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A system of superlinear elliptic equations in a cylinder

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Abstract. The article is concerned with the existence of positive solutions of a semi-linear elliptic system defined in a cylinder $\Omega = \Omega' \times (0, a) \subset \mathbb{R}^n$, where $\Omega' \subset \mathbb{R}^{n-1}$ is a bounded and smooth domain. The system couples a superlinear equation defined in the whole cylinder Ω with another superlinear (or linear) equation defined at the bottom of the cylinder $\Omega' \times \{0\}$. Possible applications for such systems are interacting substances (gas in the cylinder and fluid at the bottom) or competing species in a cylindrical habitat (insects in the air and plants on the ground). We provide a priori L^∞ bounds for all positive solutions of the system when the nonlinear terms satisfy certain growth conditions. It is interesting that due to the structure of the system our growth restrictions are weaker than those of the pioneering result by Brezis–Turner for a single equation. Using the a priori bounds and topological arguments, we prove the existence of positive solutions for these particular semi-linear elliptic systems.

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1. Introduction

For scalar equations of the form

$$\begin{cases} -\Delta u = f(x, u) & \text{in } \Omega \subset \mathbb{R}^n \\ u \quad = 0 & \text{on } \Omega \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^n$ and $f(x, u)$ behaves like u^p for u large, the question of the existence of positive solutions has been intensively studied [2, 3, 11, 17, 20, 23]. One way to obtain existence results for (1.1) is using topological arguments especially when the equation has no variational structure. The main difficulty when using a topological approach lies in the need of obtaining a priori bounds. In recent years, several approaches have been developed to deal with this problem [3, 11, 17]. Subsequently, many existence results proved by a priori estimates for the scalar Eq. (1.1) have been extended to corresponding elliptic nonlinear coupled systems [4,5,12,13,15]. For instance, the following superlinear system

$$\begin{cases} -\Delta u = v^p & \text{in } \Omega,\ u = 0 \text{ on } \partial\Omega \\ -\Delta v = u^q & \text{in } \Omega,\ v = 0 \text{ on } \partial\Omega \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $p, q > 1$, is usually referred to as the coupled Lane-Emden system and has been widely investigated in the last few years (see [10,16,24] and the references therein). Such problems arise in the study of multicomponent reaction diffusion processes and in the modeling of several physical phenomena such as pattern formation and population evolution (see [25] and the references therein). The solutions in most of the cases represent densities and thus positive solutions of the systems are of particular interest. The exponents (p, q) in system (1.2) interplay, compensating each other, which play a crucial role in the questions of existence and nonexistence of positive solutions.

In [4], Clément, de Figueiredo and Mitidieri used a method which was developed in [11] for the case of one equation to obtain L^∞ a priori bounds. For another coupled system studied by de Figueiredo and Yang [15], the difficulties of obtaining a priori bounds were due to the presence of gradients in the nonlinear terms. The authors had to use some norm with weights depending on the distance to the boundary of the domain. They obtained a priori bounds via the so called blow-up method which was introduced by Gidas–Spruck [17] for the scalar case. In [5], the authors found L^∞ a priori bounds with different exponent assumptions imposed on the nonlinear terms; the technique used in their work is based on the work of Brezis and Turner [3] for one equation, in which they combined the Hardy–Sobolev inequality with interpolation techniques. In [5] the Brezis–Turner exponent assumption was replaced by conditions that involve two curves in the (p, q) plane. We remark that the method introduced by Brezie-Turner was the first general way to obtain uniform bounds of positive solutions and has become a classical way. Many other problems like reaction-diffusion systems and Ambrosetti-Prodi type problems have been solved by this method (see [14,21]).

The objective of this paper is to study the existence of positive solutions of a particular semi-linear elliptic system defined in a cylinder $\Omega = \Omega' \times (0, a) \subset \mathbb{R}^n$, where $\Omega' \subset \mathbb{R}^{n-1}$ is a bounded and smooth domain. The system couples a superlinear equation defined in the whole cylinder Ω with another superlinear (or linear) equation defined at the bottom $\Omega' \times \{0\}$ of the cylinder. Possible applications for such systems are interacting substances (gas in the cylinder and fluid at the bottom) or competing species in a cylindrical habitat (insects in the air and plants on the ground). Extending the method of Brezis–Turner [3] to this kind of system, we provide *a priori* L^∞ bounds for all positive solutions when the nonlinear terms satisfy certain growth conditions. The approach we use consists of using the Hardy–Sobolev inequality and a suitable fixed point theorem. Unlike the setting in [3] where the nonlinear term $f(x, u)$ is defined on $\Omega \times \mathbb{R}$, in our framework f is non-local and we have to distinguish two cases, depending on the space dimension. It is interesting that due to the structure of the system our growth restrictions are weaker than those of the pioneering result by Brezis–Turner for a single equation. Using the a priori bounds and topological arguments, we prove the existence of positive solutions for these particular semi-linear elliptic systems.

2. The Main Result

In this paper we consider a system of equations on a cylindrical domain $\Omega = \Omega' \times (0,a) \subset \mathbb{R}^n (n \geq 3)$, with $x = (x', x_n) \in \Omega$ and $\Omega' \subset \mathbb{R}^{n-1}$ is smooth. The particularity of this system is that it couples two unknowns $u(x)$ and $v(x')$ which are defined on *different domains*. We can think of Ω as a jar or a cylindrical habitat containing two interacting substances or species: the substance $u(x)$ (say a gas, insects, birds...) is distributed in the interior of the jar or habitat Ω, while the substance $v(x')$ (say a fluid, plants, worms...) is located at the bottom $\Omega' \times \{0\}$ of the jar or on the ground of the habitat. A simple model of such a time independent interacting system is

$$\begin{cases} -\Delta_{(n)} u(x) = h(x) v(x')^{\gamma} & x \in \Omega \\ -\Delta_{(n-1)} v(x') = \int_0^a u^{\eta}(x', x_n)\, dx_n & x' \in \Omega' \\ u(x) = 0 \quad x \in \partial\Omega' \times [0,a]; & \partial_{\nu} u(x) = 0, \ x \in \Omega' \times \{0, a\} \\ v(x') = 0 & x' \in \partial\Omega' \end{cases} \tag{2.1}$$

where $\Delta_{(n)} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, $x' = (x_1, \ldots, x_{n-1})$, ν denotes the exterior normal to the boundary $\partial\Omega$, and γ, η are exponents with $\gamma > 1$ and $\eta \geq 1$.

Here, we assume that the vertically cumulated effect of the substance $u(x), x \in \Omega$, interacts with the substance $v(x')$ at the bottom Ω', hence the term $\int_0^a u^{\eta}(x', x_n)\, dx_n$ in the second equation; on the other hand, the substance $v(x')$ at the bottom Ω' interacts with the substance $u(x)$ via a continuous coefficient function $h : \overline{\Omega} \to \mathbb{R}^+$, which we may consider decreasing with increasing height x_n.

The operator $\Delta_{(n-1)}$ with Dirichlet boundary condition in the second equation is invertible, and we can insert the expression

$$v(x') = (-\Delta_{(n-1)})^{-1} \left(\int_0^a u^{\eta}(x', x_n)\, dx_n \right)$$

into the first equation of the system, to obtain the non-local equation

$$\begin{cases} -\Delta_{(n)} u(x) = h(x) \left[(-\Delta_{(n-1)})^{-1} (\int_0^a u^{\eta}(x', x_n)\, dx_n) \right]^{\gamma} & x \in \Omega \\ u(x) = 0 \ \text{ for } \ x \in \partial\Omega' \times [0,a] \quad \partial_{\nu} u(x) = 0 \ \text{ for } \ x \in \Omega' \times \{0, a\}. \end{cases} \tag{2.2}$$

Our aim is to prove the following result:

Theorem 2.1. *Suppose that $\Omega := \Omega' \times (0,a) \subset \mathbb{R}^n$ is a bounded open domain. Furthermore,*

1) *if* $1 \le \eta < \frac{4n}{(n-1)(n-2)}$, *then assume that* $1 < \gamma\eta \le \frac{2n+2}{n}$;

if $\eta \ge \frac{4n}{(n-1)(n-2)}$, *then assume that* $1 < \gamma\eta \le \frac{n+1}{n-1} + \frac{2n\gamma}{(n-1)^2}$.

2) $h \in C(\overline{\Omega}, \mathbb{R}^+)$, *with* $h_m := \min\{h(x), x \in \overline{\Omega}\} > 0$.

Then Eq. (2.2), *and hence system* (2.1), *has a positive solution* $u \in W^{2,q}(\Omega), 1 \le q < \infty$.

Remark 2.1. Notice that for $n = 3, 4$, we are always in case 1), since then

$$\frac{2n+2}{n} < \frac{4n}{(n-1)(n-2)}.$$

The proof follows the ideas of the influential paper by Brezis–Turner [3], in which a single equation with a super-linear non-linearity was considered. It is interesting to note that the maximal exponent in the article of Brezis–Turner was $\frac{n+1}{n-1}$. For $\eta = 1$, the maximal exponent for γ is

$$\begin{cases} \frac{2n+2}{n}, & 3 \le n \le 6, \\ \frac{n^2-1}{n^2-4n+1}, & n \ge 7 \end{cases}$$

which is larger than $\frac{n+1}{n-1}$, this is due to the regularizing effect of the inverted operator $(-\Delta_{(n-1)})^{-1}$.

We have not seen such type of coupled systems in the literature. Of course, one can consider many different versions of such couplings.

3. L^p regularity on the cylinder

The proof of Theorem 2.1 depends on a priori estimates of the solutions and a related existence theorem. The L^p theory presented here is to pave the way to get the a priori bound. In this part we will concentrate on showing that a weak solution of the equation

$$\begin{cases} -\Delta_{(n)} u = f(x) \; x \in \Omega \\ u(x', x_n) = 0, \quad x' \in \partial\Omega' \\ \partial_{x_n} u(x', x_n) = 0 \quad x_n \in \{0, a\}. \end{cases} \tag{3.1}$$

with $f \in L^p(\Omega)$ $(1 < p < \infty)$ will also be a strong solution which is twice weakly differentiable. The proof of the regularity is based on the a priori estimates below. In view of the mixed boundary conditions and the special shape of the domain, we will do an even reflection on the bottom of the cylinder to reduce the problem to a familiar case for which we can refer to the ninth chapter in [18].

3.1. L^p a priori estimate

We define the space $H^1_{cyl}(\Omega)$ as the closure in $H^1(\Omega)$ of the set $C^1_{cyl}(\Omega) = \{u \in C^1(\Omega) \mid u(x) = 0,\ x \in \partial\Omega' \times [0, a]\}$. Correspondingly, $W^{1,p}_{cyl} = \{u \in W^{1,p}(\Omega) \mid u(x) = 0, x \in \partial\Omega' \times [0, a]\ \}$.

Interior estimate:

Lemma 3.1.1. *Assume that* $u \in W^{2,p}_{loc}(\Omega) \cap L^p(\Omega)$, $1 < p < \infty$, *is a strong solution of the Eq.* (3.1), *then for* $f \in L^p(\Omega)$ *and for any open domain* $\Omega_i \subset\subset \Omega$,

$$\|u\|_{W^{2,p}(\Omega_i)} \leq C(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}), \tag{3.2}$$

where $C = C(n, p, \Omega_i, \Omega)$.

Since the interior estimate does not require the boundary condition, the proof of this lemma follows from the same proof of Theorem 9.11 [18].

Estimate on the bottom and the top:

Lemma 3.1.2. *Assume that* $u \in W^{2,p}(\Omega)$, $\Omega = \Omega' \times (0, a) \subset \mathbb{R}^n$, *where* $\Omega' \subset \mathbb{R}^{n-1}$ *is a bounded and smooth domain.* $1 < p < \infty$ *is a strong solution of* (3.1), *then for* $f \in L^p(\Omega)$ *and for any open domain* $\Omega_b \subset\subset \Omega \cup \Big\{\Omega' \times \{0\}\Big\}$ *or* $\Omega_t \subset\subset \Omega \cup \Big\{\Omega' \times \{a\}\Big\}$

$$\|u\|_{W^{2,p}(\Omega_b)} \leq C_b(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)})$$

or

$$\|u\|_{W^{2,p}(\Omega_t)} \leq C_t(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)})$$

where $C_b = C(n, p, \Omega_b, \Omega)$, $C_t = C(n, p, \Omega_t, \Omega)$.

Proof. We extend u and f to $\Omega' \times (-a, a)$ by even reflection, that is, by setting

$$u(x', x_n) = u(x', -x_n), \qquad f(x', x_n) = f(x', -x_n)$$

for $x_n < 0$. It follows that the extended functions, say $\tilde{u}$ and $\tilde{f}$, satisfy the same equation of (3.1) weakly in $\Omega' \times (-a, a)$. To prove this we take an arbitrary test function $\varphi \in C^1_{cyl}(\Omega' \times (-a, a))$, then since u is a weak solution of (3.1) on Ω, we have

$$\int_{\Omega' \times (0,a)} \nabla u \nabla \phi \, dx = \int_{\Omega' \times (0,a)} f \phi \, dx, \quad \forall\ \phi \in C^1_{cyl}(\Omega' \times (0, a)). \tag{3.3}$$

As $\varphi \in C^1$ in $\Omega' \times (0, a)$ and $\varphi = 0$ on $\partial\Omega'$, we can take $\phi = \varphi$ in $\Omega' \times (0, a)$, then

$$\int_{\Omega' \times (0,a)} \nabla u \nabla \varphi \, dx = \int_{\Omega' \times (0,a)} f \varphi \, dx. \tag{3.4}$$

On the other hand, due to the even reflection, from (3.3), we get

$$\int_{\Omega' \times (-a,0)} \nabla u \nabla \phi' \, dx = \int_{\Omega' \times (-a,0)} f \phi' \, dx, \quad \forall\ \phi' \in C^1_{cyl}(\Omega' \times (-a, 0)),$$

then taking $\phi' = \varphi$ in $\Omega' \times (-a, 0)$, so

$$\int_{\Omega' \times (-a,0)} \nabla u \nabla \varphi \, dx = \int_{\Omega' \times (-a,0)} f \varphi \, dx. \tag{3.5}$$

(3.4)+(3.5), we obtain

$$\begin{aligned}&\int_{\Omega'\times(0,a)}\nabla u\nabla\varphi\,dx+\int_{\Omega'\times(-a,0)}\nabla u\nabla\varphi\,dx=\int_{\Omega'\times(-a,a)}\nabla\tilde{u}\nabla\varphi\,dx\\&=\int_{\Omega'\times(0,a)}f\varphi\,dx+\int_{\Omega'\times(-a,0)}f\varphi\,dx\\&=\int_{\Omega'\times(-a,a)}\tilde{f}\varphi\,dx.\end{aligned}$$

Consequently, we have

$$\int_{\Omega'\times(-a,a)}\nabla\tilde{u}\nabla\varphi\,dx=\int_{\Omega'\times(-a,a)}\tilde{f}\varphi\,dx\quad\forall\varphi\in C^1_{cyl}(\Omega'\times(-a,a)).$$

Besides, $\tilde{u}=0$, $x\in\partial\Omega'\times[-a,a]$ and $\frac{\partial\tilde{u}}{\partial x_n}\big|_{x_n=-a}=-\frac{\partial\tilde{u}}{\partial x_n}\big|_{x_n=a}=-\frac{\partial u}{\partial x_n}\big|_{x_n=a}=0$, so that $\tilde{u}$ is a weak solution of (3.1) in $\Omega'\times(-a,a)$. By the evenness of $\tilde{u}$, we also have $\frac{\partial\tilde{u}}{\partial x_n}\big|_{x_n=0}=0$. Then, for any open subset $\widetilde{\Omega}_b\subset\subset\Omega'\times(-a,a)$ we are able to apply the interior estimate and thus get the desired estimate for $\widetilde{\Omega}_b$ and hence also for $\Omega_b:=\widetilde{\Omega}_b\cap\Omega$. □

Estimate on the side:

Lemma 3.1.3. *Assume that $u\in W^{2,p}(\Omega)$, $\Omega=\Omega'\times(0,a)\subset\mathbb{R}^n$, where $\Omega'\subset\mathbb{R}^{n-1}$ is a bounded and smooth domain. $1<p<\infty$ is a strong solution of (3.1) with $u=0$ on $\partial\Omega'\times[0,a]$, then for $f\in L^p(\Omega)$ and for any open domain $\Omega_s\subset\subset\{\overline{\Omega'}\times(0,a)\}$,*

$$\|u\|_{W^{2,p}(\Omega_s)}\le C(\|u\|_{L^p(\Omega)}+\|f\|_{L^p(\Omega)}),$$

where $C=C(n,p,\Omega_s,\Omega)$.

Proof. Since $u(x)=0, x'\in\partial\Omega'$, the proof follows from the boundary L^p estimate of Theorem 9.13 [18]. □

Estimate on the edge ($\partial\Omega'\times\{0,a\}$):

Lemma 3.1.4. *Assume $u\in W^{2,p}(\Omega)$, $\Omega=\Omega'\times(0,a)\subset\mathbb{R}^n$, where $\Omega'\subset\mathbb{R}^{n-1}$ is a bounded and smooth domain. $1<p<\infty$, a strong solution of (3.1) with $u=0$ on $\partial\Omega'\times[0,a]$, then for $f\in L^p(\Omega)$ and for any open domain $\Omega_e\subset\subset\Omega\cup\left\{\overline{\Omega'}\times\{0\}\right\}$,*

$$\|u\|_{W^{2,p}(\Omega_e)}\le C(\|u\|_{L^p(\Omega)}+\|f\|_{L^p(\Omega)}),\tag{3.6}$$

where $C=C(n,p,\Omega_e,\Omega)$.

Proof. In the proof of Lemma 3.1.2, we extended u and f to $\Omega'\times(-a,a)$ by even reflection, and we proved that the extended function $\tilde{u}$ is a weak solution of (3.1) in $\Omega'\times(-a,a)$ with f replaced by $\tilde{f}$. In this case, each point $x_0\in\partial\Omega'\times\{0\}$ is a boundary point of $\Omega'\times(-a,a)$ on the side, we then can proceed as in the proof of Lemma 3.1.3 with Ω_s replaced by $\Omega_S\subset\subset\{\overline{\Omega'}\times(-a,a)\}$, since $\Omega_e\subset\Omega_S$, we have

$$\begin{aligned}\|u\|_{W^{2,p}(\Omega_e)}\le\|\tilde{u}\|_{W^{2,p}(\Omega_S)}&\le C\left(\|\tilde{u}\|_{L^p(\Omega'\times(-a,a))}+\|\tilde{f}\|_{L^p(\Omega'\times(-a,a))}\right)\\&\le C\left(2\|u\|_{L^p(\Omega'\times(0,a))}+2\|f\|_{L^p(\Omega'\times(0,a))}\right).\end{aligned}$$

We therefore derive

$$\|u\|_{W^{2,p}(\Omega_e)} \le C(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}).$$

□

Combining all the estimates above, we get the following result.

Global L^p estimate and regularity:

Lemma 3.1.5. *Assume that $u \in W^{2,p}(\Omega) \cap W^{1,p}_{cyl}(\Omega)$, $1 < p < \infty$, satisfies (3.1); if $f \in L^p(\Omega)$, then*

$$\|u\|_{W^{2,p}(\Omega)} \le C(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}),$$

where $C = C(n, p, \Omega)$.

Proof. (see a similar proof of Theorem 2.2.3 [26]) From the boundary estimate we conclude that for $x_0 \in \partial\Omega$, there exists a neighborhood $U(x_0)$ such that

$$\begin{aligned}\|u\|_{W^{2,p}(U(x_0)\cap\Omega)} &\le \|u\|_{W^{2,p}(\Omega_s)} + \|u\|_{W^{2,p}(\Omega_b)} + \|u\|_{W^{2,p}(\Omega_t)} + \|u\|_{W^{2,p}(\Omega_e)} \\ &\le C(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}). \qquad (3.7)\end{aligned}$$

According to Heine–Borel theorem, there exists a finite open covering $U_1, \ldots, U_N$ to cover $\partial\Omega$. Denote $K = \Omega \setminus \overset{N}{\underset{i=1}{\cup}} U_i$, then K is a closed subset of Ω and there exists a subdomain $U_0 \subset\subset \Omega$ such that $U_0 \supset K$. Lemma 3.1.1 shows that

$$\|u\|_{W^{2,p}(U_0)} \le C(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}). \qquad (3.8)$$

Using the theorem on the partition of unity, we can choose functions $\eta_0, \eta_1, \ldots, \eta_N$ such that

$$0 \le \eta_i \le 1, \quad \forall x \in U_i \ (i = 0, 1, \ldots, N),$$
$$\sum_{i=0}^{N} \eta(x) = 1, \quad x \in \bar{\Omega}.$$

Thus

$$\begin{aligned}\|u\|_{W^{2,p}(\Omega)} = \Big\| \sum_{i=0}^{N} \eta_i u \Big\|_{W^{2,p}(\Omega)} &\le \sum_{i=0}^{N} \|\eta_i u\|_{W^{2,p}(\Omega)} \\ &\le C(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}). \qquad (3.9)\end{aligned}$$

□

In the next lemma we eliminate the dependence of u on the right.

Lemma 3.1.6. (A better a priori L^P estimate, cf. [6], Lemma 3.2.1) *Assume that $u \in W^{2,p}(\Omega) \cap W^{1,p}_{cyl}(\Omega)$, $2 \le p < \infty$, satisfies (3.1), if $f \in L^p(\Omega)$, then*

$$\|u\|_{W^{2,p}(\Omega)} \le C\|f\|_{L^p(\Omega)}, \qquad (3.10)$$

where $C = C(n, p, \Omega)$.

Proof. We argue by contradiction. If (3.10) is not true, then $\forall N$, $\exists u_N \in W^{2,p}(\Omega) \cap W^{1,p}_{cyl}(\Omega)$, $f_N \in L^p(\Omega)$, such that

$$\begin{cases} -\Delta_{(n)} u_N = f_N, \; x \in \Omega \\ u_N(x', x_n) = 0, \quad x' \in \partial\Omega' \\ \partial_{x_n} u_N(x', x_n) = 0 \quad x_n \in \{0, a\} \end{cases} \tag{3.11}$$

but

$$\|u_N\|_{W^{2,p}(\Omega)} \geq N\|f_N\|_{L^p(\Omega)}.$$

Let

$$v_N = \frac{u_N}{\|u_N\|_{L^p(\Omega)}}, \quad g_N = \frac{f_N}{\|u_N\|_{L^p(\Omega)}},$$

then

$$\begin{cases} -\Delta_{(n)} v_N = g_N, \; x \in \Omega \\ v_N(x', x_n) = 0, \quad x' \in \partial\Omega' \\ \partial_{x_n} v_N(x', x_n) = 0 \quad x_n \in \{0, a\} \end{cases} \tag{3.12}$$

and

$$\|v_N\|_{L^p(\Omega)} = 1, \quad \|v_N\|_{W^{2,p}(\Omega)} = \frac{\|u_N\|_{W^{2,p}(\Omega)}}{\|u_N\|_{L^p(\Omega)}}.$$

From the global estimate Lemma 3.1.5 we have

$$\begin{aligned} \|v_N\|_{W^{2,p}(\Omega)} &\leq C(\|g_N\|_{L^p(\Omega)} + \|v_N\|_{L^p(\Omega)}) \\ &\leq C\left(\frac{\|f_N\|_{L^p(\Omega)}}{\|u_N\|_{L^p(\Omega)}} + 1\right) \\ &\leq \frac{C}{N}\frac{\|u_N\|_{W^{2,p}(\Omega)}}{\|u_N\|_{L^p(\Omega)}} + C \\ &= \frac{C}{N}\|v_N\|_{W^{2,p}(\Omega)} + C \end{aligned}$$

taking $N > C$, then

$$\|v_N\|_{W^{2,p}(\Omega)} \leq C. \tag{3.13}$$

Following from Rellich–Kondrachov theorem (cf. [1], Theorem 6.3), $W^{2,p}(\Omega) \hookrightarrow W^{1,p}(\Omega)$ compactly. That is there exists a sub-sequence such that

$$\|v_N - v\|_{L^p(\Omega)} \to 0, \quad \|\nabla v_N - \nabla v\|_{L^p(\Omega)} \to 0. \tag{3.14}$$

Since v_N satisfies (3.12) weakly, then

$$\int_\Omega \nabla v_N \nabla \varphi \, dx = \int_\Omega g_N \varphi \, dx, \quad \forall \varphi \in C^\infty_{cyl}(\Omega). \tag{3.15}$$

From (3.14), we have $v_N \rightharpoonup v$ in $W^{1,p}_{cyl}(\Omega)$, and hence

$$\int_\Omega \nabla v_N \nabla \varphi \, dx \to \int_\Omega \nabla v \nabla \varphi \, dx, \qquad N \to \infty.$$

On the other hand, since

$$\|g_N\|_{L^p(\Omega)} = \frac{\|f_N\|_{L^p(\Omega)}}{\|u_N\|_{L^p(\Omega)}} \le \frac{1}{N}\frac{\|u_N\|_{W^{2,p}(\Omega)}}{\|u_N\|_{L^p(\Omega)}} = \frac{1}{N}\|v_N\|_{W^{2,p}(\Omega)}$$

and (3.13), we see $\|g_N\|_{L^p(\Omega)} \to 0$ as $N \to \infty$, which implies $\forall \varphi \in C^\infty_{cyl}(\Omega)$

$$\int_\Omega g_N \varphi \, dx \to 0, \qquad N \to \infty.$$

So,

$$\int_\Omega \nabla v \nabla \varphi \, dx = 0, \quad \forall \varphi \in C^\infty_{cyl}(\Omega), \quad v \in W^{1,p}_{cyl}(\Omega).$$

as $N \to \infty$ in (3.15). Hence v weakly satisfies

$$\begin{cases} -\Delta_{(n)} v = 0, \, x \in \Omega \\ v = 0, \, x' \in \partial\Omega' \\ \partial_{x_n} v = 0 \;\; x_n \in \{0, a\} \end{cases} \tag{3.16}$$

In the following we prove $v = 0$. Indeed, multiplying with v on both sides of Eq. (3.16), we get $\int_\Omega |\nabla v|^2 \, dx = 0$, so $\nabla v = 0$, combining with the boundary condition then $v = 0$, which contradicts with $\|v_N\|_{L^p(\Omega)} \to \|v\|_{L^p(\Omega)} = 1$. □

3.2. Regularity:

With the above a priori estimate we can get the following existence result:

Lemma 3.2.1. *If $f \in L^p(\Omega)$ with $2 \le p < \infty$, then the problem (3.1) has a unique strong solution $u \in W^{2,p}(\Omega)$.*

Proof. The existence of the strong solution follows as in Th.9.15 [18]. Here we present the main points of the proof. We start from the L^2 regularity.

L^2 interior regularity: If $f \in L^2(\Omega)$, $u \in H^1_{cyl}(\Omega)$ is a weak solution of (3.1), then $u \in H^2_{loc}(\Omega) \cap W^{1,p}_{cyl}(\Omega)$, and for each open subset $V \subset\subset \Omega$ we have the estimate

$$\|u\|_{H^2(V)} \le C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}), \tag{3.17}$$

the constant C depending only on V, Ω. The proof of L^2 interior regularity is the same as Theorem 1 ([8] section 6.3.1).

In order to get the boundary regularity, we extend u and f to $\Omega' \times (-a, a)$ as we did in Lemma 3.1.2. The extended function $\tilde{u}$ and $\tilde{f}$ satisfy the same equation of (2.1) weakly in $\Omega' \times (-a, a)$. Since the bottom $\Omega' \times \{0\}$ is inside of $\Omega' \times (-a, a)$ after the extension, then the proof of regularity near the bottom $\Omega' \times \{0\}$ is the same as L^2 interior regularity. Considering $u = 0$ on $\partial\Omega'$, then the regularity near the side of the cylinder is the same as Theorem 4 ([8] section 6.3.2). Thus we have:

L^2 boundary regularity: If $f \in L^2(\Omega)$, $u \in H^1_{cyl}(\Omega)$ is a weak solution of (3.1), then $u \in H^2(\Omega)$, and we have the estimate

$$\|u\|_{H^2(\Omega)} \le C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}), \tag{3.18}$$

the constant C depending only on Ω.

We are now in a position to prove Lemma 3.2.1 with $2 < p < \infty$. In fact, given that we have the same L^p a priori estimates as in chapter 9 [18], the interior regularity result follows directly from Lemma 9.16 [18]. After we did the even reflection, the case of local boundary regularity is handled similarly as the Lemma 9.16 as well.

For the uniqueness, assume $u_1, u_2 \in W^{2,p}(\Omega)$ both the strong solution of (3.1). Let $u = u_1 - u_2$, then $u \in W^{2,p}(\Omega)$ and satisfy (3.16) weakly with v replaced by u. From Lemma 3.1.6,

$$\|u\|_{W^{2,p}(\Omega)} \leq 0,$$

therefore, $u = 0$ a.e. in Ω, that is, $u_1 = u_2$. □

4. A Priori Bounds

In this section we will see that the growth conditions imposed on the nonlinear terms play an important role in acquiring a priori bounds for all positive solutions of system (2.1). These terms are embedded into different L^p spaces as the dimension n varies. Depending on the size of η (the growth of nonlinearity in u) we will find two different growth restrictions for γ (the growth of the nonlinearity in v).

First, we state the following Hardy-type estimate in $H^1_{cyl}(\Omega)$, which is the preparatory step of the technical aspects of the proof. The constant C may change from line to line, we will use C for a generic constant.

Lemma 4.1. *There exists $C > 0$ such that for any $u \in H^1_{cyl}(\Omega)$, we have*

$$\int_\Omega |\nabla u|^2 \, dx \geq C \int_\Omega \left| \frac{u}{\delta_{n-1}} \right|^2 dx$$

where $\delta_{n-1} = \delta(x')$ denotes the distance of x to $\partial\Omega'$, $\Omega' \subset \mathbb{R}^{n-1}$.

Proof. Notice that $\displaystyle\int_\Omega \frac{u^2(x)}{\delta^2(x')} \, dx = \int_0^a \int_{\Omega'} \frac{u^2(x)}{\delta^2(x')} \, dx' dx_n$; we start the proof from the inner integral. Consider $u_n \in C^\infty_{cyl}(\Omega)$; for fixed x_n, $u_n(x', x_n)$ is a function of x', then by Hardy's inequality [22] we have

$$\int_{\Omega'} \frac{u_n^2(x', x_n)}{\delta^2(x')} \, dx' \leq C \int_{\Omega'} \left| \frac{\partial u_n(x', x_n)}{\partial x'} \right|^2 dx',$$

and then integrating along the x_n direction,

$$\begin{aligned} &\int_0^a \int_{\Omega'} \frac{u_n^2(x)}{\delta^2(x')} \, dx' dx_n \\ &\quad \leq C \int_0^a \int_{\Omega'} \left| \frac{\partial u_n(x)}{\partial x'} \right|^2 dx' dx_n + \int_0^a \int_{\Omega'} \left| \frac{\partial u_n(x)}{\partial x_n} \right|^2 dx' dx_n \\ &\quad \leq C \int_\Omega \left| \frac{\partial u_n(x)}{\partial x} \right|^2 dx. \end{aligned} \tag{4.1}$$

Since $C^\infty_{cyl}(\Omega)$ is dense in $H^1_{cyl}(\Omega)$, for $u \in H^1_{cyl}(\Omega)$, there exists functions $u_n(x) \in C^\infty_{cyl}(\Omega)$ such that

$$\int_\Omega \Big|\nabla(u_n(x) - u(x))\Big|^2 dx \to 0, \quad \int_\Omega |u_n(x) - u(x)|^2 dx \to 0$$

as $n \to \infty$. This implies that $\{u_n\}$ is a Cauchy sequence in $H^1_{cyl}(\Omega)$, then there exist n_ϵ such that for $n, m \geq n_\epsilon$,

$$\int_\Omega \Big|\nabla(u_n(x) - u_m(x))\Big|^2 dx \leq \epsilon.$$

Notice $u_n - u_m \in C^\infty_{cyl}(\Omega)$, we substitute u_n with $u_n - u_m$ in (4.1), then

$$\int_0^a \int_{\Omega'} \frac{|u_n(x) - u_m(x)|^2}{\delta^2(x')} dx' dx_n \leq C \int_\Omega \Big|\nabla(u_n(x) - u_m(x))\Big|^2 dx \leq \epsilon,$$

which implies that $\{\frac{u_n(x)}{\delta(x')}\}$ is a Cauchy sequence in $L^2(\Omega)$ and hence

$$\frac{u_n(x)}{\delta(x')} \to y,$$

for some $y \in L^2(\Omega)$. It remains to show $y = \frac{u(x)}{\delta(x')}$. Since $\delta(x')$ is bounded, we have that

$$u_n(x) \to y\delta(x'), \text{ in } L^2(\Omega).$$

In fact,

$$\begin{aligned} \int_\Omega |u_n(x) - \delta(x')y|^2 dx' dx_n &= \int_\Omega \Big|\frac{u_n(x)}{\delta(x')} \cdot \delta(x') - \delta(x')y\Big|^2 dx \\ &= \int_\Omega |\delta(x')|^2 \Big|\frac{u_n(x)}{\delta(x')} - y\Big|^2 dx \\ &\leq C \int_\Omega \Big|\frac{u_n(x)}{\delta(x')} - y\Big|^2 dx \\ &\to 0, \end{aligned}$$

and since $u_n(x) \to u(x)$ in $L^2(\Omega)$, we conclude that indeed $y = \frac{u(x)}{\delta(x')}$. Then we complete the proof by letting $n \to \infty$ in (4.1). □

The next lemma is a variant of the Hardy-inequality.

Lemma 4.2. *There exists $C > 0$ such that for $n \geq 3$, and $0 \leq \tau \leq 1$, we have*

$$\Big\|\frac{u}{\delta^\tau_{n-1}}\Big\|_{L^q(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}, \quad \forall u \in H^1_{cyl}(\Omega)$$

where $\dfrac{1}{q} = \dfrac{1}{2} - \dfrac{1-\tau}{n}$.

Proof. By the Hölder inequality,

$$\Big\|\frac{u}{\delta^\tau_{n-1}}\Big\|_{L^q(\Omega)} = \Big(\int_\Omega \Big(\frac{u^\tau}{\delta^\tau_{n-1}} \cdot u^{1-\tau}\Big)^q dx\Big)^{\frac{1}{q}}$$

$$
\begin{aligned}
&\leq \left(\left(\int_\Omega \left(|\frac{u}{\delta_{n-1}}|^{\tau q}\right)^{\frac{r}{q}} dx\right)^{\frac{q}{r}}\right)^{\frac{1}{q}} \cdot \left(\left(\int_\Omega \left(|u|^{(1-\tau)q}\right)^{\frac{s}{q}} dx\right)^{\frac{q}{s}}\right)^{\frac{1}{q}} \\
&= \left\|\frac{u^\tau}{\delta_{n-1}^\tau}\right\|_{L^r(\Omega)} \cdot \left\|u^{1-\tau}\right\|_{L^s(\Omega)} \\
&= \left\|\frac{u}{\delta_{n-1}}\right\|_{L^{\tau r}(\Omega)}^\tau \|u\|_{L^{(1-\tau)s}(\Omega)}^{1-\tau} \qquad (4.2)
\end{aligned}
$$

where $\frac{1}{q} = \frac{1}{r} + \frac{1}{s}$. We choose $\tau r = 2$ and $\frac{1}{(1-\tau)s} = \frac{1}{2} - \frac{1}{n}$, thus

$$
\frac{1}{q} = \frac{1}{s} + \frac{\tau}{2} = \frac{1}{2} - \frac{1-\tau}{n}.
$$

Applying Lemma 4.1 and Sobolev's embedding theorem to the respective term in (4.2) we obtain

$$
\left\|\frac{u}{\delta_{n-1}^\tau}\right\|_{L^q(\Omega)} \leq C\|Du\|_{L^2(\Omega)}^\tau \|Du\|_{L^2(\Omega)}^{1-\tau} \tag{4.3}
$$

Then (4.3) becomes the desired inequality. □

In what follows we let J_1' denote the first positive eigenfunction satisfying

$$
\begin{cases} -\Delta_{(n-1)} J_1' = \lambda_1' J_1', & x' \in \Omega' \\ J_1'(x') = 0, & x' \in \partial\Omega' \end{cases}
$$

where λ_1' is the first eigenvalue of $-\Delta_{(n-1)}$ and J_1' is normalized so that $\int_{\Omega'} |J_1'|^2 \, dx' = 1$. Furthermore, $J_1(x)$ is the eigenfunction to the corresponding Laplacian equation in Ω, with $J_1(x', x_n) := J_1'(x'), x_n \in (0, a)$, that is $J_1(x', x_n)$ is constant in the variable x_n and satisfies

$$
\begin{cases} -\Delta_{(n)} J_1 = \lambda_1' J_1 & x \in \Omega \\ J_1(x) = 0 & x \in \partial\Omega' \times [0, a] \\ \partial_{x_n} J_1(x) = 0 & x \in \Omega' \times \{0, a\} \end{cases}
$$

Remark 4.1. It is known that $J_1'(x') > 0$ in Ω' and it follows from Hopf's Lemma that $J_1'(x') \geq C\delta_{n-1}(x')$ with $C > 0$. Note that $\int_\Omega |J_1(x)|^2 \, dx = a$.

The basic *a priori* bound we prove is the following.

Theorem 4.1. *Suppose that $h(x) \geq h_m > 0$. Furthermore,*

- *if $1 \leq \eta < \frac{4n}{(n-1)(n-2)}$, then suppose that $1 < \gamma\eta \leq \frac{2n+2}{n}$;*
- *if $\eta \geq \frac{4n}{(n-1)(n-2)}$, then suppose that $1 < \gamma\eta \leq \frac{n+1}{n-1} + \frac{2n\gamma}{(n-1)^2}$.*

Then there is a constant K such that for any $u \in H_{cyl}^1(\Omega)$ non-negative and satisfying weakly

$$
\begin{cases} -\Delta_{(n)} u = h(x)\left[(-\Delta_{(n-1)})^{-1}\left(\int_0^a u^\eta(x', x_n)\, dx_n\right)\right]^\gamma + tJ_1 & x \in \Omega \\ u(x', x_n) = 0 & x' \in \partial\Omega' \\ \partial_{x_n} u(x', x_n) = 0 & x_n \in \{0, a\} \end{cases} \tag{4.4}
$$

then we have $u \in L^\infty(\Omega)$ *and*

$$\|u\|_{L^\infty(\Omega)} \le K,$$

where K *is independent of* $t \ge 0$.

We first prove some lemmas.

Lemma 4.3. *Under the assumptions of Theorem 4.1, there is a constant* $K_1 > 0$ *such that for any non-negative* $u \in H^1_{cyl}(\Omega)$ *satisfying weakly Eq. (4.4) for some* $t \ge 0$*, then we have*

$$t \le K_1 \quad \text{and} \quad \int_\Omega f(x,u)\, \delta_{n-1}(x)\, dx \le K_1\,,$$

where $f(x,u) := h(x)\big[(-\Delta_{(n-1)})^{-1}\left(\int_0^a u^\eta(x',x_n)\, dx_n\right)\big]^\gamma$.

Proof. Since $u \in H^1_{cyl}(\Omega)$ is a weak solution of (4.5), we have

$$\int_\Omega \nabla u\, \nabla\varphi(x)\, dx = \int_\Omega f(x,u)\varphi(x)\, dx + t\int_\Omega J_1\, \varphi(x)\, dx \;\; \forall\, \varphi \in H^1_{cyl}(\Omega) \tag{4.5}$$

Taking $\varphi = J_1$ we get

$$\int_\Omega \nabla u\, \nabla J_1\, dx = \int_\Omega f(x,u) J_1\, dx + t\int_\Omega |J_1|^2\, dx.$$

Note that $\partial\Omega = (\partial\Omega' \times [0,a]) \cup (\Omega' \times \{0,a\})$. The left side of the equation yields, using that $u|_{\partial\Omega'\times[0,a]} = 0$ and $\partial_\nu J_1|_{\Omega'\times\{0,a\}} = 0$

$$\begin{aligned}
\int_\Omega \nabla u \cdot \nabla J_1\, dx &= \int_{\partial\Omega} u\, \partial_\nu J_1\, dx - \int_\Omega u\Delta_{(n)} J_1\, dx \\
&= -\int_\Omega u\Delta_{(n)} J_1\, dx \\
&= \lambda_1' \int_\Omega u J_1\, dx.
\end{aligned}$$

Since by assumption $h(x)$ has the positive lower bound h_m, then

$$\begin{aligned}
\lambda_1' \int_\Omega u J_1\, dx &= \int_\Omega f(x,u) J_1\, dx + t\int_\Omega |J_1|^2\, dx \\
&= \int_\Omega h(x)\left[(-\Delta_{(n-1)})^{-1}\left(\int_0^a u^\eta(x)\, dx_n\right)\right]^\gamma J_1\, dx + t\int_\Omega |J_1|^2\, dx \\
&\ge h_m \int_\Omega \left[(-\Delta_{(n-1)})^{-1}\left(\int_0^a u^\eta(x)\, dx_n\right)\right]^\gamma J_1\, dx + t\int_\Omega |J_1|^2\, dx \\
&= h_m \int_{\Omega\cap\{[(-\Delta_{(n-1)})^{-1}\int_0^a u^\eta(x)\, dx_n] < k\}} \bigg[(-\Delta_{(n-1)})^{-1} \\
&\qquad \left(\int_0^a u^\eta(x)\, dx_n\right)\bigg]^\gamma J_1\, dx \\
&\qquad + h_m \int_{\Omega\cap\{[(-\Delta_{(n-1)})^{-1}\int_0^a u^\eta(x)\, dx_n] \ge k\}} \bigg[(-\Delta_{(n-1)})^{-1}
\end{aligned}$$

$$\left(\int_0^a u^\eta(x)\,dx_n\right)\Bigg]^\gamma J_1\,dx$$
$$+\,t\int_\Omega |J_1|^2\,dx,$$

where $k > 0$ will be chosen below. Since we consider non-negative solutions, then $\int_0^a u^\eta(x',x_n)\,dx_n$ is non-negative, and by the maximum principle, $(-\Delta_{(n-1)})^{-1}\left(\int_0^a u^\eta(x',x_n)\,dx_n\right)$ is non-negative. Therefore

$$\begin{aligned}
\lambda_1'\int_\Omega uJ_1\,dx &\geq h_m\int_0^a dx_n\int_{\Omega'\cap\{[(-\Delta_{(n-1)})^{-1}\int_0^a u^\eta(x)\,dx_n]\geq k\}} \\
&\quad \left[(-\Delta_{(n-1)})^{-1}\left(\int_0^a u^\eta(x)\,dx_n\right)\right]^\gamma J_1'\,dx' \\
&\quad +t\int_\Omega |J_1|^2\,dx \\
&\geq h_m\cdot a\cdot k^{\gamma-1}\cdot\int_{\Omega'\cap\{[(-\Delta_{(n-1)})^{-1}\int_0^a u^\eta(x)\,dx_n]\geq k\}} \\
&\quad \left[(-\Delta_{(n-1)})^{-1}\left(\int_0^a u^\eta(x)\,dx_n\right)\right] J_1'\,dx' \\
&\quad +t\int_\Omega |J_1|^2\,dx \\
&= h_m\cdot a\cdot k^{\gamma-1}\cdot\Bigg\{\int_{\Omega'}\left[(-\Delta_{(n-1)})^{-1}\left(\int_0^a u^\eta(x)\,dx_n\right)\right] J_1'\,dx' \\
&\quad -\int_{\Omega'\cap\{[(-\Delta_{(n-1)})^{-1}\int_0^a u^\eta(x)\,dx_n]<k\}}\Bigg[(-\Delta_{(n-1)})^{-1} \\
&\quad \left(\int_0^a u^\eta(x)\,dx_n\right)\Bigg] J_1'\,dx'\Bigg\} \\
&\quad +t\int_\Omega |J_1|^2\,dx \\
&\geq h_m\cdot a\cdot k^{\gamma-1}\cdot\Bigg\{\int_{\Omega'}\Bigg[(-\Delta_{(n-1)})^{-1} \\
&\quad \left(\int_0^a u^\eta(x)\,dx_n\right)\Bigg] J_1'\,dx' - C(k)\Bigg\} \\
&\quad +t\int_\Omega |J_1|^2\,dx.
\end{aligned}$$

Next, choose k such that $h_m\cdot a\cdot k^{\gamma-1}\geq (\lambda_1')^2+1$, thus

$$\begin{aligned}
\lambda_1'\int_\Omega uJ_1\,dx &\geq [(\lambda_1')^2+1]\cdot\int_{\Omega'}\left[(-\Delta_{(n-1)})^{-1}\left(\int_0^a u^\eta(x)\,dx_n\right)\right] J_1'\,dx' - C \\
&\quad +t\int_\Omega |J_1|^2\,dx \\
&= [(\lambda_1')^2+1]\cdot\int_{\Omega'}\left[\left(\int_0^a u^\eta(x)\,dx_n\right)\right]\cdot\left[(-\Delta_{(n-1)})^{-1}J_1'\right]\,dx' - C
\end{aligned}$$

$$
\begin{aligned}
&+ t\int_\Omega |J_1|^2\,dx\\
&= [(\lambda_1')^2+1]\cdot\int_{\Omega'}\left(\int_0^a u^\eta(x)\,dx_n\right)\left[\frac{1}{\lambda_1'}J_1'\right]\,dx' + t\int_\Omega |J_1|^2\,dx - C\\
&= \frac{(\lambda_1')^2+1}{\lambda_1'}\int_\Omega u^\eta(x)J_1\,dx + t\int_\Omega |J_1|^2\,dx - C\\
&= \left(\lambda_1' + \frac{1}{\lambda_1'}\right)\int_\Omega u^\eta(x)J_1\,dx + t\int_\Omega |J_1|^2\,dx - C\\
&= \left(\lambda_1' + \frac{1}{\lambda_1'}\right)\Big\{\int_{\Omega\cap\{u\le 1\}} u^\eta(x)J_1\,dx\\
&\quad + \int_{\Omega\cap\{u>1\}} u^\eta(x)J_1\,dx\Big\} + t\int_\Omega |J_1|^2\,dx - C\\
&\ge \left(\lambda_1' + \frac{1}{\lambda_1'}\right)\int_{\Omega\cap\{u>1\}} u^\eta(x)J_1\,dx + t\int_\Omega |J_1|^2\,dx - C\\
&\ge \left(\lambda_1' + \frac{1}{\lambda_1'}\right)\int_\Omega u(x)J_1\,dx + t\int_\Omega |J_1|^2\,dx - C
\end{aligned}
$$

Hence,

$$
C \ge t\int_\Omega |J_1|^2\,dx + \frac{1}{\lambda_1'}\int_\Omega u(x)J_1\,dx
$$

which implies t is bounded, and also

$$
\int_\Omega u(x)J_1\,dx < C.
$$

Since $\lambda_1'\int_\Omega uJ_1\,dx = \int_\Omega f(x,u)J_1\,dx + t\int_\Omega |J_1|^2\,dx$, we see that also $\int_\Omega f(x,u)J_1\,dx$ is bounded, and using Remark 4.1 we obtain,

$$
\int_\Omega f(x,u)\delta_{n-1}(x')\,dx \le C\int_\Omega f(x,u)J_1\,dx < K_1
$$

This completes the proof of Lemma 4.3. $\square$

Next, we show a Poincaré type inequality in $W^{1,p}_{cyl}(\Omega)$.

Lemma 4.4. *There exists a constant $C>0$ such that*

$$
\|u\|_{L^p(\Omega)} \le C\|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W^{1,p}_{cyl}(\Omega). \tag{4.6}
$$

Proof. We may assume $u\in C^\infty_{cyl}(\Omega)$ and $(0,x_2,\dots,x_n)\in\partial\Omega'$, then

$$
\begin{aligned}
|u(x_1,x_2,\dots,x_n)| &= |u(x_1,x_2,\dots,x_n) - u(0,x_2,\dots,x_n)|\\
&= \Big|\int_0^{x_1}\frac{d}{dt}u(t,x_2,\dots,x_n)\,dt\Big|,
\end{aligned}
$$

therefore Hölder's inequality yields

$$
|u|^p = \Big|\int_0^{x_1}\frac{d}{dt}u(t,x_2,\dots,x_n)\,dt\Big|^p
$$

$$\leq \Big|\int_0^{x_1} 1^q dt\Big|^{\frac{p}{q}} \Big|\int_0^{x_1} \big|\frac{\partial u}{\partial t}(t, x_2, \ldots, x_n)\big|^p \, dt\Big|, \qquad \frac{1}{p} + \frac{1}{q} = 1$$
$$\leq C\Big|\int_0^{x_1} \big|\frac{\partial u}{\partial t}(t, x_2, \ldots, x_n)\big|^p \, dt\Big|.$$

Taking the integration over Ω on both sides, we get

$$\int_\Omega |u|^p \, dx \leq C \int_\Omega \int_0^{x_1} \big|\frac{\partial u}{\partial t}(t, x_2, \ldots, x_n)\big|^p \, dt dx,$$

and applying Fubini's theorem to the right hand side of the inequality,

$$\begin{aligned} \int_\Omega |u|^p \, dx &\leq C \int_0^{x_1} \int_\Omega \big|\frac{\partial u}{\partial x_1}(x_1, x_2, \ldots, x_n)\big|^p \, dx dt \\ &\leq C \int_0^{x_1} \int_\Omega |\nabla u|^p \, dx dt \\ &\leq C \|\nabla u\|^p_{L^p(\Omega)} \end{aligned}$$

since Ω' is bounded. Now assuming $u_n \in C^\infty_{cyl}(\Omega)$ converging to u in $W^{1,p}_{cyl}(\Omega)$, from the result above we have

$$\int_\Omega |u_n|^p \, dx \leq C \|\nabla u_n\|^p_{L^p(\Omega)} \ \ \forall \ n \in \mathbb{N}.$$

Letting n go to infinity, we conclude that

$$\int_\Omega |u|^p \, dx \leq C \|\nabla u\|^p_{L^p(\Omega)}.$$

□

In the next Lemma we prove an H^1-*a priori bound* for any weak non-negative solution of Eq. (4.4).

Lemma 4.5. *Under the assumptions of Theorem 4.1 there is a constant K_2 such that*

$$\|u\|_{H^1(\Omega)} \leq K_2$$

for every non-negative weak solution of Eq. (4.4).

Proof. Taking $\varphi = u \in H^1_{cyl}(\Omega)$ in (4.5) we obtain

$$\|\nabla u\|^2_{L^2(\Omega)} \leq \int_\Omega f(x,u) u \, dx + K_1 \int_\Omega J_1 u \, dx.$$

Applying the Hölder inequality and the Poincaré inequality in $H^1_{cyl}(\Omega)$ to the second term on the right hand side we get

$$\begin{aligned} \|\nabla u\|^2_{L^2(\Omega)} &\leq \int_\Omega f(x,u) u \, dx + K_1 \|J_1\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \\ &\leq \int_\Omega f(x,u) u \, dx + C \|\nabla u\|_{L^2(\Omega)}. \end{aligned} \tag{4.7}$$

Next, for $0 < \alpha < 1$, by Hölder's inequality we get

$$\int_\Omega f(x,u) u \, dx = \int_\Omega \left(\delta^\alpha_{n-1} f^\alpha(x,u)\right) \left(f^{1-\alpha}(x,u) \cdot \frac{u}{\delta^\alpha_{n-1}}\right) dx$$

$$\leq \left\|\delta_{n-1}^{\alpha} f^{\alpha}(x,u)\right\|_{L^{\frac{1}{\alpha}}(\Omega)} \left\|f^{1-\alpha}(x,u)\cdot \frac{u}{\delta_{n-1}^{\alpha}}\right\|_{L^{\frac{1}{1-\alpha}}(\Omega)}$$

$$= \left\|\delta_{n-1} f(x,u)\right\|_{L^1(\Omega)}^{\alpha} \Big(\int_{\Omega} f(x,u)\, \frac{u^{\frac{1}{1-\alpha}}}{\delta_{n-1}^{\frac{\alpha}{1-\alpha}}}\, dx\Big)^{1-\alpha}. \tag{4.8}$$

We now distinguish the two cases:

Case 1: $1 \leq \eta < \frac{4n}{(n-1)(n-2)}$

We first show that for each $\epsilon > 0$ there is a C_ϵ such that

$$\|f(x,u)\|_{L^\infty(\Omega)} \leq \epsilon \|u\|_{L^{s\eta}(\Omega)}^{\beta_n \eta} + C_\epsilon \tag{4.9}$$

where

$$\frac{n-1}{2} < s \leq \frac{2^*}{\eta}, \quad \beta_n := \frac{1}{\eta}\cdot\frac{2n+2}{n}. \tag{4.10}$$

In fact, since $u \in H^1_{cyl}(\Omega)$, according to the Sobolev inequality, we know that $u \in L^q(\Omega), (q \leq 2^* = \frac{2n}{n-2})$, and because

$$\begin{aligned}
\Big\| \int_0^a u^\eta\, dx_n \Big\|_{L^s(\Omega')}^s &= \int_{\Omega'} \left(\int_0^a u^\eta\, dx_n\right)^s dx' \\
&= \int_{\Omega'} \left(\int_0^a u^\eta \cdot 1\, dx_n\right)^s dx' \\
&\leq \int_{\Omega'} \left((\int_0^a u^{\eta s}\, dx_n)\cdot (\int_0^a 1^\theta\, dx_n)^{\frac{s}{\theta}}\right) dx' \\
&\leq C \int_{\Omega'} (\int_0^a u^{\eta s}\, dx_n)\, dx' \\
&= C\|u\|_{L^{s\eta}(\Omega)}^{s\eta}
\end{aligned} \tag{4.11}$$

where $\frac{1}{s} + \frac{1}{\theta} = 1, (s, \theta > 1)$ and $s\eta \leq 2^*$, we see that $\int_0^a u^\eta\, dx_n \in L^s(\Omega')$. Next, using that $(-\Delta_{(n-1)})^{-1}$ is a continuous operator from $L^s(\Omega') \to W^{2,s}(\Omega')$, $s \leq \frac{2n}{n-2}\cdot\frac{1}{\eta}$, we are able to use the Morrey embedding inequality in $\Omega' \subset \mathbb{R}^{n-1}$ and we have, for $s > \frac{n-1}{2}$,

$$\begin{aligned}
\|f(\cdot,u)\|_{L^\infty(\Omega)} &\leq \max_{x\in\bar\Omega}\{h(x)\}\, C \left\| \left[(-\Delta_{(n-1)})^{-1} \left(\int_0^a u^\eta(x)\, dx_n\right)\right]\right\|_{L^\infty(\Omega')}^\gamma \\
&\leq C \left\| \left[(-\Delta_{(n-1)})^{-1} \left(\int_0^a u^\eta(x)\, dx_n\right)\right]\right\|_{W^{2,s}(\Omega')}^\gamma \quad s > (n-1)/2 \\
&\leq C \Big\| \int_0^a u^\eta(x)\, dx_n \Big\|_{L^s(\Omega')}^\gamma \\
&= C\Big(\int_{\Omega'} \left(\int_0^a u^\eta(x)\, dx_n\right)^s dx'\Big)^{\gamma/s} \\
&\leq C\Big(\int_{\Omega'} \left(\int_0^a 1^\theta\, dx_n\right)^{\frac{s}{\theta}} \cdot \left(\int_0^a |u(x)|^{s\eta}\, dx_n\right) dx'\Big)^{\gamma/s}, \quad \frac{1}{s} + \frac{1}{\theta} = 1 \\
&\leq C\, a^{\gamma/\theta} (\int_\Omega |u(x)|^{s\eta}\, dx)^{\gamma/s} \leq C\|u\|_{L^{s\eta}(\Omega)}^{\gamma\eta}.
\end{aligned} \tag{4.12}$$

Therefore $f(x,u) \in L^\infty(\Omega)$ for fixed $u \in H^1_{cyl}(\Omega)$. Due to the condition of Theorem 2.1, it follows that $1 < \gamma < \beta_n$, and we conclude that

$$\lim_{\|u\|_{L^{s\eta}(\Omega)}\to\infty} \frac{\|f(x,u)\|_{L^\infty(\Omega)}}{\|u\|^{\beta_n\eta}_{L^{s\eta}(\Omega)}} = 0,$$

which means that for $\epsilon > 0$ small, there exists $M_\epsilon > 0$ such that $\|f(x,u)\|_{L^\infty(\Omega)} \le \epsilon\|u\|^{\beta_n\eta}_{L^{s\eta}(\Omega)}$, for $\|u\|_{L^{s\eta}(\Omega)} \ge M_\epsilon$. This shows (4.9).

Next, with the aid of Lemma 4.3 and due to $(a+b)^l \le a^l + b^l$ $(a,b \ge 0,\ 0 < l < 1)$, we deduce from (4.8)

$$\begin{aligned}
\int_\Omega f(x,u)u\,dx &\le K_1^\alpha \Big(\int_\Omega f(x,u)\,\frac{u^{\frac{1}{1-\alpha}}}{\delta_{n-1}^{\frac{\alpha}{1-\alpha}}}\,dx\Big)^{1-\alpha}\\
&\le C\,\big\|f(\cdot,u)\big\|^{1-\alpha}_{L^\infty(\Omega)} \left[\int_\Omega \frac{u^{\frac{1}{1-\alpha}}}{\delta_{n-1}^{\frac{\alpha}{1-\alpha}}}\,dx\right]^{1-\alpha}\\
&\le \epsilon\, C\,\|u\|^{\beta_n\eta(1-\alpha)}_{L^{s\eta}(\Omega)} \left[\int_\Omega \frac{u^{\frac{1}{1-\alpha}}}{\delta_{n-1}^{\frac{\alpha}{1-\alpha}}}\,dx\right]^{1-\alpha} + C_\epsilon \left[\int_\Omega \frac{u^{\frac{1}{1-\alpha}}}{\delta_{n-1}^{\frac{\alpha}{1-\alpha}}}\,dx\right]^{1-\alpha}.
\end{aligned} \tag{4.13}$$

Now we choose $0 < \alpha = \dfrac{n+2}{2n+2} < 1$, so that $\beta_n\eta + \dfrac{1}{1-\alpha} = \dfrac{2}{1-\alpha}$. From (4.7), (4.8) and (4.13), we get by the Sobolev inequality for $\Omega \subset \mathbb{R}^n$

$$\begin{aligned}
\|\nabla u\|^2_{L^2(\Omega)} &\le \epsilon\, C\,\big\|u\big\|^{\beta_n\eta(1-\alpha)}_{L^{s\eta}(\Omega)} \Big\|\frac{u}{\delta^\alpha_{n-1}}\Big\|_{L^{\frac{1}{1-\alpha}}(\Omega)} + C_\epsilon \Big\|\frac{u}{\delta^\alpha_{n-1}}\Big\|_{L^{\frac{1}{1-\alpha}}(\Omega)}\\
&\quad + C\big\|\nabla u\big\|_{L^2(\Omega)}\\
&\le \epsilon\, C\,\big\|\nabla u\big\|_{L^2(\Omega)} \Big\|\frac{u}{\delta^\alpha_{n-1}}\Big\|_{L^{\frac{1}{1-\alpha}}(\Omega)}\\
&\quad + C_\epsilon \Big\|\frac{u}{\delta^\alpha_{n-1}}\Big\|_{L^{\frac{1}{1-\alpha}}(\Omega)} + C\big\|\nabla u\big\|_{L^2(\Omega)}.
\end{aligned} \tag{4.14}$$

Applying Lemma 4.2 with $\tau = \alpha$ we have

$$\Big\|\frac{u}{\delta^\alpha_{n-1}}\Big\|_{L^q(\Omega)} \le C\|\nabla u\|_{L^2(\Omega)},$$

where $\dfrac{1}{q} = \dfrac{1}{2} - \dfrac{1-\alpha}{n}$, i.e. $q = \dfrac{1}{1-\alpha}$ by the choice of α above. We can then conclude from (4.14) that

$$\|\nabla u\|_{L^2(\Omega)} \le C,$$

and the proof of Lemma 4.5 is complete in this case since also $\|u\|_{L^2(\Omega)} \le C$ by Lemma 4.4 in $H^1_{cyl}(\Omega)$. Note that the choice of s in (4.10) is possible for $1 \le \eta < \frac{4n}{(n-1)(n-2)}$.

Case 2: $\eta \ge \frac{4n}{(n-1)(n-2)}$

We show that for $1 \le \gamma < \beta_n := \frac{n^2-1}{(n-1)^2\eta - 2n}$

$$\big\|f(\cdot,u)\big\|_{L^r(\Omega)} \le \epsilon \,\big\|u\big\|^{\beta_n \eta}_{L^{\rho\eta}(\Omega)} + C_\epsilon, \tag{4.15}$$

where

$$1 \le \rho \le \frac{2^*}{\eta}, r > 1 \text{ and } \gamma\, r \le \frac{1}{\frac{1}{\rho} - \frac{2}{n-1}} = \rho^{\star} \qquad (\frac{1}{\rho} > \frac{2}{n-1}). \tag{4.16}$$

Here $\rho^{\star} = \dfrac{\rho(n-1)}{n-1-2\rho}$ denotes the critical Sobolev exponent for the embedding $W^{2,\rho}(\Omega') \subset L^{\rho^{\star}}(\Omega')$, $\Omega' \subset \mathbb{R}^{n-1}$.

In fact, first $u \in H^1_{cyl}(\Omega)$ which implies $u \in L^\rho(\Omega)$, $1 \le \rho \le 2^* \cdot \frac{1}{\eta}$, then as in (4.11), $\int_0^a u^\eta \, dx \in L^\rho(\Omega')$. By L^p regularity in Ω', we have $v \in W^{2,\rho}(\Omega')$. Then if $\gamma r \le \rho^{\star}$, we again have $\|v\|_{L^{\gamma\, r}(\Omega')} \le C\|v\|_{W^{2,\rho}(\Omega')}$ by the Sobolev embedding theorem. After this we have

$$\begin{aligned}
\|f(\cdot,u)\|^r_{L^r(\Omega)} &= \int_\Omega (h(x))^r \cdot \bigg[(-\Delta_{(n-1)})^{-1}\bigg(\int_0^a u^\eta(x',x_n)\,dx_n\bigg)\bigg]^{\gamma\, r} dx \\
&\le C\int_{\Omega'} \bigg[(-\Delta_{(n-1)})^{-1}\bigg(\int_0^a u^\eta(x',x_n)\,dx_n\bigg)\bigg]^{\gamma\, r} dx' \\
&= C\big\|v\big\|^{\gamma\, r}_{L^{\gamma\, r}(\Omega')} \\
&\le C\|v\|^{\gamma\, r}_{W^{2,\rho}(\Omega')} \\
&\le C\Big\|\int_0^a u^\eta(x',x_n)\,dx_n\Big\|^{\gamma\, r}_{L^\rho(\Omega')} \\
&= C\Big(\int_{\Omega'}\Big(\int_0^a u^\eta(x',x_n)dx_n\Big)^\rho dx'\Big)^{\frac{\gamma r}{\rho}} \\
&\le C\Big(\int_{\Omega'}\int_0^a u^{\rho\eta}(x',x_n)dx_n dx'\Big)^{\frac{\gamma r}{\rho}} \\
&= C\big\|u\big\|^{\gamma r \eta}_{L^{\rho\eta}(\Omega)}.
\end{aligned} \tag{4.17}$$

Since $\rho \le \frac{2^*}{\eta}$, we have $f(x,u) \in L^r(\Omega)$ for fixed $u \in H^1_{cyl}(\Omega)$ and

$$\|f(\cdot,u)\|_{L^r(\Omega)} \le C\|u\|^{\gamma\eta}_{L^{\rho\eta}(\Omega)} \tag{4.18}$$

and hence, for $1 < \gamma\eta < \beta_n\eta$ and every $\epsilon > 0$ there exists C_ϵ such that

$$\|f(\cdot,u)\|_{L^r(\Omega)} \le \epsilon\,\|u\|^{\beta_n\eta}_{L^{\rho\eta}(\Omega)} + C_\epsilon.$$

This shows (4.15).

From (4.8), (4.15), Lemma 4.3 and the Hölder inequality we now deduce

$$\begin{aligned}
\int_\Omega f(x,u)\,u\,dx &\le C\Big(\int_\Omega f(x,u)\ \frac{u^{\frac{1}{1-\alpha}}}{\delta_{n-1}^{\frac{\alpha}{1-\alpha}}}\,dx\Big)^{1-\alpha} \\
&\le C\bigg(\big\|f(\cdot,u)\big\|_{L^r(\Omega)}\ \bigg\|\frac{u^{\frac{1}{1-\alpha}}}{\delta_{n-1}^{\frac{\alpha}{1-\alpha}}}\bigg\|_{L^h(\Omega)}\bigg)^{1-\alpha}
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\epsilon\|u\|_{L^{\rho\eta}(\Omega)}^{\beta_n\eta(1-\alpha)} + C_\epsilon\right) \left\| \frac{u^{\frac{1}{1-\alpha}}}{\delta_{n-1}^{\frac{\alpha}{1-\alpha}}} \right\|_{L^h(\Omega)}^{1-\alpha} \\
&= \epsilon\|u\|_{L^{\rho\eta}(\Omega)}^{\beta_n\eta(1-\alpha)} \left\| \frac{u}{\delta_{n-1}^{\alpha}} \right\|_{L^{\frac{h}{1-\alpha}}(\Omega)} + C_\epsilon \left\| \frac{u}{\delta_{n-1}^{\alpha}} \right\|_{L^{\frac{h}{1-\alpha}}(\Omega)},
\end{aligned}
\tag{4.19}
$$

where $\frac{1}{r} + \frac{1}{h} = 1, r > 1, h > 1$. Again applying the Hardy inequality with $\tau = \alpha$ we get

$$
\left\| \frac{u}{\delta_{n-1}^{\alpha}} \right\|_{L^{\frac{h}{1-\alpha}}(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}, \tag{4.20}
$$

where $\frac{1-\alpha}{h} = \frac{1}{2} - \frac{1-\alpha}{n}$, and thus $1-\alpha = \frac{nh}{2(n+h)}$. Since $0 < 1-\alpha < 1$, so $0 < \frac{nh}{2(n+h)} < 1$, which implies

$$
1 < h < \frac{2n}{n-2} = 2^*, \quad r > \frac{2n}{n+2}.
$$

Then as before, we take

$$
\beta_n\eta = \frac{1}{1-\alpha} = 2\left(\frac{1}{h} + \frac{1}{n}\right) = 2\left(1 - \frac{1}{r} + \frac{1}{n}\right). \tag{4.21}
$$

Now that $\rho \leq \frac{2^*}{\eta}$, from (4.7), (4.19), (4.20) and (4.21) we get

$$
\begin{aligned}
\|\nabla u\|_{L^2(\Omega)}^2 &\leq \epsilon\|\nabla u\|_{L^2(\Omega)} \left\| \frac{u}{\delta_{n-1}^{\alpha}} \right\|_{L^{\frac{h}{1-\alpha}}(\Omega)} + C_\epsilon \left\| \frac{u}{\delta_{n-1}^{\alpha}} \right\|_{L^{\frac{h}{1-\alpha}}(\Omega)} + C\|\nabla u\|_{L^2(\Omega)} \\
&\leq \epsilon\|\nabla u\|_{L^2(\Omega)}^2 + C_\epsilon\|\nabla u\|_{L^2(\Omega)} + C\|\nabla u\|_{L^2(\Omega)}.
\end{aligned}
\tag{4.22}
$$

We can then conclude from (4.22) that

$$
\|\nabla u\|_{L^2(\Omega)} \leq C.
$$

Now combining (4.21) with $\gamma r \leq \rho^{\star}$ and $\gamma < \beta_n$, we are going to find a best r to have the largest γ. So first we take $\rho = \frac{2^*}{\eta}$. Thus

$$
\begin{aligned}
\gamma \leq \left(\frac{2^*}{\eta}\right)^{\star} \cdot \frac{1}{r} &= \frac{1}{\frac{\eta}{2^*} - \frac{2}{n-1}} \cdot \frac{1}{r} \\
&= \frac{\frac{2n}{n-2}(n-1)}{(n-1)\eta - 2\frac{2n}{n-2}} \cdot \frac{1}{r} \\
&= \frac{2n(n-1)}{(n-1)(n-2)\eta - 4n} \cdot \frac{1}{r}.
\end{aligned}
$$

Since β_n is increasing with respect to r and the largest γ is decreasing with respect to r, we can let

$$
\frac{2n(n-1)}{(n-1)(n-2)\eta - 4n} \cdot \frac{1}{r} = \frac{1}{\eta} \cdot 2\left(1 - \frac{1}{r} + \frac{1}{n}\right)
$$

and derive

$$r = \frac{n(2n^2 - 4n + 2)\eta - 4n^2}{(n^2 - 1)(n - 2)\eta - 4n(n + 1)} \ \ (\eta \geq \frac{4n}{(n - 1)(n - 2)}), \tag{4.23}$$

and thus, from (4.21)

$$\beta_n = \frac{n^2 - 1}{(n - 1)^2\eta - 2n}.$$

Like the first case, the choice of ρ in (4.16) is possible for the second case of Theorem 2.1. Based on the above two cases ($1 \leq \eta < \frac{4n}{(n-1)(n-2)}$, $\eta \geq \frac{4n}{(n-1)(n-2)}$), the proof of Lemma 4.5 is complete. □

Proof of Theorem 4.1. Likewise, we consider two cases:
Case 1: $1 \leq \eta < \frac{4n}{(n-1)(n-2)}$, $1 < \gamma\eta \leq \frac{2n+2}{n}$.
By (4.12), we know $f(x, u) \in L^\infty(\Omega)$ for any $u \in H^1_{cyl}(\Omega)$ weak solution of (4.4). According to Lemma 3.2.1, for any fixed u, we have $u \in W^{2,p}(\Omega)$, for any $p > 1$ and since J_1 is a known smooth function, we have by Lemma 3.1.6 the estimate

$$\|u\|_{W^{2,p}(\Omega)} \leq C \|f(\cdot, u)\|_{L^p(\Omega)} + \|K_1 J_1\|_{L^p(\Omega)} \leq C \|f(\cdot, u)\|_{L^\infty(\Omega)} + C.$$

Choosing $p > \frac{n}{2}$, we get by Morrey's inequality

$$\|u\|_{L^\infty(\Omega)} \leq C\|u\|_{W^{2,p}(\Omega)} \leq C\|f(\cdot, u)\|_{L^\infty(\Omega)} + C.$$

In particular, due to (4.12) and Lemma 4.5, for $\frac{n-1}{2} < s \leq \frac{2^*}{\eta}$

$$\begin{aligned} \|u\|_{L^\infty(\Omega)} &\leq C \|u\|^{\gamma\eta}_{L^{s\eta}(\Omega)} + C \\ &\leq C \|Du\|^{\gamma\eta}_{L^2(\Omega)} + C \\ &\leq C. \end{aligned}$$

So that

$$\|u\|_{L^\infty(\Omega)} \leq K.$$

Case 2: $\eta \geq \frac{4n}{(n-1)(n-2)}$, $1 < \gamma\eta \leq \frac{n+1}{n-1} + \frac{2n\gamma}{(n-1)^2}$.
Similarly, for any fixed $u \in H^1_{cyl}(\Omega)$ weak solution of (4.4), according to (4.17), $f(x, u) \in L^r(\Omega)$, so $u \in W^{2,r}(\Omega)$ by Lemma 3.2.1, and by (3.10) with $p = r$ we have

$$\|u\|_{W^{2,r}(\Omega)} \leq C\|f(x, u) + K_1 J_1\|_{L^r(\Omega)}. \tag{4.24}$$

Next, we have by the Sobolev inequality that $u \in L^\mu(\Omega)$, for $\mu \leq r^* = \dfrac{1}{\frac{1}{r} - \frac{2}{n}} = \dfrac{nr}{n - 2r}$. By (4.17), (4.24) and the Sobolev embedding theorem

$$\begin{aligned} \|u\|_{L^\mu(\Omega)} \leq C \|u\|_{W^{2,r}(\Omega)} &\leq C \|f(\cdot, u)\|_{L^r(\Omega)} + C, \\ &\leq C \|u\|^{\gamma r\eta}_{L^{\rho\eta}(\Omega)} + C \\ &\leq C \|Du\|^{\gamma r\eta}_{L^2(\Omega)} + C \\ &\leq C. \end{aligned}$$

So finally we get

$$\|u\|_{L^{\mu}(\Omega)} \leq C, \tag{4.25}$$

where $2^* \leq \mu \leq r^*$.

For $\frac{n}{r} = 2$, we get $\eta = \frac{4n}{n^2-5n+2} > \frac{4n}{(n-1)(n-2)}$, where r is given by (4.23). We denote this η as η'. Hence when $1 \leq \eta < \eta'$, thus $2 > \frac{n}{r}$, then Morrey's embedding theorem implies $r^* = \infty$, and then we are done.

Next, suppose that $\eta' \leq \eta \leq 2^*$, it then follows that $2 \leq \frac{n}{r}$. Then we will get an improved uniform L^p bound of $f(x,u)$ by showing an improved uniform L^p bound of u. To see this we first consider

$$(\frac{r^*}{\eta})^{\star} = \frac{1}{\frac{\eta}{r^*} - \frac{2}{n-1}} = \frac{1}{\frac{\eta}{r} - \frac{2\eta}{n} - \frac{2}{n-1}} = \frac{n(n-1)r}{(n-1)(n\eta - 2\eta r) - 2nr}.$$

Similarly, in the case $2 > \dfrac{(n-1)\eta}{r^*}$, we can replace $(\frac{r^*}{\eta})^{\star}$ by $+\infty$. In the case $2 \leq \dfrac{(n-1)\eta}{r^*}$, we compute

$$\begin{aligned}
&\|f(\cdot,u)\|_{L^{\frac{(\frac{r^*}{\eta})^{\star}}{\gamma}}(\Omega)} \\
&= \left\| h(x)\left[(-\Delta_{(n-1)})^{-1}\Big(\int_0^a u^{\eta}(x',x_n)\,dx_n\Big)\right]^{\gamma}\right\|_{L^{\frac{(\frac{r^*}{\eta})^{\star}}{\gamma}}(\Omega)} \\
&= \left(\int_{\Omega} h(x)^{\frac{(\frac{r^*}{\eta})^{\star}}{\gamma}} \left[(-\Delta_{(n-1)})^{-1}\Big(\int_0^a u^{\eta}(x',x_n)\,dx_n\Big)\right]^{(\frac{r^*}{\eta})^{\star}} dx\right)^{\frac{\gamma}{(\frac{r^*}{\eta})^{\star}}} \\
&\leq C\left(\int_{\Omega'} \left[(-\Delta_{(n-1)})^{-1}\Big(\int_0^a u^{\eta}(x',x_n)\,dx_n\Big)\right]^{(\frac{r^*}{\eta})^{\star}} dx'\right)^{\frac{\gamma}{(\frac{r^*}{\eta})^{\star}}} \\
&= C\left\|\left[(-\Delta_{(n-1)})^{-1}\Big(\int_0^a u^{\eta}(x',x_n)\,dx_n\Big)\right]\right\|^{\gamma}_{L^{(\frac{r^*}{\eta})^{\star}}(\Omega')} \\
&= C\|v(x')\|^{\gamma}_{L^{(\frac{r^*}{\eta})^{\star}}(\Omega')} \\
&\leq C\|v(x')\|^{\gamma}_{w^{2,\frac{r^*}{\eta}}(\Omega')} \\
&\leq C\left\|\int_0^a u^{\eta}(x',x_n)\,dx_n\right\|^{\gamma}_{L^{\frac{r^*}{\eta}}(\Omega')} \\
&= C\left(\int_{\Omega'}\Big(\int_0^a u^{\eta}(x',x_n)dx_n\Big)^{\frac{r^*}{\eta}} dx'\right)^{\frac{\gamma\eta}{r^*}} \\
&\leq C\|u(x',x_n)\|^{\gamma\eta}_{L^{r^*}(\Omega)}.
\end{aligned} \tag{4.26}$$

From (4.25), we deduce

$$\|f(\cdot,u)\|_{L^{\frac{(\frac{r^*}{\eta})^{\star}}{\gamma}}(\Omega)} \leq C.$$

Noting that

$$\begin{aligned}\frac{(\frac{r^*}{\eta})^{\star}}{\gamma} &= \frac{1}{\frac{\eta}{r^*}-\frac{2}{n-1}}\cdot\frac{1}{\gamma}\\ &= \frac{1}{\frac{\eta}{r}-\frac{2\eta}{n}-\frac{2}{n-1}}\cdot\frac{1}{\gamma}\\ &> \frac{n(n-1)r}{(n-1)(n\eta-2\eta r)-2nr}\cdot\frac{\eta(n^2-2n+1)-2n}{n^2-1} \qquad (\gamma<\beta_n),\end{aligned}$$

hence

$$\frac{(\frac{r^*}{\eta})^{\star}}{\gamma} > \frac{n(n-1)r}{(n-1)(n\eta-2\eta r)-2nr}\cdot\frac{\eta(n^2-2n+1)-2n}{n^2-1} > r, \quad (4.27)$$

where the last inequality follows by elementary calculations, using (4.23). So we see that $f(\cdot,u)$ is bounded in an improved L^p space, if $2\le\frac{(n-1)\eta}{r^*}$. Then taking $p=\frac{(\frac{r^*}{\eta})^{\star}}{\gamma}$, by (3.10) and the Sobolev inequality, we have,

$$\|u\|_{L^{\left(\frac{(\frac{r^*}{\eta})^{\star}}{\gamma}\right)^*}(\Omega)} \le \|u\|_{W^{2,\frac{(\frac{r^*}{\eta})^{\star}}{\gamma}}(\Omega)} \le \Big\|f(\cdot,u)\Big\|_{L^{\frac{(\frac{r^*}{\eta})^{\star}}{\gamma}}(\Omega)} + C \le C, \quad (4.28)$$

where $\left(\frac{(\frac{r^*}{\eta})^{\star}}{\gamma}\right)^* = \frac{1}{\frac{\gamma}{(\frac{r^*}{\eta})^{\star}}-\frac{2}{n}}$. From (4.27), we see $\left(\frac{(\frac{r^*}{\eta})^{\star}}{\gamma}\right)^* > r^*$, which means we get a better uniform L^p bound of u. Afterwards, we repeat the computation of (4.26) and get

$$\Big\|f(x,u)\Big\|_{L^{\frac{\left(\frac{\left(\frac{(\frac{r^*}{\eta})^{\star}}{\gamma}\right)^*}{\eta}\right)^{\star}}{\gamma}}(\Omega)} \le \|u\|_{L^{\left(\frac{(\frac{r^*}{\eta})^{\star}}{\gamma}\right)^*}(\Omega)} \le C.$$

Iterating (4.26)–(4.28), finally, we will derive

$$\|u\|_{L^\infty(\Omega)} \le C.$$

Thus, we have completed the proof of Theorem 4.1. □

5. Fixed Point Theorem and Existence of the Solution

In this section we complete the proof of Theorem 2.1. We first show a maximum principle for the Poisson equation with mixed boundary conditions.

Lemma 5.1. ([7]) *Let $\Omega\subset\mathbb{R}^n$, $n\ge 3$, be the cylinder in (2.1) and let Γ_1, Γ_2 be a partition of $\partial\Omega$, with $\Gamma_1=\partial\Omega'$, $\Gamma_2=\Omega'\times\{0,a\}$. Let $g\in C_0^\infty(\Omega)$, $g\ge 0$,*

$g \not\equiv 0$, and let u denote the solution of

$$\begin{cases} -\Delta u = g \text{ in } \Omega \\ \quad u = 0 \text{ on } \Gamma_1 \\ \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_2 \end{cases} \tag{5.1}$$

where ν is the outer unit normal vector to $\partial\Omega$. Then the solution of (5.1) satisfies:

$$u \geq 0 \quad in\ \overline{\Omega}.$$

Proof. If the claim were not true, then there exists a $x_0 \in \overline{\Omega}$ such that $u(x_0) < 0$. Without loss of generality, we suppose $u(x_0) = \min_{x\in\overline{\Omega}} u(x) < 0$. By the assumption, we know $x_0 \notin \Gamma_1$. Next we show $x_0 \notin \Gamma_2$; otherwise, we may assume that $x_0 \in \Omega' \times \{0\}$ or $\Omega' \times \{a\}$, by interior regularity, since $g \in C_0^\infty(\Omega)$, we obtain $u \in C^\infty(\Omega)$ and u in $W^{2,p}(\Omega)$ $(1 \leq p < \infty)$. In addition, $W^{2,p}(\Omega) \hookrightarrow C^1(\bar{\Omega})$ (for $p > n$) ([1] Theorem 4.12, PART II), so we have $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$. Since $\Omega' \times \{0\}$ or $\Omega' \times \{a\}$ is flat, Ω satisfies the interior ball condition at x_0, and from Hopf's lemma we have $\dfrac{\partial u(x_0)}{\partial \nu} < 0$, which contradicts the assumption on Γ_2. So x_0 is an interior point of Ω. But due to the maximum principle, u cannot have a negative minimum in Ω. □

We are now in the position to complete the proof of Theorem 2.1.

Proof of Theorem 2.1. For every fixed $u \geq 0$ in $C^1(\bar{\Omega})$, by the Lax-Milgram theorem we know there exists a unique solution for Eq. (2.2), which we denote by w_u. That is, $-\Delta_{(n)} w_u = f(x,u)$, with $f(x,u) = h(x)[(-\Delta_{(n-1)})^{-1} \int_0^a u^\eta(x)\,dx_n]^\gamma$. To solve problem (2.2), we define the mapping $u \to w_u =: F(u)$. If there is a fixed point of F in $C^1(\bar{\Omega})$ such that $F(u) = u$, we are done. Now we check that F satisfies the following fixed point theorem ([9], Theorem 3.1; [19], Theorem 1).

F: $C^1(\bar{\Omega}) \to C^1(\bar{\Omega})$ a compact mapping, acting in the cone of non-negative functions, will have a fixed point u with $0 < r \leq \|u\|_{C^1(\bar{\Omega})} \leq R < \infty$ provided

1) $Fu \neq s'u, s' \geq 1$ for $\|u\|_{C^1(\bar{\Omega})} = r$ and
2) $Fu \neq u - t\tilde{J}_1, t \geq 0$, for $\|u\|_{C^1(\bar{\Omega})} = R$,

where $\tilde{J}_1 = (-\Delta_{(n)})^{-1} J_1$.

Step 1: $F : C^1(\bar{\Omega}) \to C^1(\bar{\Omega})$ is a compact mapping. It is easy to see that F is continuous, since it is a composition of continuous maps. Then, let $\mathcal{A} \subset C^1(\bar{\Omega})$ be a bounded set, for $u \in \mathcal{A}$ we have

$$\begin{aligned} \|f(x,u)\|_{L^\infty(\Omega)} &= \left\| h(x)\left[(-\Delta_{(n-1)})^{-1}\left(\int_0^a u^\eta(x',x_n)\,dx_n\right)\right]^\gamma \right\|_{L^\infty(\Omega)} \\ &\leq C \max_{x\in\bar{\Omega}}\{h(x)\} \left\|(-\Delta_{(n-1)})^{-1}\left(\int_0^a u^\eta(x)\,dx_n\right)\right\|^\gamma_{L^\infty(\Omega')} \end{aligned}$$

$$
\begin{aligned}
&\le C\left\|(-\Delta_{(n-1)})^{-1}\Big(\int_0^a u^\eta(x)\,dx_n\Big)\right\|^{\gamma}_{W^{2,s}(\Omega')} \quad s>(n-1)/2\\
&\le C\left\|\int_0^a u^\eta(x)\,dx_n\right\|^{\gamma}_{L^s(\Omega')}\\
&\le C\,\|u\|^{\gamma\eta}_{L^{s\eta}(\Omega)}\\
&\le C\,\|u\|^{\gamma\eta}_{L^{\infty}(\Omega)}\\
&\le C\,\|u\|^{\gamma\eta}_{C^1(\Omega)}\\
&\le C\,\|u\|^{\gamma\eta}_{C^1(\bar\Omega)}\\
&\le C,
\end{aligned}
\tag{5.2}
$$

thus $f(x,u)\in L^\infty(\Omega)$ and $\{f(x,u), u\in\mathcal{A}\}$ is uniformly bounded. Since $-\Delta_{(n)}w_u=f(x,u)$, by Lemmas 3.1.6 and 3.2.1, $w_u\in W^{2,q}(\Omega)$, q large enough, and lies in a bounded set in $W^{2,q}(\Omega)$. Then by Morrey's inequality, we get for $q>n$, $w_u\in C^{1,\gamma'}(\bar\Omega)$, that is

$$
\begin{aligned}
\|w_u\|_{C^{1,\gamma'}(\bar\Omega)} &\le C\|w_u\|_{W^{2,q}(\Omega)}\le C\|f(\cdot,u)\|_{L^q(\Omega)}+C\\
&\le C\|f(\cdot,u)\|_{L^\infty(\Omega)}+C\le C,
\end{aligned}
$$

where $\gamma'=1-\frac{n}{q}$. Therefore we have for every x, y in $\bar\Omega$, and $\forall u\in\mathcal{A}$

$$|Dw_u(x)-Dw_u(y)|\le C|x-y|^{\gamma'}.$$

Hence, $\forall\epsilon>0$, we take $\delta=(\frac{\epsilon}{C})^{\gamma'/1}$ then, if $|x-y|<\delta$, $\{w_u\}$ satisfies

$$|Dw_u(x)-Dw_u(y)|\le C|x-y|^{\gamma'}<\epsilon$$

which means $\{w_u, u\in\mathcal{A}\}$ is uniformly bounded and equicontinuous in $C^1(\bar\Omega)$. According to the Arzelà-Ascoli theorem, it is in a compact set in $C^1(\bar\Omega)$. Hence, F is a compact mapping from $C^1(\bar\Omega)$ to $C^1(\bar\Omega)$.

Step 2: F maps the non-negative cone in $C^1(\bar\Omega)$ into itself. For this we are going to prove that when u is fixed non-negative, then w_u is non-negative. Indeed, w_u satisfies

$$
\begin{cases}
-\Delta_{(n)}w_u(x)=f(x,u), & x\in\Omega\\
w_u(x)=0, & x\in\partial\Omega'\times[0,a]\\
\partial_{x_n}w_u(x)=0, & x\in\Omega'\times\{0,a\},
\end{cases}
\tag{5.3}
$$

where $f(x,u)=f(x)=h(x)[(-\Delta_{(n-1)})^{-1}\int_0^a u^\eta(x',x_n)dx_n]^\gamma$. By (5.2) $f\in L^\infty(\Omega)$ so that $f\in L^p(\Omega)$ for any $p>1$ when u is fixed in $C^1(\bar\Omega)$.

We assume

$$
\begin{cases}
-\Delta_{(n)}w_{u_n}(x)=f_n, & x\in\Omega\\
w_{u_n}(x)=0, & x\in\partial\Omega'\times[0,a]\\
\partial_{x_n}w_{u_n}(x)=0, & x\in\Omega'\times\{0,a\},
\end{cases}
\tag{5.4}
$$

where $f_n \in C_0^\infty(\Omega)$, $f_n \geq 0$, $\|f_n - f\|_{L^p(\Omega)} \to 0$ $(1 \leq p < \infty)$. Applying Lemma 5.1, we get

$$w_{u_n} \geq 0, \quad \forall n \in \mathbb{N}, \quad x \in \bar{\Omega}.$$

On the other hand, subtracting (5.3) from (5.4), we get

$$\begin{cases} -\Delta_{(n)}(w_{u_n}(x) - w_u(x)) = f_n - f, \, x \in \Omega \\ w_{u_n}(x) - w_u(x) = 0, & x \in \partial\Omega' \times [0, a] \\ \partial_{x_n}(w_{u_n}(x) - w_u(x)) = 0, & x \in \Omega' \times \{0, a\}. \end{cases}$$

Since $f_n - f \in L^\infty(\Omega)$, by Lemma 3.2.1, we have $w_{u_n} - w_u \in W^{2,p}(\Omega)$, p large enough. Then by Lemma 3.1.6 and Morrey's inequality we have $w_{u_n} - w_u \in C^{1,\gamma'}(\bar{\Omega})$, and $\|w_{u_n} - w_u\|_{C^{1,\gamma'}(\bar{\Omega})} \leq C\|w_{u_n} - w_u\|_{W^{2,p}(\Omega)} \leq C\|f_n - f\|_{L^p(\Omega)}$ for $p > n$. So, $\|w_{u_n} - w_u\|_{C^{1,\gamma'}(\bar{\Omega})} \leq C\|f_n - f\|_{L^p(\Omega)}$. Furthermore

$$\lim_{n\to\infty} \|w_{u_n} - w_u\|_{C^{1,\gamma'}(\bar{\Omega})} \leq C \lim_{n\to\infty} \|f_n - f\|_{L^p(\Omega)} = 0,$$

which implies

$$\lim_{n\to\infty} \{\sup_{x\in\bar{\Omega}} |(w_{u_n} - w_u)(x)| + \sup_{x\in\bar{\Omega}} |(Dw_{u_n} - Dw_u)(x)|\} = 0$$

so,

$$w_{u_n} \to w_u \qquad \forall x \in \bar{\Omega}.$$

Since $w_{u_n} \geq 0$, then $w_u \geq 0$ in $\bar{\Omega}$.

Next we verify the two conditions (1) and (2).

(1) holds for $r < (\frac{1}{C})^{\frac{1}{\gamma\eta - 1} + 1}$, where C will be determined later. If not, we suppose that there exists $s' \geq 1$ and u with $\|u\|_{C^1(\bar{\Omega})} = r$ such that $Fu = s'u$. Since $-\Delta_{(n)}F(u) = f(x, u)$, we obtain

$$-\Delta_{(n)}(Fu) = -\Delta_{(n)}(s'u) = f(x, u)$$

then

$$-\Delta_{(n)}u = \frac{1}{s'}f(x, u).$$

Multiplying by u and taking the integral over Ω on both sides, we have,

$$\int_\Omega -\Delta_{(n)}u \cdot u = \frac{1}{s'}\int_\Omega f(x, u) \cdot u \leq \int_\Omega f(x, u) \cdot u. \tag{5.5}$$

Case 1: $1 \leq \eta < \frac{4n}{(n-1)(n-2)}$, $1 < \gamma\eta \leq \frac{2n+2}{n}$; by (5.5), Hölder inequality and (4.12) we get

$$\begin{aligned} \int_\Omega |\nabla u|^2 \, dx \; &\leq \int_\Omega f(x, u) \cdot u \, dx \leq C\|f(x, u)\|_{L^\infty(\Omega)}\|u\|_{L^2(\Omega)} \\ &\leq C\|u\|^{\gamma\eta}_{L^{s\eta}(\Omega)}\|u\|_{L^2(\Omega)}. \end{aligned}$$

From (4.10), the Sobolev embedding inequality and Lemma 4.4 we derive,

$$\|Du\|^2_{L^2(\Omega)} \leq C\|Du\|^{\gamma\eta+1}_{L^2(\Omega)}. \tag{5.6}$$

and hence

$$\left(\frac{1}{C}\right)^{\frac{1}{\gamma\eta-1}} \leq \|Du\|_{L^2(\Omega)} \leq C\|Du\|_{L^\infty(\Omega)}.$$

However, by assumption

$$\left(\frac{1}{C}\right)^{\frac{1}{\gamma\eta-1}+1} > r = \|u\|_{C^1(\bar{\Omega})} \geq \|Du\|_{L^\infty(\Omega)}$$

which is a contradiction.

Case 2: $\eta \geq \frac{4n}{(n-1)(n-2)}$, $1 < \gamma\eta \leq \frac{n+1}{n-1} + \frac{2n\gamma}{(n-1)^2}$; from (4.18) and (5.5), we have

$$\begin{aligned} \int_\Omega |\nabla u|^2\,dx &\leq \int_\Omega f(x,u)\cdot u\,dx \leq C\|f(x,u)\|_{L^r(\Omega)}\|u\|_{L^h(\Omega)} \\ &\leq C\|u\|^{\gamma\eta}_{L^{2^*}(\Omega)}\|u\|_{L^h(\Omega)}, \end{aligned} \tag{5.7}$$

where $\dfrac{1}{r} + \dfrac{1}{h} = 1$. Moreover, since $r > 2$, so $h < 2 < 2^*$. Then by the Sobolev embedding inequality, we have the same result as (5.6). Thus 1) will follow by the same proof.

For 2), we show that there exists $R_1 > 0$ such that there is no solution of $F(u) = u - t\tilde{J}_1$ with $\|u\|_{C^1(\bar{\Omega})} \geq R_1, \forall t \geq 0$. Indeed, suppose $u \in H^1_{cyl}(\Omega)$ a solution of $F(u) = u - t\tilde{J}_1$, then $-\Delta_{(n)}F(u) = f(x,u)$, that is,

$$-\Delta_{(n)}u = f(x,u) + tJ_1. \tag{5.8}$$

then by Theorem 4.1, $\|u\|_{L^\infty(\Omega)} \leq K$, K independent of $t \geq 0$. We conclude that for any $1 < q < \infty$,

$$\begin{aligned} \|u\|_{C^1(\bar{\Omega})} &< \|u\|_{C^{1,\gamma'}(\bar{\Omega})} \leq C\|u\|_{W^{2,q}(\Omega)} \\ &\leq \|f(x,u)\|_{L^\infty(\Omega)} \leq C\|u\|^{\gamma\eta}_{L^\infty(\Omega)} \leq C\cdot K^{\gamma\eta} = R_1. \end{aligned}$$

So for any $R > R_1$, $F(u) \neq u - t\tilde{J}_1$. ☐

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