\Big(\int_\Omega u^{p+1} \, dx \Big)^{\frac{1}{p+1}} \Big(\int_\Omega u^{p+1+\epsilon} \, dx \Big)^{\frac{p}{p+1+\epsilon}}. \qquad (5.6)
\end{aligned}
$$

Combining (5.6) and (5.5) with (5.4) we obtain

$$
\begin{aligned}
\mathcal{J}_2 &\le 2\alpha \frac{\bar m}{|\Omega|} k_f \frac{p-1}{p} \int_\Omega u^p \, dx + k_f \frac{p-1}{p} \int_\Omega u^{p+1} \, dx \\
&\quad + 2\alpha N(N-1) c k_f \, \frac{p-1}{p} \Big(\int_\Omega u^{p+1} \, dx \Big)^{\frac{1}{p+1}} \Big(\int_\Omega u^{p+1+\epsilon} \, dx \Big)^{\frac{p}{p+1+\epsilon}} \\
&\le \frac{\tilde c_1}{p} \int_\Omega u^p \, dx + \tilde c_2 \int_\Omega u^{p+1} \, dx + \tilde c_3 \Big(\int_\Omega u^{p+1+\epsilon} \, dx \Big)^{\frac{p+1}{p+1+\epsilon}} \qquad (5.7)
\end{aligned}
$$

where, in the last term, we used Young's inequality with $\tilde c_1 = 2\alpha \frac{\bar m}{|\Omega|} k_f (p-1)$, $\tilde c_2 = k_f \frac{p-1}{p} + 2\alpha N(N-1) c k_f \, \frac{p-1}{p(p+1)}$, $\tilde c_3 = 2\alpha N(N-1) c k_f \, \frac{p-1}{p+1}$.

Thanks to the Gagliardo–Nirenberg inequality (2.6), with $\mathsf{p} = 2\frac{p+1}{p}$, $\mathsf{r} = \mathsf{q} = \mathsf{s} = 2$, $a = \theta_0 := \frac{N}{2(p+1)} \in (0,1)$ for all $p > \frac{N}{2}$, we see that

$$
\begin{aligned}
\int_\Omega u^{p+1} \, dx &= \| u^{\frac{p}{2}} \|^{2\frac{p+1}{p}}_{L^{2\frac{p+1}{p}}(\Omega)} \\
&\le C_{GN} \| \nabla u^{\frac{p}{2}} \|^{2\frac{p+1}{p}\theta_0}_{L^2(\Omega)} \| u^{\frac{p}{2}} \|^{2\frac{p+1}{p}(1-\theta_0)}_{L^2(\Omega)} + C_{GN} \| u^{\frac{p}{2}} \|^{2\frac{p+1}{p}}_{L^2(\Omega)} \\
&= C_{GN} \Big(\int_\Omega |\nabla u^{\frac{p}{2}}|^2 \, dx \Big)^{\frac{N}{2p}} \Big(\int_\Omega u^p \, dx \Big)^{\frac{2(p+1)-N}{2p}} + C_{GN} \Big(\int_\Omega u^p \, dx \Big)^{\frac{p+1}{p}}. \qquad (5.8)
\end{aligned}
$$

Applying Young's inequality at the first term of (5.8) we have

$$\int_\Omega u^{p+1}\,dx \le \frac{N}{2p}\epsilon_1 C_{GN}\int_\Omega |\nabla u^{\frac{p}{2}}|^2\,dx + C_{GN}\frac{2p-N}{2p\epsilon_1^{\frac{N}{2p-N}}}\Big(\int_\Omega u^p\,dx\Big)^{\frac{2(p+1)-N}{2p-N}} + C_{GN}\Big(\int_\Omega u^p\,dx\Big)^{\frac{p+1}{p}} \tag{5.9}$$

with $\epsilon_1 > 0$ to be choose later on, and also

$$\begin{aligned}\Big(\int_\Omega u^{p+1+\epsilon}\,dx\Big)^{\frac{p+1}{p+1+\epsilon}} &= \|u^{\frac{p}{2}}\|^{2\frac{p+1}{p}}_{L^{2\frac{p+1+\epsilon}{p}}(\Omega)}\\ &\le C_{GN}\|\nabla u^{\frac{p}{2}}\|^{2\frac{p+1}{p}\theta_\epsilon}_{L^2(\Omega)}\|u^{\frac{p}{2}}\|^{2\frac{p+1}{p}(1-\theta_\epsilon)}_{L^2(\Omega)} + C_{GN}\|u^{\frac{p}{2}}\|^{2\frac{p+1}{p}}_{L^2(\Omega)}\\ &= C_{GN}\Big(\int_\Omega |\nabla u^{\frac{p}{2}}|^2\,dx\Big)^{\frac{p+1}{p}\theta_\epsilon}\Big(\int_\Omega u^p\,dx\Big)^{\frac{p+1}{p}(1-\theta_\epsilon)}\\ &\quad + C_{GN}\Big(\int_\Omega u^p\,dx\Big)^{\frac{p+1}{p}},\end{aligned} \tag{5.10}$$

with $\mathsf{p} = 2\frac{p+1}{p}$, $\mathsf{r} = \mathsf{q} = \mathsf{s} = 2$, $a = \theta_\epsilon := \frac{N(1+\epsilon)}{2(p+1+\epsilon)} \in (0,1)$ for all $p > \frac{N}{2}$ and sufficiently small $\epsilon > 0$.

Now, in the first term of (5.10), we apply Young's inequality to obtain

$$\begin{aligned}&\Big(\int_\Omega u^{p+1+\epsilon}\,dx\Big)^{\frac{p+1}{p+1+\epsilon}}\\ &\le \tilde{c}_4\int_\Omega |\nabla u^{\frac{p}{2}}|^2\,dx + \tilde{c}_5\Big(\int_\Omega u^p\,dx\Big)^{\gamma} + C_{GN}\Big(\int_\Omega u^p\,dx\Big)^{\frac{p+1}{p}},\end{aligned} \tag{5.11}$$

with

$$\begin{aligned}\tilde{c}_4 &:= \frac{N(1+\epsilon)(p+1)}{2p(p+1+\epsilon)}\epsilon_2 C_{GN},\\ \tilde{c}_5 &:= C_{GN}\Big(\frac{2p(p+1+\epsilon) - N(p+1)(1+\epsilon)}{2p(p+1+\epsilon)}\Big)\epsilon_2^{\frac{N(1+\epsilon)}{2(p+1+\epsilon)-N(1+\epsilon)}},\\ \gamma &:= \frac{2(p+1) - \frac{N(p+1)(1+\epsilon)}{p+1+\epsilon}}{2p - \frac{N(1+\epsilon)(p+1)}{p+1+\epsilon}}, \quad \epsilon_2 > 0.\end{aligned}$$

Note that we can fix $\epsilon > 0$ such that $2p - N(1+\epsilon) > 0$.

Plugging (5.9) and (5.11) into (5.7) leads to

$$\begin{aligned}\mathcal{J}_2 &\le C\int_\Omega |\nabla u^{\frac{p}{2}}|^2\,dx + \frac{\tilde{c}_1}{p}\int_\Omega u^p\,dx + C_{GN}\Big(\int_\Omega u^p\,dx\Big)^{\frac{p+1}{p}}\\ &\quad + \hat{c}_1\Big(\int_\Omega u^p dx\Big)^{\frac{2(p+1)-N}{2p-N}} + \tilde{c}_5\Big(\int_\Omega u^p\,dx\Big)^{\gamma}\end{aligned} \tag{5.12}$$

with $C := \tilde{c}_3\cdot\tilde{c}_4$, $\hat{c}_1 := C_{GN}\frac{2p-N}{2p\epsilon_1^{\frac{N}{2p-N}}}\tilde{c}_2$, $\epsilon_1 > 0$.

Also we note that

$$\mathcal{J}_3 = \lambda\int_\Omega u^p\,dx = B_1\Psi, \quad B_1 = \lambda p. \tag{5.13}$$

Finally, combining (5.12) with (5.1) and (5.2), (5.13), neglecting the negative term $\mathcal{J}_4$ and choosing ϵ_2 such that the term containing $\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx$ vanishes, we have

$$\Psi' \le B_1\Psi + B_2\Psi^{\frac{p+1}{p}} + B_3\Psi^{\frac{2(p+1)-N}{2p-N}} + B_4\Psi^{\gamma}, \tag{5.14}$$

with $B_2 := p^{\frac{1}{p}}[pC_{GN} + \tilde{c}_1]$, $B_3 := \hat{c}_1 p^{\frac{2(p+1)-N}{2p-N}}$ and $B_4 := \tilde{c}_5 p^{\gamma}$.

Integrating (5.14) from 0 to T_{max}, we arrive at the desired lower bound (1.8) with $\gamma_1 := \frac{p+1}{p}$, $\gamma_2 := \frac{2(p+1)-N}{2p-N}$. □

Proof of Corollary 1.4. We reduce (5.14) so as to have an explicit expression of the lower bound T of T_{max}. In fact, since $\Psi(t)$ blows up at time T_{max}, there exists a time $t_1 \in (0, T_{max})$ such that $\Psi(t) \ge \Psi_0$ for all $t \in (t_1, T_{max})$. Thus, taking into account that

$$1 < \gamma_1 < \gamma_2 < \gamma$$

we have

$$\begin{cases} \Psi \le \Psi^{\gamma}\Psi_0^{1-\gamma}, \\ \Psi^{\gamma_i} \le \Psi^{\gamma}\Psi_0^{\gamma_i-\gamma}, \quad i = 1, 2. \end{cases} \tag{5.15}$$

From (5.14) and (5.15) we arrive at

$$\Psi' \le \mathcal{A}\Psi^{\gamma}, \quad \forall\, t \in (t_1, T_{max}), \tag{5.16}$$

with $\mathcal{A} := B_1\Psi_0^{1-\gamma} + B_2\Psi_0^{\gamma_1-\gamma} + B_3\Psi_0^{\gamma_2-\gamma} + B_4$, and Ψ_0 in (1.7).

Integrating (5.16) from $t = 0$ to $t = T_{max}$, we obtain

$$\frac{1}{(\gamma-1)\Psi_0^{\gamma-1}} = \int_{\Psi_0}^{\infty} \frac{d\eta}{\eta^{\gamma}} \le \mathcal{A}\int_{t_1}^{T_{max}} d\tau \le \mathcal{A}\int_0^{T_{max}} d\tau = \mathcal{A}T_{max}. \tag{5.17}$$

We conclude, by (5.17), that the solution of (1.1) is bounded in $[0, T]$ with $T := \frac{1}{\mathcal{A}(\gamma-1)\Psi_0^{\gamma-1}}$. □

6. Global existence and boundedness

The aim of this section is to prove Theorem 1.5. The proof is divided into two cases.

6.1. Case 1. $\alpha > \frac{N-2}{2(N-1)}$ and $k > 1$

As in the proof of Lemma 4.1, for any $t \in (0, T_{max})$, we set $t_0 := \max\{0, t-1\}$. From the representation formula for u we can write

$$u(\cdot,t) = e^{(t-t_0)\Delta}u(\cdot,t_0) - \int_{t_0}^{t} e^{(t-s)\Delta}\nabla\cdot\left[u(\cdot,s)f(|\nabla v(\cdot,s)|^2)\nabla v(\cdot,s)\right] ds$$
$$+ \int_{t_0}^{t} e^{(t-s)\Delta}g(u)\, ds =: u_1(\cdot,t) + u_2(\cdot,t) + u_3(\cdot,t).$$

In view of (4.2) and (4.3) as well as (4.6) we have

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \mathsf{c}_1 + \|u_2(\cdot,t)\|_{L^\infty(\Omega)}\,. \tag{6.1}$$

Since the condition $\alpha > \frac{N-2}{2(N-1)}$ implies that $(1-2\alpha)N < \frac{N}{N-1}$, we can take $q \in \left[1, \frac{N}{N-1}\right)$ such that $q > (1-2\alpha)N$, and hence we pick $r > N$ satisfying $q > (1-2\alpha)r$. Then we see from the second equation in (1.1) with mass estimate (2.3) that

$$\sup_{t\in(0,T_{max})} \|\nabla v(\cdot,t)\|_{L^q(\Omega)} \le \mathsf{c}_2.$$

Using (2.2) with $\mathrm{p} = \infty$ and $q = r$ as in (4.4), we deduce from the Hölder inequality that

$$\begin{aligned}
&\|u_2(\cdot,t)\|_{L^\infty(\Omega)}\\
&\le \mathsf{c}_3 \int_{t_0}^t (1+(t-s)^{-\frac{1}{2}-\frac{N}{2r}}) e^{-\mu_1(t-s)} \|u(\cdot,s)|\nabla v(\cdot,s)|^{1-2\alpha}\|_{L^r(\Omega)}\, ds\\
&\le \mathsf{c}_3 \int_{t_0}^t (1+(t-s)^{-\frac{1}{2}-\frac{N}{2r}}) e^{-\mu_1(t-s)} \|u(\cdot,s)\|_{L^{\frac{qr}{q-(1-2\alpha)r}}(\Omega)} \|\nabla v(\cdot,s)\|^{1-2\alpha}_{L^q(\Omega)}\, ds.
\end{aligned}$$

Putting $a := 1 - \frac{q-(1-2\alpha)r}{qr} \in (0,1)$ and recalling (2.3) again, we note that

$$\|u(\cdot,s)\|_{L^{\frac{qr}{q-(1-2\alpha)r}}(\Omega)} \le \|u(\cdot,s)\|^a_{L^\infty(\Omega)} \|u(\cdot,s)\|^{1-a}_{L^1(\Omega)} \le \mathsf{c}_4 \|u(\cdot,s)\|^a_{L^\infty(\Omega)}\,,$$

and hence,

$$\|u_2(\cdot,t)\|_{L^\infty(\Omega)} \le \mathsf{c}_2\mathsf{c}_3\mathsf{c}_4 \int_{t_0}^t (1+(t-s)^{-\frac{1}{2}-\frac{N}{2r}}) e^{-\mu_1(t-s)} \|u(\cdot,s)\|^a_{L^\infty(\Omega)}\, ds.$$

This together with (6.1) implies that for any $T \in (0, T_{max})$,

$$\begin{aligned}
&\sup_{t\in(0,T)} \|u_2(\cdot,t)\|_{L^\infty(\Omega)}\\
&\le \mathsf{c}_1 + \mathsf{c}_2\mathsf{c}_3\mathsf{c}_4 \sup_{t\in(0,T)} \|u(\cdot,t)\|^a_{L^\infty(\Omega)} \int_{t_0}^t (1+(t-s)^{-\frac{1}{2}-\frac{N}{2r}}) e^{-\mu_1(t-s)}\, ds\\
&\le \mathsf{c}_1 + \mathsf{c}_5 \Big(\sup_{t\in(0,T)} \|u(\cdot,t)\|_{L^\infty(\Omega)} \Big)^a
\end{aligned}$$

and thereby we conclude that $T_{max} = \infty$ and $\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \mathsf{c}_6$ for all $t > 0$. □

6.2. Case 2. $\alpha > 0$ and $k > 2$ in the radial setting

We will derive a uniform estimate for $\Psi(t) := \frac{1}{p}\|u(\cdot,t)\|^p_{L^p(\Omega)}$ defined in (1.7). As in the proof of Theorem 1.3 in Sect. 5, we have

$$\begin{aligned}
\Psi'(t) = &\int_\Omega u^{p-1}\Delta u\, dx - \int_\Omega u^{p-1}\nabla\cdot(uf(|\nabla v|^2\nabla v))\, dx + \lambda\int_\Omega u^p\, dx\\
&- \mu\int_\Omega u^{p+k-1}\, dx =: \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_4.
\end{aligned}$$

In view of (5.2), (5.12) and (5.13) we see that

$$\mathcal{J}_1 = -\frac{4(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2\,dx,$$
$$\mathcal{J}_2 \le \bar{c}_1\varepsilon_2\int_\Omega |\nabla u^{\frac{p}{2}}|^2\,dx + \bar{c}_2\Psi(t) + \bar{c}_3\Psi^{\frac{p+1}{p}}(t) + \bar{c}_4\Psi^{\frac{2(p+1)-N}{2p-N}}(t) + \bar{c}_5\Psi^{\gamma}(t),$$
$$\mathcal{J}_3 = \lambda p\Psi(t)$$

and the Hölder inequality yields

$$\mathcal{J}_4 \le -\bar{c}_6\Psi^{\frac{p+k-1}{p}}(t).$$

Choosing ε_2 such that the term containing $\int_\Omega |\nabla u^{\frac{p}{2}}|^2\,dx$ vanishes and noting that $k>2$ implies $\frac{p+1}{p} \in (1, \frac{p+k-1}{p})$ and

$$\frac{2(p+1)-N}{2p-N},\ \gamma \in \Big(1, \frac{p+k-1}{p}\Big)$$

for sufficiently large p because $\lim_{p\nearrow\infty}\frac{2(p+1)-N}{2p-N}\cdot\frac{p}{p+1} = 1$ and $\lim_{p\nearrow\infty}\gamma\cdot\frac{p}{p+1} = 1$, we can derive from Young's inequality that

$$\Psi'(t) \le \bar{c}_7\Psi(t) - \bar{c}_8\Psi^{\frac{p+k-1}{p}}(t)$$

and therefore ODI comparison yields uniform bound for $\Psi(t)$ with sufficiently large $p > \frac{N}{2}$. Consequently, Lemma 4.1 proves that $T_{max} = \infty$ and $\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \bar{c}_9$ for all $t>0$. □

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Declarations

Conflict of interest All the authors declare no conflict of interests.

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