



Gradient formula for transition semigroup corresponding to stochastic equation driven by a system of independent Lévy processes

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Abstract. Let (P_t) be the transition semigroup of the Markov family $(X^x(t))$ defined by SDE

$$dX = b(X)dt + dZ, \quad X(0) = x,$$

where $Z = (Z_1, \dots, Z_d)^*$ is a system of independent real-valued Lévy processes. Using the Malliavin calculus we establish the following gradient formula

$$\nabla P_t f(x) = \mathbb{E} f(X^x(t)) Y(t, x), \quad f \in B_b(\mathbb{R}^d),$$

where the random field Y does not depend on f . Moreover, in the important cylindrical α -stable case $\alpha \in (0, 2)$, where Z_1, \dots, Z_d are α -stable processes, we are able to prove sharp L^1 -estimates for $Y(t, x)$. Uniform estimates on $\nabla P_t f(x)$ are also given.

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1. Introduction

Let (P_t) be the transition semigroup of a Markov family $X = (X^x(t))$ on \mathbb{R}^d , that is

$$P_t f(x) = \mathbb{E} f(X^x(t)), \quad f \in B_b(\mathbb{R}^d), \quad t \geq 0, \quad x \in \mathbb{R}^d. \quad (1)$$

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In the paper X is given by the stochastic differential equation

$$dX^x(t) = b(X^x(t))dt + dZ(t), \quad X^x(0) = x \in \mathbb{R}^d, \quad (2)$$

where $b: \mathbb{R}^d \mapsto \mathbb{R}^d$ is a $C^2(\mathbb{R}^d, \mathbb{R}^d)$ and Lipschitz mapping and

$$Z(t) = (Z_1(t), \dots, Z_d(t))^*, \quad t \geq 0,$$

is a Lévy process in \mathbb{R}^d . We assume that Z_j , $j = 1, \dots, d$, are independent real-valued Lévy processes. We denote by m_j the Lévy measure of Z_j . Recall that

$$\int_{\mathbb{R}} (\xi^2 \wedge 1) m_j(d\xi) < +\infty.$$

We assume that each Z_j is of purely jump type

$$Z_j(t) = \int_0^t \int_{\{\xi \in \mathbb{R}: |\xi| > 1\}} \xi \Pi_j(ds, d\xi) + \int_0^t \int_{\{\xi \in \mathbb{R}: |\xi| \leq 1\}} \xi [\Pi_j(ds, d\xi) - ds m_j(d\xi)], \quad (3)$$

where $\Pi_j(ds, d\xi)$ is a Poisson random measure on $[0, +\infty) \times \mathbb{R}$ with intensity measure $ds m_j(d\xi)$.

The main aim of this article is to establish the following gradient formula

$$\nabla P_t f(x) = \mathbb{E} f(X^x(t)) Y(t, x), \quad f \in B_b(\mathbb{R}^d), \quad (4)$$

where the random field Y does not depend on f . The gradient formulae of such type date back to [5, 10] and are frequently called the *Bismut–Elworthy–Li formulae*. Note that [5] uses an approach based on the Girsanov transformation. On the other hand [10] introduces martingale methods to derive formulae like (4) in the Gaussian setting; this approach also works for jump diffusions with a non-degenerate Gaussian component (cf. Section 5 in [20]).

One important consequence of (4) is the strong Feller property of the semigroup (P_t) , e.g. [7–9, 18], which in particular motivates our interest in this topic. Moreover, such gradient formulae allow the Greeks computations for pay-off functions in mathematical finance, e.g. [6, 11]. In particular in [11] the authors apply the Malliavin calculus on the Wiener space to the sensitivity analysis for asset price dynamics models.

For Lévy-driven SDEs with a possibly degenerate Gaussian component, the Bismut–Elworthy–Li formula has been obtained in [21] under the assumption on the Lévy measure to have a density with respect to Lebesgue measure in \mathbb{R}^d ; see also [22, 23] for the Bismut–Elworthy–Li formula for an SDE driven by a subordinated Brownian motion. In our study, we are focused on the more difficult situation, where the noise is presented by a collection of one-dimensional Lévy processes, and thus is quite singular.

In plain words, the substantial complication of the problem in our case is that the class of the random vector fields, which are “admissible” for the noise in the sense that they allow the integration-by-parts formula, is much more restricted. Namely, in our case only the “coordinate axis differentiability directions” in \mathbb{R}^d are actually allowed, while in the case of the Lévy measure with a density there are no limitation on these directions. For the first advances

in the Malliavin calculus for Lévy noises, supported by (singular) collection of curves, we refer to [16].

In the important cylindrical α -stable case (i.e., when each Z_j is α -stable) with $\alpha \in (0, 2)$ we obtain the sharp estimate

$$\sup_{x \in \mathbb{R}^d} \mathbb{E} |Y(t, x)| \leq C_T t^{-\frac{1}{\alpha}}, \quad t \in (0, T]. \tag{5}$$

The method we use to obtain (5) seems to be of independent interest. It has two main steps. The first one is a bound for $\mathbb{E} |Y(t) - Y(t, x)|$, where $Y(t)$ corresponds to $Y(t, x)$ when $b = 0$ in (2), i.e., $X^x(t) = x + Z(t)$. The second step concerns with $\mathbb{E} |Y(t)|$ (see Sect. 8). Both steps require sharp estimates and are quite involved (see in particular Sects. 6.2 and 8). Formula (5) implies the bound ($\|\cdot\|_\infty$ stands for the supremum norm)

$$\|\nabla P_t f\|_\infty := \sup_{x \in \mathbb{R}^d} |\nabla P_t f(x)| \leq C_T t^{-\frac{1}{\alpha}} \|f\|_\infty, \quad f \in B_b(\mathbb{R}^d), \quad t \in (0, T]. \tag{6}$$

It seems that when $0 < \alpha \leq 1$ also estimate (6) is new; it cannot be obtained by a perturbation argument which is available when $\alpha > 1$. In fact we will establish (6) for any process Z with small jumps similar to α -stable process. Recall that estimates like (6) for $\alpha > 1$ hold even in some non-degenerate multiplicative cases (see Theorem 1.1 in [15]; in such result the Lipschitz case $\gamma = 1$ requires $\alpha > 1$). We expect that our approach should also work for SDEs with multiplicative cylindrical noise; such an extension is a subject of our ongoing research.

Let us mention that from the analytical point of view we are concerned with the gradient estimates of the solution to the following equation with a non-local operator

$$\begin{aligned} \frac{\partial u}{\partial t}(t, x) &= \langle b(x), \nabla u(t, x) \rangle \\ &+ \sum_{j=1}^d \int_{\mathbb{R}} \left(u(t, x + \xi e_j) - u(t, x) - \chi_{\{|\xi| \leq 1\}} \xi \frac{\partial u}{\partial x_j}(t, x) \right) m_j(d\xi), \quad t > 0, \end{aligned}$$

$u(0, x) = f(x)$, where $e_j, j = 1, \dots, d$, is the canonical basis of \mathbb{R}^d .

2. Main result

Let $Q_t f(x) = \mathbb{E} f(Z^x(t))$ be the transition semigroup corresponding to the Lévy proces $Z^x(t) = x + Z(t)$. The proof of the following theorem concerning BEL formulae for (P_t) and (Q_t) is postponed to Sect. 6.

Theorem 1. *Let $P = (P_t)$ be given by (1), (2). Assume that:*

- (i) $b \in C^2(\mathbb{R}^d, \mathbb{R}^d)$ has bounded derivatives $\frac{\partial b_i}{\partial \xi_j}, \frac{\partial^2 b_i}{\partial \xi_j \partial \xi_k}, i, j, k = 1, \dots, d$.
- (ii) There is a $\rho > 0$ such that

$$\liminf_{\varepsilon \downarrow 0} \varepsilon^\rho m_j \{|\xi| \geq \varepsilon\} \in (0, +\infty], \quad j = 1, \dots, d.$$

(iii) *There exists a $r > 0$ such that each m_j restricted to the interval $(-r, r)$ is absolutely continuous with respect to Lebesgue measure. Moreover, the density $\rho_j = \frac{dm_j}{d\xi}$ is of class $C^1((-r, r) \setminus \{0\})$ and there exists a $\kappa > 1$ such that for all j ,*

$$\int_{-r}^r |\xi|^\kappa \rho_j(\xi) d\xi < +\infty, \tag{7}$$

$$\int_{-r}^r |\xi|^{2\kappa} \left(\frac{\rho'_j(\xi)}{\rho_j(\xi)} \right)^2 \rho_j(\xi) d\xi < +\infty, \tag{8}$$

$$\int_{-r}^r |\xi|^{2\kappa-2} \rho_j(\xi) d\xi < +\infty. \tag{9}$$

Then there are integrable random fields $Y(t) = (Y_1(t), \dots, Y_d(t))$, and $Y(t, x) = (Y_1(t, x), \dots, Y_d(t, x))$, $t > 0$, $x \in \mathbb{R}^d$, such that for any $f \in B_b(\mathbb{R}^d)$, $t > 0$, $x \in \mathbb{R}^d$,

$$\nabla Q_t f(x) = \mathbb{E} f(Z^x(t)) Y(t)$$

and the Bismut–Elworthy–Li formula (4) for (P_t) holds. Moreover, for any $T > 0$ there is an independent of $t \in (0, T]$ and x constant C such that

$$\mathbb{E} (|Y(t)| + |Y(t, x)|) \leq Ct^{-\frac{\kappa}{\rho} + \frac{1}{2}}, \tag{10}$$

$$\mathbb{E} |Y(t) - Y(t, x)| \leq Ct^{-\frac{\kappa}{\rho} + \frac{3}{2}}. \tag{11}$$

Remark 1. Note that, what is expected, the rate $-\frac{\kappa}{\rho} + \frac{1}{2}$ depends only on the small jumps of Z .

Remark 2. In fact we have formulae for the fields appearing in Theorem 1. Namely,

$$Y_j(t) = \sum_{k=1}^d [A_{k,j}(t) D_k^* \mathbf{1}(t) - D_k A_{k,j}(t)], \tag{12}$$

$$Y_j(t, x) = \sum_{k=1}^d [A_{k,j}(t, x) D_k^* \mathbf{1}(t) - D_k A_{k,j}(t, x)],$$

where:

- the matrix-valued random fields $A(t) = [A_{k,j}(t)] \in M(d \times d)$ and $A(t, x) = [A_{k,j}(t, x)] \in M(d \times d)$ are given by

$$A(t) = [\mathbb{D}Z(t)]^{-1}, \tag{13}$$

$$A(t, x) = [\mathbb{D}X^x(t)]^{-1} \nabla X^x(t), \quad \mathbb{P}\text{-a.s.}$$

We note that the matrix $A(t)$ is diagonal with entries

$$A_{j,j}(t) = \left(\int_0^t \int_{\mathbb{R}} V_j(s, \xi_j) \Pi_j(ds, d\xi_j) \right)^{-1}.$$

- $\mathbb{D}Z(t)$ and $\mathbb{D}X^x(t)$ are the Malliavin derivatives (see Sect. 3 and formulae (27) and (29)) of $Z(t)$ and $X^x(t)$ respectively, with respect to the field $V = (V_1, \dots, V_d)$,

$$V_j(t, \xi) = \phi_\delta(\xi_j)\psi_\delta(t) = V_j(t, \xi_j). \tag{14}$$

Here $\psi_\delta \in C^\infty(\mathbb{R})$ and $\phi_\delta \in C^\infty(\mathbb{R} \setminus \{0\})$ are non-negative functions such that

$$\psi_\delta(z) = \begin{cases} 0 & \text{if } |z| \geq \delta, \\ 1 & \text{if } |z| \leq \frac{\delta}{2}, \end{cases} \quad \phi_\delta(z) = |z|^\kappa \psi_\delta(z), \tag{15}$$

with κ appearing in assumption (iii) of Theorem 1, and

$$\delta \in (0, r] \quad \text{small enough.}$$

- $\nabla X^x(t)$ is the derivative in probability of X^x with respect to the initial condition x ,
- $D_k^* \mathbf{1}(t)$ is the adjoint derivative operator calculated on the constant function $\mathbf{1}$, see Sect. 3, Lemma 3.

Remark 3. The fields $Y(t)$ and $Y(t, x)$ are not uniquely determined by the BEL formulae. In particular the BEL formula for (Q_t) holds with $Y(t)$ being replaced by $Y(t) + \eta(t)$, where $\eta(t)$ is any zero-mean random variable which is independent of $Z^x(t)$. Note that the conditional expectations $\mathbb{E}(Y(t)|Z^x(t))$ and $\mathbb{E}(Y(t, x)|X^x(t))$ are uniquely determined. On the other hand, $\mathbb{E}|Y(t)|$ and $\mathbb{E}|Y(t, x)|$ may depend on the choice of the fields.

Estimate (10) implies new uniform gradient estimates

$$\|\nabla P_t f\|_\infty \leq C_T t^{-\frac{\kappa}{\rho} + \frac{1}{2}} \|f\|_\infty, \quad t \in (0, T], \quad f \in B_b(\mathbb{R}^d). \tag{16}$$

Although (16) is quite general, it is not sharp in the relevant cylindrical α -stable case with $\alpha \in (0, 2)$. In such case $\rho = \alpha$ and κ is any real number satisfying $\kappa > 1 + \frac{\alpha}{2}$. Therefore we only get that for any $\varepsilon > 0$ and $T < +\infty$ there is a constant $C_{\varepsilon, T}$ such that for any $f \in B_b(\mathbb{R}^d)$,

$$\|\nabla P_t f\|_\infty \leq C_{\varepsilon, T} t^{-\frac{1}{\alpha} - \varepsilon} \|f\|_\infty, \quad t \in (0, T]. \tag{17}$$

We will improve the previous estimate in Sect. 8 by considering $\varepsilon = 0$. To this purpose we will also use the next remark.

Remark 4. Our main theorem provides also estimate (11) for $\mathbb{E}|Y(t, x) - Y(t)|$. This can be useful. Indeed if for some specific Lévy processes Z_j we have

$$\mathbb{E}|Y(t)| \leq C_T t^{-\eta}, \quad t \in (0, T] \tag{18}$$

or even if $\mathbb{E}|\mathbb{E}(Y(t)|X^x(t))| \leq C_T t^{-\eta}$ for some η such that

$$\frac{\kappa}{\rho} - \frac{3}{2} \leq \eta \leq \frac{\kappa}{\rho} - \frac{1}{2},$$

where κ verifies our assumptions, then we can improve (10) and get, for $t \in (0, T]$,

$$\mathbb{E}|Y(t, x)| \leq C'_T t^{-\eta}, \quad t \in (0, T]. \tag{19}$$

By (19) one deduces

$$\|\nabla P_t f\|_\infty \leq C'_T t^{-\eta} \|f\|_\infty, \quad t \in (0, T].$$

In particular when Z_j are independent real α -stable processes, $\alpha \in (0, 2)$, we will get in Sect. 8 the crucial estimate

$$\mathbb{E} |Y(t)| \leq C_T t^{-\frac{1}{\alpha}}, \quad t \in (0, T]. \tag{20}$$

Combining (11) with (20) we deduce in the cylindrical α -stable case

$$\mathbb{E} |Y(t, x)| \leq C'_T t^{-\frac{1}{\alpha}}, \quad t \in (0, T], \tag{21}$$

(where C'_T is independent of x and t) and the sharp gradient estimate

$$\|\nabla P_t f\|_\infty \leq C'_T t^{-\frac{1}{\alpha}} \|f\|_\infty, \quad t \in (0, T].$$

Remark 5. The time dependent case could be also considered. This is the case when the drift $b(x)$ is replaced by $b(t, x)$ (assuming that $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is Borel and verifies $|b(t, x)| \leq C(1 + |x|)$, $b(t, \cdot) \in C^2(\mathbb{R}^d, \mathbb{R}^d)$ with all spatial derivatives bounded uniformly in $t \in [0, T]$). In such situation one deals with a time dependent Markov semigroup (P_{st}) . Fixing $s \in [0, T)$ one could obtain a formula for $\nabla(P_{st}f)(x)$ with $s < t \leq T$, $f \in \mathcal{B}_b(\mathbb{R}^d)$ which generalizes (4). The strategy is basically the same as in this paper but the computations would be much more involved.

As mentioned in the introduction a difficulty of the Proof of Theorem 1 is also to show that the Malliavin derivative of the solution $\mathbb{D}(X^x(t))$ in the direction to a suitable random field V is invertible and the inverse is integrable with sufficiently large power. The idea (see the proof of our Lemma 5) is to show that $\mathbb{D}(X^x(t)) \approx \mathbb{D}Z(t)$, where $\mathbb{D}Z(t)$, is a diagonal matrix with the terms $\int_0^t \int_{\mathbb{R}} V_j(s, \xi_j) \Pi_j(ds, d\xi_j)$ on diagonal. Therefore the integrability of $(\mathbb{D}(X^x(t)))^{-1}$ follows from the known fact, see Sect. 5 that

$$\mathbb{E} \left[\int_0^t \int_{\mathbb{R}} V_j(s, \xi_j) \Pi_j(ds, d\xi_j) \right]^{-q} \leq C(q, T) t^{-\frac{\kappa q}{\rho}}, \quad \forall q \in (1, +\infty).$$

On the other hand, several technical difficulties arise in proving the sharp bounds for $\mathbb{E} |Y(t) - Y(t, x)|$ and $\mathbb{E} |Y(t)|$.

Finally, we mention that an attempt to prove (4) has been done in [4] by the martingale approach used in [21] (see, in particular, Lemma A.3 in [4]). However the BEL formula in [4] does not seem to be correct, since there is a gap in the proof, passing from formula (48) to (49) in page 1450 of [4], which consists in an undue application of the chain rule. It seems that the complication here is substantial, and it is difficult to adapt directly the approach used in [21] to the current setting, where because of singularity of the noise it is hard to guarantee invertibility of the Malliavin derivative w.r.t. one vector field. Exactly this crucial point is our reason to use a matrix-valued Malliavin derivative of the solution w.r.t. a vector-valued field $V = (V_1, \dots, V_d)$.

3. Malliavin calculus

In this section we adopt in a very direct way the classical concepts and results of Bass and Cranston [3] and Norris [17] to the case of $Z = (Z_1, \dots, Z_d)^*$ being a Lévy process in \mathbb{R}^d with independent coordinates Z_j . For more information on Malliavin calculus for jump processes we refer the reader to the book of Ishikawa [12] (see also [2] and the references therein).

We assume that $Z = (Z_1, \dots, Z_d)^*$ is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. By the Lévy–Itô decomposition

$$Z(t) = \int_0^t \int_{\mathbb{R}^d} \xi \bar{\Pi}(ds, d\xi),$$

where $\bar{\Pi}$ is the Poisson random measure on $E := [0, +\infty) \times \mathbb{R}^d$ with intensity measure $ds\mu(d\xi)$,

$$\begin{aligned} \bar{\Pi}(ds, d\xi) &:= \hat{\Pi}(ds, d\xi)\chi_{\{|\xi| \leq 1\}} + \Pi(ds, d\xi)\chi_{\{|\xi| > 1\}}, \\ \hat{\Pi}(ds, d\xi) &:= \Pi(ds, d\xi) - ds\mu(d\xi). \end{aligned}$$

Moreover, as the coordinates of Z are independent,

$$\begin{aligned} \mu(d\xi) &:= \sum_{j=1}^d \mu_j(d\xi), \\ \mu_j(d\xi) &:= \delta_0(d\xi_1) \dots \delta_0(d\xi_{j-1})m_j(d\xi_j)\delta_0(d\xi_{j+1}) \dots \delta_0(d\xi_d), \end{aligned} \tag{22}$$

where δ_0 is the Dirac δ -function, and $m_j(d\xi_j)$ is the Lévy measure of Z_j . Note that

$$\Pi(ds, d\xi) = \sum_{j=1}^d \Pi_j(ds, d\xi),$$

where Π_j are independent Poisson random measures each on $[0, +\infty) \times \mathbb{R}^d$ with the intensity measure $ds\mu_j$ (we use the same symbol as for the one-dimensional $\Pi_j(ds, d\xi)$ appearing in (3) when no confusion may arise).

Consider the filtration

$$\mathfrak{F}_t = \sigma(\Pi([0, s] \times \Gamma) : 0 \leq s \leq t, \Gamma \in \mathcal{B}(\mathbb{R}^d)), \quad t \geq 0.$$

The Poisson random measure Π can be treated as a random element in the space $\mathbb{Z}_+(E)$ of integer-valued measures on (E, \mathcal{B}) with the σ -field \mathcal{G} generated by the family of functions

$$\mathbb{Z}_+(E) \ni \nu \mapsto \nu(A) \in \{0, 1, 2, \dots, +\infty\}, \quad A \in \mathcal{B}.$$

Definition 1. Let $p \in (0, +\infty)$. We call a random variable $\Psi: \Omega \mapsto \mathbb{R}$ an L^p -functional of Π if there is a sequence of bounded measurable functions $\varphi_n: \mathbb{Z}_+(E) \mapsto \mathbb{R}$ such that

$$\lim_{n \rightarrow +\infty} \mathbb{E}|\Psi - \varphi_n(\Pi)|^p = 0. \tag{23}$$

A random variable $\Psi: \Omega \mapsto \mathbb{R}$ is called an L^0 -functional of Π if, instead of (23), the convergence in probability holds

$$\varphi_n(\Pi) \xrightarrow{(\mathbb{P})} \Psi. \tag{24}$$

The space of all L^p -functionals of Π is denoted by $L^p(\Pi)$. Note that for $p \geq 1$, $L^p(\Pi)$ is a Banach space with the norm $\|\Psi\|_{L^p(\Pi)} = (\mathbb{E} |\Psi|^p)^{1/p}$, and for $p \in (0, 1)$, $L^p(\Pi)$ is a Polish space with the metric $\rho_{L^p(\Pi)}(\Phi, \Psi) = \mathbb{E} |\Phi - \Psi|^p$.

Assume now that $V = (V_1, \dots, V_d): [0, +\infty) \times \mathbb{R}^d \mapsto \mathbb{R}^d$ is a field given by (14) and (15). The parameter δ appearing in (15) will be specified later. Define transformations $\mathcal{Q}_k^\varepsilon$, $\varepsilon > 0$ and $k = 1, \dots, d$, $\mathcal{Q}_k^\varepsilon: \mathbb{Z}_+(E) \mapsto \mathbb{Z}_+(E)$ as follows

$$\mathcal{Q}_k^\varepsilon \left(\sum_j \delta_{\tau^j, \xi^j} \right) = \sum_j \delta_{\tau^j, \xi^j + \varepsilon V_k(\tau^j, \xi_k^j) e_k},$$

where $e_k, k = 1, \dots, d$, is the canonical basis of \mathbb{R}^d .

Now let $\Psi \in L^0(\Pi)$. Write

$$\mathcal{Q}_k^\varepsilon \Psi = (\mathbb{P}) - \lim_{n \rightarrow +\infty} \varphi_n(\mathcal{Q}_k^\varepsilon(\Pi)),$$

where $\varphi_n: \mathbb{Z}_+(E) \mapsto \mathbb{R}$ are such that (24) holds true. It follows from Lemma 2 below that $\mathcal{Q}_k^\varepsilon \Psi$ is well defined, that is the limit exists and does not depend on the particular choice of an approximation sequence (φ_n) .

Definition 2. We call $\Psi \in L^0(\Pi)$, *differentiable* (with respect to the field $V = (V_1, \dots, V_d)$) if there exist limits in probability

$$D_k \Psi = (\mathbb{P}) - \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (\mathcal{Q}_k^\varepsilon(\Psi) - \Psi), \quad k = 1, \dots, d.$$

Here $D_k \Psi$ is the *Malliavin derivative* of Ψ along the direction $V_k e_k$.

If $\Psi \in L^0(\Pi)$ is differentiable then we set

$$\mathbb{D}\Psi = (D_1 \Psi, \dots, D_d \Psi).$$

The proof of the following chain rule is standard and left to the reader.

Lemma 1. Assume that Ψ_1, \dots, Ψ_m are differentiable functionals of Π . Then for any $f \in C_b^1(\mathbb{R}^m)$ the variable $f(\Psi_1, \dots, \Psi_m)$ is differentiable and

$$D_k f(\Psi_1, \dots, \Psi_m) = \sum_{j=1}^m \frac{\partial f}{\partial x_j}(\Psi_1, \dots, \Psi_m) D_k \Psi_j, \quad k = 1, \dots, d. \tag{25}$$

Let $\rho_k = \frac{dm_k}{dx}$ be the density of the Lévy measure m_k restricted to $(-r, r) \setminus \{0\} \subset \mathbb{R}$. We extend artificially ρ_k putting $\rho_k(0) = 1$. Given $\varepsilon \in [-1, 1]$ sufficiently small and $k = 1, \dots, d$, define

$$\lambda_k^\varepsilon(t, \xi_k) := \begin{cases} \left(1 + \varepsilon \frac{\partial V_k}{\partial \xi_k}(t, \xi_k)\right) \frac{\rho_k(\xi_k + \varepsilon V_k(t, \xi_k))}{\rho_k(\xi_k)} & , \text{if } \xi_k \in \left(-\frac{r}{2}, \frac{r}{2}\right) \setminus \{0\}, \\ 1 & , \text{otherwise,} \end{cases}$$

$$\Lambda_k^\varepsilon(t, \xi_k) := \lambda_k^\varepsilon(t, \xi_k) - 1 - \log \lambda_k^\varepsilon(t, \xi_k),$$

and

$$M_k^\varepsilon(t) := \exp \left\{ \int_0^t \int_{\mathbb{R}^d} \log \lambda_k^\varepsilon(s, \xi_k) \widehat{\Pi}_k(ds, d\xi) - \int_0^t \int_{\mathbb{R}^d} \Lambda_k^\varepsilon(s, \xi_k) \mu_k(d\xi) ds \right\},$$

where μ_k is defined in (22) and

$$\widehat{\Pi}_k(ds, d\xi) := \Pi_k(ds, d\xi) - ds\mu_k(d\xi).$$

Note that the set

$$\left\{ \xi_k \in \left(-\frac{r}{2}, \frac{r}{2} \right) : \xi_k + \varepsilon V_k(t, \xi_k) = 0 \right\}$$

is of Lebesgue measure zero.

We will need the following result (see e.g. [17] or [13]).

Lemma 2. *The process M_k^ε is a martingale and for all $T \geq 0$, and $m \in \mathbb{R}$, $\mathbb{E} [M_k^\varepsilon(T)]^m < +\infty$. Let $T \in (0, +\infty)$. Then, under the probability $d\mathbb{P}^\varepsilon = M_k^\varepsilon(T)d\mathbb{P}$, $\mathcal{Q}_k^\varepsilon(\Pi)$ restricted to $[0, T] \times \mathbb{R}^d$ is a Poisson random measure with intensity $\mu_k(d\xi)ds$.*

The following lemma provides an integration by parts formula for the derivative D_k . For the completeness we repeat some elements of a proof from [17].

Lemma 3. *For any $1 \leq q \leq 2$ and $t \in (0, +\infty)$, the random variable*

$$D_k^* \mathbf{1}(t) := - \int_0^t \int_{(-r,r) \times \mathbb{R}^{d-1}} \frac{\frac{\partial}{\partial \xi_k} (V_k(s, \xi_k) \rho_k(\xi_k))}{\rho_k(\xi_k)} \widehat{\Pi}_k(ds, d\xi) \tag{26}$$

is q -integrable. Assume that $p \geq 2$ and that $\Phi \in L^p(\Pi)$ is differentiable and \mathfrak{F}_t -measurable. Then $\mathbb{E} D_k \Phi = \mathbb{E} \Phi D_k^ \mathbf{1}(t)$.*

Proof. Note that the process $D_k^* \mathbf{1}(t)$ is well defined and q -integrable thanks to (8) and (9). The integrability follows from the fact, see e.g. [1], Theorem 4.4.23, or [19], Lemma 8.22, that one has

$$\mathbb{E} |D_k^* \mathbf{1}(t)|^2 \leq c \mathbb{E} \int_0^t \int_{(-r,r) \times \mathbb{R}} \left| \frac{\frac{\partial}{\partial \xi_k} (V_k(s, \xi_k) \rho_k(\xi_k))}{\rho_k(\xi_k)} \right|^2 ds \rho_k(\xi_k) d\xi_k.$$

By Lemma 2 we have

$$\frac{d}{d\varepsilon} \mathbb{E} (\mathcal{Q}_k^\varepsilon \Phi) M_k^\varepsilon(t) = 0.$$

Thus

$$0 = \mathbb{E} [D_k \Phi M_k^0(t) + \Phi R(t)] = \mathbb{E} [D_k \Phi + \Phi R(t)],$$

where

$$R(t) := \frac{d}{d\varepsilon} M_k^\varepsilon(t)|_{\varepsilon=0}.$$

Consequently, we need to show that $D_k^* \mathbf{1}(t) = -R(t)$.

Since

$$M_k^\varepsilon(t) = \exp \left\{ \int_0^t \int_{\mathbb{R}^d} \log \lambda_k^\varepsilon(s, \xi_k) \widehat{\Pi}_k(ds, d\xi) - \int_0^t \int_{(-r,r) \times \mathbb{R}^{d-1}} \Lambda_k^\varepsilon(s, \xi_k) \mu_k(d\xi) ds \right\},$$

we have

$$R(t) = \int_0^t \int_{\mathbb{R}^d} \frac{\frac{d}{d\varepsilon} \lambda_k^\varepsilon(s, \xi_k)}{\lambda_k^\varepsilon(s, \xi_k)} \Big|_{\varepsilon=0} \widehat{\Pi}_k(ds, d\xi) - \int_0^t \int_{(-r, r) \times \mathbb{R}^{d-1}} \frac{d}{d\varepsilon} \Lambda_k^\varepsilon(s, \xi_k) \Big|_{\varepsilon=0} \mu_k(d\xi) ds.$$

Finally

$$\frac{d}{d\varepsilon} \lambda_k^\varepsilon(s, \xi_k) \Big|_{\varepsilon=0} = \frac{\partial V_k}{\partial \xi_k}(s, \xi_k) + \frac{\rho'_k(\xi_k)}{\rho_k(\xi_k)} V_k(s, \xi_k) = \frac{\frac{d}{d\xi_k} (V_k(s, \xi_k) \rho_k(\xi_k))}{\rho_k(\xi_k)}.$$

□

4. Malliavin derivative of X^x

Let $X^x(t) = [X_1^x(t), \dots, X_d^x(t)]^* \in \mathbb{R}^d$ be the value of the solution at time t . We use the convention that the vectors in \mathbb{R}^d are columns, and the derivatives (gradients) are rows. Using the chain rule (see Lemma 1) it is easy to check that each of its coordinate is a differentiable functional of Π and the $d \times d$ -matrix valued process $\mathbb{D}X^x(t)$,

$$[\mathbb{D}X^x(t)]_{i,j} = D_j X_i^x(t)$$

satisfies the following random ODE

$$d\mathbb{D}X^x(t) = \nabla b(X^x(t)) \mathbb{D}X^x(t) dt + dZ^V(t), \quad \mathbb{D}X^x(0) = 0 \quad (27)$$

(cf. Section 5 in [3]) where $Z^V(t) = [Z_{ij}^V(t)]$, $t \geq 0$, is a $d \times d$ -matrix valued process

$$Z_{j,j}^V(t) := \int_0^t \int_{\mathbb{R}} V_j(s, \xi) \Pi_j(ds, d\xi) = D_j Z_j(t), \quad Z_{j,i}^V(t) = 0 \quad \text{if } i \neq j. \quad (28)$$

Note that $\int_{\mathbb{R}} |V_j(t, \xi)| m_j(d\xi) < +\infty$ thanks to (7), and therefore, the process Z^V is well defined and q -integrable for any $q \in [1, +\infty)$. The integrability follows from the so-called Kunita inequality (see [1]) and assumption (7). In fact the Kunita inequality ensures that for $q \geq 2$,

$$\mathbb{E} \left| \int_0^t \int_{\mathbb{R}} V_j(s, \xi_i) \Pi_j(ds, d\xi_j) \right|^q \leq C_q \left[\left(\int_0^t \int_{\mathbb{R}} V_j^2(s, \xi_j) ds \rho_j(\xi_j) d\xi_j \right)^{q/2} + \int_0^t \int_{\mathbb{R}} V_j^q(s, \xi_j) ds \rho_j(\xi_j) d\xi_j \right].$$

Clearly we have:

$$\mathbb{D}Z(t) = Z^V(t), \quad t \geq 0. \quad (29)$$

Let $\nabla X^x(t)$ be the derivative in probability of the solution with respect to the initial value

$$[\nabla X^x(t)]_{i,j} = \frac{\partial}{\partial x_j} X_i^x(t).$$

Note that, the process X^x might not be integrable. However, as the noise is additive and b has bounded derivatives, $\nabla X^x(t)$ exists, it is p -integrable, for any $p \geq 1$, and

$$d\nabla X^x(t) = \nabla b(X^x(t))\nabla X^x(t)dt, \quad \nabla X^x(0) = I.$$

Since b has bounded derivatives, we have the next result in which $\|\cdot\|$ is the operator norm on the space of real $d \times d$ -matrices.

Lemma 4. *For all $t \geq 0$ and $x \in \mathbb{R}^d$, $\nabla X^x(t)$ is an invertible matrix. Moreover, there is a constant C such that*

$$\|\nabla X^x(t)\| + \|(\nabla X^x(t))^{-1}\| \leq Ce^{Ct}, \quad \forall t \geq 0, \forall x \in \mathbb{R}^d.$$

Moreover, there is a constant C , possibly depending on T , such that

$$\|\nabla X^x(t) - I\| + \|(\nabla X^x(t))^{-1} - I\| \leq Ct, \quad \forall t \in [0, T], \forall x \in \mathbb{R}^d.$$

As a simple consequence of (27) and Lemma 4 we have

$$\mathbb{D}X^x(t) = \nabla X^x(t) \int_0^t (\nabla X^x(s))^{-1} dZ^V(s). \quad (30)$$

Let

$$M(t, x) := \int_0^t (\nabla X^x(s))^{-1} dZ^V(s). \quad (31)$$

Then $\mathbb{D}X^x(t) = \nabla X^x(t)M(t, x)$ and consequently the matrix valued process $A = [A_{k,j}(t, x)]$ given by (13) satisfies

$$A(t, x) = (\mathbb{D}X^x(t))^{-1} \nabla X^x(t) = (M(t, x))^{-1}. \quad (32)$$

The proof of the following lemma is moved to the next section (Sect. 5).

Lemma 5. *Assume that the parameter δ in (15) is small enough. Let $p \geq 1$. The Malliavin matrix $\mathbb{D}X^x(t)$ is invertible and p -integrable. Moreover, the matrix valued process $A = [A_{k,j}(t, x)]$ given by (13) or (32) is differentiable and p -integrable.*

5. Proof of Lemma 5

As before $\|\cdot\|$ denotes

the operator norm on the space of real $d \times d$ -matrices. Moreover for a random $d \times d$ -matrix B we set

$$\|B\|_{L^p} = (\mathbb{E} \|B\|^p)^{1/p}, \quad p \geq 1.$$

Lemma 6. *(i) For any $t > 0$, the matrix $Z^V(t)$ is invertible, \mathbb{P} -a.s.. Moreover, for any $p \geq 1$, $T > 0$, there is a constant $C = C(p, T)$ such that*

$$\|(Z^V(t))^{-1}\|_{L^p} \leq Ct^{-\frac{\kappa}{p}}, \quad t \in (0, T].$$

(ii) Assume that the parameter δ in (15) is small enough (possibly depending on the dimension d). Then the matrix $M(t, x)$ is invertible, \mathbb{P} -a.s.. Moreover, for any $p \geq 1$ and any $T > 0$, there is a constant $C = C(p, T)$ such that

$$\|A(t, x) - (Z^V(t))^{-1}\|_{L^p} \leq Ct^{-\frac{p}{\rho} + 1}, \quad t \in (0, T] \tag{33}$$

where $A(t, x) = (M(t, x))^{-1}$.

Proof. The first part of the lemma follows from Corollary 1 from Sect. 7 below. To show the second part note that

$$M(t, x) = Z^V(t) + \int_0^t R(s, x) dZ^V(s),$$

where $R(t, x) := (\nabla X^x(t))^{-1} - I$ is a random variable taking values in the space of $d \times d$ matrices. Note that

$$\begin{aligned} \left(\int_0^t R(s, x) dZ^V(s) \right)_{i,j} &= \int_0^t \int_{\mathbb{R}} R_{ij}(s, x) V_j(s, \xi_j) \Pi_j(ds, d\xi_j) \\ &= \left(\sum_{0 < s \leq t} R(s, x) \tilde{V}(s, \Delta Z(s)) \right)_{i,j}, \end{aligned}$$

with $\Delta Z(s) = Z(s) - Z(s-)$, where $\tilde{V}(s, z)$ is a diagonal matrix, $s \geq 0, z \in \mathbb{R}^d$, such that

$$(\tilde{V}(s, z))_{i,i} = V_i(s, z_i), \quad i = 1, \dots, d.$$

Moreover, \mathbb{P} -a.s., $Z^V(t) = \sum_{0 < s \leq t} \tilde{V}(s, \Delta Z(s))$ is convergent by (7) and it is also invertible. We write

$$M(t, x) = \left(I + \int_0^t R(s, x) dZ^V(s) (Z^V(t))^{-1} \right) Z^V(t). \tag{34}$$

We would like to obtain, for $\delta > 0$ small enough, $t > 0$,

$$A(t, x) = (Z^V(t))^{-1} \left(I + \int_0^t R(s, x) dZ^V(s) (Z^V(t))^{-1} \right)^{-1}. \tag{35}$$

To this purpose we consider

$$Q(t, x) = \int_0^t R(s, x) dZ^V(s) (Z^V(t))^{-1}.$$

Recall that (e_j) is the canonical basis of \mathbb{R}^d . We get for $j = 1, \dots, d$, \mathbb{P} -a.s.,

$$\begin{aligned} Q(t, x)e_j &= \sum_{0 < s \leq t} R(s, x) \tilde{V}(s, \Delta Z(s))e_j \left(\int_0^t \int_{\mathbb{R}} V_j(y, \xi_j) \Pi_j(dy, d\xi_j) \right)^{-1} \\ &= \sum_{0 < s \leq t} R(s, x) V_j(s, \Delta Z_j(s))e_j \left(\int_0^t \int_{\mathbb{R}} V_j(y, \xi_j) \Pi_j(dy, d\xi_j) \right)^{-1} \end{aligned}$$

and so

$$\begin{aligned}
 |Q(t, x)e_j| &\leq \sum_{0 < s \leq t} \|R(s, x)\| V_j(s, \Delta Z_j(s)) \left(\int_0^t \int_{\mathbb{R}} V_j(y, \xi_j) \Pi_j(dy, d\xi_j) \right)^{-1} \\
 &\leq \min(Ct, 1/2) \int_0^t \int_{\mathbb{R}} V_j(y, \xi_j) \Pi_j(dy, d\xi_j) \left(\int_0^t \int_{\mathbb{R}} V_j(s, \xi_j) \Pi_j(ds, d\xi_j) \right)^{-1} \\
 &= \min(Ct, 1/2), \tag{36}
 \end{aligned}$$

where C is independent of $x \in \mathbb{R}^d$, $t \geq 0$ and ω , \mathbb{P} -a.s. Above we used the second estimate of Lemma 4; $\|R(s, x)\| \leq Cs$. We will need also that $|Q(t, x)e_j| \leq 1/2$. To this end we have to consider δ sufficiently small. In fact we require $\delta C \leq 1/2$.

Therefore, as $Z^V(t)$ is invertible, the matrix $M(t, x)$ is invertible and $A(t, x) = (M(t, x))^{-1}$ satisfies (35). Moreover

$$A(t, x) = (Z^V(t))^{-1} + (Z^V(t))^{-1} \sum_{n=1}^{+\infty} (-1)^n (Q(t, x))^n. \tag{37}$$

Consequently, we have

$$\left\| A(t, x) - (Z^V(t))^{-1} \right\|_{L^p} \leq C_1 t \left\| (Z^V(t))^{-1} \right\|_{L^p}$$

and (33) follows. The proof is complete. □

Remark 6. We note that in the previous proof it is important to have a term like $\int_0^t R(s, x) dZ^V(s) (Z^V(t))^{-1}$ (cf. (34)). Such term can be estimated in a sharp way by $\min(Ct, 1/2)$. On the other hand, a term like $(Z^V(t))^{-1} \int_0^t R(s, x) dZ^V(s)$ would be difficult to estimate in a sharp way (we can estimate its L^2 -norm by $Ct^{-\frac{\kappa}{p} + \frac{3}{2}}$). On this respect see also the computations in Sect. 6.2.

5.1. Proof of Lemma 5

Since b has bounded derivatives of the first and second order, $\nabla X^x(t)$ and $(\nabla X^x(t))^{-1}$ are differentiable and p -integrable. Next, thanks to (9), the matrix valued process Z^V given by (28) is also differentiable, p -integrable, and

$$\begin{aligned}
 D_k Z_{k,k}^V(t) &= \frac{d}{d\varepsilon} \int_0^t \int_{\mathbb{R}} V_k(s, \xi_k + \varepsilon V_k(s, \xi_k)) \Pi_k(ds, d\xi_k) \Big|_{\varepsilon=0} \\
 &= \int_0^t \int_{\mathbb{R}} \psi_\delta^2(s) \phi_\delta(\xi_k) \phi'_\delta(\xi_k) \Pi_k(ds, d\xi_k).
 \end{aligned} \tag{38}$$

Therefore, as

$$d\mathbb{D}X^x(t) = \nabla b(X^x(t)) \mathbb{D}X^x(t) dt + dZ^V(t), \quad \mathbb{D}X^x(0) = 0,$$

b has bounded derivatives of the first and second order, and $dZ^V(t)$ is p -integrable and differentiable, we infer that $\mathbb{D}X^x(t)$ is p -integrable and differentiable. Clearly $\nabla X^x(t)$ is invertible. By Lemma 6, the matrix $M(t, x)$ given by (31) is invertible, p -integrable and differentiable. Since, (cf. (30) and (31)),

$$M(t, x) = (\nabla X^x(t))^{-1} \mathbb{D}X^x(t)$$

and, by Lemma 6, $A(t, x) := (M(t, x))^{-1}$ is p -integrable, we infer that $\mathbb{D}X^x(t)$ is invertible, and $(\mathbb{D}X^x(t))^{-1}$ is p -integrable.

We can show the differentiability of $(\mathbb{D}X^x(t))^{-1}$ or equivalently of $A(t, x)$ in a standard way based on the observation that

$$D_k(\mathbb{D}X^x(t))^{-1} = -(\mathbb{D}X^x(t))^{-1}(D_k\mathbb{D}X^x(t))(\mathbb{D}X^x(t))^{-1}. \quad \square$$

6. Proof of Theorem 1

By Lemma 5 the random field $Y(t, x)$ given by (12) is well defined and integrable. By an approximation argument, see e.g. [21], Corollary 3.1 and its proof given in Section 4.3, or [14], see also [18], Lemma 2.2 for gradient estimates, it is enough to show that for any $f \in C_b^1(\mathbb{R}^d)$ we have (4). To this end note that

$$\nabla P_t f(x) = \nabla \mathbb{E} f(X^x(t)) = \mathbb{E} \nabla f(X^x(t)) \nabla X^x(t).$$

Since, by Lemma 1,

$$\mathbb{D}f(X^x(t)) = \nabla f(X^x(t)) \mathbb{D}X^x(t),$$

and, by Lemma 5 the matrix $\mathbb{D}X^x(t)$ is invertible, we have

$$\begin{aligned} \nabla P_t f(x) &= \mathbb{E}(\mathbb{D}f(X^x(t))) [\mathbb{D}X^x(t)]^{-1} \nabla X^x(t) \\ &= \mathbb{E}(\mathbb{D}f(X^x(t))) A(t, x) = \sum_{j=1}^d \sum_{k=1}^d \mathbb{E} D_k f(X^x(t)) A_{k,j}(t, x) e_j^*, \end{aligned}$$

where $A(t, x)$ is given by (13) or equivalently by (32), and, as gradients are row vectors, e_j^* is the transpose of e_j . By the chain rule we have

$$\sum_{k=1}^d D_k f(X^x(t)) A_{k,j}(t, x) = \sum_{k=1}^d \{D_k [f(X^x(t)) A_{k,j}(t, x)] - f(X^x(t)) D_k A_{k,j}(t, x)\}.$$

Hence, by Lemma 3, we have (4) with Y given by (12). The same arguments can be applied to show the BEL formula for the Lévy semigroup.

The proof of (10) and (11) is more difficult, and it is divided into the following two parts.

6.1. Lévy case

Assume that $b \equiv 0$, that is $X^x(t) = Z^x(t)$. Let us fix a time horizon $T < +\infty$. We are proving estimate (10) for the process $Y(t)$ corresponding to the pure Lévy case.

We have, for $j = 1, \dots, d$,

$$Y_j(t) = \sum_{k=1}^d [A_{k,j}(t) D_k^* \mathbf{1}(t) - D_k A_{k,j}(t)],$$

where $A(t) = [\mathbb{D}Z^x(t)]^{-1} = [Z^V(t)]^{-1}$ and $Z^V(t)$ is a diagonal matrix defined in (28). Therefore

$$Y_j(t) = \frac{D_j^* \mathbf{1}(t)}{Z_{j,j}^V(t)} - D_j \frac{1}{Z_{j,j}^V(t)} = \frac{D_j^* \mathbf{1}(t)}{Z_{j,j}^V(t)} + \frac{D_j Z_{j,j}^V(t)}{(Z_{j,j}^V(t))^2},$$

where $D_j^* \mathbf{1}(t)$ and $D_j Z_{j,j}^V$ are given by (26) and (38), respectively. We have

$$\mathbb{E} \left| D_j^* \mathbf{1}(t) (Z_{j,j}^V(t))^{-1} \right| \leq \left(\mathbb{E} |D_j^* \mathbf{1}(t)|^2 \right)^{\frac{1}{2}} \left(\mathbb{E} |Z_{j,j}^V(t)|^{-2} \right)^{\frac{1}{2}}.$$

By Lemma 6, there is a constant C_1 such that $\mathbb{E} |Z_{j,j}^V(t)|^{-2} \leq C_1 t^{-\frac{2\kappa}{\rho}}$. Next there are constants C_2 and C_3 such that

$$\mathbb{E} |D_j^* \mathbf{1}(t)|^2 \leq C_2 \int_0^t \int_{-\delta}^{\delta} \left| \frac{\partial}{\partial \xi_j} (V_j(s, \xi_j) \rho_j(\xi_j)) \right|^2 \rho_j(\xi_j) d\xi_j ds \leq C_3 t, \quad (39)$$

where the last estimate follows from (8) and (9). Therefore there is a constant C_4 such that

$$\mathbb{E} \left| D_j^* \mathbf{1}(t) (Z_{j,j}^V(t))^{-1} \right| \leq C_4 t^{-\frac{\kappa}{\rho} + \frac{1}{2}}, \quad t \in (0, T]. \quad (40)$$

Let us observe now that

$$\begin{aligned} |D_j Z_{j,j}^V(t)| &= \left| \int_0^t \int_{\mathbb{R}} \psi_{\delta}^2(s) \phi_{\delta}(\xi_j) \phi'_{\delta}(\xi_j) \Pi_j(ds, d\xi_j) \right| \\ &\leq \left(\int_0^t \int_{\mathbb{R}} \psi_{\delta}^2(s) \phi_{\delta}^2(\xi_j) \Pi_j(ds, d\xi_j) \right)^{1/2} \left(\int_0^t \int_{\mathbb{R}} \psi_{\delta}^2(s) (\phi'_{\delta}(\xi_j))^2 \Pi_j(ds, d\xi_j) \right)^{1/2} \\ &\leq \int_0^t \int_{\mathbb{R}} \psi_{\delta}(s) \phi_{\delta}(\xi_j) \Pi_j(ds, d\xi_j) \left(\int_0^t \int_{\mathbb{R}} \psi_{\delta}^2(s) (\phi'_{\delta}(\xi_j))^2 \Pi_j(ds, d\xi_j) \right)^{1/2}; \end{aligned}$$

here in the last inequality we have used an elementary inequality

$$\sum_k x_k^2 \leq \left(\sum_k x_k \right)^2,$$

valid for any non-negative real numbers $\{x_k\}$. Thus

$$|D_j Z_{j,j}^V(t)| \leq Z_{j,j}^V(t) \left(\int_0^t \int_{\mathbb{R}} \psi_{\delta}^2(s) (\phi'_{\delta}(\xi_j))^2 \Pi_j(ds, d\xi_j) \right)^{1/2}. \quad (41)$$

Therefore, by Lemma 6,

$$\begin{aligned} \mathbb{E} \left| \frac{D_j Z_{j,j}^V(t)}{(Z_{j,j}^V(t))^2} \right| &\leq \mathbb{E} \frac{\left(\int_0^t \int_{\mathbb{R}} \psi_{\delta}^2(s) (\phi'_{\delta}(\xi_j))^2 \Pi_j(ds, d\xi_j) \right)^{1/2}}{\int_0^t \int_{\mathbb{R}} \psi_{\delta}(s) \phi_{\delta}(\xi_j) \Pi_j(ds, d\xi_j)} \\ &\leq \left(\mathbb{E} \int_0^t \int_{\mathbb{R}} \psi_{\delta}^2(s) (\phi'_{\delta}(\xi_j))^2 \Pi_j(ds, d\xi_j) \right)^{1/2} \\ &\quad \times \left(\mathbb{E} \left(\int_0^t \int_{\mathbb{R}} \psi_{\delta}(s) \phi_{\delta}(\xi_j) \Pi_j(ds, d\xi_j) \right)^{-2} \right)^{1/2} \leq C_5 t^{-\frac{\kappa}{\rho} + \frac{1}{2}}. \end{aligned} \quad (42)$$

Note that $\int_{\mathbb{R}} (\phi'_{\delta}(\xi_j))^2 m_j(d\xi_j) < +\infty$ thanks to (9). Summing up, we can find a constant C such that

$$\mathbb{E} |Y(t)| \leq C t^{-\frac{\kappa}{\rho} + \frac{1}{2}}, \quad (43)$$

which is the desired estimate. □

6.2. General case

Recall that M and $A = M^{-1}$ are given by (31) and (32), respectively. Let $T > 0$. We prove first that (for $\delta > 0$ small enough) there is a constant c such that for $t \in (0, T]$,

$$\mathbb{E} |D_k^* \mathbf{1}(t) A_{k,j}(t, x)| \leq ct^{-\frac{\kappa}{\rho} + \frac{1}{2}}, \tag{44}$$

$$\mathbb{E} \left| D_k^* \mathbf{1}(t) A_{k,j}(t, x) - D_k^* \mathbf{1}(t) (Z^V(t))_{k,j}^{-1} \right| \leq ct^{-\frac{\kappa}{\rho} + \frac{3}{2}}. \tag{45}$$

By Lemma 6 there is a constant $C > 0$ such that

$$\|A(t, x) - (Z^V(t))^{-1}\|_{L^q} \leq Ct^{-\frac{\kappa}{\rho} + 1}, \quad t \in (0, T], \quad q \geq 1.$$

Therefore, (45) follows from (39) by using the Cauchy–Schwarz inequality. Clearly (44) follows from (40) and (45).

It is much harder to evaluate L^1 -norm of the term

$$I(t, x) := - \sum_{j=1}^d \sum_{k=1}^d D_k A_{k,j}(t, x) e_j = \sum_{j=1}^d \sum_{k=1}^d [A(t, x) (D_k M(t, x)) A(t, x)]_{k,j} e_j. \tag{46}$$

Recall that $R(s, x) := (\nabla X^x(s))^{-1} - I$. Moreover,

$$M(t, x) = Z^V(t) + \int_0^t R(s, x) dZ^V(s)$$

is differentiable, p -integrable, and we have (see also (38)):

$$D_k M(t, x) = D_k Z^V(t) + \int_0^t R(s, x) dD_k Z^V(s) + \int_0^t D_k R(s, x) dZ^V(s). \tag{47}$$

We have $\|R(s, x)\| \leq C_1 s$, $s \in [0, T]$, and that there are non-negative random variables $\eta(s)$, integrable with an arbitrary power, such that, \mathbb{P} -a.s., $0 \leq \eta(s) \leq \eta(t)$, $0 \leq s \leq t \leq T$,

$$\|D_k R(s, x)\| \leq \eta(s), \quad \|\eta(s)\|_{L^2} \leq C_2 s^{\frac{3}{2}}, \quad s \in [0, T], \tag{48}$$

where C_2 is independent of s . Indeed, using that $d\nabla X^x(t) = \nabla b(X^x(t)) \nabla X^x(t) dt$, $\nabla X^x(0) = I$,

$$dR(t, x) = - [R(t, x) \nabla b(X^x(t)) + \nabla b(X^x(t))] dt, \quad R(0, x) = 0.$$

Since ∇b is bounded we have $\|R(s, x)\| \leq C_1 s$. After differentiation we obtain

$$dD_k R(t, x) = - \left[D_k R(t, x) \nabla b(X^x(t)) + (R(t, x) + 1) \sum_{i=1}^d \frac{\partial}{\partial x_i} \nabla b(X^x(t)) D_k X_i^x(t) \right] dt, \\ D_k R(0, x) = 0.$$

By (30), there is a constant C_3 such that for all $t \in [0, T]$, $\|D_k X^x(t)\| \leq C_3 \|Z^V(t)\|$. Therefore there is a constant C_4 such that

$$\|D_k R(t, x)\| \leq C_4 \int_0^t [\|D_k R(s, x)\| + \|Z^V(s)\|] ds,$$

and consequently

$$\|D_k R(t, x)\| \leq \eta(t) := C_5 \int_0^t \|Z^V(s)\| ds, \quad t \in [0, T].$$

We will show that $I(t, x)$ is a proper perturbation of the already estimated

$$I_0(t) := \sum_{j=1}^d \frac{D_j Z_{j,j}^V(t)}{(Z_{j,j}^V(t))^2} e_j = \sum_{j=1}^d \sum_{k=1}^d \left[(Z^V(t))^{-1} (D_k Z^V(t)) (Z^V(t))^{-1} \right]_{k,j} e_j. \quad (49)$$

The proof will be completed as soon as we can show there is a constant C_6 such that

$$\mathbb{E} |I(t, x) - I_0(t)| \leq C_6 t^{-\frac{\kappa}{\rho} + \frac{3}{2}}, \quad t \in (0, T]. \quad (50)$$

This will imply that

$$\mathbb{E} |I(t, x)| \leq \mathbb{E} |I(t, x) - I_0(t)| + \mathbb{E} |I_0(t)| \leq C_7 t^{-\frac{\kappa}{\rho} + \frac{1}{2}}. \quad (51)$$

Collecting (44) and (51) will give the estimate for $\mathbb{E} |Y(t, x)|$.

Let us prove (50). Recalling that $A(t, x) = (M(t, x))^{-1}$ we have to estimate

$$\|A(t, x)(D_k M(t, x))A(t, x) - (Z^V(t))^{-1}(D_k Z^V(t)) (Z^V(t))^{-1}\| \leq J_1 + J_2 + J_3,$$

where

$$\begin{aligned} J_1 &= \|A(t, x)(D_k M(t, x))[A(t, x) - (Z^V(t))^{-1}]\|, \\ J_2 &= \|A(t, x)(D_k M(t, x) - D_k Z^V(t)) (Z^V(t))^{-1}\|, \\ J_3 &= \|[A(t, x) - (Z^V(t))^{-1}] D_k Z^V(t) (Z^V(t))^{-1}\|. \end{aligned}$$

We have

$$\mathbb{E} [J_3] \leq \| [A(t, x) - (Z^V(t))^{-1}] \|_{L^2} \| D_k Z^V(t) (Z^V(t))^{-1} \|_{L^2}.$$

Using (33) we infer

$$\mathbb{E} [J_3] \leq C_8 t^{-\frac{\kappa}{\rho} + 1} \| D_k Z^V(t) (Z^V(t))^{-1} \|_{L^2}.$$

Since

$$\| D_k Z^V(t) (Z^V(t))^{-1} \| = \left| \frac{D_k Z_{k,k}^V(t)}{Z_{k,k}^V(t)} \right|,$$

we can use (41) and get

$$\mathbb{E} \| D_k Z^V(t) (Z^V(t))^{-1} \|^2 \leq \mathbb{E} \left(\int_0^t \int_{\mathbb{R}} \psi_\delta^2(s) (\phi'_\delta(\xi_k))^2 \Pi_k(ds, d\xi_k) \right) \leq C_9 t,$$

see (9). Hence we have

$$\mathbb{E} [J_3] \leq C_{10}t^{-\frac{\kappa}{\rho} + \frac{3}{2}}.$$

We evaluate now J_2 . By Lemma 6 we have

$$\begin{aligned} \mathbb{E} [J_2] &\leq \|A(t, x)\|_{L^2} \|(D_k M(t, x) - D_k Z^V(t)) (Z^V(t))^{-1}\|_{L^2} \\ &\leq Ct^{-\frac{\kappa}{\rho}} \|(D_k M(t, x) - D_k Z^V(t)) (Z^V(t))^{-1}\|_{L^2}. \end{aligned}$$

Next

$$\begin{aligned} &(D_k M(t, x) - D_k Z^V(t)) (Z^V(t))^{-1} \\ &= \left(\int_0^t R(s, x) dD_k Z^V(s) + \int_0^t D_k R(s, x) dZ^V(s) \right) (Z^V(t))^{-1}. \end{aligned} \tag{52}$$

We will argue as in the proof of Lemma 6. Note that

$$\left(\int_0^t R(s, x) dD_k Z^V(s) \right)_{ij} = 0 \quad \text{if } j \neq k.$$

Recall that, for δ small enough,

$$\begin{aligned} \left(\int_0^t R(s, x) dD_k Z^V(s) \right)_{ik} &= \int_0^t \psi_\delta^2(s) R_{ik}(s, x) \int_{\mathbb{R}} \phi_\delta(\xi_k) \phi'_\delta(\xi_k) \Pi_k(ds, d\xi_k) \\ &= \left(\sum_{0 < s \leq t} R(s, x) \tilde{U}(s, \Delta Z(s)) \right)_{ik}, \end{aligned}$$

where $\tilde{U}(s, z)$ is a diagonal matrix, $s \geq 0, z \in \mathbb{R}^d$, such that

$$(\tilde{U}(s, z))_{ii} = U_i(s, z) = \psi_\delta^2(s) \phi_\delta(z_i) \phi'_\delta(z_i), \quad i = 1, \dots, d.$$

Hence

$$\begin{aligned} &\left| \int_0^t R(s, x) dD_k Z^V(s) (Z^V(t))^{-1} e_k \right| \\ &= \left| \sum_{0 < s \leq t} R(s, x) \tilde{U}(s, \Delta Z(s)) e_k \left(\int_0^t \int_{\mathbb{R}} V_k(y, \xi_k) \Pi_k(dy, d\xi_k) \right)^{-1} \right| \\ &\leq \sum_{0 < s \leq t} \|R(s, x)\| U_k(s, \Delta Z_k(s)) \left(\int_0^t \int_{\mathbb{R}} V_k(y, \xi_k) \Pi_k(dy, d\xi_k) \right)^{-1} \\ &\leq C_1 t \sum_{0 < s \leq t} U_k(s, \Delta Z_k(s)) \left(\int_0^t \int_{\mathbb{R}} V_k(y, \xi_k) \Pi_k(dy, d\xi_k) \right)^{-1} = C_1 t \frac{D_k Z_{k,k}^V(t)}{Z_{k,k}^V(t)}, \end{aligned}$$

see (38). We deduce that

$$\left\| \int_0^t R(s, x) dD_k Z^V(s) (Z^V(t))^{-1} \right\|_{L^2} \leq C_{11} t^{\frac{3}{2}}.$$

Therefore, in order to estimate J_2 , it remains to consider

$$\int_0^t D_k R(s, x) dZ^V(s) (Z^V(t))^{-1} = \sum_{0 < s \leq t} D_k R(s, x) \tilde{V}(s, \Delta Z(s)) (Z^V(t))^{-1},$$

where $\tilde{V}(s, z)$ is a diagonal matrix, $s \geq 0$, $z \in \mathbb{R}^d$, such that $(\tilde{V}(s, z))_{ii} = V_i(s, z_i)$. Using the bound (48) we obtain, for $j = 1, \dots, d$, \mathbb{P} -a.s.,

$$\begin{aligned} & \left| \int_0^t D_k R(s, x) dZ^V(s) (Z^V(t))^{-1} e_j \right| \\ & \leq \sum_{0 < s \leq t} \|D_k R(s, x)\| V_j(s, \Delta Z_j(s)) \left(\int_0^t \int_{\mathbb{R}} V_j(r, \xi_j) \Pi_j(dr, d\xi_j) \right)^{-1} \\ & \leq \eta(t). \end{aligned}$$

It follows that

$$\left\| \int_0^t D_k R(s, x) dZ^V(s) (Z^V(t))^{-1} \right\|_{L^2} \leq C_{12} t^{\frac{3}{2}}.$$

Summing up we have

$$\mathbb{E}[J_2] \leq C_{13} t^{-\frac{\kappa}{\rho} + \frac{3}{2}}.$$

To treat J_1 we note that by Lemma 6 we have

$$\begin{aligned} \mathbb{E}[J_1] & \leq \|A(t, x)\|_{L^2} \|(D_k M(t, x))[A(t, x) - (Z^V(t))^{-1}]\|_{L^2} \\ & \leq C t^{-\frac{\kappa}{\rho}} \|(D_k M(t, x))[A(t, x) - (Z^V(t))^{-1}]\|_{L^2}. \end{aligned} \quad (53)$$

We write

$$\begin{aligned} & \|(D_k M(t, x))[A(t, x) - (Z^V(t))^{-1}]\| \\ & = \|(D_k M(t, x))(M(t, x))^{-1} M(t, x)[A(t, x) - (Z^V(t))^{-1}]\| \\ & \leq \|(D_k M(t, x))(M(t, x))^{-1}\| \|I - M(t, x) (Z^V(t))^{-1}\|. \end{aligned}$$

The more difficult term is

$$\begin{aligned} \|(D_k M(t, x))(M(t, x))^{-1}\| & = \|(D_k M(t, x)) (Z^V(t))^{-1} Z^V(t) (M(t, x))^{-1}\| \\ & \leq \|(D_k M(t, x)) (Z^V(t))^{-1}\| \|Z^V(t) (M(t, x))^{-1}\|. \end{aligned}$$

By (37) we have

$$\begin{aligned} Z^V(t) (M(t, x))^{-1} & = Z^V(t) A(t, x) \\ & = Z^V(t) \left((Z^V(t))^{-1} + (Z^V(t))^{-1} \sum_{n=1}^{+\infty} (-1)^n (Q(t, x))^n \right) \\ & = \sum_{n=0}^{+\infty} (-1)^n (Q(t, x))^n. \end{aligned}$$

Hence, by (36),

$$\|(Z^V(t) (M(t, x))^{-1})\| \leq \tilde{C}_1,$$

where \tilde{C}_1 is independent of x , $t \in (0, T]$ and ω , \mathbb{P} -a.s.. The term

$$\|(D_k M(t, x)) (Z^V(t))^{-1}\|$$

can be treated as the first term in (52). Therefore we have

$$\|(D_k M(t, x)) (Z^V(t))^{-1}\|_{L^2} \leq \tilde{C}_2 t^{\frac{1}{2}}.$$

Summing up we have

$$\|(D_k M(t, x))(M(t, x))^{-1}\|_{L^2} \leq \tilde{C}_3 t^{\frac{1}{2}}, \quad t \in (0, T].$$

Since

$$\|I - M(t, x) (Z^V(t))^{-1}\| = \left\| \int_0^t R(s, x) dZ^V(s) (Z^V(t))^{-1} \right\| \leq \tilde{C}_4 t,$$

where c_3 is independent of x and ω , \mathbb{P} -a.s., we have

$$\mathbb{E}[J_1] \leq \tilde{C}_5 t^{-\frac{\kappa}{\rho} + \frac{3}{2}},$$

and the proof is complete. □

7. An integrability result

Assume that \mathcal{M} is a Poisson random measure on $[0, +\infty) \times \mathbb{R}$ with intensity measure $dtm(d\xi)$. Given a measurable $h: \mathbb{R} \mapsto [0, +\infty)$ let

$$J_h(t) := \int_0^t \int_{\mathbb{R}} h(\xi) \mathcal{M}(ds, d\xi).$$

Then for any $\beta > 0$,

$$\mathbb{E} e^{-\beta J_h(t)} = \exp \left\{ -t \int_{\mathbb{R}} \left(1 - e^{-\beta h(\xi)} \right) m(d\xi) \right\}.$$

Using the identity

$$y^{-q} = \frac{1}{\Gamma(q)} \int_0^{+\infty} \beta^{q-1} e^{-\beta y} d\beta, \quad y > 0,$$

we obtain

$$\begin{aligned} \mathbb{E} J_h(t)^{-q} &= \frac{1}{\Gamma(q)} \int_0^{+\infty} \beta^{q-1} \mathbb{E} e^{-\beta J_h(t)} d\beta \\ &= \frac{1}{\Gamma(q)} \int_0^{+\infty} \beta^{q-1} \exp \left\{ -t \int_{\mathbb{R}} \left(1 - e^{-\beta h(\xi)} \right) m(d\xi) \right\} d\beta. \end{aligned}$$

Using this method one can obtain (see Norris [17]) the following result.

Lemma 7. *If for a certain $\rho > 0$,*

$$\liminf_{\varepsilon \downarrow 0} \varepsilon^\rho m\{h \geq \varepsilon\} > 0,$$

then

$$\mathbb{E} J_h(t)^{-q} \leq C t^{-\frac{q}{\rho}}, \quad q \geq 1, \quad t \in (0, 1].$$

Let $\phi_\delta \in C^\infty(\mathbb{R} \setminus \{0\})$ be given by (15). Assume that $m(d\xi)$ satisfies hypothesis (ii) of Theorem 1 and $h = \phi_\delta$. Then

$$\begin{aligned} \liminf_{\varepsilon \downarrow 0} \varepsilon^{\frac{\rho}{\kappa}} m\{\phi_\delta \geq \varepsilon\} &\geq \liminf_{\varepsilon \downarrow 0} \varepsilon^{\frac{\rho}{\kappa}} m\left\{\xi \in \left[-\frac{\delta}{2}, \frac{\delta}{2}\right] : |\xi|^\kappa \geq \varepsilon\right\} \\ &\geq \liminf_{\varepsilon \downarrow 0} \varepsilon^{\frac{\rho}{\kappa}} m\left\{\xi : \varepsilon^{\frac{1}{\kappa}} \leq |\xi| \leq \frac{\delta}{2}\right\} \\ &\geq \liminf_{\varepsilon \downarrow 0} \varepsilon^\rho m\left\{\xi : \varepsilon \leq |\xi| \leq \frac{\delta}{2}\right\} > 0. \end{aligned}$$

Consequently, by Lemma 7 we have the following result:

Corollary 1. *For any $q \geq 1$ there is a constant $C = C(q, T)$ such that*

$$\mathbb{E} J_{\phi_\delta}(t)^{-q} \leq Ct^{-\frac{\kappa q}{\rho}}, \quad t \in (0, T].$$

Moreover,

$$\mathbb{E} J_{\phi_\delta}(t) = t \int_{\mathbb{R}} \phi_\delta(\xi) m(d\xi) < +\infty.$$

8. Sharp estimates in the cylindrical α -stable case

Here we are concerned with rather general perturbation of α -stable case. Indeed in such case we can improve the estimate on $Y(t)$ given in Sect. 6.1. This estimate according to Remark 4 leads to the sharp gradient estimates (6).

Below in (54) we will strengthen hypotheses (8) and (9). In Remark 7 we clarify the validity of the new assumptions in the relevant cylindrical α -stable case.

Lemma 8. *Let $\alpha \in (0, 2)$. Suppose that all the assumptions of Theorem 1 hold with $\rho = \alpha$ and for some $\kappa > 1 + \alpha/2$. Moreover, suppose that, for the same κ ,*

$$\limsup_{u \rightarrow 0^+} u^{-2\kappa+2+\alpha} \int_{|\xi| \leq u} \left[|\xi|^{2\kappa} \left(\frac{\rho'_j(\xi)}{\rho_j(\xi)} \right)^2 + |\xi|^{2\kappa-2} \right] \rho_j(\xi) d\xi < +\infty, \quad (54)$$

and there exists $p \in (1, 2)$ such that

$$\limsup_{u \rightarrow 0^+} u^{-p\kappa+p+\alpha} \int_{u \leq |\xi| \leq r} \left[|\xi|^{p\kappa} \left| \frac{\rho'_j(\xi)}{\rho_j(\xi)} \right|^p + |\xi|^{p\kappa-p} \right] \rho_j(\xi) d\xi < +\infty. \quad (55)$$

Then the following estimate holds for the \mathbb{R}^d -valued process Y (cf. (43)) :

$$\mathbb{E} |Y(t)| \leq C_T t^{-\frac{1}{\alpha}}, \quad t \in (0, T]. \quad (56)$$

Remark 7. We provide a sufficient condition such that all the hypotheses of Lemma 8 hold. To this purpose recall that ρ_j is the C^1 -density of the Lévy measure m_j associated to the process Z_j ; such density exists on $(-r, r) \setminus \{0\}$, $r > 0$ as in (iii) of Theorem 1.

Moreover, $l_\alpha(\xi) := |\xi|^{-1-\alpha}$ denotes the density of the Lévy measure of a symmetric one-dimensional α -stable process, $\alpha \in (0, 2)$.

Assume that there is a positive constant c such that, for $\xi \in (-r, r) \setminus \{0\}$,

$$\left| \frac{\rho'_j(\xi)}{\rho_j(\xi)} \right| \leq c \left(|\xi|^{-1} + 1 \right) \quad \text{and} \quad c^{-1}l_\alpha(\xi) \leq \rho_j(\xi) \leq cl_\alpha(\xi), \quad (57)$$

$j = 1, \dots, d$. It is easy to check that (57) implies all the assumptions of Lemma 8 with arbitrary $\kappa \in (1 + \alpha/2, 1 + \alpha)$. Thus under condition (57) we obtain (56) and the sharp gradient estimates (6).

Proof. To prove the result we can assume $d = 1$ so that $Y_1 = Y$; Π is the associated Poisson random measure and we set $m_1 = \mu$ for the corresponding Lévy measure having C^1 -density $\rho_1 = \rho$ on $(-r, r)$.

It is enough to show (56) for small t , say $0 < t^{1/\alpha} \vee t \leq \delta/2 \leq 1$, where $\delta \leq r$ is small enough.

Note that $\phi_\delta(\xi) = |\xi|^\kappa$ for $|\xi| \leq \delta/2$. Moreover, recall that $\psi_\delta(t) = 1$ for $t \leq \delta/2$. Let us fix $\kappa = 1 + \frac{3}{4}\alpha$. We have

$$Y(t) = \frac{D^*\mathbf{1}(t)}{Z^V(t)} - D \frac{1}{Z^V(t)} = \frac{D^*\mathbf{1}(t)}{Z^V(t)} + \frac{DZ^V}{(Z^V(t))^2}.$$

We have

$$\begin{aligned} D^*\mathbf{1}(t) &= - \int_0^t \int_{(-\delta, \delta)} \frac{\phi'_\delta(\xi)\rho(\xi) + \phi_\delta(\xi)\rho'(\xi)}{\rho(\xi)} \widehat{\Pi}(ds, d\xi), \\ DZ^V(t) &= \int_0^t \int_{(-\delta, \delta)} \phi_\delta(\xi)\phi'_\delta(\xi)\Pi(ds, d\xi), \\ Z^V(t) &= \int_0^t \int_{(-\delta, \delta)} \phi_\delta(\xi)\Pi(ds, d\xi). \end{aligned}$$

We are showing that

$$\mathbb{E} \left| \frac{D^*\mathbf{1}(t)}{Z^V(t)} \right| \leq C_2 t^{-\frac{1}{\alpha}}. \quad (58)$$

We concentrate on $D^*\mathbf{1}(t)$:

$$\begin{aligned} D^*\mathbf{1}(t) &= I_1(t) + I_2(t), \quad I_1(t) = - \int_0^t \int_{\{t^{1/\alpha} < |\xi| < \delta\}} \frac{\phi'_\delta(\xi)\rho(\xi) + \phi_\delta(\xi)\rho'(\xi)}{\rho(\xi)} \widehat{\Pi}(ds, d\xi), \\ I_2(t) &= - \int_0^t \int_{\{|\xi| \leq t^{1/\alpha}\}} \frac{\phi'_\delta(\xi)\rho(\xi) + \phi_\delta(\xi)\rho'(\xi)}{\rho(\xi)} \widehat{\Pi}(ds, d\xi). \end{aligned}$$

Concerning $I_1(t)$ we can improve some estimates of Sect. 6.1; using the Hölder inequality (because ξ is separated from 0): for the given $p \in (1, 2)$ and $q: 1/p + 1/q = 1$ we have

$$\mathbb{E} \left| I_1(t) (Z^V(t))^{-1} \right| \leq (\mathbb{E} |I_1(t)|^p)^{1/p} \left(\mathbb{E} |Z^V(t)|^{-q} \right)^{1/q}.$$

By Corollary 1, there is a constant C_1 such that $\mathbb{E} |Z^V(t)|^{-q} \leq C_1 t^{-\frac{\kappa q}{\alpha}}$, recall that $\rho = \alpha$ now. Since $p \in (1, 2)$, there exists a positive constant c such that

$$\mathbb{E} |I_1(t)|^p \leq c \int_0^t \int_{\{t^{1/\alpha} < |\xi| < \delta\}} \left| \frac{\phi'_\delta(\xi)\rho(\xi) + \phi_\delta(\xi)\rho'(\xi)}{\rho(\xi)} \right|^p \rho(\xi) d\xi ds,$$

see e.g. Lemma 8.22 in [19]. Since $\phi_\delta(\xi) = |\xi|^\kappa \psi_\delta(\xi)$, it follows that for $|\xi| \leq \delta$, $|\phi'_\delta(\xi)| \leq C_\delta |\xi|^{\kappa-1}$.

We have by (55)

$$\int_{\{t^{1/\alpha} < |\xi| < \delta\}} \left| \frac{\phi'_\delta(\xi)\rho(\xi) + \phi_\delta(\xi)\rho'(\xi)}{\rho(\xi)} \right|^p \rho(\xi) d\xi \leq C_3 (t^{1/\alpha})^{p(\kappa-1)-\alpha} = C_3 t^{\frac{p}{\alpha}(\kappa-1)-1}$$

with some constant C_3 . Combined with the previous inequality, this gives

$$\mathbb{E} |I_1(t)|^p \leq ct \cdot C_3 t^{\frac{p}{\alpha}(\kappa-1)-1} = cC_3 t^{\frac{p}{\alpha}(\kappa-1)}.$$

Therefore by the Hölder inequality

$$\mathbb{E} \left| I_1(t) (Z^V(t))^{-1} \right| \leq C_1^{\frac{1}{q}} t^{-\frac{\kappa}{\alpha}} \cdot (cC_3)^{\frac{1}{p}} t^{\frac{1}{\alpha}(\kappa-1)} = C_4 t^{-\frac{1}{\alpha}}. \tag{59}$$

For $I_2(t)$, we proceed in a similar way. Namely, by the Cauchy inequality, isometry formula, Lemma 6 and using (54) we find

$$\begin{aligned} & \mathbb{E} |I_2(t)(Z^V(t))^{-1}| \\ & \leq \left(\mathbb{E} (I_2(t))^2 \right)^{\frac{1}{2}} \left(\mathbb{E} |Z^V(t)|^{-2} \right)^{\frac{1}{2}} \\ & \leq C_5 \left(\int_0^t \int_{\{|\xi| \leq t^{1/\alpha}\}} \left(\frac{\phi'_\delta(\xi)\rho(\xi) + \phi_\delta(\xi)\rho'(\xi)}{\rho(\xi)} \right)^2 \rho(\xi) d\xi \right)^{1/2} t^{-\frac{\kappa}{\alpha}} \tag{60} \\ & \leq C_6 \left(t(t^{1/\alpha})^{2(\kappa-1)-\alpha} \right)^{\frac{1}{2}} t^{-\frac{\kappa}{\alpha}} = C_6 t^{-\frac{1}{\alpha}}, \end{aligned}$$

which completes the proof of (58). Now we are showing that

$$\mathbb{E} \left| \frac{DZ^V(t)}{(Z^V(t))^2} \right| \leq C_7 t^{-\frac{1}{\alpha}}. \tag{61}$$

To this end note that

$$\begin{aligned} & \left| \int_0^t \int_{\{\delta/2 < |\xi| \leq \delta\}} \phi_\delta(\xi)\phi'_\delta(\xi)\Pi(ds, d\xi) \right| \\ & \leq \left[\int_0^t \int_{\{\delta/2 < |\xi| \leq \delta\}} \phi_\delta^2(\xi)\Pi(ds, d\xi) \right]^{1/2} \left[\int_0^t \int_{\{\delta/2 < |\xi| \leq \delta\}} (\phi'_\delta(\xi))^2 \Pi(ds, d\xi) \right]^{1/2} \\ & \leq \int_0^t \int_{\{\delta/2 < |\xi| \leq \delta\}} \phi_\delta(\xi)\Pi(ds, d\xi) \int_0^t \int_{\{\delta/2 < |\xi| \leq \delta\}} |\phi'_\delta(\xi)| \Pi(ds, d\xi) \\ & \leq Z^V(t) \int_0^t \int_{\{\delta/2 < |\xi| \leq \delta\}} |\phi'_\delta(\xi)| \Pi(ds, d\xi). \end{aligned}$$

Hence, as the arguments from the derivation of (59) (recall that for $|\xi| \leq \delta$, $|\phi'_\delta(\xi)| \leq C_\delta |\xi|^{k-1}$) we obtain

$$\begin{aligned} \mathbb{E} \left| \frac{\int_0^t \int_{\{\delta/2 < |\xi| < \delta\}} \phi_\delta(\xi) \phi'_\delta(\xi) \Pi(ds, d\xi)}{(Z^V(t))^2} \right| &\leq \mathbb{E} \frac{\int_0^t \int_{\{t^{1/\alpha} < |\xi| < \delta\}} |\phi'_\delta(\xi)| \Pi(ds, d\xi)}{Z^V(t)} \\ &\leq cC_3 C t^{\frac{1}{\alpha}(\kappa-1)} t^{-\frac{\kappa}{\alpha}} = C_9 t^{-\frac{1}{\alpha}}. \end{aligned}$$

Set

$$K(t) := (Z^V(t))^{-2} \int_0^t \int_{\{\delta/2 \geq |\xi| > t^{1/\alpha}\}} \phi'_\delta(\xi) \phi_\delta(\xi) \Pi(ds, d\xi)$$

and

$$H(t) := (Z^V(t))^{-2} \int_0^t \int_{\{|\xi| \leq t^{1/\alpha}\}} \phi'_\delta(\xi) \phi_\delta(\xi) \Pi(ds, d\xi).$$

Since $\phi_\delta(\xi) = |\xi|^\kappa$ if $|\xi| \leq \delta/2$, we have

$$\begin{aligned} |K(t)| &\leq \kappa (Z^V(t))^{-2} \int_0^t \int_{\{\delta/2 \geq |\xi| > t^{1/\alpha}\}} \phi_\delta^2(\xi) |\xi|^{-1} \Pi(ds, d\xi) \\ &\leq \kappa (Z^V(t))^{-2} t^{-\frac{1}{\alpha}} \int_0^t \int_{\mathbb{R}} \phi_\delta^2(\xi) \Pi(ds, d\xi) \\ &= \kappa (Z^V(t))^{-2} t^{-\frac{1}{\alpha}} \left[\left(\int_0^t \int_{\mathbb{R}} \phi_\delta^2(\xi) \Pi(ds, d\xi) \right)^{1/2} \right]^2 \\ &\leq \kappa (Z^V(t))^{-2} t^{-\frac{1}{\alpha}} \left[\int_0^t \int_{\mathbb{R}} \phi_\delta(\xi) \Pi(ds, d\xi) \right]^2 = \kappa t^{-\frac{1}{\alpha}}. \end{aligned}$$

We are dealing now with $H(t)$. Since

$$\begin{aligned} &\int_0^t \int_{\{|\xi| \leq t^{1/\alpha}\}} |\phi'_\delta(\xi) \phi_\delta(\xi)| \Pi(ds, d\xi) \\ &\leq \left(\int_0^t \int_{\{|\xi| \leq t^{1/\alpha}\}} (\phi'_\delta(\xi))^2 \Pi(ds, d\xi) \right)^{1/2} \left(\int_0^t \int_{\mathbb{R}} \phi_\delta^2(\xi) \Pi(ds, d\xi) \right)^{1/2} \\ &\leq Z^V(t) \int_0^t \int_{\{|\xi| \leq t^{1/\alpha}\}} |\phi'_\delta(\xi)| \Pi(ds, d\xi) = \kappa Z^V(t) \int_0^t \int_{\{|\xi| \leq t^{1/\alpha}\}} |\xi|^{\kappa-1} \Pi(ds, d\xi), \end{aligned}$$

we have, arguing as in (60), using again (54),

$$\mathbb{E} |H(t)| \leq \kappa \mathbb{E} \frac{\int_0^t \int_{\{|\xi| \leq t^{1/\alpha}\}} |\xi|^{\kappa-1} \Pi(ds, d\xi)}{Z^V(t)} \leq C_{10} t^{-\frac{1}{\alpha}},$$

which finishes the proof of (61). □

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