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Global attractor for 3D Dirac equation with nonlinear point interaction

Elena Kopylova

Abstract. We prove global attraction to stationary orbits for 3D Dirac equation with concentrated nonlinearity. We show that each finite energy solution converges as $t \to \pm\infty$ to the set of four-frequency "nonlinear eigenfunctions". The global attraction is caused by nonlinear energy transfer from lower harmonics to the continuous spectrum and subsequent dispersion radiation.

Mathematics Subject Classification. 35B40, 35B41.

1. Introduction

In the last decades equations with point interactions became an intensively developing field of research, and this interest is driven by the possibility of investigating nonlinear problems in the context of solvable models. These equations are useful mathematical tool for modeling many phenomena in theoretical physics, (see introduction in [11]).

The first rigorous mathematical results for equations with point interaction were obtained since 1960 by F. Berezin, L. Faddeev, D. Yafaev, E. Zeidler and others [6,16,37], and since 2000 by S. Albeverio, R. Høegh-Krohn, D. Noja, D. Yafaev and others [2,4,5,32,38]. A comprehensive overview of the results can be found in [3,15].

Our paper concerns 3D Dirac equation with nonlinear point interaction. Namely, we consider the system governed by the following equations

$$\left\{\begin{array}{c} i\dot\psi(x,t)=D_m\psi(x,t)-D_m^{-1}\zeta(t)\delta(x)\\ \lim\limits_{\varepsilon\to 0+}\lim\limits_{x\to 0}K_m^\varepsilon\Big(\psi(x,t)-\zeta(t)g(x)\Big)=F(\zeta(t))\end{array}\right| \quad x\in\mathbb{R}^3,\quad t\in\mathbb{R}. \tag{1.1}$$

Here D_m is the Dirac operator $D_m := -i\alpha\cdot\nabla + m\beta$, where $m > 0$, α_k with $k = 1, 2, 3$ and β are 4×4 Dirac matrices; $\psi(x,t)$, $\zeta(t)$ are vector functions

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with values in $\mathbb{C}^4$; $g(x)$ is the Green function of the operator $D_m^2 = -\Delta + m^2$ in $\mathbb{R}^3$,

$$g(x) := \frac{e^{-m|x|}}{4\pi|x|}, \tag{1.2}$$

and $K_m^\varepsilon = (-\Delta + m^2)^{-\varepsilon}$ is a smoothing operator, defined as and $K_m^\varepsilon = (-\Delta + m^2)^{-\varepsilon}$ is a smoothing operator, defined as

$$(K_m^\varepsilon \psi)(x) = \frac{1}{(4\pi)^3} \int e^{-i\xi\cdot x} \frac{\widehat{\psi}(\xi)\, d\xi}{(\xi^2 + m^2)^\varepsilon}, \quad \varepsilon \geq 0,$$

where $\widehat{\psi}(\xi)$ is the Fourier transform of $\psi(x)$. Obviously, $(K_m^\varepsilon\psi)(x) \to \psi(x)$ as $\varepsilon \to 0$ in $H^s(\mathbb{R}^3)$ for any $\psi \in H^s(\mathbb{R}^3)$ and any $s \in \mathbb{R}$. Hence, in the limit $\varepsilon \to 0$, the coupling in (1.1) formally depends on the value of $\psi(x,t) - \zeta(t)g(x)$ at one point $x = 0$.

We assume that the nonlinearity $F_j(\zeta) = F_j(\zeta_j)$, $j = 1, \ldots, 4$, admits a real-valued potential:

$$F_j(\zeta_j) = \partial_{\overline{\zeta}_j} U(\zeta), \qquad U(\zeta) = \sum_{j=1}^4 U_j(|\zeta_j|) \in C^2(\mathbb{C}^4), \tag{1.3}$$

where $\partial_{\zeta_j} := \frac{\partial U}{\partial \zeta_{j1}} + i\frac{\partial U}{\partial \zeta_{j2}}$ with $\zeta_{j1} := \operatorname{Re}\zeta_j$ and $\zeta_{j2} := \operatorname{Im}\zeta_j$, and

$$U(\zeta) \geq b|\zeta|^2 - a, \quad \text{for } \zeta \in \mathbb{C}^4, \quad \text{where } b > 0 \text{ and } a \in \mathbb{R}. \tag{1.4}$$

The system (1.1) is $\mathbf{U}(1)$-invariant; that is,

$$F_j(e^{i\theta}\zeta_j) = e^{i\theta}F_j(\zeta_j), \quad j = 1, \ldots, 4, \quad \zeta_j \in \mathbb{C}, \quad \theta \in \mathbb{R}. \tag{1.5}$$

Our main results are as follows. First, for initial data of type

$$\psi(x,0) = f(x) + \zeta_0 g(x), \quad f \in H^{\frac{5}{2}+}(\mathbb{R}^3) \otimes \mathbb{C}^4, \quad \zeta_0 \in \mathbb{C}^4, \tag{1.6}$$

we prove a global well-posedness of the Cauchy problem for the system (1.1) (Theorem 2.1 below).

Further, we show that the system admits four-frequencies stationary orbits (or solitary wave solutions) of the type

$$\psi(x,t) = \sum_{k=1}^4 \psi_{\omega_k}(x) e^{-i\omega_k t}, \quad \omega_k \in \mathbb{R}, \quad k = 1, \ldots, 4. \tag{1.7}$$

We obtain explicit formulas for the amplitudes $\psi_{\omega_k}(x)$.

Finally, we prove that solitary waves form a global attractor in the case when all polynomials F_j are *strictly nonlinear* [see. conditions (3.2)–(3.3)]. Namely, in this case any solution with initial data (1.6) converges to the set $\mathscr{S}$ of all solitary wave solutions:

$$\psi(\cdot,t) \longrightarrow \mathscr{S}, \qquad t \to \pm\infty, \tag{1.8}$$

where the convergence holds in local L^2- seminorms.

Let us comment on previous results on the attraction to the set of solitary waves for nonlinear $\mathbf{U}(1)$-invariant equations. The first results on asymptotic stability of solitary waves for nonlinear Schrödinger equation were obtained in

[8,35,36], and then developed in [9,12,23] and other papers. The asymptotic stability means the asymptotics of type (1.8) for solutions with initial data close to $\mathscr{S}$. Such *local attraction* for equations with nonlinear point interaction was proved in [2,7,23–25,31]. These models allow an efficient analysis of the corresponding linearized dynamics.

Global attraction of type (1.8) to the set of all stationary orbits was established

(i) in [20] for 1D Klein–Gordon equation coupled to nonlinear oscillator :

$$\ddot{\psi}(x,t) = (\partial_x^2 - m^2)\psi(x,t) + \delta(x)F(\psi(0,t)); \tag{1.9}$$

(ii) in [30] for 1D Dirac equations with more regular nonlinearity $D_m^{-1}\delta(x)F$ $(\psi(0,t))$;

(iii) in [21,22] for nD Klein–Gordon and Dirac equations with nonlinearity of type $\rho(x)F(\langle\psi,\rho\rangle)$;

(iv) in [26,27,29] for 3D wave and Klein–Gordon equations with concentrated nonlinearity.

Global attraction of type (1.8) for 3D Dirac equation with nonlinear point interaction was not considered previously.

Remark 1.1. The nonlinearity in (1.1) is more singular than the nonlinearities considered in [26,27,30]. That's why we introduced the smoothing operator K_m^ε. In Sect. C.3.1, we show that without the operator K_m^ε, the limit as $x \to 0$ in the second equation of (1.1) generally does not exist.

We note also that the 3D Schrödinger equation with concentrated nonlinearity was justified in [10] as a scaling limit of a regularized nonlinear Schrödinger dynamics. We suppose that for the Dirac equation a justification can be done by suitable modification of methods [10], but it still remains an open question.

Let us comment on our approach. For the proof of global well-posedness we develop the approach which was introduced in [26,32] in the context of the Klein–Gordon and wave equations. First, we obtain some regularity properties i) of solutions $\varphi_g(x,t)$ to the free Dirac equation with initial function $\zeta_0 g(x)$, and ii) of solutions $\psi_S(x,t)$ to the Dirac equation with zero initial function and with source $D_m^{-1}\zeta(t)\delta(x)$ (Lemma 2.2, and Propositions 2.4 and 2.5). We use these regularity properties to prove the existence of a local solution to (1.1) of the type

$$\psi(x,t) = \psi_{\text{free}}(x,t) + \psi_S(x,t), \quad \psi_{\text{free}} = \psi_f + \varphi_g,$$

where $\psi_f(x,t)$ is a solution to the free Dirac equation with initial function $f(x)$. We show that $\zeta(t)$ is a solution to a first-order nonlinear integro-differential equation driven by $\psi_{\text{free}}(0,t)$. Then we prove that conditions (1.3)–(1.4) provide the conservation law (2.2). Finally, we use the conservation law to obtain the global existence theorem.

Note that our system (1.1) gives a novel model of nonlinear point interaction which provides a conservation law and a priori estimates. The introduced smoothing operator K_m^ε leads to justification of numerous limit permutation.

We justify these limits by subtle arguments using properties of special functions (generalized hypergeometric functions ${}_1F_2$, modified Struve functions L_ν, modified Bessel functions I_ν and others) [14,33].

To prove the global attraction, we split $\psi(x,t)$ as

$$\psi(x,t) = \psi_f(x,t) + \varphi_g^+(x,t) + \psi_S^-(x,t), \quad \varphi_g^+ = \varphi_g + iD_m^{-1}\zeta_0\dot\gamma,$$
$$\psi_S^- = \psi_S - iD_m^{-1}\zeta_0\dot\gamma,$$

where $\dot\gamma(x,t)$ is defined in (2.11). We show that $\psi_f(\cdot,t)$ and $\varphi_g^+(\cdot,t)$ converge to zero as $t\to\pm\infty$ in local H^1-seminorms. Hence, it remains to prove (1.8) for $\psi_S^-(\cdot,t)$ only. The proof relies on the study of the Fourier transform in time $\widetilde{\psi_S^-}(x,\omega)$ and $\widetilde\zeta(\omega)$ and of their supports. First, we establish absolute continuity of the spectral density $\widetilde\zeta(\cdot)$ outside spectral gap $[-m,m]$. The absolute continuity is a nonlinear version of Kato's theorem on absence of embedded eigenvalues in the context of the nonlinear system (1.1).

Then we prove *the omega-limit compactness*. This means that for each sequence $s_k\to\infty$ there exists an infinite subsequence $s_{k_l}\to\infty$ such that the functions $\zeta(t+s_{k_l})$ converge to some function $\eta(t)\in\mathbb{C}^4$ uniformly in $|t|<T$ for any $T>0$. The absolute continuity of $\widetilde\zeta(\cdot)$ provides that the time-spectrum of $\widetilde\eta(\cdot)$ is contained in the spectral gap $[-m,m]$. The convergence of $\zeta(t+s_{k_l})$ implies the convergence of $\psi_S^-(x,t+s_{k_l})$ to some function $\phi_S(x,t)$ in in the topology of $C_b([-T,T],L^2_{loc}(\mathbb{R}^3))$.

Further, we apply the Titchmarsh convolution theorem ([19, Theorem 4.3.3]) to conclude that the time-spectrum of each component η_j, $j=1,\dots 4$, of function η consists of a single frequency, $\widetilde\eta_j(\omega)=C_j\delta(\omega-\omega_j)$. The Titchmarsh theorem controls the inflation of spectrum by the nonlinearity. Physically, these arguments justify the following binary mechanism of energy radiation, which is responsible for the attraction to solitary waves: (i) nonlinear energy transfer from lower to higher harmonics, and (ii) subsequent dispersion decay caused by energy radiation to infinity. We finish the proof using an integral representation of $\phi_S(x,t)$ via $\eta(t)$.

Remark 1.2. Our approach is also applicable for other interpretation of 3D Dirac equation with concentrated nonlinearities. Namely, the source $D_m^{-1}\zeta(t)\delta(x)$ in the first equation of (1.1) can be replaced by more singular delta-like source $\zeta(t)\delta(x)$. In this case, the function $\psi(x,t)$ in the second equation of (1.1) should be replaced by the function $D_m^{-1}\psi(x,t)$. For such a system, the convergence (1.8) holds in local H^{-1}-seminorms.

2. Global well-posedness

We fix a nonlinear function $F:\mathbb{C}^4\to\mathbb{C}^4$ and define the domain

$$\mathscr{D}_F = \{\psi\in L^2(\mathbb{R}^3)\otimes\mathbb{C}^4 : \psi(x)=\psi_{reg}(x)+\zeta g(x),\ \zeta\in\mathbb{C}^4,$$
$$\psi_{reg}\in H^{\frac32-}(\mathbb{R}^3)\otimes\mathbb{C}^4, \exists \lim_{\varepsilon\to0+}\lim_{x\to0} K_m^\varepsilon\psi_{reg}(x)=F(\zeta)\},$$

which generally is not a linear space. Note that the first equation of (1.1) can be written in the other form

$$i\dot{\psi}(x,t) = D_m^F\psi(x,t), \qquad D_m^F\psi(x,t) := D_m\psi_{reg}(x,t) \tag{2.1}$$

(cf. equation (1.2) in [1], equation (7) in [2]).

Everywhere below we will write L^2 and H^s instead of $L^2(\mathbb{R}^3)\otimes\mathbb{C}^4$ and $H^s(\mathbb{R}^3)\otimes\mathbb{C}^4$. Denote $\|\cdot\| = \|\cdot\|_{L^2}$. In this section we will prove the following result.

Theorem 2.1. *Let conditions (1.3) and (1.4) hold. Then*

(i) *For every initial function $\psi(x,0) = f(x) + \zeta_0 g(x) \in \mathscr{D}_F$ with $f \in H^{\frac{5}{2}+}$ the equation (1.1) has a unique solution $\psi(x,t) = \psi_{reg}(x,t) + \zeta(t)g(x) \in C(\mathbb{R},\mathscr{D}_F)$, such that $\zeta(t) \in C^1[0,\infty)$.*

(ii) *The following conservation law holds:*

$$\mathscr{H}_F(\psi(\cdot,t)) := \frac{1}{2}\|D_m\psi_{reg}(\cdot,t)\|^2 + U(\zeta(t)) = \text{const}, \quad t\in\mathbb{R}. \tag{2.2}$$

(iii) *The following a priori bound holds:*

$$|\zeta(t)| \le C(\psi(\cdot,0)), \quad t\in\mathbb{R}. \tag{2.3}$$

(iv) *The map $W : (f(\cdot),\zeta_0) \mapsto (\psi_{reg}(\cdot,\cdot),\zeta(\cdot))$ is continuous $H^{\frac{5}{2}+}\oplus\mathbb{C}^4 \to C(\mathbb{R},H^{\frac{3}{2}-})\oplus(C^1(\mathbb{R})\otimes\mathbb{C}^4)$.*

Obviously, it suffices to prove Theorem 2.1 for $t \ge 0$.
We split solutions to (1.1) as

$$\psi(x,t) = \psi_{free}(x.t) + \psi_S(x,t) = \psi_f + \varphi_g + \psi_S(x,t), \tag{2.4}$$

where $\psi_f(x,t)$ and φ_g are the unique solutions to the free Dirac equation with initial functions f and $\zeta_0 g$:

$$i\dot{\psi}_f(x,t) = D_m\psi_f(x,t), \qquad \psi_f(x,0) = f(x),$$
$$i\dot{\varphi}_g(x,t) = D_m\varphi_g(x,t), \quad \varphi_g(x,0) = \zeta_0 g(x),$$

and $\psi_S(x,t)$ is the solution to

$$\begin{cases} i\dot{\psi}_S(x,t) = D_m\psi_S(x,t) - D_m^{-1}\zeta(t)\delta(x), \\ \lambda(t) + \lim\limits_{\varepsilon\to 0+}\lim\limits_{x\to 0} K_m^\varepsilon\Big(\varphi_g(x,t) + \psi_S(x,t) - \zeta(t)g(x)\Big) = F(\zeta(t)), \\ \psi_S(x,0) = 0, \quad \zeta(0) = \zeta_0. \end{cases} \tag{2.5}$$

Evidently,

$$\psi_f(\cdot,t) \in C_b([0,\infty),H^{\frac{5}{2}+}), \quad \dot{\psi}_f(\cdot,t) \in C_b([0,\infty),H^{\frac{3}{2}+}). \tag{2.6}$$

Hence,

$$\lambda(t) := \psi_f(0,t) \in C_b^1[0,\infty)\otimes\mathbb{C}^4. \tag{2.7}$$

Moreover, the linear map $f(\cdot) \to \lambda(\cdot)$ is continuous $H^{\frac{5}{2}+} \to C_b^1[0,\infty)\otimes\mathbb{C}^4$ since

$$\|\lambda\|_{C_b^1[0,\infty)\otimes\mathbb{C}^4} \le C(\varepsilon)\|f\|_{H^{\frac{5}{2}+\varepsilon}}, \quad \varepsilon > 0. \tag{2.8}$$

Now the existence and uniqueness of the solution $\psi(\cdot,t)\in C([0,\infty),\mathscr{D}_F$ of the system (1.1) is equivalent to the existence and uniqueness of the solution $(\psi_S(\cdot,t),\zeta(t))$ to (2.5) such that $\psi_S(\cdot,t)+\varphi_g(\cdot,t)\in C([0,\infty),\mathscr{D}_F)$ and $\zeta\in C^1[0,\infty)$.

Let us obtain an explicit formula for $\varphi_g(x,t)$. Note that the function

$$\phi(x,t):=\varphi_g(x,t)-\zeta_0 g(x) \tag{2.9}$$

satisfies

$$i\dot\phi(x,t)=D_m\phi(x,t)+D_m^{-1}\zeta_0\delta(x),\quad \phi(x,0)=0.$$

Hence,

$$\phi(x,t)=(-i\partial_t-D_m)D_m^{-1}\zeta_0\gamma(x,t)=-iD_m^{-1}\zeta_0\dot\gamma(x,t)-\zeta_0\gamma(x,t), \tag{2.10}$$

where

$$\gamma(x,t)=\frac{\theta(t-|x|)}{4\pi|x|}-\frac{m}{4\pi}\int_0^t\frac{\theta(s-|x|)J_1(m\sqrt{s^2-|x|^2})}{\sqrt{s^2-|x|^2}}ds, \tag{2.11}$$

is the solution to

$$\ddot\gamma(x,t)=(\Delta-m^2)\gamma(x,t)+\delta(x),\quad \gamma(x,0)=0,\quad \dot\gamma(x,0)=0. \tag{2.12}$$

Here J_1 is the Bessel function of the first order and θ is the Heaviside function. Finally, (2.9) and (2.10) imply

$$\varphi_g(x,t)=\zeta_0 g(x)-\zeta_0\gamma(x,t)-iD_m^{-1}\zeta_0\dot\gamma(x,t)=\varphi_g^+(x,t)-iD_m^{-1}\zeta_0\dot\gamma(x,t), \tag{2.13}$$

where $\varphi_g^+(x,t):=\zeta_0(g(x)-\gamma(x,t))$.

Lemma 2.2. *For any $t>0$ there exists*

$$\lim_{\varepsilon\to0+}\lim_{x\to0}K_m^\varepsilon(g(x)-\gamma(x,t))=\mu(t):=-\frac{m}{4\pi}+\frac{m}{4\pi}\int_0^t\frac{J_1(ms)}{s}ds. \tag{2.14}$$

We prove this lemma in Sect. A. Note that the function $\mu(t)$ is continuous for $t>0$, and there exists

$$\mu(0)=\lim_{t\to0}\mu(t)=-\frac{m}{4\pi}.$$

Moreover,

$$\mu(t)\to0,\quad t\to\infty, \tag{2.15}$$

since $\int_0^\infty\frac{J_1(ms)}{s}ds=1$ by [33, Formula 10.22.43].

2.1. Reduction to integro-differential equation

Here we consider the first equation of (2.5) for ψ_S with some given function $\zeta(t)\in C^1[0,\infty)\otimes C^4$. We construct the solution and formulate its properties which will be proved later. Further, we substitute the constructed solution into the second equation of (2.5) and obtain an integro-differential equation for ζ.

Lemma 2.3. *Let $\zeta(t) \in C^1[0,\infty) \otimes C^4$. Then the unique solution $\psi_S(x,t)$ to the Dirac equation*

$$i\dot{\psi}_S(x,t) = D_m\psi_S(x,t) - D_m^{-1}\zeta(t)\delta(x), \quad \psi_S(x,0) = 0 \tag{2.16}$$

is given by

$$\psi_S(x,t) := \varphi_S(x,t) + iD_m^{-1}\zeta_0\dot{\gamma}(x,t) + iD_m^{-1}p_S(x,t), \quad \zeta_0 := \zeta(0), \tag{2.17}$$

where γ is defined in (2.11), and

$$\begin{aligned}\varphi_S(x,t) &= \frac{\theta(t-|x|)}{4\pi|x|}\zeta(t-|x|)\\ &\quad -\frac{m}{4\pi}\int_0^t \frac{\theta(s-|x|)J_1(m\sqrt{s^2-|x|^2})}{\sqrt{s^2-|x|^2}}\zeta(t-s)ds, \end{aligned}\tag{2.18}$$

$$\begin{aligned}p_S(x,t) &= \frac{\theta(t-|x|)}{4\pi|x|}\dot{\zeta}(t-|x|)\\ &\quad -\frac{m}{4\pi}\int_0^t \frac{\theta(s-|x|)J_1(m\sqrt{s^2-|x|^2})}{\sqrt{s^2-|x|^2}}\dot{\zeta}(t-s)ds. \end{aligned}\tag{2.19}$$

Proof. It is easy to verify that the function $\varphi_S(x,t)$ is the unique solution to the Klein–Gordon with δ-like source

$$\ddot{\varphi}_S(x,t) = (\Delta - m^2)\varphi_S(x,t) + \zeta(t)\delta(x), \quad \varphi_S(x,0) = 0, \quad \dot{\varphi}_S(x,0) = 0. \tag{2.20}$$

In the case $m = 0$ this is well-known formula [14, Section 175]. Hence,

$$\psi_S(x,t) = (i\partial_t + D_m)D_m^{-1}\varphi_S(x,t) = \varphi_S(x,t) + iD_m^{-1}\zeta_0\dot{\gamma}(x,t) + iD_m^{-1}p_S(x,t). \tag{2.21}$$

□

In Sects. B and C , we justify the following limits

Proposition 2.4. *For any $\zeta(t) \in C^1[0,\infty) \otimes \mathbb{C}^4$ there exists*

$$\begin{aligned}&\lim_{\varepsilon\to 0+}\lim_{x\to 0} K_m^\varepsilon\left(\varphi_S(x,t) - \zeta(t)g(x)\right)\\ &\quad = \frac{1}{4\pi}\Big(m\zeta(t) - \dot{\zeta}(t) - m\int_0^t \frac{J_1(ms)}{s}\zeta(t-s)ds\Big), \quad t > 0. \end{aligned}\tag{2.22}$$

Proposition 2.5. *For any $\zeta(t) \in C^1[0,\infty) \otimes \mathbb{C}^4$ there exists*

$$\begin{aligned}\lim_{\varepsilon\to 0+}\lim_{x\to 0} K_m^\varepsilon D_m^{-1}p_S(x,t) &= \frac{m\beta}{4\pi}\Bigg(\zeta_0\left[mt\int_{mt}^\infty \frac{J_1(u)du}{u} - J_0(mt)\right]\\ &\quad + \zeta(t) - m\int_0^t\left(\int_{ms}^\infty \frac{J_1(u)du}{u}\right)\zeta(t-s)ds\Bigg), \quad t > 0. \end{aligned}\tag{2.23}$$

Substituting these limits into the second equation of (2.5) and taking into accunt (2.14), we obtain the equation for $\zeta(t)$:

$$\lambda(t)+\zeta_0\mu(t)+\frac{1}{4\pi}\Big(m\zeta(t)-\dot\zeta(t)-m\int_0^t\frac{J_1(ms)}{s}\zeta(t-s)ds\Big)$$
$$+\frac{im\beta}{4\pi}\Big(\zeta_0\Big[tm\int_{mt}^\infty\frac{J_1(u)du}{u}-J_0(mt)\Big]+\zeta(t)$$
$$-m\int_0^t\Big(\int_{ms}^\infty\frac{J_1(u)du}{u}\Big)\zeta(t-s)ds\Big)=F(\zeta(t)),\quad \zeta(0)=\zeta_0. \tag{2.24}$$

In the next two sections we will solve system (2.5) in reverse order: first we solve the equation (2.24) for $\zeta(t)$ and then we solve the first equation of (2.21) for $\psi_S(x,t)$ with this $\zeta(t)$.

2.2. Local well-posedness

Here we prove the local well-posedness for the system (2.5). To do this, we modify the nonlinearity F so that it becomes Lipschitz continuous. Define

$$\Lambda(\psi(0))=\sqrt{(\mathscr{H}_F(\psi(0))+a)/b}, \tag{2.25}$$

where $\psi(0)=\psi(\cdot,0)\in\mathscr{D}_F$ is the initial function from Theorem 2.1 and a, b are constants from (1.4). Then we may pick a modified potential function $\widetilde U(\zeta)\in C^2(\mathbb{C}^4,\mathbb{R})$, so that

(i) the identity holds

$$\widetilde U(\zeta)=U(\zeta),\quad |\zeta|\le\Lambda(\psi(0)), \tag{2.26}$$

(ii) $\widetilde U(\zeta)$ satisfies (1.4) with the same constant a, b as $U(\zeta)$ does:

$$\widetilde U(\zeta)\ge b|\zeta|^2-a,\quad \zeta\in\mathbb{C}^4, \tag{2.27}$$

(iii) the functions $\widetilde F_j(\zeta_j)=\partial_{\overline\zeta_j}\widetilde U(\zeta)$ are Lipschitz continuous:

$$|\widetilde F_j(\zeta_j)-\widetilde F_j(\eta_j)|\le C|\zeta_j-\eta_j|,\quad \zeta_j,\eta_j\in\mathbb{C}. \tag{2.28}$$

First, we establish local well-posedness for system (2.5) with the modified nonlinearity $\widetilde F$.

Proposition 2.6. (Local well-posedness). *Let the conditions (2.26)–(2.28) hold. Then*

(i) *there exists a unique solution $(\psi_S(x,t),\zeta(t))$ to (2.5) such that*

$$\psi^-_{reg}(\cdot,t):=\psi_S(\cdot,t)+\varphi_g(\cdot,t)-\zeta(t)g(\cdot)\in C([0,\tau],H^{\frac32-}),\quad \zeta\in C^2[0,\tau]\otimes\mathbb{C}^4;$$

(ii) *the map $\zeta(\cdot)\to\psi^-_{reg}(\cdot,\cdot)$ is continuous $C^2[0,\tau]\otimes\mathbb{C}^4\to C([0,\tau],H^{\frac32-})$.*

Proof. (i) First, we solve integro-differential equation (2.24) with $\widetilde F$ instead of F:

$$\lambda(t)+\zeta_0\mu(t)+\frac{1}{4\pi}\Big(m\zeta(t)-\dot\zeta(t)-m\int_0^t\frac{J_1(ms)}{s}\zeta(t-s)ds\Big)$$

$$+\frac{im\beta}{4\pi}\Big(\zeta_0\Big[tm\int_{mt}^{\infty}\frac{J_1(u)du}{u}-J_0(mt)\Big]+\zeta(t)$$

$$-m\int_0^t\left(\int_{ms}^{\infty}\frac{J_1(u)du}{u}\right)\zeta(t-s)ds\Big)=\widetilde{F}(\zeta(t)),\quad \zeta(0)=\zeta_0,\ (2.29)$$

where λ_j, $\mu\in C^1[0,\infty)$ by (2.7), (2.14). The next lemma follows by standard contraction mapping principle.

Lemma 2.7. *Let conditions* (2.26)–(2.28) *be satisfied. Then*

(i) *for sufficiently small* $\tau>0$ *the Cauchy problem* (2.29) *has a unique solution* $\zeta\in C^1[0,\tau]\otimes\mathbb{C}^4$*;*
(ii) *the map* $\lambda_j(\cdot)\to\zeta_j(\cdot)$ *is continuous* $C^1[0,\tau]\to C^2[0,\tau]$ *for every* $j=1,\dots,4$.

Now we define the function

$$\psi_S(x,t):=\varphi_S(x,t)+iD_m^{-1}\zeta_0\dot{\gamma}(x,t)+iD_m^{-1}p_S(x,t),\quad t\in[0,\tau],$$

where $\varphi_S(x,t)$ and $p_S(x,t)$ are given by (2.18) and (2.19) with $\zeta(t)$ the solution to (2.29).

Let us show that $(\psi_S(x,t),\zeta(t)$ is the solution to (2.5). Indeed, ψ_S satisfies the first equation of (2.5). Moreover, (2.14), (2.22) and (2.23) imply for $t\in[0,\tau]$

$$\lim_{\varepsilon\to0+}\lim_{x\to0}K_m^{\varepsilon}\Big(\psi_S(x,t)+\varphi_g(x,t)-\zeta(t)g(x)\Big)$$
$$=\zeta_0\mu(t)+\lim_{\varepsilon\to0+}\lim_{x\to0}K_m^{\varepsilon}\left(\varphi_S(x,t)+iD_m^{-1}p_S(x,t)-\zeta(t)g(x)\right)$$
$$=\zeta_0\mu(t)+\frac{1}{4\pi}\left(m\zeta(t)-\dot{\zeta}(t)-m\int_0^t\frac{J_1(ms)}{s}\zeta(t-s)ds\right)$$
$$+\frac{im\beta}{4\pi}\Big(\zeta_0\Big[tm\int_{mt}^{\infty}\frac{J_1(u)du}{u}-J_0(mt)\Big]+\zeta(t)$$
$$-m\int_0^t\left(\int_{ms}^{\infty}\frac{J_1(u)du}{u}\right)\zeta(t-s)ds\Big)=\widetilde{F}(\zeta(t))-\lambda(t),\qquad (2.30)$$

since $\zeta(t)$ solves (2.29). Hence, the second equation of (2.5) with $\widetilde{F}$ holds.

Let us prove the uniqueness of this solution. Suppose that $(\widetilde{\psi}_S(\cdot,t),\widetilde{\zeta}(t))$ with $\psi_S(\cdot,t)+\varphi_g(\cdot,t)\in C([0,\tau],\mathscr{D}_{\widetilde{F}})$ and $\zeta\in C^1[0,\tau]\otimes\mathbb{C}^4$ is another solution to (2.5). Then $\widetilde{\psi}_S(x,t)$ satisfies the first equation of (2.5) with the source $D_m^{-1}\widetilde{\zeta}(t)\delta(x)$ and is given by formulas (2.17)–(2.19) with $\widetilde{\zeta}(t)$ instead of $\zeta(t)$. Hence, Propositions 2.4 and 2.5 and the second equation of (2.5) imply that $\widetilde{\zeta}(t)$ solves the Cauchy problem (2.29). The uniqueness of the solution of (2.29) implies that $\widetilde{\zeta}(t)=\zeta(t)$. Hence, $\widetilde{\psi}_S=\psi_S$.

It remains to show that the function

$$\psi_{reg}^-(x,t)=\psi_S(x,t)+\varphi_g(x,t)-\zeta(t)g(x)$$

$$= \psi(x,t) - \psi_f(x,t) - \zeta(t)g(x) = \psi_{reg}(x,t) - \psi_f(x,t)$$

satisfies

$$\psi^-_{reg}(\cdot,t) \in C([0,\tau], H^{\frac{3}{2}-}(\mathbb{R}^3)). \tag{2.31}$$

Indeed, $\psi^-_{reg}(x,t)$ is a solution to

$$i\dot\psi^-_{reg}(x,t) = D_m\psi^-_{reg}(x,t) - i\dot\zeta(t)g(x) \tag{2.32}$$

with zero initial data. Hence, $\psi^-_{reg}(x,t) = (-i\partial_t - D_m)\mathrm{w}(x,t)$, where where $\mathrm{w}(x,t)$ is the solution to

$$\ddot{\mathrm{w}}(x,t) = (\Delta - m^2)\mathrm{w}(x,t) - i\dot\zeta(t)g(x), \quad \mathrm{w}(x,0) = 0, \quad \dot{\mathrm{w}}(x,0) = 0.$$

Then, for (2.31) we need to show that

$$\mathrm{w}(\cdot,t) \in C([0,\tau], H^{\frac{5}{2}-\varepsilon}(\mathbb{R}^3)), \quad \dot{\mathrm{w}}(\cdot,t) \in C([0,\tau], H^{\frac{3}{2}-\varepsilon}(\mathbb{R}^3)), \quad \text{for any} \quad \varepsilon > 0. \tag{2.33}$$

Applying the Fourier transform, we obtain

$$\widehat{\mathrm{w}}(\xi,t) = -i\int_0^t \frac{\sin(s\sqrt{\xi^2+m^2})}{(\xi^2+m^2)^{\frac{3}{2}}}\ \dot\zeta(t-s)ds,$$

$$\widehat{\dot{\mathrm{w}}}(\xi,t) = -i\int_0^t \frac{\cos(s\sqrt{\xi^2+m^2})}{\xi^2+m^2}\ \dot\zeta(t-s)ds.$$

Hence, integration by parts gives

$$\begin{aligned}(\xi^2+m^2)^{\frac{5}{4}-\frac{\varepsilon}{2}}\,\widehat{\mathrm{w}}(\xi,t) &= -i\int_0^t \frac{\sin(s\sqrt{\xi^2+m^2})}{(\xi^2+m^2)^{\frac{1}{4}+\frac{\varepsilon}{2}}}\ \dot\zeta(t-s)ds \\ &= i\Big(\frac{\dot\zeta(0)\cos(s\sqrt{\xi^2+m^2})}{(\xi^2+m^2)^{\frac{3}{4}+\frac{\varepsilon}{2}}} \\ &\quad - \frac{\dot\zeta(t)}{(\xi^2+m^2)^{\frac{3}{4}+\frac{\varepsilon}{2}}} - \int_0^t \frac{\cos(s\sqrt{\xi^2+m^2})}{(\xi^2+m^2)^{\frac{3}{4}+\frac{\varepsilon}{2}}}\ \ddot\zeta(t-s)ds\Big),\end{aligned}$$

where $\ddot\zeta \in C[0,\tau]\otimes\mathbb{C}^4$ by (2.7), (2.14) and (2.29). Therefore,

$$|(\xi^2+m^2)^{\frac{5}{4}-\frac{\varepsilon}{2}}\,\widehat{\mathrm{w}_j}(\xi,t)| \le \frac{C(1+\tau)\|\zeta_j\|_{C^2[0,\tau]}}{(\xi^2+m^2)^{\frac{3}{4}+\frac{\varepsilon}{2}}}, \quad t\in[0,\tau], \quad j=1,\ldots,4.$$

Similarly,

$$|(\xi^2+m^2)^{\frac{3}{4}-\frac{\varepsilon}{2}}\,\widehat{\dot{\mathrm{w}}_j}(\xi,t)| \le \frac{C(1+\tau)\|\zeta_j\|_{C^2[0,\tau]}}{(\xi^2+m^2)^{\frac{3}{4}+\frac{\varepsilon}{2}}}, \quad t\in[0,\tau], \quad j=1,\ldots,4.$$

Hence, (2.33) follows.
(ii) Evidently, the linear map $\zeta(\cdot) \to \psi^-_{reg}(\cdot,\cdot)$ is continuous $C^2[0,\tau]\otimes\mathbb{C}^4 \to C([0,\tau], H^{\frac{3}{2}-})$. □

Corollary 2.8. *It is obvious that* (2.30) *can be rewritten as*

$$\lim_{\varepsilon\to 0+}\lim_{x\to 0} K^\varepsilon_m\psi_{reg}(x,t) = \widetilde{F}(\zeta(t)), \quad t\in[0,\tau]. \tag{2.34}$$

2.3. Conservation law and a priori bound

Now we prove the conservation law (2.2) on the interval $[0,\tau]$.

Lemma 2.9. *Let conditions* (2.26)–(2.28) *hold, and let* $\psi(t)\in\mathscr{D}_{\widetilde F}$, $t\in[0,\tau]$, *be a solution to* (1.1). *Then*

$$\mathscr{H}_{\widetilde F}(\psi(\cdot,t))=\|D_m\psi_{reg}(\cdot,t)\|^2+\widetilde U(\zeta(t))=\text{const},\quad t\in[0,\tau]. \tag{2.35}$$

Proof. Equations (2.32) and (2.34) imply for any $t\in(0,\tau]$

$$\begin{aligned}
&\lim_{\varepsilon\to0}\frac{d}{dt}\|K_m^\varepsilon D_m\psi_{reg}\|^2\\
&\quad=\lim_{\varepsilon\to0}\Big[\langle K_m^\varepsilon D_m\dot\psi_{reg},\,K_m^\varepsilon D_m\psi_{reg}\rangle+\langle K_m^\varepsilon D_m\psi_{reg},\,K_m^\varepsilon D_m\dot\psi_{reg}\rangle\Big]\\
&\quad=\lim_{\varepsilon\to0}\Big[\langle -iK_m^\varepsilon D_m^2\psi_{reg}-K_m^\varepsilon D_m\dot\zeta g,\,K_m^\varepsilon D_m\psi_{reg}\rangle\\
&\qquad+\langle K_m^\varepsilon D_m\psi_{reg},\,-iK_m^\varepsilon D_m^2\psi_{reg}-K_m^\varepsilon D_m\dot\zeta g\rangle\Big]\\
&\quad=\lim_{\varepsilon\to0}\Big[-\langle D_m^2\dot\zeta g,\,K_m^{2\varepsilon}\psi_{reg}\rangle-\langle K_m^{2\varepsilon}\psi_{reg},\,D_m^2\dot\zeta g\rangle\Big]\\
&\quad=\lim_{\varepsilon\to0}\Big[-\dot\zeta\cdot\langle\delta(x),K_m^{2\varepsilon}\psi_{reg}\rangle-\overline{\dot\zeta}\cdot\langle K_m^{2\varepsilon}\psi_{reg},\,\delta(x)\rangle\Big]\\
&\quad=-\dot\zeta\cdot\overline{\widetilde F}(\zeta)-\overline{\dot\zeta}\cdot\widetilde F(\zeta)=-2\frac{d}{dt}\widetilde U(\zeta).
\end{aligned}\tag{2.36}$$

Here the scalar product $\langle K_m^\varepsilon D_m^2\psi_{reg},K_m^\varepsilon D_m\psi_{reg}\rangle$ exists since $K_m^\varepsilon\psi_{reg}(\cdot,t)=K_m^\varepsilon\psi_{reg}^-(\cdot,t)+K_m^\varepsilon\psi_f(\cdot,t)\in C([0,\infty),H^{3/2})$ for any $\varepsilon>0$ due to (2.6) and (2.31). Moreover, for any $\nu>0$ and $\varepsilon\ge0$

$$\sup_{t\in[0,\tau]}\|K_m^\varepsilon D_m\psi_{reg}(\cdot,t)\|_{H^{1/2-\nu}}<\infty.$$

Hence, uniformly in $t\in[0,\tau]$, we have

$$\lim_{\varepsilon\to0}\|K_m^\varepsilon D_m\psi_{reg}(t)\|=\|D_m\psi_{reg}(t)\|.$$

Therefore,

$$\begin{aligned}
\frac{d}{dt}\|D_m\psi_{reg}(\cdot,t)\|^2&=\frac{d}{dt}\lim_{\varepsilon\to0}\|K_m^\varepsilon D_m\psi_{reg}(\cdot,t)\|^2\\
&=\lim_{\varepsilon\to0}\frac{d}{dt}\|K_m^\varepsilon D_m\psi_{reg}(\cdot,t)\|^2=-2\frac{d}{dt}\widetilde U(\zeta),\quad \tau\in[0,\tau].
\end{aligned}$$

in the sense of distributions. Then (2.35) follows. □

Corollary 2.10. *The following identity holds*

$$\widetilde U(\zeta(t))=U(\zeta(t)),\quad t\in[0,\tau]. \tag{2.37}$$

Proof. First note that

$$\mathscr{H}_F(\psi(0))\ge U(\zeta_0)\ge b|\zeta_0|^2-a.$$

Therefore, $|\zeta_0|\le\Lambda(\psi(0))$ and then $\widetilde U(\zeta_0)=U(\zeta_0)$, $\mathscr{H}_{\widetilde F}(\psi(0))=\mathscr{H}_F(\psi(0))$. Further,

$$\mathscr{H}_{\widetilde F}(\psi(t))\ge\widetilde U(\zeta(t))\ge b|\zeta(t)|^2-a,\quad t\in[0,\tau].$$

Hence, (2.35) implies the a priory bound

$$|\zeta(t)| \le \sqrt{(\mathscr{H}_{\widetilde{F}}(\psi(t)) + a)/b} = \sqrt{(\mathscr{H}_{\widetilde{F}}(\psi(0)) + a)/b}$$
$$= \sqrt{(\mathscr{H}_F(\psi(0)) + a)/b} = \Lambda(\psi(0)), \quad t \in [0,\tau]. \tag{2.38}$$

Therefore, (2.37) follows by (2.26). □

2.4. Bootstrap argument

Identity (2.37) implies that we can replace $\widetilde{F}$ by F in Proposition 2.6 and in Lemma 2.9.

Now we can finish the proof of Theorem 2.1. The unique solution $\psi_{free}(x,t)$ to the free Dirac equation with initial function $f(x) + \zeta_0 g(x)$ exists for $t \in [0,\infty)$ (see Formula (2.13)). At the same time, the solution $\zeta(t)$ to equation (2.24) exists for $0 \le t \le \tau$, where the time span τ in Lemma 2.7 depends only on $\Lambda(\psi(0))$. This solution defines the function $\psi_S(x,t)$ by formulas (2.17)–(2.19) so that $(\psi_S(x,t), \zeta(t)$ is the unique solution to the system (2.5) on the interval $[0,\tau]$. The bound (2.38) at $t=\tau$ allows us to extend the solution $\zeta(t)$ to the time interval $[\tau, 2\tau]$, and formulas (2.17)–(2.19) define $\psi_S(x,t)$ on the interval $[0,2\tau]$ then. We proceed by induction to obtain the solution for all $t \ge 0$.

3. Solitary waves and main theorem

We assume that

$$U(\zeta) = \sum_{j=1}^{4} U_j(\zeta_j), \text{ where } \quad U_j(\zeta_j) = \sum_{n=0}^{N_j} u_{n,j}|\zeta_j|^{2n},$$
$$u_{n,j} \in \mathbb{R}, \quad u_{N_j,j} > 0, \quad N_j \ge 2, \quad j = 1,\dots,4. \tag{3.1}$$

This assumption guarantees the bound (1.4), and it is crucial in our argument: it allow us to apply the Titchmarsh convolution theorem. Equality (3.1) implies that

$$F_j(\zeta_j) = \partial_{\bar{\zeta}_j} U_j(\zeta_j) = a_j(|\zeta_j|^2)\zeta_j, \quad j = 1,\dots,4, \tag{3.2}$$

where

$$a_j(|\zeta_j|^2) := \sum_{n=1}^{N_j} 2n u_{n,j}|\zeta_j|^{2n-2}. \tag{3.3}$$

Definition 3.1. (i) The solitary waves of equation (1.1) are solutions of the form

$$\psi(x,t) = \sum_k \psi_{\omega_k}(x) e^{-i\omega_k t}, \quad \omega_k \in \mathbb{R}, \quad \omega_l \ne \omega_j, \quad l \ne j, \quad \psi_{\omega_k} \in L^2(\mathbb{R}^3), \tag{3.4}$$

where the sum has a finite number of terms.

(ii) The solitary manifold is the set: $\mathscr{S} = \left\{ \sum_k \psi_{\omega_k} \colon \omega_k \in \mathbb{R},\ \omega_l \ne \omega_j,\ l \ne j \right\}$.

Below we show that the number of nonzero terms in (3.4) does not exceed 4. From (3.2) it follows that the set $\mathscr{S}$ is invariant under multiplication by $e^{i\theta}$, $\theta \in \mathbb{R}$. Note that there is a zero solitary wave, since $F(0) = 0$.
Now we derive more precise representation for solitary waves.

Proposition 3.2. *Assume that $F(\zeta)$ satisfies (3.2). Then nonzero solitary waves are given by*

$$\psi(x,t) = \phi_\Omega(x,t) + iD_m^{-1}\dot{\phi}_\Omega(x,t), \tag{3.5}$$

where $\Omega = (\omega_1,\ldots,\omega_4)$ with $|\omega_j| < m$,

$$\phi_\Omega(x,t) = (\phi_{\omega_1}(x)e^{-i\omega_1 t} \ldots, \phi_{\omega_4}(x)e^{-i\omega_4 t}), \tag{3.6}$$

$$\phi_{\omega_j}(x) = C_j \frac{e^{-\sqrt{m^2-\omega_j^2}|x|}}{4\pi|x|}, \quad j = 1,\ldots,4, \tag{3.7}$$

and $C_j = C_j(\omega_j) \in \mathbb{R}$ are solutions to

$$(m - \sqrt{m^2-\omega_j^2})(1 + \sigma_j \frac{m}{\omega_j}) = 4\pi a_j(|C_j|^2), \quad j = 1,\ldots,4 \tag{3.8}$$

with

$$\sigma_j = \begin{cases} 1, & j = 1,2, \\ -1, & j = 3,4. \end{cases} \tag{3.9}$$

Remark 3.3. In (3.5) some ω_j may be identical in contrast to (3.4).

Proof. We look for a solution $\psi(x,t)$ to (1.1) in the form (3.4):

$$\psi(x,t) = \sum_k \psi_{\omega_k}(x)e^{-i\omega_k t}, \quad \text{where} \quad \omega_k < \omega_{k+1}. \tag{3.10}$$

Consider the function

$$\chi(x,t) := \psi(x,t) - iD_m^{-1}\dot{\psi}(x,t) = \sum_k \chi_{\omega_k}(x)e^{-i\omega_k t}, \tag{3.11}$$

where

$$\chi_{\omega_k} = \psi_{\omega_k} - \omega_k D_m^{-1}\psi_{\omega_k} = D_m^{-1}(D_m - \omega_k)\psi_{\omega_k}.$$

Hence,

$$\begin{aligned}\psi_{\omega_k} &= D_m(D_m-\omega_k)^{-1}\chi_{\omega_k} = D_m(D_m+\omega_k)(D_m^2-\omega_k^2)^{-1}\chi_{\omega_k} \\ &= \chi_{\omega_k} + (\omega_k^2 + \omega_k D_m)(D_m^2-\omega_k^2)^{-1}\chi_{\omega_k}.\end{aligned} \tag{3.12}$$

Further, (3.11) implies that

$$D_m\chi(x,t) = D_m\psi(x,t) - i\dot{\psi}(x,t) = D_m^{-1}\zeta(t)\delta(x)$$

by the first equation of (1.1). Hence,

$$\sum_k e^{-i\omega_k t} D_m^2 \chi_{\omega_k}(x) = \zeta(t)\delta(x).$$

by (3.11). Therefore,

$$\chi_{\omega_k}(x) = \overline{C}_k \frac{e^{-m|x|}}{4\pi|x|}, \qquad \zeta(t) = \sum_k \overline{C}_k e^{-i\omega_k t}. \tag{3.13}$$

where $\overline{C}_k := (C_{k1}, \dots, C_{k4})$. Now we derive the explicit formulas for $\psi_{\omega_k}(x)$, using (3.12) and (3.13) only. One has

$$(D_m^2-\omega_k^2)^{-1}\,\frac{e^{-m|x|}}{4\pi|x|}=\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{e^{-i\xi x}d^3\xi}{(\xi^2+m^2)(\xi^2+m^2-\omega_k^2)}$$
$$=\frac{1}{(2\pi)^3\omega_k^2}\int_{R^3}\Big(\frac{e^{-i\xi x}}{\xi^2+m^2-\omega_k^2}-\frac{e^{-i\xi x}}{\xi^2+m^2}\Big)d^3\xi$$
$$=\frac{1}{\omega_k^2}\Big(\frac{e^{-\sqrt{m^2-\omega_k^2}|x|}}{4\pi|x|}-\frac{e^{-m|x|}}{4\pi|x|}\Big).$$

Moreover,

$$D_m(D_m^2-\omega_k^2)^{-1}\,\frac{e^{-m|x|}}{4\pi|x|}=D_m^{-1}(D_m^2-\omega_k^2)^{-1}D_m^2\,\frac{e^{-m|x|}}{4\pi|x|}=D_m^{-1}(D_m^2-\omega_k^2)^{-1}\delta(x)$$
$$=D_m^{-1}\,\frac{e^{-\sqrt{m^2-\omega_k^2}|x|}}{4\pi|x|}.$$

Substituting this into (3.12), we obtain by (3.13)

$$\psi_{\omega_k}(x)=\varphi_{\omega_k}(x)+\omega_kD_m^{-1}\varphi_{\omega_k}(x), \tag{3.14}$$

where we denote

$$\varphi_{\omega_k}(x):=\overline{C}_k\frac{e^{-\sqrt{m^2-\omega_k^2}|x|}}{4\pi|x|}.$$

Now we are able to find coefficients C_{kj}. The second equation of (1.1) together with (3.4) and (3.14) imply

$$\lim_{\epsilon\to0+}\lim_{x\to0}K_m^\epsilon\sum_k e^{-i\omega_kt}\Big(\varphi_{\omega_k}(x)+\omega_kD_m^{-1}\varphi_{\omega_k}(x)-\overline{C}_kg(x)\Big)$$
$$=\lim_{\epsilon\to0+}\lim_{x\to0}K_m^\epsilon\sum_k e^{-i\omega_kt}\Big(\varphi_{\omega_k}(x)-\overline{C}_kg(x)+\omega_km\beta D_m^{-2}\varphi_{\omega_k}(x)$$
$$-i\omega_k\alpha\cdot\nabla D_m^{-2}\varphi_{\omega_k}(x)\Big)=F(\sum_k\overline{C}_ke^{-i\omega_kt}). \tag{3.15}$$

Note, that

$$\lim_{\epsilon\to0+}\lim_{x\to0}K_m^\epsilon\Big(\varphi_{\omega_k}(x)-\overline{C}_kg(x)\Big)=\overline{C}_k\lim_{\epsilon\to0+}\lim_{x\to0}K_m^\epsilon\Big(\frac{e^{-\sqrt{m^2-\omega_k^2}|x|}}{4\pi|x|}-\frac{e^{-m|x|}}{4\pi|x|}\Big)$$
$$=\frac{\overline{C}_k}{2\pi^2}\int_0^\infty\Big(\frac{r^2}{r^2+m^2-\omega_k^2}-\frac{r^2}{r^2+m^2}\Big)dr$$
$$=\frac{\overline{C}_k}{2\pi^2}\int_0^\infty\Big(\frac{m^2}{r^2+m^2}-\frac{m^2-\omega_k^2}{r^2+m^2-\omega_k^2}\Big)dr$$
$$=\frac{\overline{C}_k}{4\pi}(m-\sqrt{m^2-\omega_k^2}). \tag{3.16}$$

Similarly,

$$\lim_{\epsilon\to0+}\lim_{x\to0}K_m^\epsilon D_m^{-2}\varphi_{\omega_k}(x)=\frac{\overline{C}_k}{2\pi^2}\int_0^\infty\frac{r^2dr}{(r^2+m^2)(r^2+m^2-\omega_k^2)}$$

$$= \frac{\overline{C}_k}{2\pi^2\omega_k^2}\int_0^\infty \Big(\frac{m^2}{r^2+m^2} - \frac{m^2-\omega_k^2}{r^2+m^2-\omega_k^2}\Big)dr$$
$$= \frac{\overline{C}_k}{4\pi\omega_k^2}\big(m-\sqrt{m^2-\omega_k^2}\big). \tag{3.17}$$

Moreover,

$$\lim_{\epsilon\to 0+}\lim_{x\to 0} K_m^\epsilon \nabla_n D_m^{-2}\varphi_{\omega_k}(x) = \frac{\overline{C}_k}{(2\pi)^3}\lim_{\epsilon\to 0+}\int_{\mathbb{R}^3}\frac{\xi_n\xi^2 d^3\xi}{(\xi^2+m^2-\omega_k^2)(\xi^2+m^2)^{1+\epsilon}} = 0,$$
$$n = 1,2,3. \tag{3.18}$$

Substituting (3.16)–(3.18) into (3.15), we get

$$\frac{1}{4\pi}\sum_k C_{kj}e^{-i\omega_k t}\left(m-\sqrt{m^2-\omega_k^2}+\sigma_j\frac{m}{\omega_k}\left(m-\sqrt{m^2-\omega_k^2}\right)\right)$$
$$= a_j(|\sum_k C_{kj}e^{-i\omega_k t}|^2)\sum_k C_{kj}e^{-i\omega_k t}, \quad j=1,\dots,4. \tag{3.19}$$

Lemma 3.4. *Let C_{kj} be solutions to (3.19). Then for each fixed $j = 1,\dots,4$ only one of the coefficients C_{kj} is nonzero.*

Proof. It suffices to consider the case $j = 1$ only. We should prove that may be no more than one nonzero $c_k := C_{k1}$. Assume, to the contrary, that $c_{k_1}, c_{k_2},\dots,c_{k_n} \neq 0$ with $k_1 < k_2 < \cdots < k_n$, where $2 \le n$. Then $\omega_{k_1} < \omega_{k_2} < \cdots < \omega_{k_n}$ by (3.10). Denote $\delta_{l,p} = \omega_{k_p} - \omega_{k_l} > 0$, $1 \le l < p \le n$. Evidently, $\delta := \delta_{1,n} = \max\limits_{1\le l<p\le n}\delta_{l,p}$. Then

$$|\sum_k c_k e^{-i\omega_k t}|^2 = a + be^{i\delta t} + \overline{b}e^{-i\delta t} + \sum_{(l,p)\neq(1,n)}(b_{l,p}e^{i\delta_{l,p}t} + \overline{b}_{l,p}e^{-i\delta_{l,p}t})$$

with some $a > 0$ and $b \neq 0$. Hence, (3.3) implies

$$a_1(|\sum_k c_k e^{-i\omega_k t}|^2) = de^{i(N_1-1)\delta t} + \overline{d}e^{-i(N_1-1)\delta t} + R,$$

where R consists of terms of the type $Ce^{i\sigma t}$ with $|\sigma| < (N_1-1)\delta$. Note that $d \neq 0$ since a_1 is a polynomial of degree $N_1 - 1 \ge 1$ due to (3.1) and (3.3). Now the right hand side of (3.19) contains the terms $e^{-i[\omega_{k_1}t-(N_j-1)\delta]t}$ and $e^{-i[\omega_{k_n}t+(N_j-1)\delta]t}$ with nonzero coefficients, which are absent on the left hand side. This contradiction proves the lemma. □

The lemma and formulas (3.4) and (3.14) imply

$$\psi_j(x,t) = \sum_k \psi_{\omega_k,j}(x)\,e^{-i\omega_k t} = \sum_k \varphi_{\omega_k,j}(x)e^{-i\omega_k t} + \left(\sum_k D_m^{-1}\omega_k\varphi_{\omega_k,j}(x)\,e^{-i\omega_k t}\right)_j$$
$$= \varphi_{\omega_{k_j},j}(x)\,e^{-i\omega_{k_j}t} + (D_m^{-1}\pi(x,t))_j, \quad j=1,\dots,4,$$

where

$$\pi_j(x,t) = \omega_{k_j}\varphi_{\omega_{k_j},j}(x)\,e^{-i\omega_{k_j}t}.$$

We can assume that $k_j = j$. Then $C_{k_j j} = C_{jj}$, $\omega_{k_j} = \omega_j$, and $\varphi_{\omega_j,j}(x) = \phi_{\omega_j}(x)$ from (3.7) with $C_j = C_{jj}$. Then (3.5) follows. It remains to note that equation (3.19) in the case when $C_{jk} = 0$ for $k \neq j$ is equation (3.8) for $C_j = C_{jj}$. Proposition is completely proved. □

The following lemma gives a sufficient condition for the existence of nonzero solitary waves.

Lemma 3.5. *Let F satisfy (3.2)–(3.3) with $M_j = -u_{1,j} > 0$, where $j \in \{1; 2; 3; 4\}$. Then there exists an open subset $I(M_j) \subset (-m, m)$ such that for any $\omega_j \in I(M_j)$ the jth equation of (3.8) has nonzero solutions $C_j = C_j(\omega_j)$. Moreover, $I(M_j) = (-m, m)$ if $M_j > m/(32\pi^2)$.*

We prove this lemma in Appendix D.
Now the solitary manifold $\mathscr{S}$ reads

$$\mathscr{S} = \left\{\Phi_\Omega + D_m^{-1}\Psi_\Omega \colon \Omega = (\omega_1, \ldots, \omega_4) \in \mathbb{R}^4\right\}, \tag{3.20}$$

where

$$\Phi_\Omega(x) = (\phi_{\omega_1}(x), \ldots, \phi_{\omega_4}(x)), \qquad \Psi_\Omega(x) = (\omega_1\phi_{\omega_1}(x), \ldots, \omega_4\phi_{\omega_4}(x)).$$

Our main result is the following theorem.

Theorem 3.6. *Let (3.1) be satisfied, and let $\psi(0) := \psi(x, 0) = f(x) + \zeta_0$ with $f \in H^{\frac{5}{2}+}$. Then the solution $\psi(x, t)$ to (1.1) with initial function $\psi(0)$ converges to solitary manifold $\mathscr{S}$ in the space $L^2_{loc}(\mathbb{R}^3)$:*

$$\lim_{t\to\pm\infty} \mathrm{dist}_{L^2_{loc}(\mathbb{R}^3)}(\psi(\cdot, t), \mathscr{S}) = 0. \tag{3.21}$$

It suffices to prove Theorem 3.6 for $t \to +\infty$.

4. Dispersive component

The following lemma states well known decay in local seminorms for the free Dirac equation.

Lemma 4.1. (cf. [22, Proposition 4.3]) *Let $\psi_f(x, t)$ be a solution to the free Dirac equation with initial function $f \in H^2(\mathbb{R}^3)$. Then $\forall R > 0$,*

$$\|\psi_f(\cdot, t)\|_{H^2(B_R)} \to 0, \qquad t \to \infty, \tag{4.1}$$

where B_R is the ball of radius R.

Corollary 4.2. *From (4.1) immediately follows that*

$$\lambda(t) = \psi_f(0, t) \to 0, \quad t \to \infty. \tag{4.2}$$

Now consider

$$\varphi_g^+(x, t) = \varphi_g(x, t) + iD_m^{-1}\zeta_0\dot{\gamma}(x, t) = \zeta_0(g(x) - \gamma(x, t)), \tag{4.3}$$

where φ_g is the solution free Dirac equation with initial function $\zeta_0 g$, given by (2.13).

Lemma 4.3. *$\varphi_g^+(x,t) = \zeta_0(g(x) - \gamma(x,t))$ decays in H^2_{loc} seminorms. That is, $\forall R > 0$*

$$\|\varphi_g^+(\cdot,t)\|_{H^2(B_R)} \to 0, \qquad t \to \infty. \tag{4.4}$$

Proof. According to (2.12) the function $h(x,t) := \gamma(x,t) - g(x)$ is the solutions to

$$\ddot{h}(x,t) = (\Delta - m^2)h(x,t), \qquad (h(x,t), \dot{h}(x,t))|_{t=0} = (-g,0). \tag{4.5}$$

Then (4.4) follows by Lemma 3.3 of [27]. □

In conclusion, let us show that

$$\varphi_g(\cdot,t) \in C_b([0,\infty), L^2). \tag{4.6}$$

Indeed, the energy conservation for equation (4.5) implies that

$$(h(\cdot,t), \dot{h}(\cdot,t)) \in C_b([0,\infty), L^2(\mathbb{R}^3) \oplus H^{-1}(\mathbb{R}^3)).$$

Hence,

$$(\gamma(\cdot,t), \dot{\gamma}(\cdot,t)) = (h(\cdot,t), \dot{h}(\cdot,t)) + (g(\cdot),0) \in C_b([0,\infty), L^2(\mathbb{R}^3) \oplus H^{-1}(\mathbb{R}^3)).$$

Then (4.6) follows by (4.3).

5. Complex Fourier–Laplace transform

The conservation low (2.2) and a priory bound (2.3) imply that $\psi(\cdot,t) \in C_b([0,\infty), L^2)$. Hence, (4.6) implies

$$\psi_S(\cdot,t) = \psi(\cdot,t) - \psi_f(\cdot,t) - \varphi_g(\cdot,t) \in C_b([0,\infty), L^2). \tag{5.1}$$

Let us analyze the Fourier–Laplace transform of $\psi_S(x,t)$:

$$\widetilde{\psi}_S(x,\omega) = \mathscr{F}_{t\to\omega}[\theta(t)\psi_S(x,t)] := \int_0^\infty e^{i\omega t}\psi_S(x,t)\,dt, \quad \omega \in \mathbb{C}^+, \quad x \in \mathbb{R}^3, \tag{5.2}$$

where $\mathbb{C}^+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Note that $\widetilde{\psi}_S(\cdot,\omega)$ is an L^2-valued analytic function of $\omega \in \mathbb{C}^+$ due to (5.1). Equation (2.16) implies that

$$-\omega\widetilde{\psi}_S(x,\omega) = D_m\widetilde{\psi}_S(x,\omega) - D_m^{-1}\widetilde{\zeta}(\omega)\delta(x), \quad \omega \in \mathbb{C}^+, \quad x \in \mathbb{R}^3, \tag{5.3}$$

where $\widetilde{\zeta}(\omega)$ is the Fourier–Laplace transform of $\zeta(t)$:

$$\widetilde{\zeta}(\omega) = \mathscr{F}_{t\to\omega}[\theta(t)\zeta(t)] = \int_0^\infty e^{i\omega t}\zeta(t)\,dt.$$

Applying the Fourier transform to (5.3), we get

$$\begin{aligned}
\hat{\widetilde{\psi}}_S(\xi,\omega) &= \frac{(\alpha\cdot\xi + m\beta)\widetilde{\zeta}(\omega)}{(\alpha\cdot\xi + m\beta + \omega)(\xi^2 + m^2)} = \Big(\frac{1}{\xi^2+m^2} - \frac{\omega}{(\alpha\cdot\xi + m\beta + \omega)(\xi^2+m^2)}\Big)\widetilde{\zeta}(\omega) \\
&= \Big(\frac{1}{\xi^2+m^2} + \frac{\omega^2}{(\xi^2+m^2-\omega^2)(\xi^2+m^2)} - \frac{\omega(\alpha\cdot\xi + m\beta)}{(\xi^2+m^2-\omega^2)(\xi^2+m^2)}\Big)\widetilde{\zeta}(\omega) \\
&= \Big(\frac{1}{\xi^2+m^2-\omega^2} + \frac{\alpha\cdot\xi + m\beta}{\omega}\Big(\frac{1}{\xi^2+m^2} - \frac{1}{\xi^2+m^2-\omega^2}\Big)\Big)\widetilde{\zeta}(\omega), \\
&\quad \xi \in \mathbb{R}^3, \quad \omega \in \mathbb{C}^+.
\end{aligned} \tag{5.4}$$

Denote

$$\varkappa(\omega) = \sqrt{\omega^2 - m^2}, \qquad \mathrm{Im}\,\varkappa(\omega) > 0, \qquad \omega \in \mathbb{C}^+. \tag{5.5}$$

The function $\varkappa(\omega)$ is analytic on $\mathbb{C}^+$, and $\widetilde{\psi}_S(x,\omega)$ is given by

$$\begin{aligned} \widetilde{\psi}_S(x,\omega) &= V(x,\omega)\widetilde{\zeta}(\omega), \quad \text{where} \\ V(x,\omega) &= \frac{e^{i\varkappa(\omega)|x|}}{4\pi|x|} + \frac{1}{\omega} D_m\Big(\frac{e^{-m|x|}}{4\pi|x|} - \frac{e^{i\varkappa(\omega)|x|}}{4\pi|x|}\Big), \qquad \omega \in \mathbb{C}^+. \end{aligned} \tag{5.6}$$

We then have, formally, for any $\varepsilon > 0$,

$$\begin{aligned} \psi_S(x,t) &= \frac{1}{2\pi}\int_{\mathrm{Im}\,\omega=\varepsilon} e^{-i\omega t} V(x,\omega)\widetilde{\zeta}(\omega)\,d\omega \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\omega t} V(x,\omega+i0)\widetilde{\zeta}(\omega+i0)\,d\omega = \mathscr{F}^{-1}_{\omega\to t}\big[V(x,\omega)\widetilde{\zeta}(\omega)\big]. \end{aligned} \tag{5.7}$$

We will justify this identities in the next section.

6. Traces on the real line

By (5.1) the Fourier transform $\widetilde{\psi}_S(\cdot,\omega) = \mathscr{F}_{t\to\omega}[\theta(t)\psi_S(\cdot,t)]$ is a tempered L^2-valued distribution of $\omega \in \mathbb{R}$. It is the boundary value of the analytic function (5.2) in the following sense:

$$\widetilde{\psi}_S(\cdot,\omega) = \lim_{\varepsilon\to 0+} \widetilde{\psi}_S(\cdot,\omega+i\varepsilon), \qquad \omega \in \mathbb{R}, \tag{6.1}$$

where the convergence holds in $\mathscr{S}'(\mathbb{R}, L^2)$. Indeed,

$$\widetilde{\psi}_S(\cdot,\omega+i\varepsilon) = \mathscr{F}_{t\to\omega}[\theta(t)\psi_S(\cdot,t)e^{-\varepsilon t}],$$

while $\theta(t)\psi_S(\cdot,t)e^{-\varepsilon t} \underset{\varepsilon\to 0+}{\longrightarrow} \theta(t)\psi_S(\cdot,t)$ in $\mathscr{S}'(\mathbb{R}, L^2)$. Therefore, (6.1) holds by the continuity of the Fourier transform $\mathscr{F}_{t\to\omega}$ in $\mathscr{S}'(\mathbb{R})$.

Similarly to (6.1), the distribution $\widetilde{\zeta}(\omega)$, $\omega \in \mathbb{R}$, is the boundary value of analytic in $\mathbb{C}^+$ function $\widetilde{\zeta}(\omega)$:

$$\widetilde{\zeta}(\omega) = \lim_{\varepsilon\to 0+} \widetilde{\zeta}(\omega+i\varepsilon), \quad \omega \in \mathbb{R}, \tag{6.2}$$

since the function $\theta(t)\zeta(t)$ is bounded. The convergence holds in the space of tempered distributions $\mathscr{S}'(\mathbb{R})$.

Let us justify that the representation (5.6) for $\widetilde{\psi}_S(x,\omega)$ is also valid when $\omega \in \mathbb{R}\backslash\{-m; m\}$. Namely,

Lemma 6.1. *$V(x,\omega)$ is a smooth function of $\omega \in \mathbb{R}\backslash\{-m; m\}$ for any fixed $x \in \mathbb{R}^3\backslash\{0\}$, and the identity*

$$\widetilde{\psi}_S(x,\omega) = V(x,\omega)\widetilde{\zeta}(\omega), \quad \omega \in \mathbb{R}\backslash\{-m; m\} \tag{6.3}$$

holds in the sense of distributions.

Proof. This lemma follows from (6.1) and (6.2) by the smoothness of $V(x,\omega)$ for $\omega \neq \pm m$. □

7. Absolutely continuous spectrum

Here we prove that the distribution $\widetilde{\zeta}(\omega) = \widetilde{\zeta}(\omega + i0)$ is absolutely continuous for real $|\omega| > m$.

Proposition 7.1. (cf. [21, Proposition 2.3]) $\widetilde{\zeta}(\omega) \in L^2_{loc}(\mathbb{R}\backslash[-m, m]) \otimes \mathbb{C}^4$.

Proof. We need to prove that

$$\int_I |\widetilde{\zeta}(\omega)|^2 \, d\omega < \infty \tag{7.1}$$

for any compact interval I such that $I \cap [-m, m] = \emptyset$. The Parseval identity applied to

$$\widetilde{\psi}_S(x, \omega + i\epsilon) = \int_0^\infty \psi_S(x, t) e^{i\omega t - \epsilon t} \, dt, \quad \epsilon > 0,$$

gives

$$\int_{\mathbb{R}} \|\widetilde{\psi}_S(\cdot, \omega + i\epsilon)\|^2_{L^2} \, d\omega = 2\pi \int_0^\infty \|\psi_S(\cdot, t)\|^2_{L^2} \, e^{-2\epsilon t} \, dt. \tag{7.2}$$

The right-hand side of (7.2) does not exceed C_0/ϵ, with some $C_0 > 0$, since $\sup_{t\geq 0} \|\psi_S(\cdot, t)\|_{L^2} < \infty$ by (5.1). Taking into account (5.6), we obtain

$$\int_{\mathbb{R}} |\widetilde{\zeta}(\omega + i\epsilon)|^2 \big\| V(\cdot, \omega + i\epsilon) \frac{\widetilde{\zeta}(\omega + i\epsilon)}{|\widetilde{\zeta}(\omega + i\epsilon)|} \big\|^2_{L^2} \, d\omega \leq \frac{C_0}{\epsilon}, \tag{7.3}$$

since for any $\varepsilon > 0$ the set of zeros of analytic function $\widetilde{\zeta}(\omega + i\varepsilon)$ has measure zero.

Lemma 7.2. *There exists C_I such that*

$$\big\| V(\cdot, \omega + i\epsilon) \frac{\widetilde{\zeta}(\omega + i\epsilon)}{|\widetilde{\zeta}(\omega + i\epsilon)|} \big\|^2_{L^2} \geq \frac{C_I}{\varepsilon}, \quad \omega \in I, \quad 0 < \varepsilon \leq |I|/2. \tag{7.4}$$

Proof. For concreteness, we will consider the case $I \subset (m, +\infty)$. Due to the middle line of (5.4), $\widehat{V}(\xi, \omega) = \widehat{V}_1(\xi) - \widehat{V}_2(\xi, \omega)$, where

$$\widehat{V}_1(\xi) = \frac{1}{\xi^2 + m^2}, \qquad \widehat{V}_2(\xi, \omega) = \frac{\omega(\alpha \cdot \xi + m\beta - \omega)}{(\xi^2 + m^2 - \omega^2)(\xi^2 + m^2)}.$$

One has

$$\begin{aligned} \big\| V_1(\cdot) \frac{\widetilde{\zeta}(\omega + i\epsilon)}{|\widetilde{\zeta}(\omega + i\epsilon)|} \big\|^2_{L^2} &= \frac{1}{(2\pi)^3} \big\| \widehat{V}_1(\cdot) \frac{\widetilde{\zeta}(\omega + i\epsilon)}{|\widetilde{\zeta}(\omega + i\epsilon)|} \big\|^2_{L^2} \\ &= \frac{1}{4\pi} \int_0^\infty \frac{\rho^2 \, d\rho}{(\rho^2 + m^2)^2} = \text{Const.} \end{aligned}$$

Hence it suffices to prove (7.4) for V_2 only.
Denote by $\Pi_\pm(\xi)$ orthogonal projections onto the eigenspaces of the operator $\widehat{D}_m(\xi) = \alpha \cdot \xi + \beta m$ corresponding to the eigenvalues $\pm\sqrt{\xi^2 + m^2}$:

$$\Pi_\pm(\xi) := \frac{1}{2}\Big(1 \pm \frac{\widehat{D}_m(\xi)}{\sqrt{\xi^2 + m^2}}\Big). \tag{7.5}$$

Denote by $e_\pm(\xi,\omega)=\Pi_\pm(\xi)\frac{\widetilde\zeta(\omega)}{|\widetilde\zeta(\omega)|}$ the eigenvectors of the operator $\alpha\cdot\xi+\beta m-\omega$. Then the function $\widehat V_2(\xi,\omega)\frac{\widetilde\zeta(\omega)}{|\widetilde\zeta(\omega)|}$ for $\omega\in\mathbb{C}^+$ can be expressed as

$$\widehat V_2(\xi,\omega)\frac{\widetilde\zeta(\omega)}{|\widetilde\zeta(\omega)|}=\omega\frac{(-\omega+\sqrt{\xi^2+m^2})e_+(\xi,\omega)+(-\omega-\sqrt{\xi^2+m^2})e_-(\xi,\omega)}{(\xi^2+m^2-\omega^2)(\xi^2+m^2)}$$
$$=\frac{\omega\, e_+(\xi,\omega)}{(\sqrt{\xi^2+m^2}+\omega)(\xi^2+m^2)}-\frac{\omega\, e_-(\xi,\omega)}{(\sqrt{\xi^2+m^2}-\omega)(\xi^2+m^2)}.$$

Using the mutual orthogonality of e_+ and e_- with respect to the L^2-product, we obtain for $\omega\in\mathbb{C}^+$

$$\Big\|V_2(\cdot,\omega)\frac{\widetilde\zeta(\omega)}{|\widetilde\zeta(\omega)|}\Big\|^2_{L^2}=\frac{|\omega|^2}{(2\pi)^3}\int\Big(\frac{|e_+(\xi,\omega)|^2}{|\sqrt{\xi^2+m^2}+\omega|^2(\xi^2+m^2)^2}+\frac{|e_-(\xi,\omega)|^2}{|\sqrt{\xi^2+m^2}-\omega|^2(\xi^2+m^2)^2}\Big)\,d\xi.$$

Hence, for $\omega\in I\subset(m,\infty)$ and $\varepsilon>0$, we have

$$\Big\|\ V_2(\cdot,\omega+i\varepsilon)\frac{\widetilde\zeta(\omega+i\varepsilon)}{|\widetilde\zeta(\omega+i\varepsilon)|}\Big\|^2_{L^2}$$
$$\ge\frac{m^2}{(2\pi)^3}\int_0^\infty\Big(\int_{|\xi|=\rho}\frac{|e_-(\xi,\omega+i\varepsilon)|^2}{((\sqrt{\xi^2+m^2}-\omega)^2+\varepsilon^2)(\xi^2+m^2)^2}\,dS\Big)\,d\rho$$
$$\ge\frac{m^2}{(2\pi)^3}\int_I\frac{Q(\omega+i\varepsilon,r)\,dr}{((r-\omega)^2+\varepsilon^2)r^3\sqrt{r^2-m^2}},\tag{7.6}$$

where

$$r=\sqrt{\rho^2+m^2},\quad\text{and}\quad Q(\omega+i\varepsilon,r):=\int_{|\xi|=\sqrt{r^2-m^2}}\big|e_-(\xi,\omega+i\varepsilon)\big|^2dS,\quad r>m.$$

Let us prove that $q(I):=\inf\limits_{\varepsilon>0}\ \inf\limits_{r,\omega'\in I}|Q(\omega'+i\varepsilon,r)|>0$. By (7.5),

$$e_-(\xi,\omega'+i\varepsilon)=\frac12\Big(1-\frac{\alpha\cdot\xi+m\beta}{\sqrt{\xi^2+m^2}}\Big)\frac{\widetilde\zeta(\omega'+i\varepsilon)}{|\widetilde\zeta(\omega'+i\varepsilon)|}$$
$$=\frac{1}{2r}(r-\alpha\cdot\xi-m\beta)\frac{\widetilde\zeta(\omega'+i\varepsilon)}{|\widetilde\zeta(\omega'+i\varepsilon)|},\quad |\xi|=\sqrt{r^2-m^2}.$$

The unit sphere S_1 and the interval I are compact sets. Hence, it suffices to show that for any vector $\mathrm{w}\in S_1$ and any $r\in I$ there exists $\xi\in S_{\sqrt{r^2-m^2}}$ such that

$$(r-\alpha\cdot\xi-m\beta)\mathrm{w}\neq0.$$

Indeed, suppose that $(r-\alpha\cdot\xi-m\beta)\mathrm{w}=0$ for some $\xi\in S_{\sqrt{r^2-m^2}}$. Then, $(\alpha\cdot\xi)\mathrm{w}=(r-m\beta)\mathrm{w}$, and for $\check\xi=-\xi$ we have

$$(r-\alpha\cdot\check\xi-m\beta)\mathrm{w}=(r-m\beta)\mathrm{w}-(\alpha\cdot\check\xi)\mathrm{w}$$
$$=(r-m\beta)\mathrm{w}+(\alpha\cdot\xi)\mathrm{w}=2(r-m\beta)\mathrm{w}\neq0$$

because of the nondegeneracy of the matrix $r - m\beta$ for $r > m$.

Now, (7.6) implies for any $\varepsilon \in (0, |I|/2)$

$$\Big\|V_2(\cdot,\omega+i\varepsilon)\frac{\widetilde{\zeta}(\omega+i\varepsilon)}{|\widetilde{\zeta}(\omega+i\varepsilon)|}\Big\|_{L^2}^2 \ge \frac{m^2 q(I)}{(2\pi)^3}\int_I \frac{dr}{((r-\omega)^2+\varepsilon^2)r^3\sqrt{r^2-m^2}}$$
$$\ge C_I \int_{I\cap[\omega-\varepsilon,\omega+\varepsilon]} \frac{dr}{2\varepsilon^2} \ge \frac{C_I}{\varepsilon}.$$

The last inequality is due to $|I \cap [\omega-\varepsilon, \omega+\varepsilon]| \ge \varepsilon$, which follows from $\omega \in I$ and $\varepsilon < |I|/2$. □

Substituting (7.4) into (7.3), we obtain

$$\int_I |\widetilde{\zeta}(\omega+i\varepsilon)|^2\, d\omega < C_0/C_I, \quad \varepsilon \in (0, |I|/2). \tag{7.7}$$

We conclude that the set of functions $g_\varepsilon(\omega) = \widetilde{\zeta}(\omega+i\varepsilon)$, $0 < \varepsilon \le \varepsilon_I$ defined for $\omega \in I$, is bounded in the Hilbert space $L^2(I)$, and, by the Banach Theorem, is weakly compact. The convergence of the distributions (6.2) implies the weak convergence $g_\varepsilon \underset{\varepsilon\to 0+}{\rightharpoonup} g$ in the Hilbert space $L^2(I)$. The limit function $g(\omega)$ coincides with the distribution $\widetilde{\zeta}(\omega)$ restricted onto I. This proves the bound (7.1) and finishes the proof of the proposition. □

8. Omega-limit compactness

Lemma 8.1. *For any sequence $s_k \to \infty$ there exists an infinite subsequence (which we also denote by s_k) such that*

$$\zeta(t+s_k) \to \eta(t), \quad k\to\infty, \quad t \in \mathbb{R}, \tag{8.1}$$

where $\eta(t)$ is some function from $C_b(\mathbb{R}) \otimes \mathbb{C}^4$. The convergence is uniform on $[-T, T]$ for any $T > 0$. Moreover, $\eta(t)$ is the solution to

$$-\dot{\eta}(t) + m\eta(t) - m\int_0^\infty \frac{J_1(ms)}{s}\eta(t-s)ds$$
$$+ im\beta\Big(\eta(t) - m\int_0^\infty \Big(\int_{ms}^\infty \frac{J_1(mu)}{u}du\Big)\eta(t-s)ds\Big) = 4\pi F(\eta(t)),\ t\in\mathbb{R}. \tag{8.2}$$

Proof. Theorem 2.1-iii), bound (2.8) and equation (2.24) imply that $\zeta \in C_b^1(\mathbb{R}) \otimes \mathbb{C}^4$. Then (8.1) follows from the Arzelá-Ascoli theorem. Further, using the asymptotics of Bessel function [33, Formula 10.7.8], we obtain

$$J_0(mt) \to 0, \quad t\int_{mt}^\infty \frac{J_1(u)du}{u}$$
$$= t\int_{mt}^\infty \Big(\frac{\cos(u-3\pi/4)}{u^{3/2}} + \mathscr{O}(u^{-5/2})\Big)du \to 0, \quad t\to\infty.$$

Moreover, for any $t \in \mathbb{R}$

$$\int_0^{t+s_k} \frac{J_1(ms)}{s}\zeta(t+s_k-s)ds \to \int_0^\infty \frac{J_1(ms)}{s}\eta(t-s)ds, \quad k\to\infty,$$

$$\int_0^{t+s_k} \Big(\int_{ms}^\infty \frac{J_1(mu)}{u}du\Big)\zeta(t+s_k-s)ds$$
$$\to \int_0^\infty \Big(\int_{ms}^\infty \frac{J_1(mu)}{u}\Big)\eta(t-s)ds, \quad j\to\infty$$

by the Lebesgue dominated convergence theorem. Then equation (2.24) for $\zeta(t)$ together with (2.15) and (4.2) implies (8.2). □

Corollary 8.2. *The distributions $\widetilde{\eta}_j(\omega)$, $j=1,\dots 4$, belongs to the space of quasimeasures which are defined as functions with bounded continuous Fourier transform.*

Lemma 8.3. $\operatorname{supp}\widetilde{\eta}\subset[-m,m]$.

Proof. Due to (8.1) and the continuity of the Fourier transform in $\mathscr{S}'(\mathbb{R})$, we have

$$\chi(\omega)\widetilde{\zeta}(\omega)e^{-i\omega s_k}\overset{\mathscr{S}'}{\longrightarrow}\chi(\omega)\eta(\omega), \quad k\to\infty.$$

for any $\chi\in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp}\chi\cap[-m,m]=\emptyset$. The products $\chi(\omega)\widetilde{\zeta}(\omega)$ are absolutely continuous measures since $\widetilde{\zeta}(\omega)$ is locally L^2 for $\omega\in\mathbb{R}\backslash[-m,m]$ by Proposition 7.1. Then $\eta(\omega)=0$ for $\omega\notin[-m,m]$ by the Riemann–Lebesgue Theorem. □

9. Spectral inclusion and the Titchmarsh theorem

Here we will prove the following identity

$$\eta_j(t)=C_je^{-i\omega_j^+t}, \quad t\in R, \quad \omega_j^+\in[-m,m], \quad j=1,\dots,4. \tag{9.1}$$

We start with an investigation of $\operatorname{supp}\widetilde{F_j(\eta_j)}$.

Lemma 9.1. *The following spectral inclusion holds:*

$$\operatorname{supp}\widetilde{F_j(\eta_j)}\subset\operatorname{supp}\widetilde{\eta}_j. \tag{9.2}$$

Proof. Applying the Fourier transform to (8.2), we get by the theory of quasimeasures (see [20]) that

$$4\pi\widetilde{F_j(\eta_j)}(\omega)=\big(i\omega+m-m\widetilde{P}(\omega)+im\sigma_j(1-m\widetilde{Q}(\omega))\big)\widetilde{\eta}_j(\omega), \quad j=1,\dots,4. \tag{9.3}$$

where σ_j is defined in (3.9), $\widetilde{P}(\omega)$ and $\widetilde{Q}(\omega)$ are the Fourier transforms of the functions $P(t)=\theta(t)\frac{J_1(mt)}{t}$ and $Q(t)=\theta(t)\int_{mt}^\infty\frac{J_1(mu)}{u}du$. Note that $P(t)$ and $Q(t)$ belong to $L^1(\mathbb{R})$. Therefore, the multiplication by $\widetilde{P}(\omega)$ and $\widetilde{Q}(\omega)$ is well-defined in the sense of quasimeasures (see Appendix B of [20]). Finally, (9.3) implies (9.2). □

The second step is the following lemma

Lemma 9.2. *For any omega-limit trajectory $\eta_j(t)$ one has*

$$|\eta_j(t)|=\text{const}, \quad t\in\mathbb{R}. \tag{9.4}$$

Proof. The assumption (3.2) implies that the function $F_j(\eta_j(t))$, $j=1,\dots,4$ admits the representation

$$F_j(\eta_j(t)) = a_j(\eta_j(t))\eta_j(t), \tag{9.5}$$

where, according to (3.3)

$$a_j(\eta_j) = \sum_{n=1}^{N_j} 2nu_{n,j}|\eta_j|^{2n-2}. \tag{9.6}$$

The functions $\eta_j(t)$ and $a_j(\eta_j(t))$ are bounded continuous functions in $\mathbb{R}$ by Lemma 8.1. Hence, $\eta_j(t)$ and $a_j(\eta_j(t))$ are tempered distributions. Moreover, $\operatorname{supp}\widetilde{\eta}_j \subset [-m,m]$ and $\operatorname{supp}\widetilde{\overline{\eta}}_j \subset [-m,m]$ according to Lemma 8.3. Hence, $\widetilde{a_j(\eta_j)}$ also has a bounded support. Denote $\mathbf{F}_j = \operatorname{supp}\widetilde{F_j(\eta_j)}$, $\mathbf{A}_j = \operatorname{supp}\widetilde{a_j(\eta_j)}$, $\mathbf{Z}_j = \operatorname{supp}\widetilde{\eta}_j$. Then the spectral inclusion (9.2) gives

$$\mathbf{F}_j \subset \mathbf{Z}_j.$$

On the other hand, applying the Titchmarsh convolution theorem [19, Theorem 4.3.3] to (9.5), we obtain

$$\inf\,\mathbf{F}_j = \inf\,\mathbf{A}_j + \inf\,\mathbf{Z}_j, \quad \sup\,\mathbf{F}_j = \sup\,\mathbf{A}_j + \sup\,\mathbf{Z}_j.$$

Hence, $\inf\,\mathbf{A}_j = \sup\,\mathbf{A}_j = 0$, and then $\mathbf{A}_j \subset \{0\}$. Thus, we conclude that $\operatorname{supp}\widetilde{a_j(\eta_j)} = \mathbf{A}_j \subset \{0\}$, and therefore the distribution $\widetilde{a_j(\eta_j)}(\omega)$ is a finite linear combination of $\delta(\omega)$ and it's derivatives. Then $a_k(\eta_j(t))$ is a polynomial in t. By Lemma 8.1, $a_j(\eta_j(t))$ is bounded then we conclude that $a_j(\eta_j(t)) = \text{const}$. Finally, (9.4) follows since $a_j(\eta_j(t))$ is a polynomial in $\eta_j(t)$, and its degree $2N-2 \geq 2$ by (3.1) and (9.6). □

Now (9.4) means that $\eta_j(t)\overline{\eta}_j(t) \equiv C = \text{const}$, and then $\widetilde{\eta}_j * \widetilde{\overline{\eta}}_j = 2\pi C\delta(\omega-\omega_j^+)$. Hence, if η_j is not identically zero, the Titchmarsh theorem implies that $\mathbf{Z}_j = \omega_j \in [-m,m]$. Indeed,

$$0 = \sup\,\mathbf{Z}_j + \sup\,(-\mathbf{Z})_j = \sup\,\mathbf{Z}_j - \inf\,\mathbf{Z}_j,$$

and hence $\inf\,\mathbf{Z}_j = \sup\,\mathbf{Z}_j$. Therefore, $\widetilde{\eta}_j$ is a finite linear combination of $\delta(\omega-\omega_j^+)$ and its derivatives. But the derivatives could not be present because of the boundedness of $\eta_j(t)$. Thus $\widetilde{\eta}_j \sim \delta(\omega-\omega_j^+)$, which implies (9.1).

10. Convergence of singular component

Denote

$$\psi_S^-(x,t) = \psi_S(x,t) - iD_m^{-1}\zeta_0\dot{\gamma}(x,t) = \varphi_S(x,t) + iD_m^{-1}p_S(x,t), \tag{10.1}$$

where $\varphi_S(x,t)$ and $p_S(x,t)$ are defined in (2.18) and (2.19). Here we prove that $\psi_S^-(x,t)$ converges to some solitary wave.

Lemma 10.1. *The convergence holds*

$$\psi_S^-(\cdot,t+s_j)\to\phi_{\Omega^+}(\cdot,t)+iD_m^{-1}\dot\phi_{\Omega^+}(\cdot,t),\quad j\to\infty \tag{10.2}$$

in the topology of $C_b([-T,T],L^2_{loc}(\mathbb{R}^3))$ for any $T>0$. Here

$$\phi_{\Omega^+,j}(x,t)=\phi_{\omega_j^+}(x)e^{-i\omega_j^+t}=C_j\frac{e^{-\sqrt{m^2-(\omega_j^+)^2}|x|}}{4\pi|x|}e^{-i\omega_j^+t},\quad j=1,\dots,4.$$

Proof. Definition (2.18) of $\varphi_S(x,t)$, Lemma 8.1 and identity (9.1) imply that for any $x\neq 0$

$$\begin{aligned}\varphi_{S,j}(x,t+s_k)\to&\ \frac{C_je^{-i\omega_j^+(t-|x|)}}{4\pi|x|}-\frac{mC_j}{4\pi}\int_0^\infty\frac{\theta(s-|x|)J_1(m\sqrt{s^2-|x|^2})}{\sqrt{s^2-|x|^2}}e^{-i\omega_j^+(t-s)}ds\\ =&\ \frac{C_je^{-i\omega_j^+t}}{4\pi}\Big(\frac{e^{i\omega_j^+|x|}}{|x|}-m\widetilde{L}(x,\omega_j^+)\Big)=C_je^{-i\omega_j^+t}\frac{e^{-\sqrt{m^2-(\omega_j^+)^2}|x|}}{4\pi|x|}\\ =&\ \phi_{\omega_j^+}(x)e^{-i\omega_j^+t},\quad k\to\infty,\quad t\in\mathbb{R}\end{aligned}$$

by the Lebesgue dominated convergence theorem. Here $\widetilde{L}(x,\omega)=\frac{1}{m|x|}\big(e^{i|x|\omega}-e^{i|x|\sqrt{\omega^2-m^2}}\big)$ is the Fourier transform of the function $L(x,t)=\frac{\theta(t-|x|)J_1(m\sqrt{t^2-|x|^2})}{\sqrt{t^2-|x|^2}}$ (see Appendix in [27]). Hence, for any $T>0$,

$$\varphi_S(\cdot,t+s_k)\to\phi_{\Omega^+}(\cdot,t),\quad k\to\infty \tag{10.3}$$

in $C_b([-T,T],L^2_{loc}(\mathbb{R}^3))$. It remains to prove that for any $T>0$

$$D_m^{-1}p_S(\cdot,t+s_k)\to D_m^{-1}\dot\phi_{\Omega^+}(\cdot,t)\quad k\to\infty \tag{10.4}$$

in $C_b([-T,T],H^1_{loc}(\mathbb{R}^3))$. Lemma 8.1 and equation (2.24) imply that

$$\dot\zeta_j(t+s_k)\to\dot\eta_j(t)=C_j(-i\omega_j^+e^{-i\omega_j^+t}),\quad k\to\infty,\quad t\in\mathbb{R} \tag{10.5}$$

uniformly on $[-T,T]$ for any $T>0$. Hence, using (2.21), we obtain similarly to (10.3) that for any $T>0$,

$$p_S(\cdot,t+s_k)\to\dot\phi_{\Omega^+}(\cdot,t),\quad k\to\infty \tag{10.6}$$

in $C_b([-T,T],L^2_{loc}(\mathbb{R}^3))$. Further, $D_m^{-2}p_{S,j}(x,t)=\displaystyle\int_{\mathbb{R}^3}\frac{e^{-m|y|}}{4\pi|y|}p_{S,j}(x-y,t)dy$, and

$$\begin{aligned}&\int_{\mathbb{R}^3}\frac{e^{-m|y|}}{4\pi|y|}|p_{S,j}(x-y,t)|dy\\ &\quad\le C\left(\int_{\mathbb{R}^3}\frac{e^{-m|y|}}{|y||x-y|}dy+\int_{\mathbb{R}^3}\frac{e^{-m|y|}}{|y|}dy\right)\le C'\le\infty,\quad x\in\mathbb{R}^3,\quad t\in\mathbb{R}\end{aligned}$$

by (2.21). Then for any $x\in\mathbb{R}^3$,

$$\lim_{k\to\infty}D_m^{-2}p_S(\cdot,t+s_k)=D_m^{-2}\lim_{k\to\infty}p_S(\cdot,t+s_k)=D_m^{-2}\dot\phi_{\Omega^+}(x,y) \tag{10.7}$$

by the Lebesgue dominated convergence theorem. Finally, from (10.6)–(10.7) it follows that for any $T>0$,

$$D_m^{-1}p_S(\cdot,t+s_k)\to D_m^{-1}\dot\phi_{\Omega^+}(\cdot,t),\quad k\to\infty$$

in $C_b([-T,T],H^1_{loc}(\mathbb{R}^3))$, which implies (10.4). □

11. Proof of global attraction

Substituting (2.13) and (2.17) into (2.4), we obtain

$$\begin{aligned}\psi(x,t) &= \psi_f(x.t)+\varphi_g(x,t)+\psi_S(x,t)\\ &= \psi_f(x.t)+\zeta_0\big(g(x)-\gamma(x,t)\big)-iD_m^{-1}\zeta_0\dot\gamma(x,t)\\ &\quad +\varphi_S(x,t)+iD_m^{-1}\zeta_0\dot\gamma(x,t)+iD_m^{-1}p_S(x,t)\\ &= \psi_f(x.t)+\varphi_g^+(x,t)+\psi_S^-(x,t)\end{aligned}$$

by (4.3) and (10.1). Due to Lemmas 4.1 and 4.3 it suffices to prove that

$$\lim_{t\to\infty}\operatorname{dist}_{L^2_{loc}(\mathbb{R}^3)}(\psi_S^-(\cdot,t),\mathscr{S})=0. \tag{11.1}$$

Assume by contradiction that there exists a sequence $s_k\to\infty$ such that

$$\operatorname{dist}_{L^2_{loc}(\mathbb{R}^3)}(\psi_S^-(\cdot,s_k),\mathscr{S})\ge\delta,\quad \forall k \tag{11.2}$$

for some $\delta>0$. According to Lemma 10.1, there exist a subsequence s_{k_n} of the sequence s_k, $\omega_j^+\in\mathbb{R}$ and functions $\phi_{\omega_j^+}$ such that

$$\psi_S^-(\cdot,t+s_{k_n})\to\phi_{\Omega^+}(\cdot,t)+iD_m^{-1}\dot\phi_{\Omega^+}(\cdot,t),\quad k_n\to\infty,\quad t\in\mathbb{R},$$

in $C_b([-T,T],L^2_{loc}(\mathbb{R}^3))$ with any $T>0$. This implies that

$$\psi_S^-(\cdot,s_{k_n})\to\Phi_{\Omega_+}(\cdot)+D_m^{-1}\Psi_{\Omega^+}(\cdot),\quad k_n\to\infty, \tag{11.3}$$

in $L^2_{loc}(\mathbb{R}^3)$. Here

$$\Phi_{\Omega_+,j}(x)=\phi_{\Omega^+,j}(x,0)=\phi_{\omega_j^+}(x),\ \ \Psi_{\Omega^+,j}(x)=i\dot\phi_{\Omega^+,j}(x,0)=\omega_j^+\psi_{\omega_j^+}(x).$$

The convergence (11.3) contradict (11.2) due to (3.20). This completes the proof of Theorem 3.6. □

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A. Proof of Lemma 2.2

Note that

$$\begin{aligned}&\lim_{x\to 0}(g(x)-\gamma(x,t))\\&\quad=\lim_{x\to 0}\left(\frac{e^{-m|x|}}{4\pi|x|}-\frac{\theta(t-|x|)}{4\pi|x|}-\frac{m}{4\pi}\int_0^t\frac{\theta(s-|x|)J_1(m\sqrt{s^2-|x|^2})}{\sqrt{s^2-|x|^2}}ds\right)\\&\quad=-\frac{m}{4\pi}+\frac{m}{4\pi}\int_0^t\frac{J_1(ms)}{s}ds.\end{aligned}$$

Hence it suffices to prove that

$$\lim_{\varepsilon\to 0+}\lim_{x\to 0}K_m^\varepsilon(g(x)-\gamma(x,t))=\lim_{x\to 0}(g(x)-\gamma(x,t)).\tag{A.1}$$

Applying the Fourier transform $\widehat{f}(\xi)=\mathscr{F}_{x\to\xi}f(x)$, we get

$$\begin{aligned}\widehat{g}(\xi,t)-\widehat{\gamma}(\xi,t)&=\frac{1}{\xi^2+m^2}-\int_0^t\frac{\sin(s\sqrt{\xi^2+m^2})}{\sqrt{\xi^2+m^2}}ds=\frac{\cos(t\sqrt{\xi^2+m^2})}{\xi^2+m^2}\\&=\frac{\cos(t\sqrt{\xi^2+m^2})-\cos(t|\xi|)}{\xi^2}\\&\quad-\frac{m^2\cos(t\sqrt{\xi^2+m^2})}{\xi^2(\xi^2+m^2)}+\frac{\cos(t|\xi|)}{\xi^2},\quad t>0.\end{aligned}\tag{A.2}$$

Then for (A.1) it suffices to justify the following permutation of limits:

$$\begin{aligned}&\lim_{\varepsilon\to 0+}\lim_{x\to 0}K_m^\varepsilon\mathscr{F}^{-1}_{\xi\to x}\Big(\frac{\cos(t\sqrt{\xi^2+m^2})-\cos(t|\xi|)}{\xi^2}-\frac{m^2\cos(t\sqrt{\xi^2+m^2})}{\xi^2(\xi^2+m^2)}+\frac{\cos(t|\xi|)}{\xi^2}\Big)\\&\quad=\lim_{x\to 0}\mathscr{F}^{-1}_{\xi\to x}\Big(\frac{\cos(t\sqrt{\xi^2+m^2})-\cos(t|\xi|)}{\xi^2}\\&\qquad-\frac{m^2\cos(t\sqrt{\xi^2+m^2})}{\xi^2(\xi^2+m^2)}+\frac{\cos(t|\xi|)}{\xi^2}\Big),\quad t>0.\end{aligned}\tag{A.3}$$

We will do it for each term in (A.3) separately.
Step i) Applying the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0+}\lim_{x\to 0}K_m^\varepsilon\mathscr{F}^{-1}_{\xi\to x}\frac{m^2\cos(t\sqrt{\xi^2+m^2})}{\xi^2(\xi^2+m^2)}&=\lim_{x\to 0}\mathscr{F}^{-1}_{\xi\to x}\frac{m^2\cos(t\sqrt{\xi^2+m^2})}{\xi^2(\xi^2+m^2)}\\&=\frac{m^2}{2\pi^2}\int_0^\infty\frac{\cos(t\sqrt{r^2+m^2})}{r^2+m^2}dr.\end{aligned}$$

Step ii) Let us prove that

$$\lim_{\varepsilon\to 0+}\lim_{x\to 0}K_m^\varepsilon\mathscr{F}^{-1}_{\xi\to x}\frac{\cos(t|\xi|)}{\xi^2}=\lim_{x\to 0}\mathscr{F}^{-1}_{\xi\to x}\frac{\cos(t|\xi|)}{\xi^2}=0,\quad t>0.$$

Indeed, for $\rho:=|x|<t$,

$$\begin{aligned}\lim_{x\to 0}\mathscr{F}^{-1}_{\xi\to x}\frac{\cos(t|\xi|)}{\xi^2}&=\lim_{\rho\to 0+}\frac{1}{2\pi^2}\int_0^\infty\frac{\sin(r\rho)\cos(tr)}{r\rho}dr\\&=\lim_{\rho\to 0+}\frac{1}{4\pi^2\rho}\int_0^\infty\frac{\sin(r(t+\rho))-\sin(r(t-\rho))}{r}dr=0,\end{aligned}$$

(A.4)

since

$$\int_0^\infty \frac{\sin(r\alpha)}{r}dr = \int_0^\infty \frac{\sin u}{u}du, \quad \alpha > 0.$$

On the other hand,

$$\begin{aligned}
&\lim_{x\to 0} K_m^\varepsilon \mathscr{F}^{-1}_{\xi\to x}\frac{\cos(t|\xi|)}{\xi^2}\\
&\quad= \lim_{x\to 0}\mathscr{F}^{-1}_{\xi\to x}\frac{\cos(t|\xi|)}{\xi^2(\xi^2+m^2)^\varepsilon} = \lim_{\rho\to 0+}\frac{1}{2\pi^2\rho}\int_0^\infty \frac{\sin(r\rho)\cos(tr)}{r(r^2+m^2)^\varepsilon}dr\\
&\quad= \frac{1}{2\pi^2}\int_0^\infty \frac{\cos(tr)\,dr}{(r^2+m^2)^\varepsilon} = \frac{1}{2\pi^2}\Big(\frac{2m}{t}\Big)^{\frac12-\varepsilon}\frac{\sqrt{\pi}\mathbf{K}_{\varepsilon-\frac12}(mt)}{\Gamma(\varepsilon)}
\end{aligned}\tag{A.5}$$

by [14, Formula 1.3.(7)]. Here $\mathbf{K}_\nu$ is the modified Bessel function, and Γ is the gamma function. One can justify the last limit, splitting the integral into a sum of integrals over the intervals $[0,1]$ and $[1,\infty)$, and integrating by parts in the second one. Further,

$$\lim_{\varepsilon\to 0+}\lim_{x\to 0} K_m^\varepsilon \mathscr{F}^{-1}_{\xi\to x}\frac{\cos(t|\xi|)}{\xi^2} = \lim_{\varepsilon\to 0+}\frac{1}{2\pi^2}\Big(\frac{2m}{t}\Big)^{\frac12-\varepsilon}\frac{\sqrt{\pi}\mathbf{K}_{\varepsilon-\frac12}(mt)}{\Gamma(\varepsilon)} = 0, \quad t>0,$$

since

$$\lim_{\varepsilon\to 0+}\frac{1}{\Gamma(\varepsilon)} = 0, \qquad \lim_{\varepsilon\to 0+}\mathbf{K}_{\varepsilon-\frac12}(mt) = \mathbf{K}_{-\frac12}(mt) = \Big(\frac{\pi}{2mt}\Big)^{\frac12}e^{-mt} \tag{A.6}$$

by [33, Formulas 5.7.1 and 10.39.2].

Step iii) It remains to check that

$$\begin{aligned}
&\lim_{\varepsilon\to 0+}\lim_{x\to 0} K_m^\varepsilon \mathscr{F}^{-1}_{\xi\to x}\frac{\cos(t\sqrt{\xi^2+m^2})-\cos(t|\xi|)}{\xi^2}\\
&\quad= \lim_{x\to 0}\mathscr{F}^{-1}_{\xi\to x}\frac{\cos(t\sqrt{\xi^2+m^2})-\cos(t|\xi|)}{\xi^2}, \quad t>0.
\end{aligned}\tag{A.7}$$

One has

$$\begin{aligned}
&K_m^\varepsilon \mathscr{F}^{-1}_{\xi\to x}\frac{\cos(t\sqrt{\xi^2+m^2})-\cos(t|\xi|)}{\xi^2}\\
&\quad= \frac{1}{2\pi^2}\int_0^{2m}\frac{\big(\cos(t\sqrt{r^2+m^2})-\cos(tr)\big)\sin(\rho r)}{\rho r(r^2+m^2)^\varepsilon}dr\\
&\qquad+\sum_\pm \frac{1}{4\pi^2}\int_{2m}^\infty \frac{\big(e^{\pm it\sqrt{r^2+m^2}}-e^{\pm itr}\big)\sin(\rho r)}{\rho r(r^2+m^2)^\varepsilon}dr, \quad \rho=|x|.
\end{aligned}\tag{A.8}$$

Evidently,

$$\begin{aligned}
&\lim_{\varepsilon\to 0+}\lim_{\rho\to 0+}\int_0^{2m}\frac{\big(\cos(t\sqrt{r^2+m^2})-\cos(tr)\big)\sin(\rho r)}{\rho r(r^2+m^2)^\varepsilon}dr\\
&\quad= \lim_{\rho\to 0+}\int_0^{2m}\frac{\big(\cos(t\sqrt{r^2+m^2})-\cos(tr)\big)\sin(\rho r)}{\rho r}dr.
\end{aligned}$$

Hence, (A.7) will follow from

$$\lim_{\varepsilon\to 0+}\lim_{\rho\to 0+}\int_{2m}^{\infty}\frac{(e^{\pm it\sqrt{r^2+m^2}}-e^{\pm itr})\sin(\rho r)}{\rho r(r^2+m^2)^{\varepsilon}}dr$$
$$=\lim_{\rho\to 0+}\int_{2m}^{\infty}\frac{(e^{\pm it\sqrt{r^2+m^2}}-e^{\pm itr})\sin(\rho r)}{\rho r}dr. \tag{A.9}$$

One has

$$e^{\pm it\sqrt{r^2+m^2}}-e^{\pm itr}=e^{\pm itr}\Big(\pm\frac{itm^2}{2r}+R_{\pm}(r,t)\Big),\quad r\ge 2m,\quad t>0, \tag{A.10}$$

where

$$|R_{\pm}(r,t)|\le C(m)(1+t)^2/r^2,\qquad |r|\ge 2m. \tag{A.11}$$

The last estimate implies

$$\lim_{\varepsilon\to 0+}\lim_{\rho\to 0+}\int_{2m}^{\infty}\frac{e^{\pm itr}R_{\pm}(r,t)\sin(\rho r)}{\rho r(r^2+m^2)^{\varepsilon}}dr$$
$$=\lim_{\rho\to 0+}\int_{2m}^{\infty}\frac{e^{\pm itr}R_{\pm}(r,t)\sin(\rho r)}{\rho r}dr=\int_{2m}^{\infty}e^{\pm itr}R_{\pm}(r,t)\,dr. \tag{A.12}$$

Moreover,

$$\lim_{\varepsilon\to 0+}\lim_{\rho\to 0+}\int_{2m}^{\infty}\frac{e^{\pm itr}\sin(r\rho)\,dr}{\rho r^2(r^2+m^2)^{\varepsilon}}$$
$$=\lim_{\rho\to 0+}\int_{2m}^{\infty}\frac{e^{\pm itr}\sin(r\rho)\,dr}{\rho r^2}=\int_{2m}^{\infty}\frac{e^{\pm itr}\,dr}{r},\quad t>0, \tag{A.13}$$

that easily follows by means of integration by parts. Finally, (A.12)–(A.13) imply (A.9).

B. Proof of Proposition 2.4

For any $t>0$,

$$\lim_{x\to 0}(\varphi_S(x,t)-\zeta(t)g(x))=\lim_{x\to 0}\Big(\frac{\theta(t-|x|)}{4\pi|x|}\zeta(t-|x|)-\zeta(t)\frac{e^{-m|x|}}{4\pi|x|}$$
$$-\frac{m}{4\pi}\int_0^t\frac{\theta(s-|x|)J_1(m\sqrt{s^2-|x|^2})}{\sqrt{s^2-|x|^2}}\zeta(t-s)ds\Big)$$
$$=\frac{1}{4\pi}\Big(m\zeta(t)-\dot\zeta(t)-m\int_0^t\frac{J_1(ms)}{s}\zeta(t-s)ds\Big),$$

Hence, it suffices to prove that

$$\lim_{\varepsilon\to 0+}\lim_{x\to 0}K_m^{\varepsilon}(\varphi_S(x,t)-\zeta(t)g(x))=\lim_{x\to 0}(\varphi_S(x,t)-\zeta(t)g(x)). \tag{B.1}$$

The Fourier transform of $\varphi_S(x,t)-\zeta(t)g(x)$ for any $t>0$ reads

$$\widehat{\varphi}_S(\xi,t)-\zeta(t)\widehat{g}(\xi)=\int_0^t\frac{\sin(s\sqrt{\xi^2+m^2})}{\sqrt{\xi^2+m^2}}\zeta(t-s)ds-\frac{\zeta(t)}{\xi^2+m^2}$$

$$= -\frac{\cos(t\sqrt{\xi^2+m^2})}{\xi^2+m^2}\zeta(0) + \int_0^t \frac{\cos(s\sqrt{\xi^2+m^2})}{\xi^2+m^2}\dot\zeta(t-s)ds.$$

Due to (A.2)–(A.3),

$$\lim_{\varepsilon\to 0+}\lim_{x\to 0} K_m^\varepsilon \mathscr{F}^{-1}_{\xi\to x}\frac{\cos(t\sqrt{\xi^2+m^2})}{\xi^2+m^2} = \lim_{x\to 0}\mathscr{F}^{-1}_{\xi\to x}\frac{\cos(t\sqrt{\xi^2+m^2})}{\xi^2+m^2}.$$

Therefore, because of the continuity of the Fourier transform in $\mathscr{S}'$, it remains to prove that

$$\lim_{\varepsilon\to 0+}\lim_{x\to 0}\int_0^t \mathscr{F}^{-1}_{\xi\to x}\frac{\cos(s\sqrt{\xi^2+m^2})}{(\xi^2+m^2)^{1+\varepsilon}}\dot\zeta(t-s)ds$$
$$= \lim_{x\to 0}\int_0^t \mathscr{F}^{-1}_{\xi\to x}\frac{\cos(s\sqrt{\xi^2+m^2})}{\xi^2+m^2}\dot\zeta(t-s)ds. \quad (B.2)$$

Due to splitting (A.2), equality (B.2) will follow from the following three equalities

$$\lim_{\varepsilon\to 0+}\lim_{x\to 0}\int_0^t \mathscr{F}^{-1}_{\xi\to x}\frac{\cos(s\sqrt{\xi^2+m^2})}{\xi^2(\xi^2+m^2)^{1+\varepsilon}}\dot\zeta(t-s)ds$$
$$= \lim_{x\to 0}\int_0^t \mathscr{F}^{-1}_{\xi\to x}\frac{\cos(s\sqrt{\xi^2+m^2})}{\xi^2(\xi^2+m^2)}\dot\zeta(t-s)ds. \quad (B.3)$$

$$\lim_{\varepsilon\to 0+}\lim_{x\to 0}\int_0^t \mathscr{F}^{-1}_{\xi\to x}\frac{\cos(s\sqrt{\xi^2+m^2})-\cos(t|\xi|)}{\xi^2(\xi^2+m^2)^{\varepsilon}}\dot\zeta(t-s)ds$$
$$= \lim_{x\to 0}\int_0^t \mathscr{F}^{-1}_{\xi\to x}\frac{\cos(s\sqrt{\xi^2+m^2})-\cos(s|\xi|)}{\xi^2}\dot\zeta(t-s)ds. \quad (B.4)$$

$$\lim_{\varepsilon\to 0+}\lim_{x\to 0}\int_0^t \mathscr{F}^{-1}_{\xi\to x}\frac{\cos(s|\xi|)}{\xi^2(\xi^2+m^2)^{\varepsilon}}\dot\zeta(t-s)ds$$
$$= \lim_{x\to 0}\int_0^t \mathscr{F}^{-1}_{\xi\to x}\frac{\cos(s|\xi|)}{\xi^2}\dot\zeta(t-s)ds. \quad (B.5)$$

Step i) For any $s\in\mathbb{R}$ and $\varepsilon\geq 0$ the function $\widehat{T}_\varepsilon(\xi,s) := \frac{\cos(s\sqrt{\xi^2+m^2})}{\xi^2(\xi^2+m^2)^{1+\varepsilon}} \in L^1(\mathbb{R})$. Hence $T_\varepsilon(x,s) = \mathscr{F}^{-1}_{\xi\to x}\widehat{T}_\varepsilon(\xi,s)$ is continuous in x. Moreover it is uniformly continuous in s. Therefore,

$$\lim_{\varepsilon\to 0+}\lim_{x\to 0}\int_0^t T_\varepsilon(x,s)\dot\zeta(t-s)ds$$
$$= \lim_{\varepsilon\to 0+}\int_0^t T_\varepsilon(0,s)\dot\zeta(t-s)ds$$
$$= \int_0^t T_0(0,s)\dot\zeta(t-s)ds = \lim_{x\to 0}\int_0^t T_0(x,s)\dot\zeta(t-s)ds. \quad (B.6)$$

by uniformly continuity of $T_\varepsilon(0,s)$ in ε.

Step ii) Let us prove (B.4). Due to (A.8)

$$\mathscr{F}^{-1}_{\xi\to x}\frac{\cos(s\sqrt{\xi^2+m^2})-\cos(s|\xi|)}{\xi^2(\xi^2+m^2)^\varepsilon}=\frac{1}{2\pi^2}S_\varepsilon(\rho,s)+\frac{1}{4\pi^2}\sum_\pm P^\pm_\varepsilon(\rho,s),$$

where $\rho=|x|$, and

$$S_\varepsilon(\rho,s)=\int_0^{2m}\frac{\big(\cos(s\sqrt{r^2+m^2})-\cos(sr)\big)\sin(\rho r)}{\rho r(r^2+m^2)^\varepsilon}dr,$$
$$P^\pm_\varepsilon(\rho,s)=\int_{2m}^\infty\frac{\big(e^{\pm is\sqrt{r^2+m^2}}-e^{\pm isr}\big)\sin(\rho r)}{\rho r(r^2+m^2)^\varepsilon}dr,$$

Applying the Lebesgue dominated convergence theorem, we obtain

$$\lim_{\varepsilon\to0+}\lim_{\rho\to0}\int_0^t S_\varepsilon(\rho,s)\dot\zeta(t-s)ds=\int_0^t S_0(0,s)\dot\zeta(t-s)ds$$
$$=\lim_{\rho\to0}\int_0^t S_0(\rho,s)\dot\zeta(t-s)ds$$

where $S_\varepsilon(0,s)=\lim\limits_{\rho\to0}S_\varepsilon(\rho,s)=\int_0^{2m}\frac{\cos(s\sqrt{r^2+m^2})-\cos(sr)}{(r^2+m^2)^\varepsilon}dr$, $\varepsilon\ge0$. Further, (A.10) implies

$$P^\pm_\varepsilon(\rho,s)=\pm\frac{im^2}{2}\int_{2m}^\infty\frac{se^{\pm isr}\sin(r\rho)\,dr}{\rho r^2(r^2+m^2)^\varepsilon}+\int_{2m}^\infty\frac{e^{\pm isr}R_\pm(r,s)\sin(\rho r)}{\rho r(r^2+m^2)^\varepsilon}dr$$
$$=\frac{m^2}{2}\int_{2m}^\infty\frac{\sin(r\rho)\,de^{\pm isr}}{\rho r^2(r^2+m^2)^\varepsilon}+\widetilde R^\pm_\varepsilon(\rho,s),$$

where

$$\lim_{\varepsilon\to0+}\lim_{\rho\to0}\int_0^t\widetilde R^\pm_\varepsilon(\rho,s)\dot\zeta(t-s)ds=\int_0^t\widetilde R^\pm_0(0,s)\dot\zeta(t-s)ds$$
$$=\lim_{\rho\to0}\int_0^t\widetilde R^\pm_0(\rho,s)\dot\zeta(t-s)ds$$

by (A.11) and the Lebesgue theorem. Finally,

$$\lim_{\varepsilon\to0+}\lim_{\rho\to0}\int_0^t\Big(\int_{2m}^\infty\frac{\sin(r\rho)}{\rho r^2(r^2+m^2)^\varepsilon}de^{\pm isr}\Big)\dot\zeta(t-s)ds$$
$$=\lim_{\rho\to0}\int_0^t\Big(\int_{2m}^\infty\frac{\sin(r\rho)}{\rho r^2}de^{\pm isr}\Big)\dot\zeta(t-s)ds,$$

which easily follows by means of integration by parts.
Step iii) Let us prove (B.5). Due to (A.5) we need to prove that

$$\lim_{\varepsilon\to0+}\lim_{\rho\to0+}\int_0^t\Big(\int_0^\infty\frac{\sin(r\rho)\cos(sr)}{\rho r(r^2+m^2)^\varepsilon}dr\Big)\dot\zeta(t-s)ds$$
$$=\lim_{\rho\to0+}\int_0^t\Big(\int_0^\infty\frac{\sin(r\rho)\cos(sr)}{\rho r}dr\Big)\dot\zeta(t-s)ds. \tag{B.7}$$

Taking into account (A.4), we obtain

$$\lim_{\rho\to 0+}\int_0^t\left(\int_0^\infty \frac{\sin(r\rho)\cos(sr)}{\rho r}dr\right)\dot\zeta(t-s)ds$$
$$=\lim_{\rho\to 0+}\int_0^\rho\left(\int_0^\infty \frac{\sin(r\rho)\cos(sr)}{\rho r}dr\right)\dot\zeta(t-s)ds$$
$$=\lim_{\rho\to 0+}\frac{1}{2\rho}\int_0^\rho\left(\int_0^\infty \frac{\sin(r(s+\rho))+\sin(r(\rho-s))}{r}dr\right)\dot\zeta(t-s)ds$$
$$=\frac{\pi}{2}\lim_{\rho\to 0+}\frac{1}{\rho}\int_0^\rho \dot\zeta(t-s)ds=\frac{\pi}{2}\dot\zeta(t),\quad t>0,$$

since $\int_0^\infty \frac{\sin u}{u}=\frac{\pi}{2}$. Hence (B.7) is equivalent to

$$\lim_{\varepsilon\to 0+}\lim_{\rho\to 0+}\int_0^t\Big(\int_0^\infty \frac{\sin(r\rho)\cos(sr)}{\rho r(r^2+m^2)^\varepsilon}dr\Big)\dot\zeta(t-s)ds=\frac{\pi}{2}\dot\xi(t). \tag{B.8}$$

By [14, Formula 2.3.(28)],

$$\int_0^\infty \frac{\sin(r\rho)\cos(sr)}{r(r^2+m^2)^\varepsilon}dr=\int_0^\infty \frac{\sin(r(s+\rho))-\sin(r(s-\rho))}{2r(r^2+m^2)^\varepsilon}dr$$
$$=\frac{1}{2}\begin{cases} G_1(\varepsilon,s+\rho)+G_2(\varepsilon,s+\rho)-G_1(\varepsilon,s-\rho)-G_2(\varepsilon,s-\rho), & \rho<s\\ G_1(\varepsilon,s+\rho)+G_2(\varepsilon,s+\rho)+G_1(\varepsilon,\rho-s)+G_2(\varepsilon,\rho-s), & s<\rho\end{cases} \tag{B.9}$$

where

$$G_1(\varepsilon,z)=\frac{\sqrt{\pi}}{2}m^{1-2\varepsilon}\frac{\Gamma(-\frac12+\varepsilon)}{\Gamma(\varepsilon)}\ {}_1F_2(\frac12,\frac32-\varepsilon,\frac32,\frac14 m^2z^2)z,$$
$$G_2(\varepsilon,z)=\frac{\sqrt{\pi}2^{-2\varepsilon}}{2}\frac{\Gamma(\frac12-\varepsilon)}{\Gamma(1+\varepsilon)}\ {}_1F_2(\varepsilon,1+\varepsilon,\frac12+\varepsilon,\frac14 m^2z^2)z^{2\varepsilon},$$

and ${}_1F_2(a,b,c;x)$ is the generalized hypergeometric function:

$${}_1F_2(a,b,c;x)=\sum_{k=0}^\infty\frac{(a)_k}{(b)_k(c)_k}\frac{x^k}{k!},\qquad (a)_k=\frac{\Gamma(a+k)}{\Gamma(a)}$$

where the series converges for all finite values of $x\in\mathbb{C}$ and defines an entire function (see [33, § 16.2 (ii)]). It is easy to see that

$$\frac{|G_1(\varepsilon,s+\rho)-G_1(\varepsilon,s-\rho)|}{\rho}\le\frac{C(t)}{\Gamma(\varepsilon)},\quad \rho<s<t,\quad 0<\varepsilon<1.$$

Hence,

$$\lim_{\varepsilon\to 0+}\lim_{\rho\to 0+}\int_\rho^t\frac{G_1(\varepsilon,s+\rho)-G_1(\varepsilon,s-\rho)}{\rho}\dot\zeta(t-s)ds=0. \tag{B.10}$$

Further,

$$\frac{1}{\rho}\int_\rho^t|\big((s+\rho)^{2k+2\varepsilon}-(s-\rho)^{2k+2\varepsilon}\big)\dot\zeta(t-s)|ds$$

$$= \rho^{2k+2\varepsilon} \int_1^{\frac{t}{\rho}} \left((\tau+1)^{2k+2\varepsilon} - (\tau-1)^{2k+2\varepsilon}\right)|\dot\zeta(t-\tau\rho)|d\tau$$
$$\le \max_{s\in[0,t]} |\dot\zeta(s)| \frac{\rho^{2k+2\varepsilon}}{2k+\varepsilon+1}\left((\tau+1)^{2k+2\varepsilon+1} - (\tau-1)^{2k+2\varepsilon+1}\right)\Big|_1^{\frac{t}{\rho}}$$
$$\le 2 \max_{s\in[0,t]} |\dot\zeta(s)| \Big(\frac{(2\rho)^{2k+2\varepsilon}}{2k+\varepsilon+1} + (t+\rho)^{2k+2\varepsilon}\Big),\ k=0,1,2,\ldots \quad \text{(B.11)}$$

Hence,

$$\lim_{\rho\to 0+} \Big|\int_\rho^t \frac{\widetilde G_2(\varepsilon,s+\rho) - \widetilde G_2(\varepsilon,s-\rho)}{\rho}\dot\zeta(t-s)ds\Big| \le \frac{C(t)}{\Gamma(\varepsilon)} \to 0, \quad \varepsilon\to 0, \tag{B.12}$$

where we denote

$$\widetilde G_2(\varepsilon,z) = \frac{\sqrt{\pi}2^{-2\varepsilon}}{\sqrt{2}}\frac{\Gamma(\frac12-\varepsilon)}{\Gamma(1+\varepsilon)}\sum_{k=1}^\infty \frac{\Gamma(\varepsilon+k)}{\Gamma(\varepsilon)(1+\varepsilon)_k(\frac12+\varepsilon)_k}\frac{(mz)^{2k}z^{2\varepsilon}}{4^k k!}$$

In the case $k=0$, taking into account (B.11), we obtain for any small $\nu>0$

$$\lim_{\varepsilon\to 0+}\lim_{\rho\to 0+}\frac{1}{\rho}\int_\rho^t \left((s+\rho)^{2\varepsilon} - (s-\rho)^{2\varepsilon}\right)\dot\zeta(t-s)ds = \lim_{\varepsilon\to 0+}\int_0^t 4\varepsilon s^{2\varepsilon-1}\dot\zeta(t-s)ds$$
$$= \lim_{\varepsilon\to 0+}\Big(\int_0^\nu \ldots + \int_\nu^t \ldots\Big) = \lim_{\varepsilon\to 0+}\int_0^\delta 4\varepsilon s^{2\varepsilon-1}\dot\zeta(t-s)ds$$
$$= \dot\zeta(t-\eta(\nu))\lim_{\varepsilon\to 0+}\int_0^\nu 4\varepsilon s^{2\varepsilon-1}ds = 2\dot\zeta(t-\eta(\nu)),$$

where $\eta(\nu)\in[0,\nu]$. Because of the arbitrariness of $\nu>0$, we get

$$\lim_{\varepsilon\to 0+}\lim_{\rho\to 0+}\frac{1}{\rho}\int_\rho^t \left((s+\rho)^{2\varepsilon} - (s-\rho)^{2\varepsilon}\right)\dot\zeta(t-s)ds = 2\dot\zeta(t).$$

Together with (B.9), (B.10) and (B.12), this gives

$$\lim_{\varepsilon\to 0+}\lim_{\rho\to 0+}\int_\rho^t \left(\int_0^\infty \frac{\sin(r\rho)\cos(sr)}{\rho r(r^2+m^2)^\varepsilon}dr\right)\dot\zeta(t-s)ds = \frac{\pi}{2}\dot\zeta(t).$$

Now for (B.8) it remains to prove that

$$\lim_{\varepsilon\to 0+}\lim_{\rho\to 0+}\int_0^\rho \left(\int_0^\infty \frac{\sin(r\rho)\cos(sr)}{\rho r(r^2+m^2)^\varepsilon}dr\right)\dot\zeta(t-s)ds = 0 \tag{B.13}$$

One has

$$|G_1(\varepsilon,\rho+s) + G_1(\varepsilon,\rho-s)| \le \frac{C}{\Gamma(\varepsilon)}\rho,$$
$$G_2(\varepsilon,\rho+s) + G_2(\varepsilon,\rho-s) \le C\rho^{2\varepsilon}, \quad 0<s<\rho<1, \quad 0<\varepsilon<1,$$

Hence, (B.9) implies that

$$\Big|\int_0^\rho \left(\int_0^\infty \frac{\sin(r\rho)\cos(sr)}{\rho r(r^2+m^2)^\varepsilon}dr\right)\dot\zeta(t-s)ds\Big|$$

$$\le C\rho^{2\varepsilon}\frac{1}{\rho}\int_0^\rho|\dot\zeta(t-s)|ds\to 0,\quad \rho\to 0+,\quad 0<\varepsilon<1.$$

Therefore (B.13) follows.

C. Proof of Proposition 2.5

Note that $D_m^{-1}=D_mD_m^{-2}=-i\alpha\cdot\nabla D_m^{-2}+m\beta D_m^{-2}$. Then the Proposition 2.5 will follow from the next three lemmas:

Lemma C.1. *The following equality holds,*

$$\lim_{\varepsilon\to 0+}\lim_{x\to 0}K_m^\varepsilon D_m^{-2}p_S(x,t)=\lim_{x\to 0}D_m^{-2}p_S(x,t),\quad t>0. \tag{C.1}$$

Lemma C.2. *The following limit holds,*

$$\lim_{x\to 0}D_m^{-2}p_S(x,t)=\frac{1}{4\pi}\Bigg(\zeta_0\Bigg[mt\int_{mt}^\infty\frac{J_1(u)du}{u}-J_0(mt)\Bigg] +\zeta(t)-m\int_0^t\Bigg(\int_{ms}^\infty\frac{J_1(u)du}{u}\Bigg)\zeta(t-s)ds\Bigg),\quad t>0. \tag{C.2}$$

Lemma C.3. *The following limit holds,*

$$\lim_{x\to 0}\nabla K_m^\varepsilon D_m^{-2}p_S(x,t)=0,\quad 0<\varepsilon<1,\quad t>0. \tag{C.3}$$

C.1. Proof of Lemma C.1

Note that $p_S(x,t)$ is a solution to (2.20) with $\dot\zeta(t)$ instead of $\zeta(t)$. Hence,

$$\widehat{p_S}(\xi,t)=\int_0^t\frac{\sin(s\sqrt{\xi^2+m^2})}{\sqrt{\xi^2+m^2}}\dot\zeta(t-s)ds,\quad \widehat{D_m^{-2}p_S}(\xi,t) =\int_0^t\frac{\sin(s\sqrt{\xi^2+m^2})}{(\xi^2+m^2)\sqrt{\xi^2+m^2}}\dot\zeta(t-s)ds,\quad t>0, \tag{C.4}$$

and for (C.1) it suffices to prove that

$$\lim_{\varepsilon\to 0+}\lim_{x\to 0}\int_0^t\mathscr{F}^{-1}_{\xi\to x}\frac{\sin(s\sqrt{\xi^2+m^2}}{(\xi^2+m^2)^{1+\varepsilon}\sqrt{\xi^2+m^2}}\dot\zeta(t-s)ds =\lim_{x\to 0}\int_0^t\mathscr{F}^{-1}_{\xi\to x}\frac{\sin(s\sqrt{\xi^2+m^2})}{(\xi^2+m^2)\sqrt{\xi^2+m^2}}\dot\zeta(t-s)ds,\quad t>0. \tag{C.5}$$

We split the integrand in (C.5) as

$$\frac{\sin(s\sqrt{\xi^2+m^2})}{(\xi^2+m^2)\sqrt{\xi^2+m^2}}\dot\zeta(t-s)=\Bigg(\frac{\sin(s\sqrt{\xi^2+m^2})-\sin(s|\xi|)}{\xi^2\sqrt{\xi^2+m^2}} -\frac{m^2\sin(s\sqrt{\xi^2+m^2})}{\xi^2(\xi^2+m^2)\sqrt{\xi^2+m^2}}+\frac{\sin(s|\xi|)}{\xi^2\sqrt{\xi^2+m^2}}\Bigg)\dot\zeta(t-s), \tag{C.6}$$

and justify the permutation of the limits (C.5) for integrals of each terms in the RHS of (C.6) separately.
The proof of equality

$$\lim_{\varepsilon\to 0+}\lim_{x\to 0}\int_0^t \mathscr{F}^{-1}_{\xi\to x}\frac{m^2\sin(s\sqrt{\xi^2+m^2})}{\xi^2(\xi^2+m^2)^{1+\varepsilon}\sqrt{\xi^2+m^2}}\dot\zeta(t-s)ds$$
$$=\lim_{x\to 0}\int_0^t \mathscr{F}^{-1}_{\xi\to x}\frac{m^2\sin(s\sqrt{\xi^2+m^2})}{\xi^2(\xi^2+m^2)\sqrt{\xi^2+m^2}}\dot\zeta(t-s)ds$$

is similar to the proof of equality (B.3). By the same arguments,

$$\lim_{\varepsilon\to 0+}\lim_{x\to 0}\int_0^t \mathscr{F}^{-1}_{\xi\to x}\frac{\sin(s\sqrt{\xi^2+m^2})-\sin(s|\xi|)}{\xi^2(\xi^2+m^2)^{\varepsilon}\sqrt{\xi^2+m^2}}\ddot\zeta(t-s)ds$$
$$=\lim_{x\to 0}\int_0^t \mathscr{F}^{-1}_{\xi\to x}\frac{\sin(s\sqrt{\xi^2+m^2})-\sin(s|\xi|)}{\xi^2\sqrt{\xi^2+m^2}}\dot\zeta(t-s)ds$$

since

$$|\sin(t\sqrt{\xi^2+m^2})-\sin(t|\xi|)|=2|\sin\frac{t(\sqrt{\xi^2+m^2}+|\xi|)}{2}\sin\frac{t(\sqrt{\xi^2+m^2}-|\xi|)}{2}|$$
$$\le\frac{tm^2}{\sqrt{\xi^2+m^2}+|\xi|}. \tag{C.7}$$

It remains to prove that

$$\lim_{\varepsilon\to 0+}\lim_{x\to 0}\int_0^t \mathscr{F}^{-1}_{\xi\to x}\frac{\sin(s|\xi|)}{\xi^2(\xi^2+m^2)^{\varepsilon}\sqrt{\xi^2+m^2}}\dot\zeta(t-s)ds$$
$$=\lim_{x\to 0}\int_0^t \mathscr{F}^{-1}_{\xi\to x}\frac{\sin(s|\xi|)}{\xi^2\sqrt{\xi^2+m^2}}\dot\zeta(t-s)ds. \tag{C.8}$$

Applying the Lebesgue theorem, we obtain

$$\lim_{\varepsilon\to 0+}\lim_{x\to 0}\int_0^t \mathscr{F}^{-1}_{\xi\to x}\frac{\sin(s|\xi|)}{\xi^2(\xi^2+m^2)^{\varepsilon}\sqrt{\xi^2+m^2}}\dot\zeta(t-s)ds$$
$$=\frac{1}{2\pi^2}\lim_{\varepsilon\to 0+}\lim_{\rho\to 0+}\int_0^t\left(\int_0^\infty\frac{\sin(\rho r)\sin(sr)}{\rho r(r^2+m^2)^{\frac12+\varepsilon}}\dot\zeta(t-s)dr\right)ds$$
$$=\frac{1}{2\pi^2}\lim_{\varepsilon\to 0+}\int_0^t\left(\int_0^\infty\frac{\sin(sr)}{(r^2+m^2)^{\frac12+\varepsilon}}dr\right)\dot\zeta(t-s)ds$$
$$=\frac{\sqrt{\pi}}{4\pi^2}\Gamma(\frac12)\lim_{\varepsilon\to 0+}\int_0^t s^{\varepsilon}[\mathrm{I}_\varepsilon(ms)-\mathbf{L}_{-\varepsilon}(ms)]\dot\zeta(t-s)ds$$
$$=\frac{1}{4\pi}\int_0^t[\mathrm{I}_0(ms)-\mathbf{L}_0(ms)]\dot\zeta(t-s)ds \tag{C.9}$$

by [14, Formula 2.3.(6)]. Here $\mathrm{I}_\varepsilon(z)$ is the modified Bessel function, and $\mathbf{L}_{-\varepsilon}(z)$ is the modified Struve function, satisfying

$$\mathrm{I}_\varepsilon(z)\sim(\frac12 z)^{\varepsilon}/\Gamma(\varepsilon+1),\quad \mathbf{L}_{-\varepsilon}(z)\sim(\frac12 z)^{-\varepsilon+1},\quad z\to 0 \tag{C.10}$$

by Formulas (10.30.1) and (11.2.2) of [33]. On the other hand,

$$\lim_{x\to 0}\int_0^t \mathscr{F}^{-1}_{\xi\to x}\frac{\sin(s|\xi|)}{\xi^2\sqrt{\xi^2+m^2}}\dot\zeta(t-s)ds$$
$$=\lim_{\rho\to 0+}\frac{1}{2\pi^2}\int_0^t\left(\int_0^\infty\frac{\sin(sr)\sin(\rho r)}{\rho r\sqrt{r^2+m^2}}dr\right)\dot\zeta(t-s)ds$$
$$=\lim_{\rho\to 0+}\frac{1}{2\pi^2}\int_0^t\left(\int_0^\infty\frac{\sin(sr)\sin(\rho r)dr}{\rho(r^2+m^2)}\right)\dot\zeta(t-s)ds$$
$$+\lim_{\rho\to 0+}\frac{1}{2\pi^2}\int_0^t\left(\int_0^\infty\frac{\sin(sr)\sin(\rho r)dr}{\rho r(r^2+m^2)(r+\sqrt{r^2+m^2})}\right)\dot\zeta(t-s)ds. \quad \text{(C.11)}$$

By the Lebesgue theorem

$$\lim_{\rho\to 0+}\int_0^t\left(\int_0^\infty\frac{\sin(sr)\sin(\rho r)dr}{\rho r(r^2+m^2)(r+\sqrt{r^2+m^2})}\right)\dot\zeta(t-s)ds$$
$$=\int_0^t\left(\int_0^\infty\frac{\sin(sr)dr}{(r^2+m^2)(r+\sqrt{r^2+m^2})}\right)\dot\zeta(t-s)ds \quad \text{(C.12)}$$

Further, applying [18, Formula 3.742 (1)], we obtain

$$\lim_{\rho\to 0+}\int_0^t\left(\int_0^\infty\frac{\sin(sr)\sin(\rho r)dr}{\rho(r^2+m^2)}\right)\dot\zeta(t-s)ds$$
$$=\lim_{\rho\to 0+}\frac{\pi}{4m}\int_0^t\frac{e^{-|s-\rho|m}-e^{-(s+\rho)m}}{\rho}\dot\zeta(t-s)ds$$
$$=\frac{\pi}{4m}\lim_{\rho\to 0+}\frac{e^{-\rho m}}{\rho}\int_0^\rho(e^{sm}-e^{-sm})\dot\zeta(t-s)ds$$
$$+\frac{\pi}{4m}\lim_{\rho\to 0+}\int_\rho^t\frac{e^{\rho m}-e^{-\rho m}}{\rho}e^{-sm}\dot\zeta(t-s)ds=\frac{\pi}{2}\int_0^t e^{-sm}\dot\zeta(t-s)ds.$$

Moreover,

$$\int_0^t\left(\int_0^\infty\frac{\sin(sr)rdr}{r^2+m^2}\right)\dot\zeta(t-s)ds=\frac{\pi}{2}\int_0^t e^{-sm}\dot\zeta(t-s)ds$$

by [14, Formula 2.2.(15)]. Therefore,

$$\lim_{\rho\to 0+}\int_0^t\left(\int_0^\infty\frac{\sin(sr)\sin(\rho r)dr}{\rho r(r^2+m^2)}\right)\dot\zeta(t-s)ds$$
$$=\int_0^t\left(\int_0^\infty\frac{\sin(sr)dr}{(r^2+m^2)}\right)\dot\zeta(t-s)ds \quad \text{(C.13)}$$

Now (C.11), (C.12), (C.13) and [14, Formula 2.2.(26)] imply

$$\lim_{x\to 0}\int_0^t \mathscr{F}^{-1}_{\xi\to x}\frac{\sin(s|\xi|)}{\xi^2\sqrt{\xi^2+m^2}}\dot\zeta(t-s)ds$$

$$= \frac{1}{2\pi^2}\int_0^t \left(\int_0^\infty \frac{\sin(sr)\,dr}{\sqrt{r^2+m^2}}\right)\dot\zeta(t-s)ds$$

$$= \frac{1}{4\pi}\int_0^t [\mathrm{I}_0(ms)-\mathbf{L}_0(ms)]\dot\zeta(t-s)ds,$$

which coincides with the right hand side of (C.9). Hence, (C.8) follows.

C.2. Proof of Lemma C.2

Integrating by parts in (C.4), we obtain

$$\widehat{D_m^{-2}p_S}(\xi,t) = -\frac{\sin(t\sqrt{\xi^2+m^2})}{(\xi^2+m^2)\sqrt{\xi^2+m^2}}\zeta(0) + \int_0^t \frac{\cos(s\sqrt{\xi^2+m^2})}{\xi^2+m^2}\zeta(t-s)ds. \tag{C.14}$$

Let us calculate the inverse Fourier transform of $\frac{\cos(s\sqrt{\xi^2+m^2})}{\xi^2+m^2}$ and of $\frac{\sin(t\sqrt{\xi^2+m^2})}{(\xi^2+m^2)\sqrt{\xi^2+m^2}}$. In the sense of distributions, we obtain

$$\begin{aligned}
&\mathscr{F}^{-1}_{\xi\to x}\frac{\cos(s\sqrt{\xi^2+m^2})}{\xi^2+m^2}\\
&\quad = \mathscr{F}^{-1}_{\xi\to x}\left(\frac{1}{\xi^2+m^2} - \int_0^s \frac{\sin(u\sqrt{\xi^2+m^2})}{\sqrt{\xi^2+m^2}}du\right)\\
&\quad = \frac{e^{-m|x|}}{4\pi|x|} - \int_0^s \left(\frac{\delta(u-|x|)}{4\pi|x|} - \frac{m}{4\pi}\frac{\theta(u-|x|)J_1(m\sqrt{u^2-x^2})}{\sqrt{u^2-x^2}}\right)du\\
&\quad = \frac{e^{-m|x|}}{4\pi|x|} - \frac{\theta(s-|x|)}{4\pi|x|} + \frac{m}{4\pi}\int_0^s \frac{\theta(u-|x|)J_1(m\sqrt{u^2-x^2})}{\sqrt{u^2-x^2}}du.
\end{aligned} \tag{C.15}$$

$$\begin{aligned}
\mathscr{F}^{-1}_{\xi\to x}\ \frac{\sin(t\sqrt{\xi^2+m^2})}{(\xi^2+m^2)\sqrt{\xi^2+m^2}} &= \mathscr{F}^{-1}_{\xi\to x}\left(\int_0^t \frac{\cos(s\sqrt{\xi^2+m^2})}{\xi^2+m^2}ds\right)\\
= \int_0^t \Bigg(\frac{e^{-m|x|}}{4\pi|x|} - \frac{\theta(s-|x|)}{4\pi|x|}&\\
+\frac{m}{4\pi}\int_0^s \frac{\theta(u-|x|)J_1(m\sqrt{u^2-|x|^2})}{\sqrt{u^2-|x|^2}}du\Bigg)ds&
\end{aligned} \tag{C.16}$$

Hence, (C.14) -(C.16) imply for $t>0$ and $|x|\le t$

$$\begin{aligned}
D_m^{-2}p_S(x,t) = -\zeta_0\Bigg(t\frac{e^{-m|x|}-1}{4\pi|x|} + \frac{1}{4\pi} + \frac{m}{4\pi}\int_{|x|}^t \left(\int_{|x|}^s \frac{J_1(m\sqrt{u^2-x^2})}{\sqrt{u^2-x^2}}du\right)ds\Bigg)&\\
+\int_0^t \left(\frac{e^{-m|x|}}{4\pi|x|} - \frac{\theta(s-|x|)}{4\pi|x|}\right)\zeta(t-s)ds&
\end{aligned}$$

$$+\frac{m}{4\pi}\int_{|x|}^{t}\left(\int_{|x|}^{s}\frac{J_1(m\sqrt{u^2-|x|^2})}{\sqrt{u^2-|x|^2}}du\right)\zeta(t-s)ds. \tag{C.17}$$

One has

$$\begin{aligned}\lim_{|x|\to 0}\int_0^t\left(\frac{e^{-m|x|}}{4\pi|x|}-\frac{\theta(s-|x|)}{4\pi|x|}\right)\zeta(t-s)ds &= \lim_{|x|\to 0}\frac{e^{-m|x|}-1}{4\pi|x|}\int_0^t\zeta(t-s)ds\\ &\quad+\lim_{|x|\to 0}\frac{1}{4\pi|x|}\int_0^{|x|}\zeta(t-s)ds\\ &=-\frac{m}{4\pi}\int_0^t\zeta(s)ds+\frac{1}{4\pi}\zeta(t).\end{aligned}$$

Further, changing the order of integration gives

$$\begin{aligned}\lim_{|x|\to 0}\int_{|x|}^t\left(\int_{|x|}^s\frac{J_1(m\sqrt{u^2-|x|^2})}{\sqrt{u^2-|x|^2}}du\right)ds &= \int_0^t\left(\int_0^s\frac{J_1(mu)}{u}du\right)ds\\ &= t\int_0^t\frac{J_1(mu)du}{u}-\int_0^t J_1(mu)\,du\\ &= t\int_0^t\frac{J_1(mu)du}{u}+\frac{1}{m}J_0(mt)-\frac{1}{m}.\end{aligned}$$

Substituting this into (C.17), we obtain

$$\begin{aligned}4\pi\lim_{|x|\to 0}D_m^{-2}p_S(x,t) &= \zeta_0\left(tm-tm\int_0^t\frac{J_1(mu)du}{u}-J_0(mt)\right)\\ &\quad -m\int_0^t\zeta(s)ds+\zeta(t)+m\int_0^t\left(\int_0^s\frac{J_1(mu)}{u}du\right)\zeta(t-s)ds\\ &=\zeta_0\left(tm\int_{mt}^{\infty}\frac{J_1(u)du}{u}-J_0(mt)\right)+\zeta(t)\\ &\quad +m\int_0^t\left(\int_{ms}^{\infty}\frac{J_1(u)}{u}du\right)\zeta(t-s)ds,\end{aligned}$$

since $\displaystyle\int_0^\infty\frac{J_1(u)du}{u}=1$ by [18, Formula 6.561(17)].

C.3. Proof of Lemma C.3

Note that

$$\begin{aligned}\frac{\sin(s\sqrt{\xi^2+m^2})}{(\xi^2+m^2)^{3/2+\varepsilon}} &= \frac{\sin(s|\xi|)}{(\xi^2+m^2)^{3/2+\varepsilon}}+\frac{\sin(s\sqrt{\xi^2+m^2})-\sin(s|\xi|)}{(\xi^2+m^2)^{3/2+\varepsilon}}\\ &=\frac{\sin(s|\xi|)}{|\xi|(\xi^2+m^2)^{1+\varepsilon}}-\frac{m^2\sin(s|\xi|)}{|\xi|(\xi^2+m^2)^{3/2+\varepsilon}(\sqrt{\xi^2+m^2}+|\xi|)}\\ &\quad+\frac{\sin(s\sqrt{\xi^2+m^2})-\sin(s|\xi|)}{(\xi^2+m^2)^{3/2+\varepsilon}}\\ &=Q_1(|\xi|,s,\varepsilon)+Q_2(|\xi|,s,\varepsilon)+Q_3(|\xi|,s,\varepsilon).\end{aligned}$$

Taking into account (C.7), we obtain

$$\lim_{x\to 0+}\nabla\mathscr{F}^{-1}_{\xi\to x}Q_2(|\xi|,s,\varepsilon)=0,\qquad \lim_{x\to 0+}\nabla\mathscr{F}^{-1}_{\xi\to x}Q_3(|\xi|,s,\varepsilon)=0,\quad \varepsilon>0,\quad s>0.$$

Hence, for (C.4) it remains to prove that

$$\lim_{x\to 0}\nabla\int_0^t\mathscr{F}^{-1}_{\xi\to x}\frac{\sin(s|\xi|)}{|\xi|(\xi^2+m^2)^{1+\varepsilon}}\dot\zeta(t-s)ds=0,\quad \varepsilon>0,\quad t>0. \tag{C.18}$$

Formula 1.3(7) of [14], and formulas 10.27.4 and 10.25.2 of [33] imply

$$\begin{aligned}
&\mathscr{F}^{-1}_{\xi\to x}\frac{\sin(s|\xi|)}{|\xi|(\xi^2+m^2)^{1+\varepsilon}}=\frac{1}{2\pi^2|x|}\int_0^\infty\frac{\sin(sr)\sin(|x|r)}{(r^2+m^2)^{1+\varepsilon}}dr\\
&=\frac{1}{4\pi^2|x|}\int_0^\infty\frac{\cos((s-|x|)r)-\cos((s+|x|)r)}{(r^2+m^2)^{1+\varepsilon}}dr\\
&=\frac{\sqrt{\pi}}{4\pi^2|x|\Gamma(1+\varepsilon)(2m)^{\frac12+\varepsilon}}\Big[|s-|x||^{\frac12+\varepsilon}\mathbf{K}_{\frac12+\varepsilon}(m|s-|x||)\\
&\quad-(s+|x|)^{\frac12+\varepsilon}\mathbf{K}_{\frac12+\varepsilon}(m(s+|x|))\Big]\\
&=\frac{a_\varepsilon}{|x|}\Big[|s-|x||^{\frac12+\varepsilon}\Big(\mathbf{I}_{-\frac12-\varepsilon}(m|s-|x||)-\mathbf{I}_{\frac12+\varepsilon}(m|s-|x||)\Big)\\
&\quad-(s+|x|)^{\frac12+\varepsilon}\Big(\mathbf{I}_{-\frac12-\varepsilon}(m(s+|x|))-\mathbf{I}_{\frac12+\varepsilon}(m(s+|x|))\Big)\Big]\\
&=\frac{a_\varepsilon}{m^{\frac12+\varepsilon}|x|}\Big(\Pi_{-\frac12-\varepsilon}(m(s-|x|))-\Pi_{-\frac12-\varepsilon}(m(s+|x|))\Big)\\
&\quad+\frac{a_\varepsilon m^{\frac12+\varepsilon}}{|x|}\Big((s+|x|)^{1+2\varepsilon}\Pi_{\frac12+\varepsilon}(m(s+|x|))-|s-|x||^{1+2\varepsilon}\Pi_{\frac12+\varepsilon}(m|s-|x||)\Big),
\end{aligned}$$

where we denote

$$a_\varepsilon=\frac{1}{8\sqrt{\pi}\Gamma(1+\varepsilon)\sin((\frac12+\varepsilon)\pi)(2m)^{1+2\varepsilon}},$$

$$\Pi_\nu(y)=(\frac12 y)^{-\nu}\mathbf{I}_\nu(y)=\sum_{k=0}^\infty\frac{y^{2k}}{4^k k!\Gamma(\nu+k+1)}.$$

One has

$$\begin{aligned}
&\frac{\Pi_{-\frac12-\varepsilon}(m(s-|x|))-\Pi_{-\frac12-\varepsilon}(m(s+|x|))}{|x|}\\
&=\sum_{k=0}^\infty\frac{m^{2k}\Big((s-|x|)^{2k}-(s+|x|)^{2k}\Big)}{4^k k!\Gamma(\frac12-\varepsilon+k)|x|}
\end{aligned}$$

$$= -\sum_{k=1}^{\infty} \frac{2m^{2k}\left(\binom{1}{2k} s^{2k-1} + \binom{3}{2k} s^{2k-3}|x|^2 + \cdots + \binom{2k-1}{2k} s|x|^{2k-2}\right)}{4^k k!\Gamma(\frac{1}{2} - \varepsilon + k)}.$$

Hence

$$\lim_{x\to 0} \int_0^t \nabla \frac{\Pi_{-\frac{1}{2}-\varepsilon}(m(s-|x|)) - \Pi_{-\frac{1}{2}-\varepsilon}(m(s+|x|))}{|x|} \dot{\zeta}(t-s)ds = 0, \quad \varepsilon \geq 0, \quad t > 0. \tag{C.19}$$

Further,

$$\frac{(s+|x|)^{1+2\varepsilon}\Pi_{\frac{1}{2}+\varepsilon}(m(s+|x|)) - |s-|x||^{1+2\varepsilon}\Pi_{\frac{1}{2}+\varepsilon}(|s-|x||)}{|x|}$$
$$= \sum_{k=0}^{\infty} \frac{m^{2k\varepsilon}\left((s+|x|)^{2k+1+2\varepsilon} - |s-|x||^{2k+1+2\varepsilon}\right)}{4^k k!\Gamma(\frac{3}{2} + \varepsilon + k)|x|}.$$

Denoting $\alpha_k := 2k + 2\varepsilon$. In the case $\tau := \frac{s}{|x|} \geq 1$, we obtain

$$\begin{aligned}\nabla_j \frac{(s+|x|)^{\alpha_k+1} - (s-|x|)^{\alpha_k+1}}{|x|} &= \frac{x_j}{|x|^2}\Bigg[(\alpha_k+1)\Big((s+|x|)^{\alpha_k} + (s-|x|)^{\alpha_k}\Big) \\ &\quad - \frac{(s+|x|)^{\alpha_k+1} - (s-|x|)^{\alpha_k+1}}{|x|}\Bigg] \\ &= \frac{x_j|x|^{\alpha_k}}{|x|^2}\Bigg[(\alpha_k+1)\Big((\tau+1)^{\alpha_k} + (\tau-1)^{\alpha_k}\Big) \\ &\quad -(\tau+1)^{\alpha_k+1} + (\tau-1)^{\alpha_k+1}\Bigg].\end{aligned} \tag{C.20}$$

One has

$$|(\alpha_k+1)((\tau+1)^{\alpha_k} + (\tau-1)^{\alpha_k}\Big) - (\tau+1)^{\alpha_k+1} + (\tau-1)^{\alpha_k+1}|$$
$$\leq \begin{cases} C_1(k), & 1 \leq \tau \leq 2 \\ C_2(k)(\tau+1)|\alpha_k - 1|^{\alpha_k-2}, & \tau \geq 2 \end{cases}$$

where $C_1(k) = (\alpha_k+2)(3^{\alpha_k}+1)$, $C_2(k) = \left((\frac{3}{2})^{\alpha_k-2} + \frac{1}{3}\right)(\alpha_k+1)\alpha_k$. Hence,

$$|\int_{|x|}^t \nabla_j \frac{(s+|x|)^{\alpha_k+1} - (s-|x|)^{\alpha_k+1}}{|x|}\dot{\zeta}(t-s)ds|$$
$$= \frac{|x_j||x|^{\alpha_k}}{|x|}|\int_1^{\frac{t}{|x|}} \Big((\alpha_k+1)\big((\tau+1)^{\alpha_k} + (\tau-1)^{\alpha_k}\Big)$$
$$-(\tau+1)^{\alpha_k+1} + (\tau-1)^{\alpha_k+1}\Big)\dot{\zeta}(t-\tau|x|)d\tau|$$

$$\le |x|^{\alpha_k}\max_{[0,t]}|\dot\zeta(t)|\Bigg(C_1(k)+C_2(k)\Big[2^{\alpha_k-1}+|x|^{1-\alpha_k}(t+|x|)^{\alpha_k-1}\Big]\Bigg).$$

Therefore,

$$\lim_{x\to 0}\int_{|x|}^t \nabla\frac{(s+|x|)^{1+2\varepsilon}\Pi_{\frac12+\varepsilon}(m(s+|x|))-|s-|x||^{1+2\varepsilon}\Pi_{\frac12+\varepsilon}(|s-|x||)}{|x|}\dot\zeta(t-s)ds=0,$$
$$\varepsilon>0,\quad t>0.$$

It remains to prove that

$$\lim_{x\to 0}\int_0^{|x|} \nabla\frac{(s+|x|)^{1+2\varepsilon}\Pi_{\frac12+\varepsilon}(m(s+|x|))-|s-|x||^{1+2\varepsilon}\Pi_{\frac12+\varepsilon}(|s-|x||)}{|x|}\dot\zeta(t-s)ds=0,$$
$$\varepsilon>0,\quad t>0. \tag{C.21}$$

In the case $s<|x|$, we obtain similarly to (C.20)

$$\begin{aligned}|\nabla_j\frac{(|x|+s)^{\alpha_k+1}-(|x|-s)^{\alpha_k+1}}{|x|}| &= |\frac{x_j|x|^{\alpha_k}}{|x|^2}\Bigg((\alpha_k+1)\Bigg((1+\tau)^{\alpha_k}-(1-\tau)^{\alpha_k}\Bigg)\\ &\qquad -(1+\tau)^{\alpha_k+1}+(1-\tau)^{\alpha_k+1}\Bigg)|\\ &\le 4\frac{|x|^{\alpha_k}}{|x|}(\alpha_k+1)2^{\alpha_k},\quad \tau=\frac{s}{|x|}<1.\end{aligned}$$

Hence, for small $|x|$,

$$\begin{aligned}&|\int_0^{|x|}\nabla_j\frac{(|x|+s)^{\alpha_k+1}-(|x|-s)^{\alpha_k+1}}{|x|}\dot\zeta(t-s)ds|\\ &\le |x|^{\alpha_k}|\int_0^1\Bigg(\Big((1+\tau)^{\alpha_k}+(1-\tau)^{\alpha_k}\Big)-(1+\tau)^{\alpha_k+1}\\ &\quad +(1-\tau)^{\alpha_k+1}\Bigg)\dot\zeta(t-\tau|x|)d\tau|\le C(t)|x|^{\alpha_k}(\alpha_k+1)2^{\alpha_k},\end{aligned}$$

which implies (C.21).

C.3.1. The case $\varepsilon=0$. Note, that the limit (C.18) does not exist for $\varepsilon=0$ i.e. without the smoothing operator K_m^ε. Namely, in this case

$$\begin{aligned}&\mathscr{F}^{-1}_{\xi\to x}\frac{\sin(s|\xi|)}{|\xi|(\xi^2+m^2)}\\ &=\frac{1}{2\pi^2|x|}\int_0^\infty\frac{\sin(sr)\sin(|x|r)}{(r^2+m^2)}dr=\frac{1}{8\pi m|x|}(e^{-m|s-|x||}-e^{-m(s+|x|)})\\ &=\frac{1}{2\pi^2|x|}\begin{cases}e^{-ms}(e^{m|x|}-e^{-m|x|}), & s>|x|,\\ e^{-m|x|}(e^{ms}-e^{-ms}), & s<|x|.\end{cases}\end{aligned}$$

by [18, Formula 3.741(1)]. Hence,

$$\lim_{x\to 0}\nabla_j\int_0^t\mathscr{F}^{-1}_{\xi\to x}\frac{\sin(s|\xi|)}{|\xi|(\xi^2+m^2)}\dot\zeta(t-s)ds$$

$$= \frac{1}{2\pi^2} \lim_{x\to 0} \left(\nabla_j \frac{e^{m|x|} - e^{-m|x|}}{|x|} \right) \int_{|x|}^{t} e^{-ms} \dot\zeta(t-s) ds$$
$$- \frac{m}{2\pi^2} \lim_{x\to 0} \frac{x_j}{|x|^2} \int_0^{|x|} (e^{ms} - e^{-ms}) \dot\zeta(t-s) ds$$
$$- \frac{1}{2\pi^2} \lim_{x\to 0} \frac{x_j}{|x|^3} \int_0^{|x|} (e^{ms} - e^{-ms}) \dot\zeta(t-s) ds$$

Evidently, the first limit in the RHS is zero. Further, L'Hopital's rule implies

$$\lim_{\rho\to 0} \frac{\int_0^\rho (e^{ms} - e^{-ms}) \dot\zeta(t-s) ds}{\rho^2} = \lim_{\rho\to 0} \frac{(e^{m\rho} - e^{-m\rho}) \dot\zeta(t-\rho)}{2\rho} = m\dot\zeta(t)$$

Hence, the second limit in the RHS is zero, and the third limit does not exist.

D. Existence of nonzero solitary waves

Here we prove the lemma 3.5. It suffices to consider the case $j = 1$ only, since the equation (3.8) for $j = 2$ is the same, and for $j = 3, 4$ is similar. We rewrite the equation (3.8) with $j = 1$ as

$$\frac{\omega_1(\omega_1 + m)}{m + \sqrt{m^2 - \omega_1^2}} = 4\pi a_1(|C_1|^2), \qquad \omega_1 \in (-m, m), \tag{D.1}$$

where

$$a_1(|\zeta|^2) = M - \sum_{n=2}^{N_1} 2n u_{n,1} |\zeta|^{2n-2}, \quad N_1 \ge 2, \quad M > 0, \quad u_{N_1,1} > 0.$$

Necessarily, equation (D.1) has nonzero solutions $C_1 = C_1(\omega_1)$ for $\omega_1 \in (-m, m)$, satisfying the condition

$$\frac{\omega_1(\omega_1 + m)}{m + \sqrt{m^2 - \omega_1^2}} < 4\pi M. \tag{D.2}$$

Obviously, (D.2) holds for any $M > 0$ and ω_1 sufficiently close to 0 or to $-m$. Moreover, (D.2) holds for any $M > 0$ and for $\omega_1 \in (-m, m)$ such that

$$\omega_1(\omega_1 + m) < 4\pi M m,$$

which is equivalent to

$$-m < \omega_1 < \min\left\{ m;\ \frac{-m + \sqrt{m^2 + 64\pi^2 M m}}{2} \right\}.$$

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Elena Kopylova
Vienna University
Vienna
Austria
e-mail: elena.kopylova@univie.ac.at

and

Institute for Information Transmission Problems
Moscow
Russia

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