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Existence results for nonlocal problems governed by the regional fractional Laplacian

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Abstract. The aim of the present paper is to study existence results of minimizers of the critical fractional Sobolev constant on bounded domains. Under some values of the fractional parameter we show that the best constant is achieved. If moreover the underlying domain is a ball, we obtain positive radial minimizers for all possible values of the fractional parameter in higher dimension, while we impose a positive mass condition in low dimension.

Keywords. Minimizers, Critical fractional Sobolev constant, Regional fractional Laplacian.

1. introduction and main results

Let Ω be a Lipschitz open set of $\mathbb{R}^N$, $s \in (1/2, 1)$ and $N > 2s$. The purpose of this paper is to study the existence of minimizers to the best Sobolev critical constant

$$S_{N,s}(\Omega) = \inf_{\substack{u \in H_0^s(\Omega) \\ u \neq 0}} \frac{Q_{N,s,\Omega}(u)}{\|u\|^2_{L^{2_s^*}(\Omega)}}, \tag{1.1}$$

where $H_0^s(\Omega)$ is the completion of $C_c^\infty(\Omega)$ with respect to the $H^s(\Omega)$-norm, $2_s^* := \frac{2N}{N-2s}$ is the so-called fractional critical Sobolev exponent and $Q_{N,s,\Omega}(\cdot)$ is a nonnegative quadratic form defined on $H_0^s(\Omega)$ by

$$Q_{N,s,\Omega}(u) := \frac{c_{N,s}}{2} \int_\Omega \int_\Omega \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}}\, dxdy.$$

We notice that for $s \in (0, 1/2]$ and Ω bounded, the constant function 1 belongs to $H_0^s(\Omega)$, and thus, the above Sobolev constant is zero in this case. We refer the reader to Appendix A below for more details and the definition of Lipschitz domains in this paper.

We recall that nonnegative minimizers of the constant $S_{N,s}(\Omega)$ are weak solutions to nonlinear Dirichlet problem

$$\begin{cases} (-\Delta)^s_\Omega u = u^{2^*_s-1} & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

where $(-\Delta)^s_\Omega$ is the *regional fractional Laplacian* defined as

$$(-\Delta)^s_\Omega u(x) = c_{N,s} P.V. \int_\Omega \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy, \;\; x \in \Omega.$$

Here, $c_{N,s}$ is the usual positive normalization constant of $(-\Delta)^s$ and *P.V.* stands for the principal value of the integral.

In the theory of partial differential equations, the existence of solutions of nonlinear equations appears as a natural question. This strongly depends on the type of nonlinearities that are considered. For instance, nonlinear equations involving subcritical power nonlinearities, say $f(t) = |t|^{p-1}$ with $p < 2^*_s$, are quite well-understood and due to compactness, the existence of solutions can be easily established by using for example the Mountain Pass theorem. One can also study the corresponding minimization problem and prove that a minimizer exists. Besides, at the critical exponent $p = 2^*_s$ we lose compactness and therefore standard argument of calculus of variation cannot be applied to derive the existence of solutions. As a typical example, when Ω is a star-shaped bounded domain, it has been proved that the Dirichlet problem

$$(-\Delta)^s u = u^{2^*_s-1}, \qquad u > 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega \tag{1.3}$$

does not admit a solution. Such a nonexistrence result was first proved in [11] and later in [17,18] by means of a fractional Pohozaev type identity. However, (1.2) can have a solution even if Ω is star-shaped and smooth. It is therefore interesting to understand the type of domains and exponents for which (1.2) does not admit a solution.

In the case where $\Omega = \mathbb{R}^N$ or $\Omega = \mathbb{R}^N_+$, the infinimum $S_{N,s}(\Omega) > 0$ for all $s \in (0,1)$. Moreover, see e.g. [2,16] all minimizers of $S_{N,s}(\mathbb{R}^N)$ are of the form

$$u(x) = a\Big(\frac{1}{b^2 + |x-x_0|^2}\Big)^{\frac{N-2s}{2}}, \qquad x \in \mathbb{R}^N \tag{1.4}$$

where a, b are positive constants and $x_0 \in \mathbb{R}^N$.

Problem of type (1.2) is less understood in contrast with (1.3). The only paper investigating it is [12]. Precisely, the authors in [12] considered the equivalent minimization problem and obtain existence of minimizers under some assumptions on Ω and the range of the parameter s. In particular, it is proved in [12] that if a portion of $\partial\Omega$ lies on a hyperplane and $N \geq 4s$, then $S_{N,s}(\Omega)$ is achieved.

Our first main result removes this assumption on Ω provided s is close to 1/2.

Theorem 1.1. *Let $N \geq 2$ and $\Omega \subset \mathbb{R}^N$ be a bounded C^1 open set. Then there exists $s_0 \in (1/2, 1)$ such that for all $s \in (1/2, s_0)$, the infimum $S_{N,s}(\Omega)$ is achieved by a positive function $u \in H_0^s(\Omega)$ satisfying (1.2).*

The main ingredient to prove Theorem 1.1 is to show that $S_{N,s}(\Omega) < S_{N,s}(\mathbb{R}^N_+)$ for s closed to $1/2$. In fact, this strict inequality allows for a sort of compactness. We achieve this by showing that $S_{N,1/2}(\Omega) = 0$ provided Ω is a bounded Lipschitz open set. We notice here that our notion of Lipschitz open set is that $\partial\Omega$ is locally given by the restriction of a bi-Lipschitz map. This is strictly weaker than the *strongly* Lipschitz property, meaning that $\partial\Omega$ is locally given by a graph of a Lipschitz function, see Definiton A.2 and Remark A.3 below.

Next, let $\mathcal{B}$ denote the unit centered ball in $\mathbb{R}^N$. We consider the minimization problem (1.1) on the space $H^s_{0,rad}(\mathcal{B})$, the completion of the space of radial functions belonging to $C_c^\infty(\mathcal{B})$ with respect to the norm $H_0^s(\mathcal{B})$. More precisely, we consider the infinimum problem, for $h \in L^\infty(\mathcal{B})$ being radial,

$$S_{N,s,rad}(\mathcal{B}, h) = \inf_{\substack{u \in H^s_{0,rad}(\mathcal{B}) \\ u \neq 0}} \frac{Q_{N,s,\mathcal{B}}(u) + \int_{\mathcal{B}} hu^2 dx}{\|u\|^2_{L^{2^*_s}(\mathcal{B})}}. \tag{1.5}$$

Our next result is related to the existence of minimizers for the infimum $S_{N,s,rad}(\mathcal{B}, 0)$ in high dimension $N \geq 4s$. Our second main result is the following.

Theorem 1.2. *Let $s \in (1/2, 1)$ and $N \geq 4s$. Then the infimum*

$$S_{N,s,rad}(\mathcal{B}, 0) = \inf_{\substack{u \in H^s_{0,rad}(\mathcal{B}) \\ u \neq 0}} \frac{Q_{N,s,\mathcal{B}}(u)}{\|u\|^2_{L^{2^*_s}(\mathcal{B})}} \tag{1.6}$$

is achieved by a positive function $u \in H^s_{0,rad}(\mathcal{B})$, satisfying

$$(-\Delta)^s_{\mathcal{B}} u = u^{2^*_s - 1} \quad \text{in} \quad \mathcal{B}, \qquad u = 0 \quad \text{on} \quad \partial\mathcal{B}.$$

We now turn our attention to the minimization problem $S_{N,s,rad}(\mathcal{B}, h)$ in low dimension $N < 4s$. This Sobolev constant is related to the Schrödinger operator $(-\Delta)^s_{\mathcal{B}} + h$. As a necessary condition for the existence of positive minimizers, it is important to assume that $(-\Delta)^s_{\mathcal{B}} + h$ defines a coercive bilinear form on $H^s_{0,rad}(\mathcal{B})$.

Before stating our next result, we need to introduce the mass of $\mathcal{B}$ at 0 associated to the Schrödinger operator $(-\Delta)^s + h$, where $(-\Delta)^s$ is the standard fractional Laplacian. Indeed, let $G(x, y)$ be the Green function of the operator $(-\Delta)^s + h$ on $\mathcal{B}$ and $\mathcal{R}$ be the fundamental solution of $(-\Delta)^s$ on $\mathbb{R}^N$. Then the function $x \mapsto \mathbf{k}(x) = G(x, 0) - \mathcal{R}(x)$ is continuous in $\mathcal{B}$. The *mass* of the operator $(-\Delta)^s + h$ at 0 is given by $\mathbf{k}(0)$. Our next existence result is a consequence of the fact that the mass is positive, see [13,19].

Theorem 1.3. *Let $s \in (1/2, 1)$, $2 \leq N < 4s$, $h \in L^\infty_{rad}(\mathcal{B})$ and suppose that $S_{N,s,rad}(\mathcal{B}, h) > 0$. Assume that $\mathbf{k}(0) > 0$. Then $S_{N,s,rad}(\mathcal{B}, h)$ is achieved by a positive function $u \in H^s_{0,rad}(\mathcal{B})$, satisfying*

$$(-\Delta)^s_{\mathcal{B}} u + hu = u^{2^*_s - 1} \quad \text{in} \quad \mathcal{B}, \qquad u = 0 \quad \text{on} \quad \partial\mathcal{B}.$$

The role of the mass in proving the existence of minimizers (for Sobolev constant) in low dimensions is very crucial. As we will see later, it helps us to restore the compactness. Indeed, the strict positivity $\mathbf{k}(0) > 0$ implies that the Sobolev constant in $\mathcal{B}$ is strictly less than that of $\mathbb{R}^N$, and thereby produces the existence of minimizers.

An interesting question that arises is whether symmetry breaking occurs? More generally, for $p \geq 1$, is every positive solution to $u \in H^s_0(\mathcal{B})$ to

$$(-\Delta)^s_{\mathcal{B}} u = u^p \quad \text{in} \quad \mathcal{B}, \qquad u = 0 \quad \text{on} \quad \partial\mathcal{B},$$

is radial? We conjecture that the answer to this question is no.

In Proposition 2.3 we obtain a priori L^∞-bounds of minimizers. Hence, by the ineterior regularity theory and standard boostrap arguments, they belong to $C^\infty(\Omega)$, provided $h \in C^\infty(\Omega)$. In addition, the boundary regularity result in [4,10] implies that minimizers are actually $C^{2s-1}(\overline{\Omega})$.

The rest of the paper is organized as follows. in Sect. 2 we give some preliminaries that will be useful throughout this paper. In Sect. 3 we prove Theorems 1.1. In Sect. 4 we collect some useful results needed to prove Theorems 1.2 and 1.3 whereas in Sect. 5 we establish Theorems 1.2 and 1.3. Finally in the Appendix A we prove that the constant function 1 belongs to $H^s_0(\Omega)$ for $s \in (0, 1/2]$.

2. Preliminary

In this section, we introduce some preliminary properties which will be useful in this work. For all $s \in (0, 1)$, the fractional Sobolev space $H^s(\Omega)$ is defined as the set of all measurable functions u such that

$$[u]^2_{H^s(\Omega)} := \frac{c_{N,s}}{2} \int_\Omega \int_\Omega \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}} \, dxdy$$

is finite. It is a Hilbert space endowed with the norm

$$\|u\|^2_{H^s(\Omega)} = \|u\|^2_{L^2(\Omega)} + [u]^2_{H^s(\Omega)}.$$

We refer to [7] for more details on this fractional Sobolev spaces. Next, we denote by $H^s_0(\Omega)$ the completion of $C^\infty_c(\Omega)$ under the norm $\|\cdot\|_{H^s(\Omega)}$. Moreover, for $s \in (1/2, 1)$, $H^s_0(\Omega)$ is a Hilbert space equipped with the norm

$$\|u\|^2_{H^s_0(\Omega)} = \frac{c_{N,s}}{2} \int_\Omega \int_\Omega \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}} \, dxdy$$

which is equivalent to the usual one in $H^s(\Omega)$ thanks to Poincaré inequality. We define the Hilbert space

$$\mathcal{H}^s_0(\Omega) = \{u \in H^s(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega\}$$

endowed with the norm $\|\cdot\|_{H^s(\mathbb{R}^N)}$, which is the completion of $C^\infty_c(\Omega)$ with respect to the norm $\|\cdot\|_{H^s(\mathbb{R}^N)}$. In the sequel, $H^s_{0,rad}(\Omega)$ and $\mathcal{H}^s_{0,rad}(\Omega)$ are respectively the space of radially symmetric functions of $H^s_0(\Omega)$ and $\mathcal{H}^s_0(\Omega)$. We denote by $L^\infty_{rad}(\Omega)$ the space of radial functions u belonging to $L^\infty(\Omega)$.

Given $x \in \Omega$ and $r > 0$, we denote by $B_r(x)$ the open ball centered at x with radius r. When the center is not specified, we will understand that it's the origin, e.g. $B_2(0) = B_2$. The upper half-ball centered at x with radius r is denoted by $B_r^+(x)$. We will always use $\delta_\Omega(x) = \text{dist}(x, \partial\Omega)$ for the distance from x to the boundary. For every set $A \subset \mathbb{R}^N$, we denote by $\mathbb{1}_A$ its characteristic function.

Proposition 2.1. *(see [5, 7]) The embedding $H_0^s(\Omega) \hookrightarrow L^p(\Omega)$ is continuous for any $p \in [2, 2_s^*]$, and compact for any $p \in [2, 2_s^*)$.*

The next proposition gives an elementary result regarding the role of convex functions applied to $(-\Delta)_\Omega^s$.

Proposition 2.2. *Assume that $\phi : \mathbb{R} \to \mathbb{R}$ is a Lipschitz convex function such that $\phi(0) = 0$. Then if $u \in H_0^s(\Omega)$ we have*

$$(-\Delta)_\Omega^s \phi(u) \le \phi'(u)(-\Delta)_\Omega^s u \quad \textit{weakly in} \quad \Omega. \tag{2.1}$$

Proof. The proof of the above lemma is standard. In fact, using that every convex ϕ satisfies $\phi(a) - \phi(b) \le \phi'(a)(a - b)$ for all $a, b \in \mathbb{R}$, the proof follows. □

We conclude this section showing in proposition below, the boundedness of any nonnegative solution of (1.2). The argument uses Moser's iteration method. A similar result has been established in [1] for the case of fractional Laplacian.

Proposition 2.3. *Let $u \in H_0^s(\Omega)$ be a nonnegative solution to problem (1.2). Then $u \in L^\infty(\Omega)$.*

Proof. For $\beta \ge 1$ and $T > 0$ large, we define the following convex function

$$\phi_{T,\beta}(t) = \begin{cases} 0, & \text{if} \quad t \le 0 \\ t^\beta, & \text{if} \quad 0 < t < T \\ \beta T^{\beta-1}(t - T) + T^\beta, & \text{if} \quad t \ge T. \end{cases}$$

Throughout the proof, we will use $\phi_{T,\beta} =: \phi$ for the sake of simplicity. Since ϕ is Lipschitz, with constant $\Lambda_\phi = \beta T^{\beta-1}$, and $\phi(0) = 0$, then $\phi(u) \in H_0^s(\Omega)$ and by the convexity of ϕ, we have, according to Proposition 2.2 that

$$(-\Delta)_\Omega^s \phi(u) \le \phi'(u)(-\Delta)_\Omega^s u. \tag{2.2}$$

By Proposition 2.1 and inequality (2.2) we have that

$$\begin{aligned} \|\phi(u)\|_{L^{2_s^*}(\Omega)}^2 \le C\|\phi(u)\|_{H_0^s(\Omega)}^2 &= C \int_\Omega \phi(u)(-\Delta)_\Omega^s \phi(u)\, dx \\ &\le C \int_\Omega \phi(u)\phi'(u)(-\Delta)_\Omega^s u\, dx \\ &= C \int_\Omega \phi(u)\phi'(u) u^{2_s^*-1}\, dx. \end{aligned}$$

Moreover, since $u\phi'(u) \leq \beta\phi(u)$, we have that

$$\|\phi(u)\|^2_{L^{2^*_s}(\Omega)} \leq C\beta \int_\Omega (\phi(u))^2 u^{2^*_s-2}\, dx. \tag{2.3}$$

We point out that the integral on the right-hand side of the above inequality is finite. Indeed, using that $\beta \geq 1$ and $\phi(u)$ is linear when $u \geq T$, we have from a quick computation that

$$\begin{aligned}\int_\Omega (\phi(u))^2 u^{2^*_s-2}\, dx &= \int_{\{u\leq T\}} (\phi(u))^2 u^{2^*_s-2}\, dx + \int_{\{u>T\}} (\phi(u))^2 u^{2^*_s-2}\, dx \\ &\leq T^{2\beta-2} \int_\Omega u^{2^*_s}\, dx + C\int_\Omega u^{2^*_s}\, dx < \infty.\end{aligned}$$

We now choose β in (2.3) so that $2\beta - 1 = 2^*_s$. Denoting by β_1 such a value, then we can equivalently write

$$\beta_1 := \frac{2^*_s + 1}{2}. \tag{2.4}$$

Let $K > 0$ be a positive number whose value will be fixed later on. Then applying Hölder's inequality with exponents $q := 2^*_s/2$ and $q' := 2^*_s/(2^*_s - 2)$ in the integral on the right-hand side of inequality (2.3), we find that

$$\begin{aligned}&\int_\Omega (\phi(u))^2 u^{2^*_s-2}\, dx = \int_{\{u\leq K\}} (\phi(u))^2 u^{2^*_s-2}\, dx + \int_{\{u>K\}} (\phi(u))^2 u^{2^*_s-2}\, dx \\ &\leq \int_{\{u\leq K\}} \frac{(\phi(u))^2}{u} K^{2^*_s-1}\, dx + \left(\int_\Omega (\phi(u))^{2^*_s}\, dx\right)^{2/2^*_s} \left(\int_{\{u>K\}} u^{2^*_s}\, dx\right)^{\frac{2^*_s-2}{2^*_s}}.\end{aligned} \tag{2.5}$$

Now, thanks to Monotone Convergence Theorem, we can choose K as big as we wish so that

$$\left(\int_{\{u>K\}} u^{2^*_s}\, dx\right)^{\frac{2^*_s-2}{2^*_s}} \leq \frac{1}{2C\beta_1}, \tag{2.6}$$

where C is the positive constant appearing in (2.3). Therefore, by taking into account (2.6) in (2.5) and by using also (2.4), we deduce from (2.3) that

$$\|\phi(u)\|^2_{L^{2^*_s}(\Omega)} \leq 2C\beta_1 \left(K^{2^*_s-1} \int_\Omega \frac{(\phi(u))^2}{u}\, dx\right).$$

Since $\phi(u) \leq u^{\beta_1}$ and recalling (2.4), and by letting $T \to \infty$, we get that

$$\left(\int_\Omega u^{2^*_s\beta_1}\, dx\right)^{2/2^*_s} \leq 2C\beta_1 \left(K^{2^*_s-1} \int_\Omega u^{2^*_s}\, dx\right) < \infty,$$

and therefore

$$u \in L^{2^*_s\beta_1}(\Omega). \tag{2.7}$$

Suppose now that $\beta > \beta_1$. Thus, using that $\phi(u) \leq u^\beta$ in the right hand side of (2.3) and letting $T \to \infty$ we get

$$\left(\int_\Omega u^{2_s^*\beta}\, dx\right)^{2/2_s^*} \leq C\beta\left(\int_\Omega u^{2\beta+2_s^*-2}\, dx\right). \tag{2.8}$$

Therefore,

$$\left(\int_\Omega u^{2_s^*\beta}\, dx\right)^{\frac{1}{2_s^*(\beta-1)}} \leq (C\beta)^{\frac{1}{2(\beta-1)}}\left(\int_\Omega u^{2\beta+2_s^*-2}\, dx\right)^{\frac{1}{2(\beta-1)}}. \tag{2.9}$$

We are now in position to use an iterative argument as in [1, Proposition 2.2]. For that, we define inductively the sequence $\beta_{m+1},\ m \geq 1$ by

$$2\beta_{m+1} + 2_s^* - 2 = 2_s^*\beta_m,$$

from which we deduce that,

$$\beta_{m+1} - 1 = \left(\frac{2_s^*}{2}\right)^m (\beta_1 - 1).$$

Now by using β_{m+1} in place of β, in (2.9), it follows that

$$\left(\int_\Omega u^{2_s^*\beta_{m+1}}\, dx\right)^{\frac{1}{2_s^*(\beta_{m+1}-1)}} \leq (C\beta_{m+1})^{\frac{1}{2(\beta_{m+1}-1)}}\left(\int_\Omega u^{2_s^*\beta_m}\, dx\right)^{\frac{1}{2_s^*(\beta_m-1)}}.$$

For the sake of clarity, we set

$$C_{m+1} := (C\beta_{m+1})^{\frac{1}{2(\beta_{m+1}-1)}} \quad \text{and} \quad A_m := \left(\int_\Omega u^{2_s^*\beta_m}\, dx\right)^{\frac{1}{2_s^*(\beta_m-1)}}$$

so that

$$A_{m+1} \leq C_{m+1}A_m,\ m \geq 1. \tag{2.10}$$

Then iterating the above inequality, we find that

$$A_{m+1} \leq \prod_{i=2}^{m+1} C_i A_1,$$

which implies that

$$\begin{aligned}\log A_{m+1} &\leq \sum_{i=2}^{m+1} \log C_i + \log A_1\\ &\leq \sum_{i=2}^{\infty} \log C_i + \log A_1.\end{aligned}$$

Since $\beta_{m+1} = (\beta_1 - 1/2)^m(\beta_1 - 1) + 1$ then the serie $\sum_{i=2}^\infty \log C_i$ converges. Also, since $u \in L^{2_s^*\beta_1}(\Omega)$ (see (2.7)), then $A_1 \leq C$. From this, we find that

$$\log A_{m+1} \leq C_0 \tag{2.11}$$

with being $C_0 > 0$ a positive constant independent of m. By letting $m \to \infty$, it follows that

$$\|u\|_{L^\infty(\Omega)} \le C_0' < \infty.$$

This completes the proof. □

3. Existence of minimizers for s close to $1/2$

We aim to study the existence of nontrivial solutions of (1.2). As pointed point out in the introduction the embedding $H_0^s(\Omega) \hookrightarrow L^{2_s^*}(\Omega)$ fails to be compact and due to this, the functional energy associated to (1.2) does not satisfy the Palais-Smale compactness condition. Hence finding the critical points by standard variational methods become a very tough task. Therefore, a natural question arises:

$(\mathcal{Q})$ *Does problem* (1.2) *admits a nontrivial solution*?

In other words, we are looking at whether the quantity

$$S_{N,s}(\Omega) = \inf_{\substack{u \in H_0^s(\Omega) \\ u \neq 0}} \frac{Q_{N,s,\Omega}(u)}{\|u\|^2_{L^{2_s^*}(\Omega)}} \tag{3.1}$$

is attained or not. Here $Q_{N,s,\Omega}(\cdot)$ is a nonnegative quadratic form define on $H_0^s(\Omega)$ by

$$Q_{N,s,\Omega}(u) := \frac{c_{N,s}}{2} \int_\Omega \int_\Omega \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}}\, dxdy.$$

As a quick comment on the above question, Frank et al. [12, Theorem 4] gave a positive answer in the special case of a class of C^1 open sets whose boundary has a flat part, that is C^1 domains Ω with the shape $B_r^+(z) \subset \Omega \subset \mathbb{R}^N_+$ for some $r > 0$ and $z \in \partial\mathbb{R}^N_+$, and such that $\mathbb{R}^N_+ \setminus \Omega$ has nonempty interior. This flatness assumption on the boundary of Ω allows the authors in [12] to obtain the strict inequality $S_{N,s}(\Omega) < S_{N,s}(\mathbb{R}^N_+)$, which is the crucial ingredient for the proof of Theorem 4 in there. Notice that in [12], the question remains open for a larger class of sets.

In the sequel, we give a positive affirmation to the above question in the case of arbitrary open sets with C^1 boundary, provided that s is close to $1/2$. As a consequence, one has in contrast with the fractional Laplacian that the above question has a positive answer even if Ω is convex and of class C^∞.

For the reader's convenience, we restate our main result in the following.

Theorem 3.1. *Let $N \ge 2$ and $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz open set. There exists $s_0 \in (1/2, 1)$ such that for all $s \in (1/2, s_0)$, any minimizing sequence for $S_{N,s}(\Omega)$, normalized in $H_0^s(\Omega)$ is relatively compact in $H_0^s(\Omega)$. In particular, the infimum is achieved.*

The proof of the above main theorem is a direct consequence of the key proposition below (see Proposition 3.2), in which we examine the asymptotic behavior of the Sobolev critical constant $S_{N,s}(\Omega)$ as s tends to $1/2^+$, by showing that the latter goes to zero. The proof of this only requires the domain to be Lipschitz. Our key proposition is stated as follows.

Proposition 3.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz open set. Then*

$$\lim_{s \searrow 1/2} S_{N,s}(\Omega) = 0. \tag{3.2}$$

We now collect some interesting results that are needed to complete the proof of Proposition 3.2 above. Let us start with the following upper semicontinuous lemma.

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz open set. Fix $s_0 \in [1/2, 1)$. Then*

$$\limsup_{s \searrow s_0} S_{N,s}(\Omega) \leq S_{N,s_0}(\Omega). \tag{3.3}$$

Proof. For $t \in \mathbb{R}$, we recall the elementary inequality

$$|e^t - 1| \leq \sum_{k=1}^{+\infty} \frac{|t|^k}{k!} \leq \sum_{k=1}^{+\infty} \frac{|t|^k}{(k-1)!} \leq |t| e^{|t|}. \tag{3.4}$$

For all $r, \gamma > 0$, we also recall the following growth regarding the logarithmic function:

$$|\log|z|| \leq \frac{1}{e\gamma}|z|^{-\gamma} \text{ if } |z| \leq r \text{ and } |\log|z|| \leq \frac{1}{e\gamma}|z|^{\gamma} \text{ if } |z| \geq r. \tag{3.5}$$

Let $\varepsilon > 0$ and let $u_\varepsilon \in C_c^\infty(\Omega)$ such that $\|u_\varepsilon\|_{L^{2_s^*}(\Omega)} = 1$ and $Q_{N,s_0,\Omega}(u_\varepsilon) \leq S_{N,s_0}(\Omega) + \varepsilon$. Then $S_{N,s}(\Omega) \leq Q_{N,s,\Omega}(u_\varepsilon)$. From this, we obtain that

$$S_{N,s}(\Omega) - S_{N,s_0}(\Omega) \leq Q_{N,s,\Omega}(u_\varepsilon) - Q_{N,s_0,\Omega}(u_\varepsilon) + \varepsilon. \tag{3.6}$$

On the other hand,

$$\begin{aligned}
&|Q_{N,s,\Omega}(u_\varepsilon) - Q_{N,s_0,\Omega}(u_\varepsilon)| \\
&\quad \leq \frac{1}{2}|c_{N,s} - c_{N,s_0}| \int_\Omega \int_\Omega \frac{(u_\varepsilon(x) - u_\varepsilon(y))^2}{|x-y|^{N+2s_0}}\, dxdy \\
&\qquad + \frac{c_{N,s}}{2} \int_\Omega \int_\Omega \frac{(u_\varepsilon(x) - u_\varepsilon(y))^2}{|x-y|^{N+2s_0}} ||x-y|^{2(s_0-s)} - 1|\, dxdy \\
&\quad \leq \frac{1}{c_{N,s_0}}(S_{N,s_0}(\Omega) + \varepsilon)|c_{N,s} - c_{N,s_0}| \\
&\qquad + \frac{c_{N,s}}{2} \int_\Omega \int_\Omega \frac{(u_\varepsilon(x) - u_\varepsilon(y))^2}{|x-y|^{N+2s_0}} ||x-y|^{2(s_0-s)} - 1|\, dxdy.
\end{aligned}$$

Next, from (3.4) we have that

$$\begin{aligned}
||x-y|^{2(s_0-s)} - 1| &= |e^{2(s_0-s)\log|x-y|} - 1| \\
&\leq 2|s_0 - s||\log|x-y||e^{2|s_0-s||\log|x-y||} \\
&= 2|s_0 - s||\log|x-y|||x-y|^{2|s_0-s|}.
\end{aligned}$$

Taking this into account and using the regularity of u_ε and the property (3.5) (with $\gamma < 2(1-s_0)$), we have, with

$$A_\Omega := \{(x,y) \in \Omega \times \Omega : |x-y| \le 1\} \quad \text{and}$$
$$B_\Omega := \{(x,y) \in \Omega \times \Omega : |x-y| > 1\},$$

the estimate

$$\begin{aligned}
&\int_\Omega \int_\Omega \frac{(u_\varepsilon(x)-u_\varepsilon(y))^2}{|x-y|^{N+2s_0}} \big||x-y|^{2(s_0-s)} - 1\big| \, dxdy \\
&= 2|s_0-s| \int_\Omega \int_\Omega \frac{(u_\varepsilon(x)-u_\varepsilon(y))^2}{|x-y|^{N+2s_0}} |\log|x-y|| |x-y|^{2|s_0-s|} \, dxdy \\
&\le 2|s_0-s| \mathrm{diam}(\Omega)^{2|s_0-s|} \int_\Omega \int_\Omega \frac{(u_\varepsilon(x)-u_\varepsilon(y))^2}{|x-y|^{N+2s_0}} |\log|x-y|| \, dxdy \\
&= 2|s_0-s| \mathrm{diam}(\Omega)^{2|s_0-s|} \left(\iint_{A_\Omega} \cdots + \iint_{B_\Omega} \cdots \right) \\
&\frac{(u_\varepsilon(x)-u_\varepsilon(y))^2}{|x-y|^{N+2s_0}} |\log|x-y|| \, dxdy \\
&\le 2(e\gamma)^{-1} |s_0-s| \mathrm{diam}(\Omega)^{2|s_0-s|} \left(\iint_{A_\Omega} \frac{(u_\varepsilon(x)-u_\varepsilon(y))^2}{|x-y|^{N+2s_0+\gamma}} \right. \\
&\qquad \left. + \mathrm{diam}(\Omega)^\gamma \iint_{B_\Omega} (u_\varepsilon(x)-u_\varepsilon(y))^2 \right) dxdy \\
&\le 2(e\gamma)^{-1} |s_0-s| \mathrm{diam}(\Omega)^{2|s_0-s|} \left(\|u_\varepsilon\|^2_{C^1(\Omega)} \iint_{A_\Omega} |x-y|^{2-N-2s_0-\gamma} \, dxdy \right. \\
&\qquad \left. + 4|\Omega| \mathrm{diam}(\Omega)^\gamma \right) \\
&= C \mathrm{diam}(\Omega)^{2|s_0-s|} |s_0-s|
\end{aligned}$$

where $\mathrm{diam}(\Omega) = \sup\{|x-y| : x,y \in \Omega\}$ is the diameter of Ω and $C = C(N, s_0, \gamma, \Omega, u_\varepsilon) > 0$ is a positive constant.

From the above estimate, we find that

$$\begin{aligned}
&|Q_{N,s,\Omega}(u_\varepsilon) - Q_{N,s_0,\Omega}(u_\varepsilon)| \\
&\quad \le \frac{1}{c_{N,s_0}} (S_{N,s_0}(\Omega) + \varepsilon) |c_{N,s} - c_{N,s_0}| + \frac{C c_{N,s}}{2} \mathrm{diam}(\Omega)^{2|s_0-s|} |s_0-s| \qquad (3.7)
\end{aligned}$$

and from this, we deduce from (3.6) that

$$\limsup_{s \searrow s_0} S_{N,s}(\Omega) \le S_{N,s_0}(\Omega) + \varepsilon. \qquad (3.8)$$

Since ε can be chosen arbitrarily small, (3.3) follows. This finishes the proof. □

We have the following proposition. While this result is known (see e.g. [14]) and since we could not find a detailed proof, we include its proof in

Appendix A. The idea of proof is to construct a sequence of functions with compact support in Ω and approximate the constant function 1. This allows us to deduce that $1 \in H_0^{1/2}(\Omega)$ and thus $S_{N,1/2}(\Omega) = 0$.

Proposition 3.4. *Let Ω be a bounded Lipschitz open set of $\mathbb{R}^N$. Then*

$$S_{N,1/2}(\Omega) = 0. \tag{3.9}$$

We can now give the proof of our key proposition.

Proof of Proposition 3.2. Since $S_{N,s}(\Omega) > 0$ then if follows that

$$\liminf_{s \searrow 1/2} S_{N,s}(\Omega) \geq 0. \tag{3.10}$$

On the other hand, applying Lemma 3.3 together with Proposition 3.4, we have that

$$\limsup_{s \searrow 1/2} S_{N,s}(\Omega) \leq S_{N,1/2}(\Omega) = 0, \tag{3.11}$$

Now, from (3.10) and (3.11) we deduce (3.2), and this ends the proof of Proposition 3.2. □

Having the above key tools in mind, we can now give the proof of Theorem 3.1.

Proof of Theorem 3.1. Let $s \in (1/2, 1)$ with s close to $1/2$. Then by Proposition 3.2, we have that $S_{N,s}(\Omega) \to 0$ as $s \searrow 1/2$. Consequently, for s close to $1/2$, and since $S_{N,s}(\mathbb{R}^N_+) > 0$ for all $s \in (0, 1)$ (see e.g. [9, Lemma 2.1]), we deduce that

$$0 < S_{N,s}(\Omega) < S_{N,s}(\mathbb{R}^N_+) \quad \text{for all} \quad s \in (1/2, s_0) \tag{3.12}$$

for some $s_0 \in (1/2, 1)$. With the above key inequality, we complete the proof by following closely the argument developed by Frank et al. [12] for the proof of Theorem 4 in there. □

4. The radial problem

In the present section, we consider the existence of minimizers to quotient

$$S_{N,s,rad}(\mathcal{B}, h) := \inf_{u \in C^\infty_{c,rad}(\mathcal{B})} \frac{[u]^2_{H^s(\mathcal{B})} + \int_{\mathcal{B}} hu^2 dx}{\|u\|^2_{L^{2^*_s}(\mathcal{B})}}. \tag{4.1}$$

Here and in the following, we consider the class of radial potentials $h \in L^\infty(\mathcal{B})$ such that

$$S_{N,s,rad}(\mathcal{B}, h) > 0. \tag{4.2}$$

We observe that if $h(x) \equiv -\lambda$ with $\lambda < \lambda_1(\mathcal{B})$, the first eigenvalue of $(-\Delta)^s_{\mathcal{B}}$, then (4.2) holds. The aim of this section is to provide situations in which $S_{N,s,rad}(\mathcal{B}, h) < S_{N,s}(\mathbb{R}^N)$.

Remark 4.1. We observe that if h satisfies (4.2), then if $u \in H^s_0(\mathcal{B})$ satisfies, weakly, $(-\Delta)^s_{\mathcal{B}} u + hu = f$ in $\mathcal{B}$ with $f \in L^p(\mathcal{B})$, for some $p > \frac{N}{2s}$, then $u \in C(\mathcal{B}) \cap L^\infty(\mathcal{B})$. This follows from the argument of Proposition 2.3 and the interior regularity.

We start recalling the following result from [12].

Proposition 4.2. *([12, Proposition 7]) Let $s \in (1/2, 1)$ and $N \geq 4s$. Then*

$$S_{N,s,rad}(\mathcal{B}, 0) < S_{N,s}(\mathbb{R}^N). \tag{4.3}$$

The following result plays a crucial role for the existence theorems.

Proposition 4.3. *Let $1/2 < s < 1$ and $N \geq 2$. Then there is a constant $C = C(N, s) > 0$ such that for all $u \in H^s_{0,rad}(\mathcal{B})$,*

$$Q_{N,s,\mathcal{B}}(u) \geq S_{N,s}(\mathbb{R}^N) \|u\|^2_{L^{2^*_s}(\mathcal{B})} - C_{\mathcal{B}} \|u\|^2_{L^2(\mathcal{B})}. \tag{4.4}$$

For this, we need the following two lemmas.

Lemma 4.4. *For every $\rho \in (0, 1)$, there exists $K_\rho > 0$ with the property that*

$$Q_{N,s,\mathcal{B}}(u) \geq S_{N,s}(\mathbb{R}^N) \|u\|^2_{L^{2^*_s}(\mathcal{B})} - K_\rho \|u\|^2_{L^2(\mathcal{B})}$$

for every $u \in H^s_{0,rad}(\mathcal{B})$ with supp $u \subset B_\rho$.

Proof. Let $u \in H^s_{0,rad}(\mathcal{B})$ with supp $u \subset B_\rho$. We have

$$\begin{aligned} Q_{N,s,\mathcal{B}}(u) = Q_{N,s,\mathbb{R}^N}(u) - \int_{\mathcal{B}} \kappa_{\mathcal{B}}(x) u(x)^2 \, dx \geq S_{N,s}(\mathbb{R}^N) \|u\|^2_{L^{2^*_s}(\mathcal{B})} \\ - \int_{\mathcal{B}} \kappa_{\mathcal{B}}(x) u(x)^2 \, dx, \end{aligned}$$

with being $\kappa_{\mathcal{B}}$ the killing measure for $\mathcal{B}$ defined as $\kappa_{\mathcal{B}}(x) = c_{N,s} \int_{\mathbb{R}^N \setminus \mathcal{B}} \frac{1}{|x-y|^{N+2s}} \, dy, \; x \in \mathcal{B}$. On the other hand, since supp$u \subset B_\rho$, then

$$\int_{\mathcal{B}} \kappa_{\mathcal{B}}(x) u(x)^2 \, dx = \int_{B_\rho} \kappa_{\mathcal{B}}(x) u(x)^2 \, dx$$

and for every $x \in B_\rho$,

$$\kappa_{\mathcal{B}}(x) = c_{N,s} \int_{\mathbb{R}^N \setminus \mathcal{B}} \frac{dy}{|x-y|^{N+2s}} \leq c_{N,s} \int_{|z| \geq 1-\rho} |z|^{-N-2s} \, dz = a_{N,s} (1-\rho)^{-2s}.$$

Taking this into account, we find that

$$\int_{\mathcal{B}} \kappa_{\mathcal{B}}(x) u(x)^2 \, dx \leq a_{N,s} (1-\rho)^{-2s} \int_{B_\rho} u(x)^2 \, dx \leq K_\rho \|u\|^2_{L^2(B_\rho)} \leq K_\rho \|u\|^2_{L^2(\mathcal{B})},$$

with $K_\rho = a_{N,s}(1-\rho)^{-2s}$. From this, we get that

$$Q_{N,s,\mathcal{B}}(u) \geq S_{N,s}(\mathbb{R}^N) \|u\|^2_{L^{2^*_s}(\mathcal{B})} - K_\rho \|u\|^2_{L^2(\mathcal{B})},$$

concluding the proof. □

Lemma 4.5. *For every $M, \rho > 0$ there exists $C_{\rho,M} > 0$ with*

$$Q_{N,s,\mathcal{B}}(u) \geq M\|u\|^2_{L^{2^*_s}(\mathcal{B})} - C_{\rho,M}\|u\|^2_{L^2(\mathcal{B})} \quad \textit{for every } u \in H^s_{0,rad}(\mathcal{B}) \\ \textit{with } u \equiv 0 \textit{ in } B_\rho.$$

Proof. We first recall that for $s \in (1/2, 1)$, $H^s_0(\mathcal{B}) = \mathcal{H}^s_0(\mathcal{B})$. Therefore, for every $u \in H^s_{0,rad}(\mathcal{B}) \subset H^s_0(\mathcal{B}) = \mathcal{H}^s_0(\mathcal{B})$, we have $u \in \mathcal{H}^s_{0,rad}(\mathcal{B})$. Thus, combining the fractional version of the Strauss radial lemma (see [6, Lemma 2.5]) and the Hardy inequality (see [8]) we get that

$$\begin{aligned} |u(x)|^2 &\leq \gamma_{N,s}|x|^{-(N-2s)}Q_{N,s,\mathbb{R}^N}(u) \\ &= \gamma_{N,s}|x|^{-(N-2s)}\left(Q_{N,s,\mathcal{B}}(u) + \int_{\mathcal{B}} \kappa_{\mathcal{B}}(x)u(x)^2\, dx\right) \\ &\leq \gamma_{N,s}|x|^{-(N-2s)}\left(Q_{N,s,\mathcal{B}}(u) + \gamma_{N,s,\mathcal{B}}\int_{\mathcal{B}} \delta_{\mathcal{B}}(x)^{-2s}u(x)^2\, dx\right) \\ &\leq d_{N,s,\mathcal{B}}|x|^{-(N-2s)}Q_{N,s,\mathcal{B}}(u), \end{aligned} \tag{4.5}$$

which implies that

$$\|u\|^2_{L^\infty(\mathcal{B}\setminus B_\rho)} \leq d_{N,s,\mathcal{B}}\rho^{-(N-2s)}Q_{N,s,\mathcal{B}}(u) \\ \text{for every } u \in H^s_{0,rad}(\mathcal{B}) \text{ with } u \equiv 0 \text{ in } B_\rho. \tag{4.6}$$

Consequently, using interpolation and Young's inequality with exponents $p = 2/\alpha$ and $p' = 2/(2-\alpha)$, we find that, for all $M > 0$,

$$\begin{aligned} \|u\|^2_{L^{2^*_s}(\mathcal{B}\setminus B_\rho)} &\leq C\|u\|^\alpha_{L^2(\mathcal{B}\setminus B_\rho)}\|u\|^{2-\alpha}_{L^\infty(\mathcal{B}\setminus B_\rho)} \\ &\leq \frac{1}{Md_{N,s,\mathcal{B}}\rho^{-(N-2s)}}\|u\|^2_{L^\infty(\mathcal{B}\setminus B_\rho)} + \frac{C_{\rho,M}}{M}\|u\|^2_{L^2(\mathcal{B}\setminus B_\rho)} \end{aligned}$$

with suitable constants $\alpha \in (0, 2)$ and $C_{\rho,M} > 0$, and hence

$$\begin{aligned} M\|u\|^2_{L^{2^*_s}(\mathcal{B}\setminus B_\rho)} &\leq \frac{1}{d_{N,s,\mathcal{B}}\rho^{-(N-2s)}}\|u\|^2_{L^\infty(\mathcal{B}\setminus B_\rho)} + C_{\rho,M}\|u\|^2_{L^2(\mathcal{B}\setminus B_\rho)} \\ &\leq Q_{N,s,\mathcal{B}}(u) + C_{\rho,M}\|u\|^2_{L^2(\mathcal{B})} \end{aligned}$$

for every $u \in H^s_{0,rad}(\mathcal{B})$ with $u \equiv 0$ in B_ρ. The claim follows. □

In the following, we give the

Proof of Proposition 4.3. We choose $0 < \rho_2 < \rho_1 < 1$. Moreover, let $\chi_1, \chi_2 \in C^\infty_c(\mathbb{R}^N)$ with $0 \leq \chi_i \leq 1$, $\chi_1^2 + \chi_2^2 \equiv 1$ in $\mathcal{B}$ and supp $\chi_1 \subset B_{\rho_1}$, supp $\chi_2 \subset \mathbb{R}^N \setminus \overline{B_{\rho_2}}$. Then we can write $u = \chi_1^2 u + \chi_2^2 u$ in $\mathcal{B}$.

Applying $Q_{N,s,\mathcal{B}}(\cdot)$ to $u = \sum_{i=1}^2 \chi_i^2 u$, we easily find that

$$Q_{N,s,\mathcal{B}}(u) = \sum_{i=1}^2 Q_{N,s,\mathcal{B}}(\chi_i u) \\ - \frac{c_{N,s}}{2}\sum_{i=1}^2 \int_{\mathcal{B}}\int_{\mathcal{B}} \frac{(\chi_i(x) - \chi_i(y))^2}{|x-y|^{N+2s}}u(x)u(y)\, dxdy. \tag{4.7}$$

By the regularity of χ_i, we observe that there is no singularity in the double integral and therefore it follows from the Schur test that there exists a positive constant $C > 0$ such that

$$\sum_{i=1}^{2} \int_{\mathcal{B}} \int_{\mathcal{B}} \frac{(\chi_i(x) - \chi_i(y))^2}{|x-y|^{N+2s}} u(x)u(y)\ dxdy \leq C \int_{\mathcal{B}} u^2\ dx. \tag{4.8}$$

In fact, we can write

$$\int_{\mathcal{B}} \int_{\mathcal{B}} \frac{(\chi_i(x) - \chi_i(y))^2}{|x-y|^{N+2s}} u(x)u(y)\ dxdy \leq C \int_{\mathcal{B}} \int_{\mathcal{B}} K(x,y)u(x)u(y)\ dxdy \tag{4.9}$$

$$= C \int_{\mathcal{B}} Tu(x)u(x)\ dx \tag{4.10}$$

where

$$Tu(x) = \int_{\mathcal{B}} K(x,y)u(y)\ dy \qquad \text{with} \qquad K(x,y) = |x-y|^{2-N-2s}.$$

Moreover, by Hölder inequality,

$$\int_{\mathcal{B}} Tu(x)u(x)\ dx \leq \|Tu\|_{L^2(\mathcal{B})} \|u\|_{L^2(\mathcal{B})}. \tag{4.11}$$

Now, the Schur test implies that there is $C > 0$ such that

$$\|Tu\|_{L^2(\mathcal{B})} \leq C\|u\|_{L^2(\mathcal{B})}. \tag{4.12}$$

Therefore, inequality (4.8) follows by combining (4.9), (4.11) and (4.12).
On the other hand, by Lemmas 4.4 and 4.5, there exists a positive constant $C > 0$, depending on ρ_1 and ρ_2 with the property that

$$Q_{N,s,\mathcal{B}}(\chi_i u) \geq S_{N,s}(\mathbb{R}^N)\|\chi_i u\|^2_{L^{2^*_s}(\mathcal{B})} - C\|\chi_i u\|^2_{L^2(\mathcal{B})}. \tag{4.13}$$

Plugging (4.8) and (4.13) into (4.7), we find that

$$Q_{N,s,\mathcal{B}}(u) \geq S_{N,s}(\mathbb{R}^N) \sum_{i=1}^{2} \|\chi_i u\|^2_{L^{2^*_s}(\mathcal{B})} - C \sum_{i=1}^{2} \|\chi_i u\|^2_{L^2(\mathcal{B})}. \tag{4.14}$$

Next, since $\sum_{i=1}^{2} \chi_i^2 = 1$, we have

$$\sum_{i=1}^{2} \|\chi_i u\|^2_{L^{2^*_s}(\mathcal{B})} = \sum_{i=1}^{2} \left\|\chi_i^2 u^2\right\|_{L^{\frac{N}{N-2s}}(\mathcal{B})} \geq \left\| \sum_{i=1}^{2} \chi_i^2 u^2 \right\|_{L^{\frac{N}{N-2s}}(\mathcal{B})}$$
$$= \|u^2\|_{L^{\frac{N}{N-2s}}(\mathcal{B})} = \|u\|^2_{L^{2^*_s}(\mathcal{B})}.$$

Using this in (4.13), it follows that

$$Q_{N,s,\mathcal{B}}(u) \geq S_{N,s}(\mathbb{R}^N)\|u\|^2_{L^{2^*_s}(\mathcal{B})} - C\|u\|^2_{L^2(\mathcal{B})},$$

completing the proof. □

4.1. The case $2s < N < 4s$

We now let $G(x, y)$ be the Green function of $(-\Delta)^s + h$, with zero exterior Dirichlet boundary data. Letting $G(x) = G(x, 0)$, we have that

$$\begin{cases} (-\Delta)^s G(x) + h(x)G(x) = \delta_0(x) & \text{in } \mathcal{B} \\ G(x) = 0 & \text{in } \mathbb{R}^N \setminus \mathcal{B}, \end{cases} \tag{4.15}$$

where δ_0 is the Dirac mass at 0 and $h \in L^\infty(\mathcal{B})$ a radial function. We recall that G is a radial function. In fact this follows from the construction and uniqueness of Green function. We let $\mathcal{R}(x) = t_{N,s}|x|^{2s-N}$ be the fundamental solution of $(-\Delta)^s$ on $\mathbb{R}^N$. It satisfies

$$(-\Delta)^s \mathcal{R}(x) = \delta_0(x), \tag{4.16}$$

where $t_{N,s} := \pi^{-\frac{N}{2}} 2^{-s} \frac{\Gamma((N-s)/2)}{\Gamma(s/2)}$. We now define $\overline{\mathbf{k}} \in L^1(\mathcal{B})$, by

$$\overline{\mathbf{k}}(x) := G(x) - \mathcal{R}(x). \tag{4.17}$$

It then follows, from (4.15), that

$$(-\Delta)^s \overline{\mathbf{k}}(x) + h(x)\overline{\mathbf{k}}(x) = -h(x)\mathcal{R}(x). \tag{4.18}$$

Since $N < 4s$, we have that $\overline{\mathbf{k}} \in L^2(\mathcal{B})$ and $h\mathcal{R} \in L^p(\mathcal{B}) \cap L^2(\mathcal{B})$, for some $p > \frac{N}{2s}$. Therefore, by regularity theory, $\overline{\mathbf{k}} \in C(\overline{\mathcal{B}})$. Recall that $\overline{\mathbf{k}}(y)$ is the *mass* of $\mathcal{B}$ associated to the operator $\mathcal{L}_{\mathbb{R}^N} := (-\Delta)^s + h(x)$. We remark that if $\chi \in C_c^\infty(\mathcal{B})$, with $\chi = 1$ in a neighborhood of 0, then letting

$$\mathbf{k}(x) := G(x) - \chi(x)\mathcal{R}(x),$$

then, by continuity, $\mathbf{k}(y) = \overline{\mathbf{k}}(y)$, for all $y \in \mathcal{B}$. This follows from the fact that $(-\Delta)^s \mathbf{k} + h\mathbf{k} \in L^p(\mathcal{B})$, for some $p > \frac{N}{2s}$ and thus $\mathbf{k} \in C(\mathcal{B})$.

Remark 4.6. It would be interesting to find potential h for which $\mathbf{k}(0) > 0$.

First, for $\varepsilon > 0$ we set

$$u_\varepsilon(x) = \gamma_0 \left(\frac{\varepsilon}{\varepsilon^2 + |x|^2} \right)^{\frac{N-2s}{2}},$$

where γ_0 is a positive constant (independent of ε) such that $\|u_\varepsilon\|_{L^{2_s^*}(\mathbb{R}^N)} = 1$. It is known that u_ε satisfies the Euler-Lagrange equation

$$(-\Delta)^s u_\varepsilon = S_{N,s} u_\varepsilon^{2_s^*-1} \quad \text{in} \quad \mathbb{R}^N. \tag{4.19}$$

Our next result shows that in low dimension $N < 4s$, the positive mass implies existence of minimizers.

Lemma 4.7. *Suppose that* $2s < N < 4s$*. Suppose that* $\mathbf{k}(0) > 0$*. Then*

$$S_{N,s,rad}(\mathcal{B}, h) < S_{N,s} := S_{N,s}(\mathbb{R}^N). \tag{4.20}$$

Proof. For $r \in (0,1/4)$, we let $\eta \in C_c^\infty(B_{2r})$ be radial, with $\eta = 1$ on B_r. We define the test function $v_\varepsilon \in H^s_{0,rad}(\mathcal{B})$ given by

$$\begin{aligned} v_\varepsilon(x) &= \eta(x)u_\varepsilon(x) + \varepsilon^{\frac{N-2s}{2}}\frac{\gamma_0}{t_{N,s}}\left(G(x) - \eta(x)\mathcal{R}(x)\right) \\ &= \eta(x)u_\varepsilon(x) + \varepsilon^{\frac{N-2s}{2}}\frac{\gamma_0}{t_{N,s}}\mathbf{k}(x). \end{aligned} \tag{4.21}$$

We define $W_\varepsilon := \eta u_\varepsilon - \varepsilon^{\frac{N-2s}{2}}\frac{\gamma_0}{t_{N,s}}\eta\mathcal{R}$ and $a_s := \frac{\gamma_0}{t_{N,s}}$.

Note that $\varepsilon^{-\frac{N-2s}{2}}W_\varepsilon \to 0 \in C_{loc}(\mathbb{R}^N \setminus \{0\}) \cap L^1(\mathcal{B})$ and $|\varepsilon^{-\frac{N-2s}{2}}u_\varepsilon(x)| \leq \gamma_0|x|^{2s-N}$. Hence, since $N < 4s$, we deduce that $|x|^{2(2s-N)} \in L^1_{loc}(\mathbb{R}^N)$ and thus by the dominated convergence theorem,

$$\int_{\mathcal{B}} u_\varepsilon(x)h(x)W_\varepsilon(x)\,dx = o(\varepsilon^{N-2s}). \tag{4.22}$$

We then have

$$\begin{aligned} &[v_\varepsilon]^2_{H^s(\mathcal{B})} + \int_{\mathcal{B}} hv_\varepsilon^2\,dx \leq [v_\varepsilon]^2_{H^s(\mathbb{R}^N)} + \int_{\mathcal{B}} hv_\varepsilon^2\,dx = \int_{\mathcal{B}} v_\varepsilon(x)\mathcal{L}_{\mathbb{R}^N}v_\varepsilon(x)\,dx \\ &\leq \varepsilon^{\frac{N-2s}{2}}a_s\int_{\mathcal{B}} v_\varepsilon(x)\mathcal{L}_{\mathbb{R}^N}G(x)\,dx + \int_{\mathcal{B}} v_\varepsilon(x)\mathcal{L}_{\mathbb{R}^N}W_\varepsilon(x)\,dx \\ &\leq \varepsilon^{\frac{N-2s}{2}}a_s u_\varepsilon(0) + \varepsilon^{N-2s}a_s^2\mathbf{k}(0) + \int_{\mathcal{B}} \eta u_\varepsilon(x)(-\Delta)^s W_\varepsilon(x)\,dx \\ &\quad + \varepsilon^{\frac{N-2s}{2}}a_s\int_{\mathcal{B}} \mathbf{k}(x)\mathcal{L}_{\mathbb{R}^N}W_\varepsilon(x)\,dx + o(\varepsilon^{N-2s}) \\ &\leq \varepsilon^{\frac{N-2s}{2}}a_s u_\varepsilon(0) + \varepsilon^{N-2s}a_s^2\mathbf{k}(0) + \int_{\mathcal{B}} \eta u_\varepsilon(x)(-\Delta)^s(\eta u_\varepsilon)(x)\,dx \\ &\quad - \varepsilon^{\frac{N-2s}{2}}a_s\int_{\mathcal{B}} \eta u_\varepsilon(x)(-\Delta)^s(\eta\mathcal{R})(x)\,dx \\ &\quad + \varepsilon^{\frac{N-2s}{2}}a_s\int_{\mathcal{B}} \mathbf{k}(x)\mathcal{L}_{\mathbb{R}^N}W_\varepsilon(x)\,dx + o(\varepsilon^{N-2s}) \\ &\leq \varepsilon^{\frac{N-2s}{2}}a_s u_\varepsilon(0) + \varepsilon^{N-2s}a_s^2\mathbf{k}(0) \\ &\quad + \int_{\mathbb{R}^N} \eta u_\varepsilon(x)(-\Delta)^s(\eta u_\varepsilon)(x)\,dx - \varepsilon^{\frac{N-2s}{2}}a_s\int_{\mathbb{R}^N} \eta u_\varepsilon(x)(-\Delta)^s(\eta\mathcal{R})(x)\,dx \\ &\quad + \varepsilon^{\frac{N-2s}{2}}a_s\int_{\mathbb{R}^N} \mathbf{k}(x)\mathcal{L}_{\mathbb{R}^N}W_\varepsilon(x)\,dx + o(\varepsilon^{N-2s}). \end{aligned}$$

Letting $\overline{W}_\varepsilon = u_\varepsilon - \varepsilon^{\frac{N-2s}{2}}a_s\mathcal{R}(x)$, since $N < 4s$, we have that

$$\varepsilon^{-\frac{N-2s}{2}}\overline{W}_\varepsilon \to 0 \qquad \text{in } C^1_{loc}(\mathbb{R}^N \setminus \{0\}) \cap \mathcal{L}^1_s \cap L^2_{loc}(\mathbb{R}^N). \tag{4.23}$$

Therefore, using that $(-\Delta)^s\mathcal{R} = \delta_0$ and $(-\Delta)^s u_\varepsilon = S_{N,s} u_\varepsilon^{2_s^*-1}$, we get

$$\begin{aligned}
&\varepsilon^{\frac{N-2s}{2}} a_s u_\varepsilon(0) + \int_{\mathbb{R}^N} \eta u_\varepsilon(x)(-\Delta)^s(\eta u_\varepsilon)(x)\,dx \\
&\quad - \varepsilon^{\frac{N-2s}{2}} a_s \int_{\mathbb{R}^N} \eta u_\varepsilon(x)(-\Delta)^s(\eta\mathcal{R})(x)\,dx \\
&\quad = \varepsilon^{\frac{N-2s}{2}} a_s u_\varepsilon(0) + \int_{\mathbb{R}^N} \eta^2 u_\varepsilon(x)(-\Delta)^s u_\varepsilon(x)\,dx \\
&\quad - \varepsilon^{\frac{N-2s}{2}} a_s \int_{\mathbb{R}^N} \eta u_\varepsilon(x)(-\Delta)^s\mathcal{R}(x)\,dx \\
&\quad + \int_{\mathbb{R}^N} \eta u_\varepsilon(x)\overline{W}_\varepsilon(x)(-\Delta)^s\eta(x)\,dx - \int_{B_{2r}} \eta u_\varepsilon(x) J_\varepsilon(x) dx \\
&\quad = S_{N,s}\int_{\mathbb{R}^N} \eta^2 u_\varepsilon^{2_s^*} + \int_{\mathbb{R}^N} \eta u_\varepsilon(x)\overline{W}_\varepsilon(x)(-\Delta)^s\eta(x)\,dx - \int_{B_{2r}} \eta u_\varepsilon(x) J_\varepsilon(x) dx \\
&\quad = S_{N,s}\int_{\mathbb{R}^N} \eta^2 u_\varepsilon^{2_s^*} + o(\varepsilon^{N-2s}) - \int_{B_{2r}} \eta u_\varepsilon(x) J_\varepsilon(x) dx,
\end{aligned}$$

where $J_\varepsilon(x) := c_{N,s}\int_{\mathbb{R}^N} \frac{(\overline{W}_\varepsilon(x)-\overline{W}_\varepsilon(y))(\eta(x)-\eta(y))}{|x-y|^{N+2s}}\,dy$. To estimate J_ε, we consider first $x \in B_{r/2}$ and thus

$$J_\varepsilon(x) = c_{N,s}\int_{|y|>r} \frac{(\overline{W}_\varepsilon(x)-\overline{W}_\varepsilon(y))(\eta(x)-\eta(y))}{|x-y|^{N+2s}}\,dy = o(\varepsilon^{\frac{N-2s}{2}})O(|x|^{\frac{N-2s}{2}}).$$

If now $|x| \geq r/2$, we estimate

$$\begin{aligned}
|J_\varepsilon(x)| &\leq c_{N,s}\int_{|y|<r/4} \frac{|(\overline{W}_\varepsilon(x)-\overline{W}_\varepsilon(y))(\eta(x)-\eta(y))|}{|x-y|^{N+2s}}\,dy \\
&\quad + c_{N,s}\int_{|y|>r/4} \frac{|(\overline{W}_\varepsilon(x)-\overline{W}_\varepsilon(y))(\eta(x)-\eta(y))|}{|x-y|^{N+2s}}\,dy \\
&\leq o(\varepsilon^{\frac{N-2s}{2}}) + \|\nabla\eta\|_{L^\infty(\mathbb{R}^N)} \int_{4r>|y|>r/4} \frac{\sup_{t\in[0,1]} |\nabla\overline{W}_\varepsilon(\gamma_{x,y}(t))||\gamma_{x,y}'(t)|}{|x-y|^{N+2s-1}}\,dy \\
&= o(\varepsilon^{\frac{N-2s}{2}})
\end{aligned}$$

where $\gamma_{x,y} : [0,1] \to B_{r/2}\setminus B_{r/4}$ is the C^1 shortest curve satisfying $\gamma_{x,y}(0) = x$, $\gamma_{x,y}(1) = y$ and $\sup_{t\in[0,1]} |\gamma_{x,y}'(t)| \leq C|x-y|$.

Since $N < 4s$, by (4.18) and (4.23), we have

$$\left|\int_{\mathbb{R}^N} \mathbf{k}(x)\mathcal{L}_{\mathbb{R}^N} W_\varepsilon(x)\,dx\right| \leq \left|\int_{B_{2r}} |\mathcal{L}_{\mathbb{R}^N}\mathbf{k}(x)||W_\varepsilon(x)|\,dx\right| = o(\varepsilon^{\frac{N-2s}{2}}).$$

We thus conclude that

$$\begin{aligned}
[v_\varepsilon]_{H^s(\mathcal{B})}^2 + \int_{\mathcal{B}} h v_\varepsilon^2\,dx &\leq S_{N,s}\int_{\mathbb{R}^N} \eta^2 u_\varepsilon^{2_s^*} \\
&\quad + \varepsilon^{N-2s} a_s^2 \mathbf{k}(0) + o(\varepsilon^{N-2s}) + O(\varepsilon^{N-2s}) o_r(1) \\
&\leq S_{N,s} + \varepsilon^{N-2s} a_s^2 \mathbf{k}(0) + o(\varepsilon^{N-2s}) + O(r^{4s-N}\varepsilon^{N-2s}).
\end{aligned} \tag{4.24}$$

Since $2_s^* > 2$, there exists a positive constant $C(N,s)$ such that

$$||a+b|^{2_s^*} - |a|^{2_s^*} - 2_s^* ab|a|^{2_s^*-2}| \le C(N,s)\left(|a|^{2_s^*-2}b^2 + |b|^{2_s^*}\right) \quad \text{for all } a,b\in\mathbb{R}.$$

As a consequence, with $a = \eta(x)u_\varepsilon(x)$ and $b = \varepsilon^{\frac{N-2s}{2}} a_s \mathbf{k}(x)$, we obtain

$$\begin{aligned}
&\int_{\mathcal{B}} v_\varepsilon^{2_s^*} - \int_{\mathbb{R}^N} (\eta u_\varepsilon)^{2_s^*} = 2_s^* \varepsilon^{\frac{N-2s}{2}} a_s \int_{\mathcal{B}} (\eta u_\varepsilon)^{2_s^*-1}\mathbf{k}(x)\,dx \\
&\quad + o(\varepsilon^{N-2s}) + O\left(\varepsilon^{N-2s}\int_{\mathbb{R}^N} |\eta(x)u_\varepsilon(x)|^{2_s^*-2}\mathbf{k}^2(x)dx\right) \\
&\quad = 2_s^*\varepsilon^{\frac{N-2s}{2}} \frac{a_s}{S_{N,s}} \int_{\mathcal{B}} \eta^{2_s^*-1}\mathbf{k}(x)(-\Delta)^s u_\varepsilon\,dx + o(\varepsilon^{N-2s}) \\
&\qquad + \varepsilon^{N-2s} O\left(\|\eta u_\varepsilon\|^{2_s^*-2}_{L^{2_s^*}(B_{2r})} \|\mathbf{k}\|^2_{L^{2_s^*}(B_{2r})}\right). \\
&\quad = 2_s^*\varepsilon^{\frac{N-2s}{2}} \frac{a_s}{S_{N,s}} \int_{\mathcal{B}} \mathbf{k}(x)(-\Delta)^s \overline{W}_\varepsilon\,dx \\
&\qquad + 2_s^*\varepsilon^{\frac{N-2s}{2}} \frac{a_s}{S_{N,s}} \int_{\mathcal{B}} (\eta^{2_s^*-1}-1)\mathbf{k}(x)(-\Delta)^s \overline{W}_\varepsilon\,dx \\
&\qquad + 2_s^*\varepsilon^{N-2s} \frac{a_s^2}{S_{N,s}}\mathbf{k}(0) + o(\varepsilon^{N-2s}) + O(\varepsilon^{N-2s} r^{N-2s}) \\
&\quad = 2_s^*\varepsilon^{\frac{N-2s}{2}} \frac{a_s}{S_{N,s}} \int_{\mathcal{B}} \overline{W}_\varepsilon(x)\mathcal{L}_{\mathbb{R}^N}\mathbf{k}(x)\,dx \\
&\qquad + 2_s^*\varepsilon^{\frac{N-2s}{2}} \frac{a_s}{S_{N,s}} \int_{\mathcal{B}} (\eta^{2_s^*-1}-1)\mathbf{k}(x)(-\Delta)^s \overline{W}_\varepsilon\,dx \\
&\qquad + 2_s^*\varepsilon^{N-2s} \frac{a_s^2}{S_{N,s}}\mathbf{k}(0) + o(\varepsilon^{N-2s}) + O(\varepsilon^{N-2s} r^{N-2s}) \\
&\quad = 2_s^*\varepsilon^{N-2s} \frac{a_s^2}{S_{N,s}} O\left(\int_{|x|<2r} |x|^{2s-N}\left(\frac{1}{(\varepsilon^2+|x|^2)^{\frac{N-2s}{2}}} - \frac{1}{|x|^{N-2s}}\right)dx\right) \\
&\qquad + 2_s^*\varepsilon^{\frac{N-2s}{2}} \frac{a_s}{S_{N,s}} \int_{\mathcal{B}} (\eta^{2_s^*-1}-1)\mathbf{k}(x)(-\Delta)^s \overline{W}_\varepsilon\,dx \\
&\qquad + 2_s^*\varepsilon^{N-2s} \frac{a_s^2}{S_{N,s}}\mathbf{k}(0) + o(\varepsilon^{N-2s}) + O(\varepsilon^{N-2s}) o_r(1).
\end{aligned}$$

We estimate

$$\begin{aligned}
&\int_{\mathcal{B}} (\eta^{2_s^*-1}-1)\mathbf{k}(x)(-\Delta)^s\overline{W}_\varepsilon\,dx = \int_{\mathcal{B}} (\eta^{2_s^*-1}-1)\mathbf{k}(x)(-\Delta)^s(\eta_{r/4}\overline{W}_\varepsilon)\,dx + o(\varepsilon^{\frac{N-2s}{2}}) \\
&\quad = c_{N,s}\int_{|x|\ge r}(1-\eta^{2_s^*-1}(x))\mathbf{k}(x)\int_{|y|<r/2} \frac{\eta_{r/4}(y)\overline{W}_\varepsilon(y)\,dy}{|x-y|^{N+2s}}\,dy + o(\varepsilon^{\frac{N-2s}{2}}) = o(\varepsilon^{\frac{N-2s}{2}}).
\end{aligned}$$

Here, from the definition of η, we define $\eta_{r/4} \in C_c^\infty(B_{r/2})$ with $\eta_{r/4} = 1$ on $B_{r/4}$. From the above estimates, we then obtain

$$\int_{\mathcal{B}} v_\varepsilon^{2_s^*} = \int_{\mathbb{R}^N} (\eta u_\varepsilon)^{2_s^*} + 2_s^* \varepsilon^{N-2s} \frac{a_s^2}{S_{N,s}} \mathbf{k}(0) + o(\varepsilon^{N-2s}) + O(\varepsilon^{N-2s}) o_r(1)$$
$$= 1 + 2_s^* \varepsilon^{N-2s} \frac{a_s^2}{S_{N,s}} \mathbf{k}(0) + o(\varepsilon^{N-2s}) + O(\varepsilon^{N-2s}) o_r(1).$$

Combining this with (4.24), we finally get

$$\frac{[v_\varepsilon]^2_{H^s(\mathcal{B})} + \int_{\mathcal{B}} h v_\varepsilon^2 \, dx}{\|v_\varepsilon\|^2_{L^{2_s^*}(\mathcal{B})}} \leq S_{N,s} - \varepsilon^{N-2s} a_s^2 \mathbf{k}(0) + o(\varepsilon^{N-2s}) + O(\varepsilon^{N-2s}) o_r(1).$$

This finishes the proof. □

5. Existence of radial minimizers

The goal of this section is to investigate the existence of a radial solution of problem (1.2) in the case when $\Omega = \mathcal{B}$ is the unit ball of $\mathbb{R}^N$, $N > 2s$. More precisely, we aim to analyze the attainability of the following radial critical level

$$S_{N,s,rad}(\mathcal{B}, h) = \inf_{\substack{u \in H^s_{0,rad}(\mathcal{B}) \\ u \neq 0}} \frac{Q_{N,s,\mathcal{B}}(u) + \int_{\mathcal{B}} h u^2 \, dx}{\|u\|^2_{L^{2_s^*}(\mathcal{B})}}. \tag{5.1}$$

To this end, we make use of the method of missing mass as in [12]. The idea is to prove that a minimizing sequence for $S_{N,s,rad}(\mathcal{B}, h)$ does not concentrate at the origin. For that, we will exploit inequalities (4.3) and (4.20) respectively for high ($N \geq 4s$) and low ($2s < N < 4s$) dimensions.

For the reader's convenience, we restate the main result of this subsection in the following.

Theorem 5.1. *Let $s \in (1/2, 1)$, $N > 2s$ and $h \in L^\infty(\mathcal{B})$ be a radial function. Suppose that $0 < S_{N,s,rad}(\mathcal{B}, h) < S_{N,s}(\mathbb{R}^N)$. Then any minimizing sequence for $S_{N,s,rad}(\mathcal{B}, h)$, normalized in $H^s_{0,rad}(\mathcal{B})$ is relatively compact in $H^s_{0,rad}(\mathcal{B})$. In particular, the infimum is achieved.*

To prove the above theorem, we first collect some useful results. Let's introduce

$$S^*_{N,s,rad}(\mathcal{B}) := \inf \Big\{ \liminf_{k \to \infty} \|u_k\|^{-2}_{L^{2_s^*}(\mathcal{B})} : Q_{N,s,\mathcal{B}}(u_k)$$
$$= 1,\ u_k \rightharpoonup 0 \text{ in } H^s_{0,rad}(\mathcal{B}) \Big\}. \tag{5.2}$$

As we will see in the sequel, the infimum $S^*_{N,s,rad}(\mathcal{B})$ is crucial in showing that normalized minimizing sequences that weakly converge to zero in $H^s_{0,rad}(\mathcal{B})$ move away from the origin in such a way that the concentration at the origin is excluded.

We have the following interesting one-sided inequality.

Proposition 5.2. *Let $1/2 < s < 1$ and $N \geq 2$. Then*

$$S^*_{N,s,rad}(\mathcal{B}) \geq S_{N,s}(\mathbb{R}^N). \tag{5.3}$$

Proof. Let $(u_k) \subset H^s_{0,rad}(\mathcal{B})$ with $Q_{N,s,\mathcal{B}}(u_k) = 1$ and $u_k \rightharpoonup 0$ in $H^s_{0,rad}(\mathcal{B})$. Then by Proposition 4.3 there is $C_{\mathcal{B}} > 0$ such that

$$Q_{N,s,\mathcal{B}}(u_k) \geq S_{N,s}(\mathbb{R}^N)\|u_k\|^2_{L^{2^*_s}(\mathcal{B})} - C_{\mathcal{B}}\|u_k\|^2_{L^2(\mathcal{B})}.$$

By the compact embedding $H^s_{0,rad}(\mathcal{B}) \hookrightarrow L^2(\mathcal{B})$, we have $u_k \to 0$ in $L^2(\mathcal{B})$. Using this and by passing to the limit in the above inequality, we find that

$$1 \geq S_{N,s}(\mathbb{R}^N) \limsup_{k\to\infty} \|u_k\|^2_{L^{2^*_s}(\mathcal{B})},$$

that is,

$$\liminf_{k\to\infty} \|u_k\|^{-2}_{L^{2^*_s}(\mathcal{B})} \geq S_{N,s}(\mathbb{R}^N).$$

From the above inequality, we conclude the proof. □

Having collected the above results, we are ready to prove our main result.

Proof of Theorem 5.1. Let (u_k) be a minimizing sequence for $S_{N,s,rad}(\mathcal{B}, h)$, which is normalized in $H^s_{0,rad}(\mathcal{B})$. Then after passing to a subsequence, there is $u \in H^s_{0,rad}(\mathcal{B})$ such that

$$\begin{aligned} u_k &\rightharpoonup u \quad \text{weakly in } H^s_{0,rad}(\mathcal{B}) \\ u_k &\to u \quad \text{strongly in } L^2(\mathcal{B}) \\ u_k &\to u \quad \text{a.e. in } \mathcal{B}. \end{aligned} \tag{5.4}$$

Now, by setting $w_k = u_k - u$, it follows that $w_k \rightharpoonup 0$ weakly in $H^s_{0,rad}(\mathcal{B})$. Using this, we have that

$$\begin{aligned} 1 = Q_{N,s,\mathcal{B},h}(u_k) &:= Q_{N,s,\mathcal{B}}(u_k) + \int_{\mathcal{B}} h u_k^2 \, dx \\ &= Q_{N,s,\mathcal{B},h}(u) + Q_{N,s,\mathcal{B}}(w_k) + o(1), \end{aligned} \tag{5.5}$$

where $Q_{N,s,\mathcal{B},h}(u) := Q_{N,s,\mathcal{B}}(u) + \int_{\mathcal{B}} hu^2 \, dx$. From the above identities, we see that $Q_{N,s,\mathcal{B}}(w_k)$ converges, say, to R_1, which satisfies according to the above equality,

$$1 = Q_{N,s,\mathcal{B},h}(u) + R_1. \tag{5.6}$$

Moreover, using that $u_k \to u$ a.e. in $\mathcal{B}$ and the Brezis-Lieb lemma [3], we get that

$$\begin{aligned} S_{N,s,rad}(\mathcal{B}, h)^{-\frac{N}{N-2s}} + o(1) &= \|u_k\|^{\frac{2N}{N-2s}}_{L^{2^*_s}(\mathcal{B})} \\ &= \|u\|^{\frac{2N}{N-2s}}_{L^{2^*_s}(\mathcal{B})} + \|w_k\|^{\frac{2N}{N-2s}}_{L^{2^*_s}(\mathcal{B})} + o(1), \end{aligned} \tag{5.7}$$

from which we deduce that $\int_{\mathcal{B}} |w_k|^{\frac{2N}{N-2s}} \, dx$ converges, say, to R_2 satisfying

$$S_{N,s,rad}(\mathcal{B}, h)^{-\frac{N}{N-2s}} = \|u\|^{\frac{2N}{N-2s}}_{L^{2^*_s}(\mathcal{B})} + R_2. \tag{5.8}$$

Now by Proposition 5.2 we easily see that

$$R_1 \geq S_{N,s}(\mathbb{R}^N) R_2^{\frac{N-2s}{N}}. \tag{5.9}$$

Indeed, (5.9) follows immediately if $R_2 = 0$. Otherwise, if $R_2 > 0$, then it suffices to use $\tilde{w}_k := w_k/Q_{N,s,\mathcal{B}}(w_k)^{1/2}$ in the definition of $S^*_{N,s,rad}(\mathcal{B})$ since $\tilde{w}_k \rightharpoonup 0$ weakly in $H^s_{0,rad}(\mathcal{B})$ and $Q_{N,s,\mathcal{B}}(\tilde{w}_k) = 1$ as well.

From (5.6), (5.8), (5.9) and by using the elementary inequality [1]

$$(a-b)^\alpha \geq a^\alpha - b^\alpha \quad \text{for } 0 \leq \alpha \leq 1,\ a \geq b \geq 0 \tag{5.10}$$

with $\alpha = (N-2s)/N$, we find that

$$\begin{aligned}
1 &= Q_{N,s,\mathcal{B},h}(u) + R_1 \\
&\geq Q_{N,s,\mathcal{B},h}(u) + S_{N,s}(\mathbb{R}^N) R_2^{\frac{N-2s}{N}} \\
&= Q_{N,s,\mathcal{B},h}(u) + (S_{N,s}(\mathbb{R}^N) - S_{N,s,rad}(\mathcal{B},h)) R_2^{\frac{N-2s}{N}} \\
&\quad + S_{N,s,rad}(\mathcal{B}) \Big(S_{N,s,rad}(\mathcal{B},h)^{-\frac{N}{N-2s}} - \|u\|_{L^{2^*_s}(\mathcal{B})}^{\frac{2N}{N-2s}} \Big)^{\frac{N-2s}{N}} \\
&\geq Q_{N,s,\mathcal{B},h}(u) + (S_{N,s}(\mathbb{R}^N) - S_{N,s,rad}(\mathcal{B},h)) R_2^{\frac{N-2s}{N}} \\
&\quad + S_{N,s,rad}(\mathcal{B},h) \Big(S_{N,s,rad}(\mathcal{B},h)^{-1} - \|u\|^2_{L^{2^*_s}(\mathcal{B})} \Big) \\
&= Q_{N,s,\mathcal{B},h}(u) + (S_{N,s}(\mathbb{R}^N) - S_{N,s,rad}(\mathcal{B},h)) R_2^{\frac{N-2s}{N}} \\
&\quad + 1 - S_{N,s,rad}(\mathcal{B},h) \|u\|^2_{L^{2^*_s}(\mathcal{B})}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&Q_{N,s,\mathcal{B},h}(u) - S_{N,s,rad}(\mathcal{B},h)\|u\|^2_{L^{2^*_s}(\mathcal{B})} + (S_{N,s}(\mathbb{R}^N) \\
&\quad - S_{N,s,rad}(\mathcal{B},h)) R_2^{\frac{N-2s}{N}} \leq 0.
\end{aligned} \tag{5.11}$$

Since $Q_{N,s,\mathcal{B},h}(u) \geq S_{N,s,rad}(\mathcal{B},h)\|u\|^2_{L^{2^*_s}(\mathcal{B})}$ and $S_{N,s}(\mathbb{R}^N) > S_{N,s,rad}(\mathcal{B},h)$ by assumption, it follows from (5.11) that $R_2 = 0$ which implies that $u \not\equiv 0$ thanks to (5.8). Therefore,

$$Q_{N,s,\mathcal{B},h}(u) \leq S_{N,s,rad}(\mathcal{B},h)\|u\|^2_{L^{2^*_s}(\mathcal{B})},$$

which implies that u is an optimizer. Therefore, instead of the inequality (5.9), we have equality, yielding $R_1 = 0$. This implies that $Q_{N,s,\mathcal{B},h}(u) = 1$ and from this, we conclude that (u_k) converges strongly in $H^s_{0,rad}(\mathcal{B})$. The proof is therefore finished. □

Proof of Theorem 1.2 and Theorem 1.3 (completed). The proof of Theorem 1.2 and Theorem 1.3 are immediate consequences of Theorem 5.1, Lemma 4.7 and Proposition 4.2. □

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[1] $0 \leq b \leq a \Rightarrow 0 \leq b/a \leq 1$ and then $0 \leq b/a \leq (b/a)^\alpha \leq 1$ for all $0 \leq \alpha \leq 1$. Hence,

$$\frac{a^\alpha - b^\alpha}{(a-b)^\alpha} = \frac{1-(b/a)^\alpha}{(1-(b/a))^\alpha} \leq \frac{1-(b/a)}{(1-(b/a))^\alpha} \leq 1.$$

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A. Appendix

In this section, we prove that the constant function 1 belongs to $H_0^s(\Omega)$ for $s \in (0, 1/2]$. By Sobolev embedding, it is enough to treat the case $s = 1/2$.

For every $k \in \mathbb{N}$, we define $\chi_k \in C^{0,1}(\mathbb{R}_+)$ by

$$\chi_k(t) = \begin{cases} 0 & \text{if} \quad t \leq \dfrac{1}{k^2}, \\ \dfrac{\log k^2 t}{|\log 1/k|} & \text{if} \quad \dfrac{1}{k^2} \leq t \leq \dfrac{1}{k}, \\ 1 & \text{if} \quad t \geq \dfrac{1}{k}. \end{cases} \tag{A.1}$$

We wish now to approximate the constant function 1 with respect to the $H^{1/2}(\Omega)$-norm. The general strategy is to build an approximation sequence with χ_k together with a partition of unity. Before going further in our analysis, we need first of all a *one*-dimensional approximation argument.

Lemma A.1. *We have*

$$\chi_k \to 1 \qquad \text{in } H^{1/2}(\mathbb{R}_+) \text{ as } k \to \infty. \tag{A.2}$$

Proof. Clearly, by definition $\chi_k \to 1$ a.e. in $\mathbb{R}_+$. The goal is to show that

$$\|\chi_k - 1\|_{H^{1/2}(\mathbb{R}_+)} \to 0 \qquad \text{as } k \to \infty. \tag{A.3}$$

We start by proving that

$$\|\chi_k - 1\|_{L^2(\mathbb{R}_+)} \to 0 \qquad \text{as } k \to \infty. \tag{A.4}$$

We have

$$\begin{aligned}
\|\chi_k - 1\|^2_{L^2(\mathbb{R}_+)} &= \int_0^\infty (\chi_k - 1)^2 \, dt = \int_0^{1/k^2} (\chi_k - 1)^2 \, dt + \int_{1/k^2}^{1/k} (\chi_k - 1)^2 \, dt \\
&= \frac{1}{k^2} + \int_{1/k^2}^{1/k} \Big(\frac{\log k^2 t}{\log k} - 1\Big)^2 \, dt = \frac{1}{k^2} + \frac{1}{k^2} \int_1^k \Big(\frac{\log t}{\log k} - 1\Big)^2 \, dt \\
&= \frac{1}{k^2} + \frac{1}{k \log^2 k} \int_{1/k}^1 \log^2 t \, dt \\
&= \frac{1}{k^2} + \frac{1}{k^2 \log^2 k}\Big(2 - \frac{\log^2 k}{k} - \frac{2\log k}{k} - \frac{2}{k}\Big).
\end{aligned}$$

From the estimate above, (A.4) follows.

Next, we also prove that

$$[\chi_k - 1]_{H^{1/2}(\mathbb{R}_+)} \to 0 \qquad \text{as } k \to \infty. \tag{A.5}$$

We have

$$\begin{aligned}
[\chi_k - 1]^2_{H^{1/2}(\mathbb{R}_+)} &= \frac{c_{1,1/2}}{2} \int_0^\infty \int_0^\infty \frac{(\chi_k(x) - \chi_k(y))^2}{(x-y)^2} \, dxdy \\
&= c\bigg(\int_0^{1/k} \int_0^{1/k} \cdots + 2 \int_0^{1/k} \int_{1/k}^\infty \cdots + \int_{1/k}^\infty \int_{1/k}^\infty \cdots \bigg) \\
&\quad \times \frac{(\chi_k(x) - \chi_k(y))^2}{(x-y)^2} \, dxdy.
\end{aligned}$$

Since $\chi_k(x) = \chi_k(y) = 1$ for $(x, y) \in (1/k, \infty) \times (1/k, \infty)$ then the third integral in the above equality vanishes. Therefore,

$$[\chi_k - 1]^2_{H^{1/2}(\mathbb{R}_+)} = c \int_0^\infty \int_0^\infty \frac{(\chi_k(x) - \chi_k(y))^2}{(x-y)^2} \, dxdy = c(I_k + J_k)$$

where

$$\begin{aligned}
I_k &:= \int_0^{1/k} \int_0^{1/k} \frac{(\chi_k(x) - \chi_k(y))^2}{(x-y)^2} \, dxdy \quad \text{and} \quad J_k \\
&:= 2 \int_0^{1/k} \int_{1/k}^\infty \frac{(\chi_k(x) - \chi_k(y))^2}{(x-y)^2} \, dxdy.
\end{aligned}$$

Estimate of J_k. We have

$$\begin{aligned}
&\int_0^{1/k} \int_{1/k}^\infty \frac{(\chi_k(x) - \chi_k(y))^2}{(x-y)^2} \, dxdy \\
&= \bigg(\int_0^{1/k^2} \int_{1/k}^\infty \cdots + \int_{1/k^2}^{1/k} \int_{1/k}^\infty \cdots \bigg) \frac{(\chi_k(x) - \chi_k(y))^2}{(x-y)^2} \, dxdy \\
&= J_k^1 + J_k^2
\end{aligned}$$

where

$$J_k^1 := \int_0^{1/k^2} \int_{1/k}^{\infty} \frac{(\chi_k(x) - \chi_k(y))^2}{(x-y)^2}\, dxdy \quad \text{and}$$
$$J_k^2 := \int_{1/k^2}^{1/k} \int_{1/k}^{\infty} \frac{(\chi_k(x) - \chi_k(y))^2}{(x-y)^2}\, dxdy.$$

Regarding J_k^1, we have from the definition of χ_k that

$$\begin{aligned} J_k^1 &= \int_0^{1/k^2} \int_{1/k}^{\infty} \frac{1}{(x-y)^2}\, dxdy \overset{\tau=\frac{x}{y}}{=} \int_0^{1/k^2} \frac{1}{y} \int_{1/ky}^{\infty} \frac{1}{(\tau-1)^2}\, d\tau dy \\ &= \int_0^{1/k^2} \frac{k}{1-ky}\, dy = -\log\Big(1 - \frac{1}{k}\Big). \end{aligned} \tag{A.6}$$

For J_k^2, we also use the definition of χ_k to see that

$$\begin{aligned} J_k^2 &= \int_{1/k^2}^{1/k} \int_{1/k}^{\infty} \frac{\left(1 - \frac{\log k^2 x}{\log k}\right)^2}{(x-y)^2}\, dxdy = \frac{1}{\log^2 k} \int_{1/k^2}^{1/k} \int_{1/k}^{\infty} \frac{(\log k - \log k^2 x)^2}{(x-y)}\, dxdy \\ &= \frac{1}{\log^2 k} \int_{1/k^2}^{1/k} \int_{1/k}^{\infty} \frac{(\log kx)^2}{(x-y)^2}\, dxdy \overset{\substack{\tau=kx \\ t=ky}}{=} \frac{1}{\log^2 k} \int_{1/k}^{1} \int_1^{\infty} \frac{\log^2 \tau}{(\tau - t)^2}\, d\tau dt \\ &= \frac{1}{\log^2 k} \int_1^{\infty} \Big(\frac{1}{(\tau - \frac{1}{k})} - \frac{1}{(\tau - 1)} \Big) \log^2 \tau\, d\tau \\ &= \frac{1}{\log^2 k} \int_1^{\infty} \frac{\frac{1}{k} - 1}{(\tau - \frac{1}{k})(\tau - 1)} \log^2 \tau\, d\tau. \end{aligned} \tag{A.7}$$

Using that $\log \tau \sim \tau - 1$ as $\tau \to 1$ and $\frac{\log^2 \tau}{(\tau - \frac{1}{k})(\tau - 1)} \sim \frac{\log^2 \tau}{\tau^2} \leq \frac{c}{\tau^{2-\varepsilon}}$ as $\tau \to \infty$, for every $\varepsilon > 0$, then the above integral is convergence for k sufficiently large. This implies that

$$J_k^2 = o(1) \qquad \text{as } k \to \infty. \tag{A.8}$$

Combining (A.6) and (A.7), and by using (A.8), we find that

$$\begin{aligned} J_k &= 2\Bigg(-\log\Big(1 - \frac{1}{k}\Big) + \frac{1}{\log^2 k} \int_1^{\infty} \frac{\frac{1}{k} - 1}{(\tau - \frac{1}{k})(\tau - 1)} \log^2 \tau\, d\tau \Bigg) \\ &\to 0 \qquad \text{as } k \to \infty. \end{aligned} \tag{A.9}$$

Estimate of I_k. We have

$$\begin{aligned} I_k &= \Bigg(\int_0^{1/k^2} \int_0^{2/k^2} \cdots + \int_0^{1/k^2} \int_{2/k^2}^{1/k} \cdots \\ &\quad + \int_{1/k^2}^{1/k} \int_0^{2/k^2} \cdots + \int_{1/k^2}^{1/k} \int_{2/k^2}^{1/k} \cdots \Bigg) \frac{(\chi_k(x) - \chi_k(y))^2}{(x-y)^2}\, dxdy \\ &= I_k^1 + I_k^2 + I_k^3 \end{aligned}$$

where

$$I_k^1 := \int_0^{1/k^2} \int_0^{2/k^2} \frac{(\chi_k(x) - \chi_k(y))^2}{(x-y)^2}\, dxdy,$$
$$I_k^2 := \int_{1/k^2}^{1/k} \int_{2/k^2}^{1/k} \frac{(\chi_k(x) - \chi_k(y))^2}{(x-y)^2}\, dxdy$$

and

$$I_k^3 := \left(\int_0^{1/k^2} \int_{2/k^2}^{1/k} \cdots + \int_{1/k^2}^{1/k} \int_0^{2/k^2} \cdots \right) \frac{(\chi_k(x) - \chi_k(y))^2}{(x-y)^2}\, dxdy.$$

It now suffices to estimate I_k^1, I_k^2 and I_k^3.
Concerning I_k^1, we have

$$\begin{aligned} I_k^1 &= \int_0^{1/k^2} \int_{1/k^2}^{2/k^2} \frac{\chi_k(x)^2}{(x-y)^2}\, dxdy = \frac{1}{\log^2 k} \int_0^{1/k^2} \int_{1/k^2}^{2/k^2} \frac{(\log k^2 x)^2}{(x-y)^2}\, dxdy \\ &\overset{\substack{\tau = k^2 x \\ t = k^2 y}}{=} \frac{1}{\log^2 k} \int_0^1 \int_1^2 \frac{\log^2 \tau}{(\tau - t)^2}\, d\tau dt = \frac{1}{\log^2 k} \int_0^1 \int_1^2 \frac{(\log \tau - \log 1)^2}{(\tau - t)^2}\, d\tau dt \\ &\le \frac{c}{\log^2 k} \int_0^1 \int_1^2 \frac{(\tau - 1)^2}{(\tau - t)^2}\, d\tau dt = \frac{c}{\log^2 k} \int_1^2 \int_0^1 \frac{(\tau - 1)^2}{(\tau - t)^2}\, dt d\tau \\ &= \frac{c}{\log^2 k} \int_1^2 (\tau - 1)^2 \Big(\frac{1}{\tau - 1} - \frac{1}{\tau} \Big) = \frac{c'}{\log^2 k}. \end{aligned} \tag{A.10}$$

Next, as regards I_k^2, the change of variables $\tau = k^2 x$ and $t = k^2 y$ gives

$$\begin{aligned} I_k^2 &= \int_{1/k^2}^{1/k} \int_{2/k^2}^{1/k} \frac{(\log k^2 x - \log k^2 y)^2}{(x-y)^2}\, dxdy = \frac{1}{\log^2 k} \int_1^k \int_2^k \frac{(\log \tau - \log t)^2}{(\tau - t)^2}\, d\tau dt \\ &= \frac{1}{\log^2 k} \int_1^k \int_2^k \frac{(\log(\tau/t))^2}{(\tau - t)^2}\, d\tau dt \overset{r = \tau/t}{=} \frac{1}{\log^2 k} \int_1^k \frac{1}{t} \int_{2/t}^{k/t} \frac{\log^2 r}{(r-1)^2}\, drdt \\ &\le \frac{1}{\log^2 k} \int_1^k \frac{dt}{t} \int_0^\infty \frac{\log^2 r}{(r-1)^2}\, dr = \frac{c}{\log k}. \end{aligned} \tag{A.11}$$

For I_k^3, we have

$$\begin{aligned} I_k^3 &\le 2 \int_0^{2/k^2} \int_{1/k^2}^{1/k} \frac{(\chi_k(x) - \chi_k(y))^2}{(x-y)^2}\, dxdy \\ &= 2 \int_0^{2/k^2} \int_{1/k^2}^{1/k} \frac{(\chi_k(x) - \chi_k(y))^2}{(x-y)^2}\, dxdy \\ &= 2 \int_0^{1/k^2} \int_{1/k^2}^{1/k} \frac{(\chi_k(x) - \chi_k(y))^2}{(x-y)^2}\, dxdy \\ &\quad + 2 \int_{1/k^2}^{2/k^2} \int_{1/k^2}^{1/k} \frac{(\chi_k(x) - \chi_k(y))^2}{(x-y)^2}\, dxdy. \end{aligned}$$

Now,

$$\int_0^{1/k^2}\int_{1/k^2}^{1/k}\frac{(\chi_k(x)-\chi_k(y))^2}{(x-y)^2}\,dxdy=\frac{1}{\log^2 k}\int_0^{1/k^2}\int_{1/k^2}^{1/k}\frac{(\log k^2x)^2}{(x-y)^2}\,dxdy$$

$$\overset{\substack{\tau=k^2x\\ t=k^2y}}{=}\frac{1}{\log^2 k}\int_0^1\int_1^k\frac{\log^2\tau}{(\tau-t)^2}\,d\tau dt=\frac{1}{\log^2 k}\int_1^k\Big(\frac{1}{(\tau-1)^2}-\frac{1}{\tau^2}\Big)\log^2\tau\,d\tau$$

$$\leq\frac{1}{\log^2 k}\int_1^\infty\Big(\frac{1}{(\tau-1)^2}-\frac{1}{\tau^2}\Big)\log^2\tau\,d\tau=\frac{c}{\log^2 k}. \tag{A.12}$$

Arguing as in the case of I_k^2, we have that

$$\int_{1/k^2}^{2/k^2}\int_{1/k^2}^{1/k}\frac{(\chi_k(x)-\chi_k(y))^2}{(x-y)^2}\,dxdy=\frac{1}{\log^2 k}\int_1^k\int_1^2\frac{(\log t-\log\tau)^2}{(t-\tau)^2}\,dtd\tau$$

$$\overset{r=t/\tau}{=}\frac{1}{\log^2 k}\int_1^k\frac{d\tau}{\tau}\int_{1/\tau}^{2/\tau}\frac{\log^2 r}{(r-1)^2}\,dr\leq\frac{1}{\log^2 k}\int_1^k\frac{d\tau}{\tau}\int_1^\infty\frac{\log^2 r}{(r-1)^2}\,dr$$

$$=\frac{c}{\log k}. \tag{A.13}$$

Putting together (A.10), (A.11), (A.12) and (A.13), we find that

$$I_k\leq\frac{c}{\log^2 k}+\frac{c}{\log k}\to 0\qquad\text{as } k\to\infty. \tag{A.14}$$

From (A.9) and (A.14), we conclude that

$$[\chi_k-1]_{H^{1/2}(\mathbb{R}_+)}\to 0\qquad\text{as } k\to\infty. \tag{A.15}$$

Now, (A.3) follows by combining (A.4) and (A.15). As wanted. □

Definition A.2. We say that an open subset Ω of $\mathbb{R}^N$ is Lipschitz if for each $q\in\partial\Omega$, there exist a tangent hyperplane H_q, a normal N_q of H_q, $r_q>0$, open r_q-balls $B_{r_q}\subset H_q$ and a function $\Phi_q:B_{r_q}\times I\to\mathbb{R}^N$ such that

(i) $\Phi_q(B_{r_q}\cap H_q^+)\subset\Omega$

(ii) $\Phi_q(B_{r_q}\cap\partial H_q^+)\subset\partial\Omega$

(iii) $C^{-1}|x-y|\leq|\Phi_q(x)-\Phi_q(y)|\leq C|x-y|,\ C>1,\ x,y\in B_{r_q}\times I,\ I\subset\mathbb{R}$.

Here, H_q^+ is the upper half-tangent hyperplane containing N_q. Put $Q_q:=B_{r_q}\times(-r_q,r_q)$ and we recall that B_{r_q} is a $(N-1)$-ball.

Remark A.3. We would like to make the following observation. It is well-known that a domain Ω is said to be *strongly* Lipschitz if its boundary can be seen as a local graph of a Lipschitz function $\phi:\mathbb{R}^{N-1}\to\mathbb{R}$. Moreover, by mean of a vectorfield η (with $|\eta|=1$ on $\partial\Omega$) which is globally transversal [2] to $\partial\Omega$, one can construct a bi-Lipschitz mapping via ϕ. In particular, Ω fulfills properties (i)-(iii). However, every Lipschitz domain in the sense of definition (i)-(iii) is not necessarily a local graph of a Lipschitz function, see [15] for a counterexample.

[2] η is said to be globally tranversal to $\partial\Omega$ if there is $\kappa>0$ such that $\eta\cdot\nu\geq\kappa$ a.e. on $\partial\Omega$. Here ν is the unit normal vector to $\partial\Omega$.

Clearly, there exists $\beta > 0$ such that

$$\overline{\Omega_\beta} := \{0 \le \delta_\Omega(x) \le \beta\} \subset \cup_{q\in\partial\Omega}\Phi_q(Q_q). \tag{A.16}$$

We recall that Ω_β is the so-called *inner tubular neighbourhood* of Ω. By compactness, there exists $m \in \mathbb{N}$ such that

$$\overline{\Omega_\beta} := \{0 \le \delta_\Omega(x) \le \beta\} \subset \cup_{j=1}^{m}\Phi_{q_j}(Q_{q_j}). \tag{A.17}$$

We will write j in the place of q_j provided there is no ambiguity. For $j = 1,\dots,m$, let u_k^j be a sequence define by

$$u_k^j(\Phi_j(x)) = \chi_k(x_N),\ \forall x \in Q_j,$$

where χ_k is defined in (A.1). Equivalently, u_k^j can be defined as

$$u_k^j(x) = \chi_k(\Phi_j^{-1}(x)\cdot N_j),\ \forall x \in \Omega. \tag{A.18}$$

Define $\mathcal{O}_j := \Phi_j(Q_j)$ and $\mathcal{O}_{m+1} = \Omega \setminus \overline{\Omega_\beta}$. We also write $Q_j^+ := B_{r_j} \times (0, r_j)$.

We have the following.

Lemma A.4. *For all $j = 1,\dots,m$ there exists a positive constant $C > 0$ depending only on j, m, Ω and N such that*

$$\|u_k^j - \mathbb{1}_\Omega\|_{H^{1/2}(\mathcal{O}_j\cap\Omega)} \le C\|\chi_k - 1\|_{H^{1/2}(0,r_j)}. \tag{A.19}$$

Proof. For $j = 1,\dots,m$, by using the change of variables $x = \Phi_j(z)$ and $y = \Phi_j(\overline{z})$, we get

$$\begin{aligned}
&\int_{\mathcal{O}_j\cap\Omega}\int_{\mathcal{O}_j\cap\Omega}\frac{(u_k(x)-u_k(y))^2}{|x-y|^{N+1}}\,dxdy\\
&\quad= \int_{Q_j^+}\int_{Q_j^+}\frac{(u_k(\Phi_j(z))-u_k(\Phi_j(\overline{z})))^2}{|\Phi_j(z)-\Phi_j(\overline{z})|^{N+1}}\,dzd\overline{z}\\
&\quad= \int_{Q_j^+}\int_{Q_j^+}\frac{(\chi_k(z_N)-\chi_k(\overline{z}_N))^2}{|\Phi_j(z)-\Phi_j(\overline{z})|^{N+1}}\,dzd\overline{z}\\
&\quad\le C\int_{Q_j^+}\int_{Q_j^+}\frac{(\chi_k(z_N)-\chi_k(\overline{z}_N))^2}{|z-\overline{z}|^{N+1}}\,dzd\overline{z}\\
&\quad\le C\int_{B_{r_j}}\int_{B_{r_j}}\int_0^{r_j}\int_0^{r_j}\frac{(\chi_k(z_N)-\chi_k(\overline{z}_N))^2}{|z-\overline{z}|^{N+1}}\,dzd\overline{z}\\
&\quad\le C\int_{B_{r_j}}dz'\int_{H_j}d\overline{z}'\int_0^{r_j}\int_0^{r_j}\frac{(\chi_k(z_N)-\chi_k(\overline{z}_N))^2}{(|z'-\overline{z}'|^2+|z_N-\overline{z}_N|^2)^{\frac{N+1}{2}}}\,dz_Nd\overline{z}_N.
\end{aligned} \tag{A.20}$$

By translation and rotation, we have

$$\begin{aligned}&\int_{B_{r_j}} dz' \int_{H_j} d\overline{z}' \int_0^{r_j}\int_0^{r_j} \frac{(\chi_k(z_N)-\chi_k(\overline{z}_N))^2}{(|z'-\overline{z}'|^2+|z_N-\overline{z}_N|^2)^{\frac{N+1}{2}}}\, dz_N d\overline{z}_N\\ &\quad = \int_{B_{r_j}} dz' \int_{\mathbb{R}^{N-1}} d\overline{z}' \int_0^{r_j}\int_0^{r_j} \frac{(\chi_k(z_N)-\chi_k(\overline{z}_N))^2}{(|z'-\overline{z}'|^2+|z_N-\overline{z}_N|^2)^{\frac{N+1}{2}}}\, dz_N d\overline{z}_N\\ &\quad \le CA \int_0^{r_j}\int_0^{r_j} \frac{(\chi_k(z_N)-\chi_k(\overline{z}_N))^2}{|z_N-\overline{z}_N|^2}\, dz_N d\overline{z}_N,\end{aligned}$$

where $A=\int_{\mathbb{R}^{N-1}}\frac{dl}{(1+|l|^2)^{(N+1)/2}}\le C$ and B_{r_j} is a bounded open subset of $\mathbb{R}^{N-1}$. Therefore, since the estimate of the L^2 norm follows easily, this and (A.20) give (A.19), concluding the proof. □

Consider $0\le\psi_j\in C_c^\infty(\mathcal{O}_j)$ a partitioning of unity subordinated to $\{\mathcal{O}_j\}_{j=1,\ldots,m+1}$. Define

$$u_k := \sum_{j=1}^{m+1}\psi_j u_k^j \in C_c^{0,1}(\Omega), \tag{A.21}$$

where $u_k^{m+1}\equiv 1$ on Ω. We have the following approximation.

Lemma A.5. *There holds*

$$\|u_k-\mathbb{1}_\Omega\|_{H^{1/2}(\Omega)}\to 0 \qquad \text{as } k\to\infty. \tag{A.22}$$

Proof. We estimate

$$\begin{aligned}[u_k-\mathbb{1}_\Omega]^2_{H^{1/2}(\Omega)} &\le \left(\sum_{j=1}^{m+1}[\psi_j u_k^j-\psi_j]_{H^{1/2}(\Omega)}\right)^2 \le m\sum_{j=1}^{m}[\psi_j u_k^j-\psi_j]^2_{H^{1/2}(\Omega)}\\ &\le C\sum_{j=1}^{m}\int_{\mathcal{O}_j\cap\Omega\times\mathcal{O}_j\cap\Omega}\ldots\, dxdy + C\sum_{j=1}^{m}\int_{\Omega\setminus\mathcal{O}_j\times\Omega\cap\mathcal{O}_j}\ldots\, dxdy\\ &=: CI_1(k)+CI_2(k).\end{aligned}$$

We now estimate $I_1(k)$ and $I_2(k)$. Let us start with $I_2(k)$.

We have

$$\begin{aligned}I_2(k) &= \sum_{j=1}^{m}\int_{\Omega\setminus\mathcal{O}_j\times\Omega\cap\mathcal{O}_j}\frac{[(\psi_j u_k^j-\psi_j)(x)-(\psi_j u_k^j-\psi_j)(y)]^2}{|x-y|^{N+1}}dxdy\\ &= \sum_{j=1}^{m}\int_{\Omega\setminus\mathcal{O}_j}\frac{dx}{|x-y|^{N+1}}\int_{\Omega\cap\mathrm{Supp}\psi_j}(\psi_j u_k^j-\psi_j)(y)^2dy\\ &\le C\sum_{j=1}^{m}\mathrm{dist}(\mathrm{Supp}\psi_j,\partial\mathcal{O}_j)^{-N-1}\int_{\Omega\cap\mathcal{O}_j}\psi_j^2|u_k^j(y)-1|^2dy\\ &\le C(N)\max_{1\le j\le m}\mathrm{dist}(\mathrm{Supp}\psi_j,\partial\mathcal{O}_j)^{-N-1}\sum_{j=1}^{m}\|u_k^j-\mathbb{1}_\Omega\|^2_{L^2(\Omega\cap\mathcal{O}_j)}.\end{aligned} \tag{A.23}$$

Now regarding $I_1(k)$, we have

$$\begin{aligned}
I_1(k) &= \sum_{j=1}^{m} \int_{\mathcal{O}_j\cap\Omega}\int_{\mathcal{O}_j\cap\Omega} \frac{[\psi_j(x)(u_k^j(x)-1)-\psi_j(y)(u_k^j(y)-1)]^2}{|x-y|^{N+1}}\,dxdy \\
&= \sum_{j=1}^{m} \int_{\mathcal{O}_j\cap\Omega}\int_{\mathcal{O}_j\cap\Omega} \\
&\quad \frac{[\psi_j(x)((u_k^j(x)-1)-(u_k^j(y)-1))+(\psi_j(x)-\psi_j(y))(u_k^j(y)-1)]^2}{|x-y|^{N+1}}\,dxdy \\
&\le 2\sum_{j=1}^{m} \int_{\mathcal{O}_j\cap\Omega}\int_{\mathcal{O}_j\cap\Omega} \frac{\psi_j(x)^2[(u_k^j(x)-1)-(u_k^j(y)-1)]^2}{|x-y|^{N+1}}\,dxdy \\
&\quad + 2\sum_{j=1}^{m} \int_{\mathcal{O}_j\cap\Omega}\int_{\mathcal{O}_j\cap\Omega} \frac{(\psi_j(x)-\psi_j(y))^2(u_k^j(y)-1)^2}{|x-y|^{N+1}}\,dxdy \\
&= I_1^1(k)+I_1^2(k),
\end{aligned}$$

where

$$\begin{aligned}
I_1^1(k) &= 2\sum_{j=1}^{m} \int_{\mathcal{O}_j\cap\Omega}\int_{\mathcal{O}_j\cap\Omega} \frac{\psi_j(x)^2[(u_k^j(x)-1)-(u_k^j(y)-1)]^2}{|x-y|^{N+1}}\,dxdy \\
&\le 2\sum_{j=1}^{m} \int_{\mathcal{O}_j\cap\Omega}\int_{\mathcal{O}_j\cap\Omega} \frac{[(u_k^j(x)-1)-(u_k^j(y)-1)]^2}{|x-y|^{N+1}}dxdy \\
&\qquad \text{(since } 0\le\psi_j\le 1) \\
&= c\sum_{j=1}^{m}[u^j-\mathbb{1}_\Omega]^2_{H^{1/2}(\mathcal{O}_j\cap\Omega)}
\end{aligned} \tag{A.24}$$

and

$$I_1^2(k) = 2\sum_{j=1}^{m} \int_{\mathcal{O}_j\cap\Omega}\int_{\mathcal{O}_j\cap\Omega} \frac{(\psi_j(x)-\psi_j(y))^2(u_k^j(y)-1)^2}{|x-y|^{N+1}}\,dxdy.$$

Using that ψ_j is Lipschitz, we get

$$\begin{aligned}
&2\int_{\mathcal{O}_j\cap\Omega}\int_{\mathcal{O}_j\cap\Omega} \frac{(\psi_j(x)-\psi_j(y))^2(u_k^j(y)-1)^2}{|x-y|^{N+1}}\,dxdy \\
&\quad \le c(j)^2 \iint_{|x-y|<1} \frac{(u_k^j(y)-1)^2|x-y|^2}{|x-y|^{N+1}}\,dxdy + 8\iint_{|x-y|\ge 1} \frac{(u_k^j(y)-1)^2}{|x-y|^{N+1}}\,dxdy \\
&\quad \le \tilde{c}(j)\|u_k^j-\mathbb{1}_\Omega\|^2_{L^2(\mathcal{O}_j\cap\Omega)}
\end{aligned}$$

which implies that

$$I_1^2(k) \le \max_{1\le j\le m}\tilde{c}(j)\sum_{j=1}^{m}\|u_k^j-\mathbb{1}_\Omega\|^2_{L^2(\mathcal{O}_j\cap\Omega)}. \tag{A.25}$$

Finally, (A.23), (A.24) and (A.25) yield

$$\begin{aligned}\|u_k - \mathbb{1}_\Omega\|^2_{H^{1/2}(\Omega)} &= \|u_k - \mathbb{1}_\Omega\|^2_{L^2(\Omega)} + [u_k - \mathbb{1}_\Omega]^2_{H^{1/2}(\Omega)} \\ &\le c\sum_{j=1}^m \|u_k^j - \mathbb{1}_\Omega\|^2_{L^2(\mathcal{O}_j\cap\Omega)} + CI_1(k) + CI_2(k) \\ &= \tilde{c}\sum_{j=1}^m \|u_k^j - \mathbb{1}_\Omega\|^2_{H^{1/2}(\mathcal{O}_j\cap\Omega)} \\ &\le C(N,m)\sum_{j=1}^m \|\chi_k - 1\|^2_{H^{1/2}(0,r_j)}. \end{aligned} \tag{A.26}$$

In the latter inequality, we used Lemma A.4. Now, since from Lemma A.1 there holds $\|\chi_k - 1\|^2_{H^{1/2}(0,r_j)} \to 0$ as $k \to \infty$, we complete the proof by letting $k \to \infty$ in the inequality (A.26). □

As a direct consequence of the above approximation results, we have the following.

Proposition A.6. *Let $N \ge 2$, $s \in (0, 1/2]$ and let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Then*

$$S_{N,s}(\Omega) = 0. \tag{A.27}$$

Before proving the proposition above, we mention that our result extends to $s = 1/2$ the one obtained in [12, Lemma 16]. Below, we give the

Proof of Proposition A.6. By definition

$$S_{N,s}(\Omega) = \inf_{\substack{u\in H_0^s(\Omega)\\ u\neq 0}} \frac{Q_{N,s,\Omega}(u)}{\|u\|^2_{L^{2_s^*}(\Omega)}} = \inf_{\substack{u\in C_c^{0,1}(\Omega)\\ u\neq 0}} \frac{Q_{N,s,\Omega}(u)}{\|u\|^2_{L^{2_s^*}(\Omega)}}, \tag{A.28}$$

where $C_c^{0,1}(\Omega)$ is the space of Lipschitz functions with compact support. Now by Lemma A.5, we get

$$\begin{aligned} 0 \le S_{N,s}(\Omega) &\le \frac{Q_{N,s,\Omega}(u_k)}{\|u_k\|^2_{L^{2_s^*}(\Omega)}} \le C(N,s)\frac{Q_{N,1/2,\Omega}(u_k)}{\|u_k\|^2_{L^{2_s^*}(\Omega)}} \\ &= C(N,s)\frac{[u_k - \mathbb{1}_\Omega]_{H^{1/2}(\Omega)}}{\|u_k\|^2_{L^{2_s^*}(\Omega)}} \to 0 \text{ as } k \to \infty, \end{aligned} \tag{A.29}$$

where u_k is defined by (A.21), which satisfies $\liminf_{k\to\infty} \|u_k\|^2_{L^{2_s^*}(\Omega)} > 0$. □

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