Temporal-spatial decays for the Stokes flows in half spaces

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Abstract. In this paper, we prove new estimates on temporal-spatial decays in L^1 for solutions to the Stokes equations in the half spaces. Our approach is based on the Ukai's solution formula of the Stokes equations and heat kernel estimates.

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1. Introduction

In this paper, we consider the Stokes equation

$$\begin{cases} \partial_t u - \Delta u + \nabla p = 0 & \text{in } \mathbb{R}_+^n \times (0, \infty), \\ \nabla \cdot u = 0 & \text{in } \mathbb{R}_+^n \times (0, \infty), \\ u(x, t) = 0 & \text{on } \partial \mathbb{R}_+^n \times (0, \infty), \\ u(x, 0) = u_0 & \text{in } \mathbb{R}_+^n. \end{cases} \quad (1)$$

where

- $n \geq 2$, and $\mathbb{R}_+^n = \{x = (x', x_n) : x' \in \mathbb{R}^{n-1}, x_n > 0\}$ is the upper-half space of $\mathbb{R}^n$;
- $u = (u_1(x, t), u_2(x, t), \dots, u_n(x, t))$ and $p = p(x, t)$ are unknown velocity vector and the pressure, respectively;
- the initial data $u_0 = u_0(x)$ is assumed to satisfy a compatibility condition: $\nabla \cdot u_0 = 0$ in $\mathbb{R}_+^n$ and the normal component of u_0 equals to zero on $\partial \mathbb{R}_+^n$.

The estimates of the solution to the Stokes equation play an important role in the study of the temporal and the spatial decays for weak solutions to the Navier–Stokes equation. See [6, 10, 12, 13, 15] and the references therein.

Denote by A the Stokes operator $-P\Delta$ in $\mathbb{R}_+^n$, where P is the projection: $L^r(\mathbb{R}_+^n) \rightarrow L_\sigma^r(\mathbb{R}_+^n)$, $1 < r < \infty$. Then the solution to (1) can be written as $u(t, x) = e^{-tA}u_0$. In the whole space $\mathbb{R}^n$, the flow e^{-tA} behaves similarly to the heat Laplacian semigroup $e^{t\Delta}$; hence,

$$\|e^{-tA}u_0\|_{L^q(\mathbb{R}^n)} \lesssim t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|u_0\|_{L^p(\mathbb{R}^n)}$$

for all $1 \leq p \leq q \leq \infty$. However, this is not the case in the half-space $\mathbb{R}_+^n$. Precisely, in the half-space case, we have

$$\|e^{-tA}u_0\|_{L^q(\mathbb{R}_+^n)} \lesssim t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|u_0\|_{L^p(\mathbb{R}_+^n)} \quad (2)$$

for all $1 < p \leq q \leq \infty$ or $1 \leq p < q \leq \infty$. See [10]. It is worth noticing that unlike the whole space case, (2) does not hold true for $p = q = 1$. In fact, it was proved in [7] that there exists a function $u_0 \in L^1(\mathbb{R}_+^n)$ such that $e^{-tA}u_0 \notin L^1(\mathbb{R}_+^n)$. In order to deal with $L^1(\mathbb{R}_+^n)$ estimates, we need certain extra condition on the initial data u_0 . We first recall the main result in [1, Theorem 1.1].

Theorem A. Assume that the initial data u_0 satisfies $\nabla \cdot u = 0$ in $\mathbb{R}_+^n$ and

$$\int_{\mathbb{R}^{n-1}} u_0(y', y_n) dy' = 0 \quad \text{a.e. } y_n > 0. \quad (3)$$

Then, there is a constant C independent of u_0 such that

$$\|e^{-tA}u_0\|_{\mathcal{H}^1(\mathbb{R}_+^n)} \leq Ct^{-1} \int_{\mathbb{R}_+^n} y_n |y'| |u_0(y)| dy \quad (4)$$

and

$$\|e^{-tA}u_0\|_{\mathcal{H}^1(\mathbb{R}_+^n)} \leq Ct^{-1/2} \int_{\mathbb{R}_+^n} |y'| |u_0(y)| dy, \quad (5)$$

where

$$\mathcal{H}^1(\mathbb{R}_+^n) = \inf \left\{ \|F\|_{\mathcal{H}^1(\mathbb{R}^n)} : F \in \mathcal{H}^1(\mathbb{R}^n), F|_{\mathbb{R}_+^n} = f \right\}$$

with $\mathcal{H}^1(\mathbb{R}^n)$ being the Hardy space on $\mathbb{R}^n$. (See (19) for the precise definition of the Hardy space).

Then in [12] a stronger condition than (3) is considered. A vector $a = (a_1, a_2, \dots, a_n)$ on $\mathbb{R}_+^n$, $n \geq 2$ satisfies the tangential parity condition if

- (i) for $1 \leq j, k \leq n-1$ ($n \geq 3$) with $j \neq k$, $a_j(x_1, x_2, \dots, x_{n-1}, x_n)$ is odd in x_j and even in x_k ; and $a_1(x_1, x_2)$ is odd in x_1 if $n = 2$;
- (ii) for $1 \leq j \leq n-1$, $a_n(x_1, x_2, \dots, x_{n-1}, x_n)$ is even in each x_j .

Then it was proved that

Theorem B. [12] Assume that the initial data $u_0 = (u_0^1, u_0^2, \dots, u_0^n)$ satisfies $\nabla \cdot u = 0$ in $\mathbb{R}_+^n$ and $u_0^n|_{\partial\mathbb{R}_+^n} = 0$. Suppose further that u_0 satisfies the tangential parity condition. Then, for any $0 < \alpha \leq 1$ and $t > 0$

$$\|e^{-tA}u_0\|_{L^1(\mathbb{R}_+^n)} \leq Ct^{-\frac{\alpha}{2}} \int_{\mathbb{R}_+^n} |x|^\alpha |u_0(x)| dx. \quad (6)$$

It is important to note that the method in [1] used to obtain the estimates (4) and (5) can not be applied to derive the estimate (6). This is a reason why

in [12], the tangential parity condition is assumed to obtain the estimate (6). We would like to emphasize that the tangential parity condition is stronger than condition (3). It is worth noticing that such estimates (4) and (5) are not useful to study the long-time behaviour of solutions to the corresponding Navier–Stokes equation under the tangential parity condition. See for example [12]. This shows the motivation of the estimate (6).

The main aim of this paper is to prove the following general estimate.

Theorem 1.1. *Assume that the initial data u_0 satisfies $\nabla \cdot u_0 = 0$ in $\mathbb{R}_+^n$ and the condition (3). Then for any $t > 0$, and $0 < \alpha \leq 1$ and $0 \leq \beta \leq 1$,*

$$\|e^{-tA}u_0\|_{\mathcal{H}^1(\mathbb{R}_+^n)} \leq Ct^{-\frac{\alpha+\beta}{2}} \int_{\mathbb{R}_+^n} |y'|^\alpha y_n^\beta |u_0(y)| dy. \quad (7)$$

Some comments are in order:

- (a) Since the tangential parity condition is stronger than (3), Theorem 1.1 is still true under the tangential parity condition. Therefore it is also interesting to study the L^1 decay estimates for the solutions to the Navier–Stokes equations. However, we do not pursue this problem in this paper. Such a result should be elsewhere.
- (b) It is easy to see that the estimate (7) turns out to be the estimates (4) and (5) when we choose $(\alpha, \beta) = (1, 1)$ and $(\alpha, \beta) = (1, 0)$. Moreover, using the inequality $|y'| \leq |y|$ and $y_n \leq |y|$, the estimate (7) deduces, for $\alpha \in (0, 2]$

$$\|e^{-tA}u_0\|_{L^1(\mathbb{R}_+^n)} \leq Ct^{-\frac{\alpha}{2}} \int_{\mathbb{R}_+^n} |x|^\alpha |u_0(x)| dx,$$

which extends the range of $\alpha \in (0, 1]$ in (6) under the weaker assumption (3).

We would like to point out that the approaches in [1, 12] can not be applicable to derive the estimate (7). More importantly, although the decay estimates in [1, 12] hold true, there are gaps in their proofs. For example (please see the definitions of Λ and $E(t)$ in Sect. 3), both papers used the fact that $eE(t)u_0 \in \mathcal{H}^1(\mathbb{R}^n)$, but this statement was not proved and it is not clear if this is true. See [1, p.17] and [12, p.1542]. Moreover, one of the key ingredients is proving the following $\partial_j \Lambda^{-1} eE(t)u_0 \in \mathcal{H}^1(\mathbb{R}^n)$. However, what was proved in [1, 12] is $\partial_j \Lambda^{-1} E(t)u_0 \in \mathcal{H}^1(\mathbb{R}^n)$. See the proof of [1, Lemma 3.3] and [12, Lemma 2.2]. Therefore, our approach fills in the gaps of the proofs of Theorem A and Theorem B in [1, 12].

The organization of the paper is as follows. In Sect. 2, we prove some kernel estimates which play an essential role in proving temporal-spatial decay estimates for the Stokes equation. The proof of Theorem 1.1 will be given in Sect. 3.

Throughout the paper, we always use C and c to denote positive constants that are independent of the main parameters involved but whose values may differ from line to line. We will write $A \lesssim B$ if there is a universal constant C so that $A \leq CB$ and $A \sim B$ if $A \lesssim B$ and $B \lesssim A$.

2. Heat kernel estimates

In this section, we prove a number of kernel estimates which play an important role in the proof of the main result.

Lemma 2.1. *Let $d \in \mathbb{N}$ and $\alpha > 0$. Then we have*

$$\int_{\mathbb{R}^d} \sup_{s>0} \frac{1}{(s+t)^{d+\alpha}} \left(1 + \frac{|x-y|}{s+t}\right)^{-(d+\alpha)} dx \lesssim t^{-\alpha}$$

for all $y \in \mathbb{R}^d$ and $t > 0$. As a consequence,

$$\int_{\mathbb{R}^d} \sup_{s>0} \frac{1}{(s+t)^{\frac{d+\alpha}{2}}} e^{-\frac{|x-y|^2}{c(s+t)}} dx \lesssim t^{-\alpha/2}.$$

Proof. We have

$$\begin{aligned} & \int_{\mathbb{R}^d} \sup_{s>0} \frac{1}{(s+t)^{d+\alpha}} \left(1 + \frac{|x-y|}{s+t}\right)^{-(d+\alpha)} dx \\ &= \int_{|x-y|<t} \sup_{s>0} \frac{1}{(s+t)^{d+\alpha}} \left(1 + \frac{|x-y|}{s+t}\right)^{-(d+\alpha)} dx \\ & \quad + \sum_{k=1}^{\infty} \int_{2^{k-1}t \leq |x-y| < 2^k t} \sup_{s>0} \frac{1}{(s+t)^{d+\alpha}} \left(1 + \frac{|x-y|}{s+t}\right)^{-(d+\alpha)} dx. \end{aligned} \quad (8)$$

It is obvious that

$$\begin{aligned} \int_{|x-y|<t} \sup_{s>0} \frac{1}{(s+t)^{d+\alpha}} \left(1 + \frac{|x-y|}{s+t}\right)^{-(d+\alpha)} dx &\leq \int_{|x-y|<t} \frac{1}{t^{d+\alpha}} dx \\ &\leq t^{-\alpha}. \end{aligned}$$

It remains to estimate the sum in (8). To this end, we write

$$\begin{aligned} & \sum_{k=1}^{\infty} \int_{2^{k-1}t \leq |x-y| < 2^k t} \sup_{s>0} \frac{1}{(s+t)^{d+\alpha}} \left(1 + \frac{|x-y|}{s+t}\right)^{-(d+\alpha)} dx \\ & \lesssim \sum_{k=1}^{\infty} \int_{2^{k-1}t \leq |x-y| < 2^k t} \frac{1}{|x-y|^{d+\alpha}} dx \\ & \lesssim \sum_{k=1}^{\infty} (2^k t)^{-\alpha} \sim t^{-\alpha}. \end{aligned}$$

This proves the first inequality.

The second inequality follows directly from the first one and the following inequality

$$e^{-\frac{|x-y|^2}{c(s+t)}} \lesssim \left(1 + \frac{|x-y|}{s+t}\right)^{-(d+\alpha)}.$$

This completes our proof. $\square$

Lemma 2.2. *Let $\epsilon, \alpha, \gamma > 0$. Then there exists C such that*

(i) for every $y \in \mathbb{R}^d$ and $t > 0$,

$$\int_{\mathbb{R}^d} \frac{1}{t^d} \left(\frac{t}{t + |x-y|}\right)^{d+\epsilon} dx \leq C;$$

(ii) for every $M, t > 0$,

$$\int_0^\infty \frac{s^\alpha}{(s+t)^{\alpha+\gamma}} \left(1 + \frac{M}{s+t}\right)^{-(\alpha+\gamma)} \frac{ds}{s} \leq \frac{C}{(t+M)^\gamma}.$$

Proof. (i) The proof of this item is simple and we omit the details.

(ii) We write

$$\begin{aligned} \int_0^\infty \frac{s^\alpha}{(s+t)^{\alpha+\gamma}} \left(1 + \frac{M}{s+t}\right)^{-(\alpha+\gamma)} \frac{ds}{s} &= \int_0^{t+M} \dots + \int_{t+M}^\infty \dots \\ &= E_1 + E_2. \end{aligned}$$

For the term E_1 , we have

$$\begin{aligned} E_1 &\lesssim \int_0^{t+M} \frac{s^\alpha}{(s+t)^{\alpha+\gamma}} \left(\frac{s+t}{t+M}\right)^{\alpha+\gamma} \frac{ds}{s} \\ &\sim \left(\frac{1}{t+M}\right)^\gamma. \end{aligned}$$

For the term E_2 , it is obvious that

$$\begin{aligned} E_2 &\lesssim \int_{t+M}^\infty \frac{s^\alpha}{(s+t)^{\alpha+\gamma}} \frac{ds}{s} \\ &\lesssim \int_{t+M}^\infty \frac{1}{s^\gamma} \frac{ds}{s} \\ &\sim \left(\frac{1}{t+M}\right)^\gamma. \end{aligned}$$

This completes our proof. $\square$

Lemma 2.3. Let $d \in \mathbb{N}$, $\alpha \in (0, 1]$ and $\gamma \in [0, \alpha)$. Then for any $y \in \mathbb{R}^d$ and $t > 0$, we have

$$\int_{\mathbb{R}^d} \sup_{s>0} \frac{1}{(s+t)^{d/2-\gamma/2}} \left| e^{-\frac{|x-y|^2}{4(s+t)}} - e^{-\frac{|x|^2}{4(s+t)}} \right| dx \lesssim t^{-\alpha/2+\gamma/2} |y|^\alpha.$$

Proof. We have

$$\begin{aligned} \int_{\mathbb{R}^d} \sup_{s>0} \frac{1}{(s+t)^{d/2-\gamma/2}} \left| e^{-\frac{|x-y|^2}{4(s+t)}} - e^{-\frac{|x|^2}{4(s+t)}} \right| dx &\lesssim \int_{\mathbb{R}^d} \sup_{s:s+t < |y|^2} \dots + \int_{\mathbb{R}^d} \sup_{s:s+t \geq |y|^2} \dots \\ &=: E + F. \end{aligned}$$

For the term E , using Lemma 2.1,

$$\begin{aligned} E &\leq \int_{\mathbb{R}^d} \sup_{s>0} \frac{1}{(s+t)^{d/2-\gamma/2}} \frac{|y|^\alpha}{(s+t)^{\alpha/2}} \left| e^{-\frac{|x-y|^2}{4(s+t)}} - e^{-\frac{|x|^2}{4(s+t)}} \right| dx \\ &\leq \int_{\mathbb{R}^d} \sup_{s>0} \frac{1}{(s+t)^{d/2-\gamma/2}} \frac{|y|^\alpha}{(s+t)^{\alpha/2}} e^{-\frac{|x-y|^2}{4(s+t)}} dx \\ &\quad + \int_{\mathbb{R}^d} \sup_{s>0} \frac{1}{(s+t)^{d/2}} \frac{|y|^\alpha}{(s+t)^{\alpha/2-\gamma/2}} e^{-\frac{|x|^2}{4(s+t)}} dx \\ &\leq \frac{|y|^\alpha}{t^{\alpha/2-\gamma/2}}. \end{aligned}$$

For the second term F , we have

$$\begin{aligned} F &\leq \int_{\mathbb{R}^d} \sup_{s:t \geq |y|^2} \frac{1}{(s+t)^{d/2-\gamma/2}} \left| e^{-\frac{|x-y|^2}{4(s+t)}} - e^{-\frac{|x|^2+|y|^2}{4(s+t)}} \right| dx \\ &\quad + \int_{\mathbb{R}^d} \sup_{s:s+t \geq |y|^2} \frac{1}{(s+t)^{d/2-\gamma/2}} \left| e^{-\frac{|x|^2+|y|^2}{4(s+t)}} - e^{-\frac{|x|^2}{4(s+t)}} \right| dx \\ &= F_1 + F_2. \end{aligned}$$

Using the inequality $e^u - 1 \lesssim u^\alpha$ for $\alpha, u \in [0, 1]$, we have

$$\begin{aligned} F_2 &= \int_{\mathbb{R}^d} \sup_{s:s+t \geq |y|^2} \frac{1}{(s+t)^{d/2-\gamma/2}} e^{-\frac{|x|^2+|y|^2}{4(s+t)}} \left[e^{\frac{|y|^2}{4(s+t)}} - 1 \right] dx \\ &\lesssim \int_{\mathbb{R}^d} \sup_{s:s+t \geq |y|^2} \frac{1}{(s+t)^{d/2-\gamma/2}} \frac{|y|^\alpha}{(s+t)^{\alpha/2}} e^{-\frac{|x|^2+|y|^2}{4(s+t)}} dx \\ &\lesssim \frac{|y|^\alpha}{t^{\alpha/2-\gamma/2}}, \end{aligned}$$

where in the last inequality we used Lemma 2.1.

In order to take case of the term F_1 , we write

$$\begin{aligned} F_1 &= \int_{\Omega_1(y)} \sup_{s:t \geq |y|^2} \frac{1}{(s+t)^{d/2-\gamma/2}} \left| e^{-\frac{|x-y|^2}{4(s+t)}} - e^{-\frac{|x|^2+|y|^2}{4(s+t)}} \right| dx \\ &\quad + \int_{\Omega_2(y)} \sup_{s:s+t \geq |y|^2} \frac{1}{(s+t)^{d/2-\gamma/2}} \left| e^{-\frac{|x-y|^2}{4(s+t)}} - e^{-\frac{|x|^2+|y|^2}{4(s+t)}} \right| dx \\ &= F_{11} + F_{12}, \end{aligned}$$

where

$$\Omega_1(y) = \{x : \langle x, y \rangle \geq 0\} \quad \text{and} \quad \Omega_2(y) = \{x : \langle x, y \rangle < 0\}.$$

Hence, it suffices to show that

$$F_{11} + F_{12} \lesssim \frac{|y|^\alpha}{t^{\alpha/2-\gamma/2}}.$$

To do this, we consider these two terms separately.

Estimate of F_{11}

We have

$$\begin{aligned} F_{11} &\lesssim \int_{\Omega_1(y)} \sup_{s:t \geq \max\{|y|^2, \langle x, y \rangle / 2\}} \frac{1}{(s+t)^{d/2-\gamma/2}} \left| e^{-\frac{|x-y|^2}{4(s+t)}} - e^{-\frac{|x|^2+|y|^2}{4(s+t)}} \right| dx \\ &\quad + \int_{\Omega_1(y)} \sup_{s:|y|^2 \leq s+t < \langle x, y \rangle / 2} \frac{1}{(s+t)^{d/2-\gamma/2}} \left| e^{-\frac{|x-y|^2}{4(s+t)}} - e^{-\frac{|x|^2+|y|^2}{4(s+t)}} \right| dx \\ &=: G_1 + G_2. \end{aligned}$$

Using $e^u - 1 \lesssim u^\alpha$ for $\alpha, u \in [0, 1]$,

$$\begin{aligned} G_1 &= \int_{\Omega_1(y)} \sup_{s: s+t \geq \max\{|y|^2, \langle x, y \rangle / 2\}} \frac{1}{(s+t)^{d/2-\gamma/2}} e^{-\frac{|x|^2+|y|^2}{4(s+t)}} \left[e^{\frac{\langle x, y \rangle}{2(s+t)}} - 1 \right] dx \\ &\lesssim \int_{\Omega_1(y)} \sup_{s>0} \frac{1}{(s+t)^{d/2-\gamma/2}} \frac{\langle x, y \rangle^\alpha}{(s+t)^\alpha} e^{-\frac{|x|^2+|y|^2}{4(s+t)}} dx \\ &\lesssim \int_{\Omega_1(y)} \sup_{s>0} \frac{1}{(s+t)^{d/2-\gamma/2}} \frac{(|x||y|)^\alpha}{(s+t)^\alpha} e^{-\frac{|x|^2+|y|^2}{4(s+t)}} dx. \end{aligned}$$

Owing to the following inequality

$$\frac{(|x||y|)^\alpha}{(s+t)^\alpha} e^{-\frac{|x|^2+|y|^2}{4(s+t)}} \lesssim \frac{|y|^\alpha}{(s+t)^{\alpha/2}} e^{-\frac{|y|^2}{8(s+t)}},$$

we obtain

$$\begin{aligned} G_1 &\lesssim \int_{\Omega_1(y)} \sup_{s>0} \frac{1}{(s+t)^{d/2-\gamma/2}} \frac{|y|^\alpha}{(s+t)^\alpha} e^{-\frac{|x|^2}{8(s+t)}} dx \\ &\lesssim \frac{|y|^\alpha}{t^{\alpha/2-\gamma/2}}, \end{aligned}$$

where in the last inequality we used Lemma 2.1.

For the term G_2 , we have

$$\begin{aligned} G_2 &\lesssim \int_{\Omega_1(y)} \sup_{s: s+t < \langle x, y \rangle / 2} \frac{1}{(s+t)^{d/2-\gamma/2}} e^{-\frac{|x-y|^2}{4(s+t)}} dx \\ &\quad + \int_{\Omega_1(y)} \sup_{s>0} \frac{1}{(s+t)^{d/2-\gamma/2}} \frac{\langle x, y \rangle^\alpha}{(s+t)^\alpha} e^{-\frac{|x|^2+|y|^2}{4(s+t)}} dx \\ &=: G_{21} + G_{22}. \end{aligned}$$

Similarly to the estimate of G_1 , we have

$$G_{22} \lesssim \frac{|y|^\alpha}{t^{\alpha/2-\gamma/2}}.$$

For the term G_{21} , we write

$$\begin{aligned} G_{21} &= \int_{\Omega_1(y) \cap \{|x| \leq 2|y|\}} \sup_{s: s+t < \langle x, y \rangle / 2} \frac{1}{(s+t)^{d/2-\gamma/2}} e^{-\frac{|x-y|^2}{4(s+t)}} dx \\ &\quad + \int_{\Omega_1(y) \cap \{|x| > 2|y|\}} \sup_{s: s+t < \langle x, y \rangle / 2} \frac{1}{(s+t)^{d/2-\gamma/2}} e^{-\frac{|x-y|^2}{4(s+t)}} dx \\ &\lesssim \int_{\Omega_1(y) \cap \{|x| \leq 2|y|\}} \sup_{s>0} \frac{1}{(s+t)^{d/2-\gamma/2}} \frac{\langle x, y \rangle^{\alpha/2}}{(s+t)^{\alpha/2}} e^{-\frac{|x-y|^2}{4(s+t)}} dx \\ &\quad + \int_{\Omega_1(y) \cap \{|x| > 2|y|\}} \sup_{s>0} \frac{1}{(s+t)^{d/2-\gamma/2}} \frac{\langle x, y \rangle^\alpha}{(s+t)^\alpha} e^{-\frac{|x-y|^2}{4(s+t)}} dx \\ &= H_1 + H_2. \end{aligned}$$

Using Lemma 2.1,

$$\begin{aligned} H_1 &\lesssim \int_{\Omega_1(y) \cap \{|x| \leq 2|y|\}} \sup_{s>0} \frac{1}{(s+t)^{d/2-\gamma/2}} \frac{(|x||y|)^{\alpha/2}}{(s+t)^{\alpha/2}} e^{-\frac{|x-y|^2}{4(s+t)}} dx \\ &\lesssim \int_{\mathbb{R}^d} \sup_{s>0} \frac{1}{(s+t)^{d/2-\gamma/2}} \frac{|y|^\alpha}{(s+t)^{\alpha/2}} e^{-\frac{|x-y|^2}{4(s+t)}} dx \\ &\lesssim \frac{|y|^\alpha}{t^{\alpha/2-\gamma/2}}. \end{aligned}$$

For the term H_2 , note that $|x-y| \sim |x|$ as $|x| > 2|y|$. Hence, by Lemma 2.1,

$$\begin{aligned} H_2 &\lesssim \int_{\Omega_1(y) \cap \{|x| > 2|y|\}} \sup_{s>0} \frac{1}{(s+t)^{d/2-\gamma/2}} \frac{(|x||y|)^\alpha}{(s+t)^\alpha} e^{-\frac{|x-y|^2}{8(s+t)}} e^{-\frac{|x|^2}{c(s+t)}} dx \\ &\lesssim \int_{\Omega_1(y) \cap \{|x| > 2|y|\}} \sup_{s>0} \frac{1}{(s+t)^{d/2-\gamma/2}} \frac{|y|^\alpha}{(s+t)^{\alpha/2}} e^{-\frac{|x-y|^2}{8(s+t)}} dx \\ &\lesssim \frac{|y|^\alpha}{t^{\alpha/2-\gamma/2}}, \end{aligned}$$

where in the second inequality we used the inequality

$$\frac{|x|^\alpha}{(s+t)^{\alpha/2}} e^{-\frac{|x|^2}{c(s+t)}} \lesssim 1.$$

Collecting the estimates of H_1, H_2, G_{21}, G_{22} , we come up with

$$G_2 \lesssim \frac{|y|^\alpha}{t^{\alpha/2-\gamma/2}}.$$

Taking this and the estimate of G_1 into account, we have

$$F_{11} \lesssim \frac{|y|^\alpha}{t^{\alpha/2-\gamma/2}}.$$

Estimate of F_{12}

We have

$$\begin{aligned} F_{12} &\lesssim \int_{\Omega_2(y)} \sup_{s: s+t \geq \max\{|y|^2, |\langle x, y \rangle|/2\}} \frac{1}{(s+t)^{d/2-\gamma/2}} \left| e^{-\frac{|x-y|^2}{4(s+t)}} - e^{-\frac{|x|^2+|y|^2}{4(s+t)}} \right| dx \\ &\quad + \int_{\Omega_2(y)} \sup_{s: |y|^2 \leq s+t < |\langle x, y \rangle|/2} \frac{1}{(s+t)^{d/2-\gamma/2}} \left| e^{-\frac{|x-y|^2}{4(s+t)}} - e^{-\frac{|x|^2+|y|^2}{4(s+t)}} \right| dx \\ &=: M_1 + M_2. \end{aligned}$$

We take care of M_2 first. Since $\langle x, y \rangle < 0$ in this situation, we have

$$\begin{aligned} M_2 &\leq \int_{\Omega_2(y)} \sup_{s: |y|^2 \leq s+t < |\langle x, y \rangle|/2} \frac{1}{(s+t)^{d/2-\gamma/2}} e^{-\frac{|x-y|^2}{4(s+t)}} dx \\ &\lesssim \int_{\Omega_2(y) \cap \{|x| \leq 2|y|\}} \sup_{s: |y|^2 \leq s+t < |\langle x, y \rangle|/2} \frac{1}{(s+t)^{d/2-\gamma/2}} \frac{|\langle x, y \rangle|^{\alpha/2}}{(s+t)^{\alpha/2}} e^{-\frac{|x-y|^2}{4(s+t)}} dx \\ &\quad + \int_{\Omega_2(y) \cap \{|x| > 2|y|\}} \sup_{s: |y|^2 \leq s+t < |\langle x, y \rangle|/2} \frac{1}{(s+t)^{d/2-\gamma/2}} \frac{|\langle x, y \rangle|^\alpha}{(s+t)^\alpha} e^{-\frac{|x-y|^2}{4(s+t)}} dx. \end{aligned}$$

At this stage, arguing similarly to the estimates of H_1 and H_2 , we have

$$M_2 \lesssim \frac{|y|^\alpha}{t^{\alpha/2-\gamma/2}}.$$

It remains to estimate M_1 . We have

$$M_1 = \int_{\Omega_2(y)} \sup_{s:t \geq \max\{|y|^2, |\langle x, y \rangle|/2\}} \frac{1}{(s+t)^{d/2-\gamma/2}} e^{-\frac{|x-y|^2}{4(s+t)}} \left[e^{\frac{|\langle x, y \rangle|}{2(s+t)}} - 1 \right] dx.$$

This, along with the following inequality $e^u - 1 \lesssim u^\alpha$ for $\alpha, u \in [0, 1]$, yields

$$\begin{aligned} M_1 &\lesssim \int_{\Omega_2(y) \cap \{|x| \leq 2|y|\}} \sup_{s>0} \frac{1}{(s+t)^{d/2-\gamma/2}} \frac{|\langle x, y \rangle|^{\alpha/2}}{(s+t)^{\alpha/2}} e^{-\frac{|x-y|^2}{4(s+t)}} dx \\ &\quad + \int_{\Omega_2(y) \cap \{|x| > 2|y|\}} \sup_{s>0} \frac{1}{(s+t)^{d/2-\gamma/2}} \frac{|\langle x, y \rangle|^\alpha}{(s+t)^\alpha} e^{-\frac{|x-y|^2}{4(s+t)}} dx. \end{aligned}$$

At this stage, using the argument in the estimations of H_1 and H_2 , we arrive at

$$M_1 \lesssim \frac{|y|^\alpha}{t^{\alpha/2-\gamma/2}}.$$

This, along with the estimate of M_2 , implies

$$F_{12} \lesssim \frac{|y|^\alpha}{t^{\alpha/2-\gamma/2}}.$$

This completes our proof. $\square$

Lemma 2.4. *For every $\alpha \in (0, 1]$ and $\gamma \in [0, \alpha)$, we have*

$$\int_{\mathbb{R}} \sup_{s>0} \frac{1}{(s+t)^{1/2-\gamma/2}} \left| e^{-\frac{|x-y|^2}{4(s+t)}} - e^{-\frac{|x+y|^2}{4(s+t)}} \right| dx \lesssim y^\alpha t^{-\alpha/2+\gamma/2}$$

for all $y > 0$.

Proof. We write

$$\begin{aligned} &\int_{\mathbb{R}} \sup_{s>0} \frac{1}{(s+t)^{1/2-\gamma/2}} \left| e^{-\frac{|x-y|^2}{4(s+t)}} - e^{-\frac{|x+y|^2}{4(s+t)}} \right| dx \\ &\leq \int_{\mathbb{R}} \sup_{s>0} \frac{1}{(s+t)^{1/2-\gamma/2}} \left| e^{-\frac{|x-y|^2}{4(s+t)}} - e^{-\frac{|x|^2}{4(s+t)}} \right| dx \\ &\quad + \int_{\mathbb{R}} \sup_{s>0} \frac{1}{(s+t)^{1/2-\gamma/2}} \left| e^{-\frac{|x+y|^2}{4(s+t)}} - e^{-\frac{|x|^2}{4(s+t)}} \right| dx. \end{aligned}$$

The desired estimate then follows from Lemma 2.3. $\square$

Lemma 2.5. *Let $d \in \mathbb{N}$, $\alpha \in (0, 1]$, $\gamma \in [0, 1]$ and $j = 1, \dots, d$. Then we have, for any $y \in \mathbb{R}^d$,*

$$\int_{\mathbb{R}^d} \sup_{s>0} \frac{1}{(s+t)^{d/2-\gamma/2}} \left| \partial_{x_j} \left[e^{-\frac{|x-y|^2}{4(s+t)}} - e^{-\frac{|x|^2}{4(s+t)}} \right] \right| dx \lesssim t^{-\frac{\alpha+1-\gamma}{2}} |y|^\alpha.$$

Proof. We have

$$\begin{aligned} \partial_{x_j} \left[e^{-\frac{|x-y|^2}{4(s+t)}} - e^{-\frac{|x|^2}{4(s+t)}} \right] &= \partial_{x_j} \left[e^{-\frac{|x-y|^2}{4(s+t)}} - e^{-\frac{|x|^2+|y|^2}{4(s+t)}} \right] + \partial_{x_j} \left[e^{-\frac{|x|^2+|y|^2}{4(s+t)}} - e^{-\frac{|x|^2}{4(s+t)}} \right] \\ &= -\frac{x_j - y_j}{2(s+t)} \left[e^{-\frac{|x-y|^2}{4(s+t)}} - e^{-\frac{|x|^2+|y|^2}{4(s+t)}} \right] + \frac{y_j}{2(s+t)} e^{-\frac{|x|^2+|y|^2}{4(s+t)}} \\ &\quad - \frac{x_j}{2(s+t)} \left[e^{-\frac{|x|^2+|y|^2}{4(s+t)}} - e^{-\frac{|x|^2}{4(s+t)}} \right] \end{aligned}$$

As a consequence,

$$\int_{\mathbb{R}^d} \sup_{s>0} \frac{1}{(s+t)^{d/2-\gamma/2}} \left| \partial_{x_j} \left[e^{-\frac{|x-y|^2}{4(s+t)}} - e^{-\frac{|x|^2}{4(s+t)}} \right] \right| dx \leq E_1 + E_2 + E_3,$$

where

$$\begin{aligned} E_1 &= \int_{\mathbb{R}^d} \sup_{s>0} \frac{1}{(s+t)^{d/2-\gamma/2}} \frac{|x_j - y_j|}{2(s+t)} \left| e^{-\frac{|x-y|^2}{4(s+t)}} - e^{-\frac{|x|^2+|y|^2}{4(s+t)}} \right| dx, \\ E_2 &= \int_{\mathbb{R}^d} \sup_{s>0} \frac{1}{(s+t)^{d/2-\gamma/2}} \frac{|y_j|}{2(s+t)} e^{-\frac{|x|^2+|y|^2}{4(s+t)}} dx, \end{aligned}$$

and

$$E_3 = \int_{\mathbb{R}^d} \sup_{s>0} \frac{1}{(s+t)^{d/2-\gamma/2}} \frac{|x_j|}{2(s+t)} \left| e^{-\frac{|x|^2+|y|^2}{4(s+t)}} - e^{-\frac{|x|^2}{4(s+t)}} \right| dx.$$

By making use of the following inequalities

$$\frac{|x_j - y_j|}{2(s+t)} e^{-\frac{|x-y|^2}{4(s+t)}} \lesssim \frac{1}{\sqrt{s+t}} e^{-\frac{|x-y|^2}{8(s+t)}}, \quad \frac{|x_j - y_j|}{2(s+t)} e^{-\frac{|x|^2+|y|^2}{4(s+t)}} \lesssim \frac{1}{\sqrt{s+t}} e^{-\frac{|x|^2+|y|^2}{8(s+t)}},$$

and

$$\frac{|x_j|}{2(s+t)} e^{-\frac{|x|^2+|y|^2}{4(s+t)}} \lesssim \frac{1}{\sqrt{s+t}} e^{-\frac{|x|^2+|y|^2}{8(s+t)}}, \quad \frac{|x_j|}{2(s+t)} e^{-\frac{|x|^2}{4(s+t)}} \lesssim \frac{1}{\sqrt{s+t}} e^{-\frac{|x|^2}{8(s+t)}},$$

the estimates of E_1 and E_3 can be obtained by the argument used in the proof of Lemma 2.3. Hence,

$$E_1 + E_3 \lesssim t^{-\frac{\alpha+1-\gamma}{2}} |y|^\alpha.$$

It remains to estimate E_2 . To do this, using the following inequality

$$e^{-\frac{|x|^2+|y|^2}{4(s+t)}} \lesssim \frac{(s+t)^{\frac{1-\alpha}{2}}}{(|x|^2+|y|^2)^{\frac{1-\alpha}{2}}} e^{-\frac{|x|^2+|y|^2}{8(s+t)}} \lesssim \frac{(s+t)^{\frac{1-\alpha}{2}}}{|y|^{1-\alpha}} e^{-\frac{|x|^2+|y|^2}{8(s+t)}},$$

we have

$$\begin{aligned} E_2 &\lesssim \int_{\mathbb{R}^d} \sup_{s>0} \frac{1}{(s+t)^{d/2-\gamma/2}} \frac{(s+t)^{\frac{1-\alpha}{2}}}{|y|^{1-\alpha}} \frac{|y_j|}{2(s+t)} e^{-\frac{|x|^2+|y|^2}{4(s+t)}} dx \\ &\lesssim \int_{\mathbb{R}^d} \sup_{s>0} \frac{1}{(s+t)^{d/2-\gamma/2}} \frac{|y|^\alpha}{2(s+t)^{\frac{\alpha+1}{2}}} e^{-\frac{|x|^2+|y|^2}{4(s+t)}} dx \\ &\lesssim t^{-\frac{\alpha+1-\gamma}{2}} |y|^\alpha, \end{aligned}$$

where in the last inequality we used Lemma 2.1. $\square$

Lemma 2.6. *Let $\alpha \in [0, 1]$ and $c_0, c_1 > 0$. Then we have*

$$\int_0^\infty \frac{1}{\sqrt{st}} e^{-\frac{|x-z|^2}{c_1 s}} \left[e^{-\frac{|y-z|^2}{c_0 t}} - e^{-\frac{|y+z|^2}{c_0 t}} \right] dz \lesssim t^{-\alpha/2} |y|^\alpha \frac{1}{\sqrt{s+t}} e^{-\frac{|x-y|^2}{c(s+t)}}$$

for every $s, t > 0$, $x \in \mathbb{R}$ and $y > 0$, and for some $c > 0$.

Proof. In order to prove the lemma. We will employ the following simple inequality whose proof will be omitted: let $a > 0$. Then there exists $c > 0$ such that for any $s, t > 0$ and $x, y \in \mathbb{R}$, we have

$$\int_{\mathbb{R}} \frac{1}{\sqrt{st}} e^{-\frac{|x-z|^2}{as}} e^{-\frac{|y-z|^2}{at}} dz \lesssim \frac{1}{\sqrt{s+t}} e^{-\frac{|x-y|^2}{c(s+t)}}. \quad (9)$$

We consider two cases: $t < |y|^2$ and $t \geq |y|^2$.

Case 1: $t < |y|^2$

In this situation, we have

$$\begin{aligned} \int_0^\infty \frac{1}{\sqrt{st}} e^{-\frac{|x-z|^2}{c_1 s}} \left[e^{-\frac{|y-z|^2}{c_0 t}} - e^{-\frac{|y+z|^2}{c_0 t}} \right] dz &\leq \int_{\mathbb{R}} \frac{1}{\sqrt{st}} e^{-\frac{|x-z|^2}{c_1 s}} e^{-\frac{|y-z|^2}{c_0 t}} dz \\ &\lesssim \frac{1}{\sqrt{s+t}} e^{-\frac{|x-y|^2}{c(s+t)}} \\ &\lesssim \frac{y^\alpha}{t^{\alpha/2}} \frac{1}{\sqrt{s+t}} e^{-\frac{|x-y|^2}{c(s+t)}} \end{aligned}$$

for some $c > 0$, where we used (9) in the second inequality.

Case 2: $t \geq |y|^2$

In this case, we have

$$\begin{aligned} \int_{\mathbb{R}_+} \frac{1}{\sqrt{st}} e^{-\frac{|x-z|^2}{c_1 s}} \left| e^{-\frac{|y-z|^2}{c_0 t}} - e^{-\frac{|y+z|^2}{c_0 t}} \right| dz &= \int_{\mathbb{R}_+ \cap \{z: yz \geq 4t\}} \cdots + \int_{\mathbb{R}_+ \cap \{z: yz < 4t\}} \cdots \\ &=: E_1 + E_2. \end{aligned}$$

For E_2 , we write

$$E_2 = \int_{\mathbb{R}_+ \cap \{z: yz < 4t\}} \frac{1}{\sqrt{st}} e^{-\frac{|x-z|^2}{c_1 s}} e^{-\frac{(y+z)^2}{c_0 t}} \left[e^{\frac{4yz}{c_0 t}} - 1 \right] dz.$$

Using the inequality $e^u - 1 \lesssim u^\alpha$ for $\alpha \in [0, 1]$ and $u \lesssim 1$, we have

$$\begin{aligned} E_2 &\lesssim \int_{\mathbb{R}_+} \frac{1}{\sqrt{st}} e^{-\frac{|x-z|^2}{c_1 s}} e^{-\frac{(y+z)^2}{c_0 t}} \left[\frac{yz}{t} \right]^\alpha dz \\ &\lesssim \int_{\mathbb{R}_+} \frac{1}{\sqrt{st}} e^{-\frac{|x-z|^2}{c_1 s}} e^{-\frac{(y+z)^2}{2c_0 t}} e^{-\frac{z^2}{2c_0 t}} \left[\frac{yz}{t} \right]^\alpha dz \\ &\lesssim \frac{y^\alpha}{t^{\alpha/2}} \int_{\mathbb{R}_+} \frac{1}{\sqrt{st}} e^{-\frac{|x-z|^2}{c_1 s}} e^{-\frac{|y-z|^2}{2c_0 t}} dz \\ &\lesssim \frac{y^\alpha}{t^{\alpha/2}} \frac{1}{\sqrt{s+t}} e^{-\frac{|x-y|^2}{c(s+t)}} \end{aligned}$$

for some $c > 0$, where we used $e^{-\frac{(y+z)^2}{c_0 t}} \leq e^{-\frac{(y+z)^2}{2c_0 t}} e^{-\frac{z^2}{2c_0 t}}$ (since $y, z \geq 0$) in the second inequality and (9) in the last inequality.

For the term E_1 , we notice that $yz > 4t \geq 4y^2$, and hence $z > 4y$. In this situation,

$$|y - z| \sim y + z$$

so that

$$\left| e^{-\frac{|y-z|^2}{c_0 t}} - e^{-\frac{|y+z|^2}{c_0 t}} \right| \lesssim e^{-\frac{(y+z)^2}{c' t}}$$

for some $c' > 0$.

Therefore,

$$\begin{aligned} E_2 &\lesssim \int_{\mathbb{R}_+ \cap \{z: yz \geq 4t\}} \frac{1}{\sqrt{st}} e^{-\frac{|x-z|^2}{c_1 s}} e^{-\frac{(y+z)^2}{c' t}} \left[\frac{yz}{t} \right]^\alpha dz \\ &\lesssim \int_{\mathbb{R}_+} \frac{1}{\sqrt{st}} e^{-\frac{|x-z|^2}{c_1 s}} e^{-\frac{(y+z)^2}{2c' t}} e^{-\frac{z^2}{2c' t}} \left[\frac{yz}{t} \right]^\alpha dz \\ &\lesssim \frac{y^\alpha}{t^{\alpha/2}} \int_{\mathbb{R}_+} \frac{1}{\sqrt{st}} e^{-\frac{|x-z|^2}{c_1 s}} e^{-\frac{|y-z|^2}{2c' t}} dz \\ &\lesssim \frac{y^\alpha}{t^{\alpha/2}} \frac{1}{\sqrt{s+t}} e^{-\frac{|x-y|^2}{c(s+t)}} \end{aligned}$$

for some $c > 0$, where we used the following estimate

$$e^{-\frac{z^2}{2c' t}} \left[\frac{yz}{t} \right]^\alpha \lesssim \frac{y^\alpha}{t^{\alpha/2}}$$

in the third inequality and used (9) in the last inequality.

This completes our proof. $\square$

Lemma 2.7. *Let $d \in \mathbb{N}$, $\alpha \in (0, 1]$ and $a > 0$. Then we have, for every $x, y \in \mathbb{R}^d$,*

$$\begin{aligned} \int_0^\infty \frac{\sqrt{s}}{(t+s)^{\frac{d}{2}}} \left| \partial_j \left[e^{-\frac{|x-y|^2}{a(t+s)}} - e^{-\frac{|x|^2}{a(t+s)}} \right] \right| \frac{ds}{s} &\lesssim \frac{|y|^\alpha}{t^{(d+\alpha)/2}} \left[\left(1 + \frac{|x-y|}{\sqrt{t}} \right)^{-(d+\alpha)} \right. \\ &\quad \left. + \left(1 + \frac{|x|}{\sqrt{t}} \right)^{-(d+\alpha)} \right] \end{aligned}$$

Proof. We write

$$\begin{aligned} \int_0^\infty \frac{\sqrt{s}}{(t+s)^{\frac{d}{2}}} \left| \partial_j \left[e^{-\frac{|x-y|^2}{a(t+s)}} - e^{-\frac{|x|^2}{a(t+s)}} \right] \right| \frac{ds}{s} &= \int_{\{s: s+t < |y|^2\}} \dots + \int_{\{s: s+t \geq |y|^2\}} \dots \\ &=: E_1 + E_2. \end{aligned}$$

Using the fact that

$$\left| \partial_j \left[e^{-\frac{|x-y|^2}{a(t+s)}} - e^{-\frac{|x|^2}{a(t+s)}} \right] \right| \lesssim \frac{1}{\sqrt{s+t}} \left[e^{-\frac{|x-y|^2}{c(t+s)}} + e^{-\frac{|x|^2}{c(t+s)}} \right] \quad \text{for some } c > 0,$$

and Lemma 2.2 we have

$$\begin{aligned}
 E_1 &\lesssim \int_{\{s:s+t < |y|^2\}} \frac{|y|^\alpha \sqrt{s}}{(t+s)^{\frac{d+\alpha+1}{2}}} \left| e^{-\frac{|x-y|^2}{c(t+s)}} + e^{-\frac{|x|^2}{c(t+s)}} \right| \frac{ds}{s} \\
 &\lesssim \int_0^\infty \frac{|y|^\alpha \sqrt{s}}{(t+s)^{\frac{d+\alpha+1}{2}}} \left| e^{-\frac{|x-y|^2}{c(t+s)}} + e^{-\frac{|x|^2}{c(t+s)}} \right| \frac{ds}{s} \\
 &\lesssim \frac{|y|^\alpha}{t^{(d+\alpha)/2}} \left[\left(1 + \frac{|x-y|}{\sqrt{t}}\right)^{-(d+\alpha)} + \left(1 + \frac{|x|}{\sqrt{t}}\right)^{-(d+\alpha)} \right].
 \end{aligned}$$

For the second term, we note that

$$\partial_j \left[e^{-\frac{|x-y|^2}{a(t+s)}} - e^{-\frac{|x|^2}{a(t+s)}} \right] = -\frac{2(x_j - y_j)}{a(t+s)} \left[e^{-\frac{|x-y|^2}{a(t+s)}} - e^{-\frac{|x|^2}{a(t+s)}} \right] - \frac{2y_j}{a(t+s)} e^{-\frac{|x|^2}{a(t+s)}}.$$

Hence, for $s+t > |y|^2$, we consider two cases.

Case 1: If $|x| \leq 2|y|$, then we have

$$\begin{aligned}
 \left| \partial_j \left[e^{-\frac{|x-y|^2}{a(t+s)}} - e^{-\frac{|x|^2}{a(t+s)}} \right] \right| &\lesssim \frac{|y|}{t+s} e^{-\frac{|x-y|^2}{c(t+s)}} + \frac{|y_j|}{t+s} e^{-\frac{|x|^2}{c(t+s)}} \\
 &\lesssim \frac{1}{\sqrt{t+s}} \frac{|y|^\alpha}{(t+s)^{\alpha/2}} \left[e^{-\frac{|x-y|^2}{c(t+s)}} + e^{-\frac{|x|^2}{c(t+s)}} \right].
 \end{aligned}$$

Case 2: If $|x| \geq 2|y|$, then we have $|x-y| \sim |y|$. If $|x||y| \geq (s+t)^2$, then we have we have

$$\begin{aligned}
 \left| \partial_j \left[e^{-\frac{|x-y|^2}{a(t+s)}} - e^{-\frac{|x|^2}{a(t+s)}} \right] \right| &\lesssim \frac{|x-y|}{t+s} \frac{(|x||y|)^\alpha}{(t+s)^\alpha} e^{-\frac{|x-y|^2}{c(t+s)}} + \frac{|y_j|}{t+s} e^{-\frac{|x|^2}{c(t+s)}} \\
 &\lesssim \frac{|x-y|^{1+\alpha}}{(t+s)^{1+\alpha/2}} \frac{|y|^\alpha}{(t+s)^{\alpha/2}} e^{-\frac{|x-y|^2}{c(t+s)}} + \frac{|y_j|}{t+s} e^{-\frac{|x|^2}{c(t+s)}} \\
 &\lesssim \frac{1}{\sqrt{t+s}} \frac{|y|^\alpha}{(t+s)^{\alpha/2}} \left[e^{-\frac{|x-y|^2}{2c(t+s)}} + e^{-\frac{|x|^2}{c(t+s)}} \right].
 \end{aligned}$$

Otherwise, if $|x||y| \geq (s+t)^2$, then we write

$$\begin{aligned}
 \left| e^{-\frac{|x-y|^2}{a(t+s)}} - e^{-\frac{|x|^2}{a(t+s)}} \right| &= e^{-\frac{|x-y|^2}{a(t+s)}} \left| 1 - e^{-\frac{\langle x, 2x-y \rangle}{a(t+s)}} \right| \\
 &\lesssim e^{-\frac{|x-y|^2}{a(t+s)}} \frac{(|2x-y||y|)^\alpha}{(t+s)^\alpha} \\
 &\sim e^{-\frac{|x-y|^2}{a(t+s)}} \frac{(|x-y||y|)^\alpha}{(t+s)^\alpha},
 \end{aligned}$$

where in the second inequality we used the following inequality $|1 - e^x| \lesssim |x|^\alpha$ for all x with $|x| \lesssim 1$.

Therefore,

$$\begin{aligned}
 \left| \partial_j \left[e^{-\frac{|x-y|^2}{a(t+s)}} - e^{-\frac{|x|^2}{a(t+s)}} \right] \right| &\lesssim \frac{|x-y|^{1+\alpha}}{(t+s)^{1+\alpha/2}} \frac{|y|^\alpha}{(t+s)^{\alpha/2}} e^{-\frac{|x-y|^2}{c(t+s)}} + \frac{|y_j|}{t+s} e^{-\frac{|x|^2}{c(t+s)}} \\
 &\lesssim \frac{1}{\sqrt{t+s}} \frac{|y|^\alpha}{(t+s)^{\alpha/2}} \left[e^{-\frac{|x-y|^2}{2c(t+s)}} + e^{-\frac{|x|^2}{c(t+s)}} \right].
 \end{aligned}$$

So, we have proved that

$$\left| \partial_j \left[e^{-\frac{|x-y|^2}{a(t+s)}} - e^{-\frac{|x|^2}{a(t+s)}} \right] \right| \lesssim \frac{1}{\sqrt{t+s}} \frac{|y|^\alpha}{(t+s)^{\alpha/2}} \left[e^{-\frac{|x-y|^2}{2c(t+s)}} + e^{-\frac{|x|^2}{c(t+s)}} \right].$$

This, along with Lemma 2.2, implies

$$E_2 \lesssim \frac{|y|^\alpha}{t^{(d+\alpha)/2}} \left[\left(1 + \frac{|x-y|}{\sqrt{t}} \right)^{-(d+\alpha)} + \left(1 + \frac{|x|}{\sqrt{t}} \right)^{-(d+\alpha)} \right]$$

as desired.

This completes our proof. $\square$

Lemma 2.8. *Let $d \geq 2$, $\alpha \in (0, 1]$ and $c, c' > 0$. Then, for $x = (x', x_d), y = (y', y_d) \in \mathbb{R}^n$ and $j = 1, \dots, d-1$, we have*

$$\begin{aligned} \int_0^\infty \frac{\sqrt{s}}{(t+s)^{\frac{d}{2}}} \left| \partial_j \left[e^{-\frac{|x'-y'|^2}{c(t+s)}} - e^{-\frac{|x'|^2}{c(t+s)}} \right] \right| e^{-\frac{|x_d-y_d|^2}{c'(t+s)}} \frac{ds}{s} \\ \lesssim \frac{|y'|^\alpha}{t^{(d+\alpha)/2}} \left[\left(1 + \frac{|x-y|}{\sqrt{t}} \right)^{-(d+\alpha)} + \left(1 + \frac{|x-(0', y_d)|}{\sqrt{t}} \right)^{-(d+\alpha)} \right], \end{aligned}$$

where $(0', y_d) = (0, \dots, 0, y_d) \in \mathbb{R}^d$.

Proof. The proof of this Lemma is similar to that of Lemma 2.7 and we would like to leave the details for the interested reader. $\square$

3. Temporal-spatial decay estimates for the Stokes equations

This section is devoted to prove the temporal-spatial decay estimates for the Stokes equations in Theorem 1.1.

We first make some conventions of notations. For $x \in \mathbb{R}^n$, we denote its tangential components $(x_1, \dots, x_{n-1})$ by $x' \in \mathbb{R}^{n-1}$, so that $x = (x', x_n)$. We set $\partial_j = \partial/\partial x_j$, $\nabla' = (\partial_1, \dots, \partial_{n-1})$, $\Delta' = \sum_{j=1}^{n-1} \partial_j^2$, and $\Delta_n = \partial_n^2$.

The Riesz transforms R_j ($j = 1, \dots, n$), S_j ($j = 1, \dots, n-1$), and the operator Λ are defined by

$$\begin{aligned} \mathcal{F}(R_j f)(\xi) &= \frac{i\xi_j}{|\xi|} \mathcal{F}\{f\}(\xi), \\ \mathcal{F}(S_j f)(\xi) &= \frac{i\xi_j}{|\xi'|} \mathcal{F}\{f\}(\xi), \\ \mathcal{F}(\Lambda f)(\xi) &= |\xi'| \mathcal{F}\{f\}(\xi). \end{aligned}$$

We set $R' = (R_1, \dots, R_{n-1})$, $S = (S_1, \dots, S_{n-1})$, and define

$$Uf = rR' \cdot S(R' \cdot S + R_n)ef,$$

where r is the restriction operator from $\mathbb{R}^n$ to $\mathbb{R}_+^n$, and e is the zero extension operator from $\mathbb{R}_+^n$ to $\mathbb{R}^n$ defined by

$$ef = \begin{cases} f & \text{for } x_n \geq 0, \\ 0 & \text{for } x_n = 0. \end{cases}$$

We also define the operators $E(t)$ and $F(t)$ by

$$E(t)f(x) = \int_{\mathbb{R}_+^n} [H_t(x-y) - H_t(x'-y', x_n+y_n)] f(y)dy, \quad (10)$$

$$F(t)f(x) = \int_{\mathbb{R}_+^n} [H_t(x-y) + H_t(x'-y', x_n+y_n)] f(y)dy, \quad (11)$$

where G_t is the Gauss kernel $H_t(x) = (4\pi t)^{-n/2}e^{-|x|^2/4t}$. Note that $E(t)$ and $F(t)$ are the semigroups generated by the Dirichlet Laplacian and Neumann Laplacian, respectively.

We recall the following result in [19] regarding the solution formula of the Stokes equation (1).

Theorem 3.1. *The solution to (1) can be expressed as*

$$u^n = UE(t)V_1u_0, \quad (12)$$

$$u' = E(t)V_2u_0 - SUE(t)V_1u_0, \quad (13)$$

where $V_1u_0 = -S \cdot u'_0 + u_0^n$, and $V_2u_0 = u'_0 + Su_0^n$.

It is easy to see that the solution u defined in Theorem 3.1 is given as a restriction $r\bar{u}$ of $\bar{u} = (\bar{u}^1, \dots, \bar{u}^{n-1}, \bar{u}^n) := (\bar{u}', \bar{u}^n)$ of the form

$$\bar{u}^n = R' \cdot S(R' \cdot S + R_n)eE(t)V_1u_0, \quad (14)$$

$$\bar{u}' = eE(t)V_2u_0 - SR' \cdot S(R' \cdot S + R_n)eE(t)V_1u_0. \quad (15)$$

In order to prove Theorem 1.1 we need the following representations for $\bar{u}$ in [1, 12].

Proposition 3.2. *Let $\bar{u}$ is given by (14) and (15). Then we have*

$$\begin{aligned} \bar{u}^n &= \sum_{j=1}^{n-1} R_n R_j eE(t)u_0^j - \sum_{j=1}^{n-1} R_j^2 eE(t)u_0^n + \sum_{j,k=1}^{n-1} R_k^2 \partial_j \Lambda^{-1} eE(t)u_0^j \\ &\quad + \sum_{j=1}^{n-1} R_n R_j \partial_j \Lambda^{-1} eE(t)u_0^n, \end{aligned} \quad (16)$$

$$\begin{aligned} \bar{u}' &= \tilde{E}(t)u'_0 + R_n^2 \nabla' \cdot \Lambda^{-1} eE(t)u_0^n + \sum_{k=1}^{n-1} R' \cdot R_k eE(t)u'_0 \\ &\quad - \sum_{k=1}^{n-1} R_n R_k \nabla' \Lambda^{-1} eE(t)u_0^k \sum_{k=1}^{n-1} R_k^2 \nabla' \Lambda^{-1} eE(t)u_0^n + R_n R' eE(t)u_0^n, \end{aligned} \quad (17)$$

where $\tilde{E}(t)$ is an operator defined by

$$\tilde{E}(t)f(x) = \int_{\mathbb{R}_+^n} [H_t(x-y) - H_t(x'-y', x_n+y_n)] f(y)dy \quad (18)$$

for all $x \in \mathbb{R}^n$ and $f \in L^1(\mathbb{R}_+^n)$.

We now recall the definition of the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$. See [18]. A function $f \in L^1(\mathbb{R}^n)$ is said to be in Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ if

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^n)} = \left\| \sup_{t>0} |H_t * f| \right\|_{L^1(\mathbb{R}^n)} < \infty, \quad (19)$$

where H_t is the heat kernel of the Laplacian defined by

$$H_t(x) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}.$$

In the sequel, we denote

$$H_{n-1,t}(x') = (4\pi t)^{-\frac{n-1}{2}} e^{-\frac{|x'|^2}{4t}}, \quad x' \in \mathbb{R}^{n-1},$$

and

$$H_{1,t}(x_n) = (4\pi t)^{-\frac{1}{2}} e^{-\frac{x_n^2}{4t}}, \quad x_n \in \mathbb{R}.$$

Then we have

$$H_t(x) = H_{n-1,t}(x') H_{1,t}(x_n), \quad x = (x', x_n) \in \mathbb{R}^n.$$

In order to prove Theorem 1.1 we need the following auxiliary estimates.

Proposition 3.3. *Assume that $a \in L^1(\mathbb{R}_+^n)$ satisfies*

$$\int_{\mathbb{R}^{n-1}} a(y', y_n) dy' = 0 \quad \text{a.e. } y_n > 0, \quad (20)$$

then for any $1 \leq j \leq n-1$, $t > 0$, $\alpha \in (0, 1]$ and $\beta \in [0, 1]$,

$$\|\tilde{E}(t)a\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq Ct^{-\frac{\alpha+\beta}{2}} \int_{\mathbb{R}_+^n} |y'|^\alpha y_n^\beta |a(y)| dy, \quad (21)$$

$$\|\partial_j \Lambda^{-1} eE(t)a\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq Ct^{-\frac{\alpha+\beta}{2}} \int_{\mathbb{R}^n} |y'|^\alpha y_n^\beta |a(y)| dy, \quad (22)$$

and

$$\|R_j eE(t)a\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq Ct^{-\frac{\alpha+\beta}{2}} \int_{\mathbb{R}^n} |y'|^\alpha y_n^\beta |a(y)| dy. \quad (23)$$

Proof. Proof of (21)

Denote by a^* the odd extension of a from $\mathbb{R}_+^n$ to $\mathbb{R}^n$, i.e.

$$a^*(x', x_n) = \begin{cases} a(x', x_n) & \text{if } x_n \geq 0, \\ -a(x', -x_n) & \text{if } x_n < 0. \end{cases}$$

Then we have

$$\tilde{E}(t)a(x) = e^{t\Delta} a^*(x), \quad (24)$$

which implies that for $s > 0$,

$$e^{s\Delta} \tilde{E}(t)a(x) = e^{(s+t)\Delta} a^*(x).$$

Therefore,

$$\begin{aligned}
& \left\| \sup_{s>0} |e^{s\Delta} \tilde{E}(t) a(x)| \right\|_{L^1(\mathbb{R}^n)} \\
&= \left\| \sup_{s>0} |e^{(s+t)\Delta} a^*(x)| \right\|_{L^1(\mathbb{R}^n)} \\
&= \int_{\mathbb{R}^n} \sup_{s>0} \left| \int_{\mathbb{R}^{n-1}} \int_0^\infty H_{n-1,s+t}(x' - y') [H_{1,s+t}(x_n - y_n) \right. \\
&\quad \left. - H_{1,s+t}(x_n + y_n)] a(y', y_n) dy_n dy' \right| dx.
\end{aligned}$$

Using the fact that $\int_{\mathbb{R}^{n-1}} a(y', y_n) dy' = 0$,

$$\begin{aligned}
& \left\| \sup_{s>0} |e^{s\Delta} \tilde{E}(t) a(x)| \right\|_{L^1(\mathbb{R}^n)} \\
&= \int_{\mathbb{R}^n} \sup_{s>0} \left| \int_{\mathbb{R}^{n-1}} \int_0^\infty F_{n-1,s+t}(x', y') G_{1,s+t}(x_n, y_n) a(y', y_n) dy_n dy' \right| dx,
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
G_{n-1,s+t}(x', y') &= H_{n-1,s+t}(x' - y') - H_{n-1,s+t}(x') \\
&= \frac{1}{[4\pi(s+t)]^{\frac{n-1}{2}}} \left[e^{-\frac{|x'-y'|^2}{4(s+t)}} - e^{-\frac{|x'|^2}{4(s+t)}} \right],
\end{aligned} \tag{26}$$

and

$$\begin{aligned}
F_{1,s+t}(x_n, y_n) &= H_{1,s+t}(x_n - y_n) - H_{1,s+t}(x_n + y_n) \\
&= \frac{1}{[4\pi(s+t)]^{1/2}} \left[e^{-\frac{|x_n-y_n|^2}{4(s+t)}} - e^{-\frac{|x_n+y_n|^2}{4(s+t)}} \right].
\end{aligned} \tag{27}$$

We consider two cases: $\beta > 0$ and $\beta = 0$.

Case 1: $\beta > 0$

We have

$$\begin{aligned}
& \left\| \sup_{s>0} |e^{s\Delta} E(t) a(x)| \right\|_{L^1(\mathbb{R}^n)} \\
&\lesssim \int_{\mathbb{R}_+^n} |y'|^\alpha y_n^\beta |a(y)| dy \times \sup_{y'} \int_{\mathbb{R}^{n-1}} \sup_{s>0} |y'|^{-\alpha} |G_{n-1,s+t}(x', y')| dx' \\
&\quad \times \sup_{y_n} \int_{\mathbb{R}} \sup_{s>0} y_n^{-\beta} |F_{1,s+t}(x_n, y_n)| dx_n.
\end{aligned}$$

Then applying Lemmas 2.3 and 2.4,

$$\left\| \sup_{s>0} |e^{s\Delta} E(t) a(x)| \right\|_{L^1(\mathbb{R}^n)} \lesssim t^{-\frac{\alpha+\beta}{2}} \int_{\mathbb{R}_+^n} |y'|^\alpha y_n^\beta |a(y)| dy.$$

Case 2: $\beta = 0$

Take $\gamma \in (0, \alpha)$. Then we have, from (25),

$$\begin{aligned} & \left\| \sup_{s>0} |e^{s\Delta} E(t)a(x)| \right\|_{L^1(\mathbb{R}^n)} \\ & \lesssim \int_{\mathbb{R}_+^n} |y'|^\alpha |a(y)| dy \times \sup_{y'} \int_{\mathbb{R}^{n-1}} \sup_{s>0} |y'|^{-\alpha} (s+t)^\gamma |G_{n-1,s+t}(x', y')| dx' \\ & \quad \times \sup_{y_n} \int_{\mathbb{R}} \sup_{s>0} (s+t)^{-\gamma} |F_{1,s+t}(x_n, y_n)| dx_n. \end{aligned}$$

Applying Lemma 2.4 to the first supremum and Lemma 2.1 for the second supremum, we find that

$$\begin{aligned} \left\| \sup_{s>0} |e^{s\Delta} E(t)a(x)| \right\|_{L^1(\mathbb{R}^n)} & \lesssim t^{-\frac{\alpha-\gamma}{2}-\frac{\gamma}{2}} \int_{\mathbb{R}_+^n} |y'|^\alpha |a(y)| dy \\ & \lesssim t^{-\frac{\alpha}{2}} \int_{\mathbb{R}_+^n} |y'|^\alpha |a(y)| dy. \end{aligned}$$

This completes the proof of (21).

Proof of (22) Note that $\Lambda^{-1} = (-\Delta')^{-\frac{1}{2}}$. Hence, by the subordination formula,

$$\Lambda^{-1} = c \int_0^\infty \sqrt{u} e^{u\Delta'} \frac{du}{u}.$$

It follows that, for $j = 1, \dots, n-1$,

$$e^{s\Delta} \partial_j \Lambda^{-1} e E(t)a(x) = c \int_0^\infty \sqrt{u} \partial_j e^{s\Delta} e^{u\Delta'} e E(t)a(x) \frac{du}{u}.$$

This, in combination with (20), implies

$$\begin{aligned} e^{s\Delta} \partial_j \Lambda^{-1} e E(t)a(x) &= \int_0^\infty \int_{\mathbb{R}_+^n} \sqrt{u} \partial_j [H_{n-1,s+u+t}(x', y') - H_{n-1,s+u+t}(x', 0)] \\ & \quad \times \int_0^\infty H_{1,s}(x_n, z_n) F_{1,t}(z_n, y_n) a(y', y_n) dz_n dy_n dy' \frac{du}{u}, \end{aligned}$$

where we recall that $F_{1,t}(z_n, y_n) = H_{1,t}(z_n - y_n) - H_{1,t}(z_n + y_n)$.

By Lemmas 2.7 and 2.6,

$$\begin{aligned} & \int_0^\infty \sqrt{u} |\partial_j [H_{n-1,s+u+t}(x', y') - H_{n-1,s+u+t}(x', 0)]| \frac{du}{u} \\ & \lesssim \frac{|y'|^\alpha}{(s+t)^{\frac{n+\alpha-1}{2}}} \left[\left(1 + \frac{|x' - y'|}{\sqrt{s+t}}\right)^{-(n+\alpha-1)} + \left(1 + \frac{|x'|}{\sqrt{s+t}}\right)^{-(n+\alpha-1)} \right] \end{aligned}$$

and

$$\int_0^\infty H_{1,s}(x_n, z_n) F_{1,t}(z_n, y_n) dz_n \lesssim \frac{|y_n|^\beta}{t^{\beta/2}} \frac{1}{\sqrt{s+t}} e^{-\frac{|x_n - y_n|^2}{c(s+t)}}.$$

Consequently,

$$\begin{aligned}
 |e^{s\Delta}\partial_j\Lambda^{-1}eE(t)a(x)| &\lesssim \int_{\mathbb{R}_+^n} \frac{|y'|^\alpha y_n^\beta}{t^{\beta/2}(s+t)^{\frac{n+\alpha}{2}}} |a(y', y_n)| \\
 &\quad \times \left[\left(1 + \frac{|x' - y'|}{\sqrt{s+t}}\right)^{-(n+\alpha-1)} + \left(1 + \frac{|x'|}{\sqrt{s+t}}\right)^{-(n+\alpha-1)} \right] \\
 &\quad e^{-\frac{|x_n - y_n|^2}{c(s+t)}} dy_n dy'. \tag{28}
 \end{aligned}$$

From Lemma 2.1,

$$\sup_y \int_{\mathbb{R}^{n-1}} \sup_{s>0} \frac{1}{(s+t)^{\frac{n-1}{2} + \frac{\alpha}{4}}} \left[\left(1 + \frac{|x' - y'|}{\sqrt{s+t}}\right)^{-(n+\alpha-1)} + \left(1 + \frac{|x'|}{\sqrt{s+t}}\right)^{-(n+\alpha-1)} \right] dx' \lesssim t^{-\frac{\alpha}{4}}$$

and

$$\sup_y \int_{\mathbb{R}} \sup_{s>0} \frac{1}{(s+t)^{\frac{1}{2} + \frac{\alpha}{4}}} e^{-\frac{|x_n - y_n|^2}{c(s+t)}} dx_n \lesssim t^{-\frac{\alpha}{4}}.$$

These two inequalities and (28) imply that

$$\|e^{s\Delta}\partial_j\Lambda^{-1}eE(t)a\|_{\mathcal{H}^1(\mathbb{R}^n)} \lesssim t^{-\frac{\alpha+\beta}{2}} \int_{\mathbb{R}_+^n} |y'|^\alpha y_n^\beta |a(y', y_n)| dy$$

Proof of (23)

Using the subordination formula,

$$R_j = c \int_0^\infty \sqrt{u} \partial_j e^{u\Delta} \frac{du}{u}.$$

we have, for $j = 1, \dots, n-1$,

$$e^{s\Delta} R_j eE(t)a(x) = c \int_0^\infty \sqrt{u} \partial_j e^{s\Delta} e^{u\Delta} eE(t)a(x) \frac{du}{u}.$$

We then apply (20) to write

$$\begin{aligned}
 e^{s\Delta} R_j eE(t)a(x) &= \int_0^\infty \int_{\mathbb{R}_+^n} \sqrt{u} \partial_j [H_{n-1, s+u+t}(x', y') - H_{n-1, s+u+t}(x', 0)] \\
 &\quad \times \int_0^\infty H_{1, s+u}(x_n, z_n) F_{1, t}(z_n, y_n) a(y', y_n) dz_n dy_n dy' \frac{du}{u}.
 \end{aligned}$$

Applying Lemma 2.6,

$$\int_0^\infty H_{1, s+u}(x_n, z_n) F_{1, t}(z_n, y_n) dz_n \lesssim \frac{|y_n|^\beta}{t^{\beta/2}} \frac{1}{\sqrt{s+u+t}} e^{-\frac{|x_n - y_n|^2}{c(s+u+t)}}.$$

Therefore,

$$\begin{aligned}
 |e^{s\Delta} R_j eE(t)a(x)| &= \int_0^\infty \int_{\mathbb{R}_+^n} \frac{|y_n|^\beta}{t^{\beta/2}} \frac{\sqrt{u}}{\sqrt{s+u+t}} \left| \partial_j [H_{n-1, s+u+t}(x', y') \right. \\
 &\quad \left. - H_{n-1, s+u+t}(x', 0)] \right| \times e^{-\frac{|x_n - y_n|^2}{c(s+u+t)}} |a(y', y_n)| dy_n dy' \frac{du}{u}.
 \end{aligned}$$

Then applying Lemma 2.8,

$$\begin{aligned}
 |e^{s\Delta} R_j e E(t) a(x)| &\lesssim \int_{\mathbb{R}_+^n} \frac{|y'|^\alpha y_n^\beta}{t^{\beta/2} (s+t)^{\frac{n+\alpha}{2}}} |a(y', y_n)| \\
 &\quad \times \left[\left(1 + \frac{|x-y|}{\sqrt{s+t}}\right)^{-(n+\alpha)} + \left(1 + \frac{|x-(0', y_n)|}{\sqrt{s+t}}\right)^{-(n+\alpha)} \right] \\
 &\quad e^{-\frac{|x_n - y_n|^2}{c(s+t)}} dy_n dy'. \tag{29}
 \end{aligned}$$

From Lemma 2.1,

$$\sup_y \int_{\mathbb{R}^n} \sup_{s>0} \frac{1}{(s+t)^{\frac{n}{2} + \frac{\alpha}{4}}} \left[\left(1 + \frac{|x-y|}{\sqrt{s+t}}\right)^{-(n+\alpha)} + \left(1 + \frac{|x-(0', y_n)|}{\sqrt{s+t}}\right)^{-(n+\alpha)} \right] dx \lesssim t^{-\frac{\alpha}{4}}.$$

This, along with (29), implies that

$$\|e^{s\Delta} R_j e E(t) a\|_{\mathcal{H}^1(\mathbb{R}^n)} \lesssim t^{-\frac{\alpha+\beta}{2}} \int_{\mathbb{R}_+^n} |y'|^\alpha y_n^\beta |a(y', y_n)| dy$$

□

We are now ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1.: Using Proposition 3.2, we have

$$\begin{aligned}
 \|\bar{u}^n\|_{\mathcal{H}^1(\mathbb{R}^n)} &\leq \sum_{j=1}^{n-1} \|R_n\|_{\mathcal{H}^1(\mathbb{R}^n)} \|R_j e E(t) u_0^j\|_{\mathcal{H}^1(\mathbb{R}^n)} \\
 &\quad + \sum_{j=1}^{n-1} \|R_j\|_{\mathcal{H}^1(\mathbb{R}^n)} \|R_j e E(t) u_0^n\|_{\mathcal{H}^1(\mathbb{R}^n)} \\
 &\quad + \sum_{j,k=1}^{n-1} \|R_k^2\|_{\mathcal{H}^1(\mathbb{R}^n)} \|\partial_j \Lambda^{-1} e E(t) u_0^j\|_{\mathcal{H}^1(\mathbb{R}^n)} \\
 &\quad + \sum_{j=1}^{n-1} \|R_n\|_{\mathcal{H}^1(\mathbb{R}^n)} \|R_j\|_{\mathcal{H}^1(\mathbb{R}^n)} \|\partial_j \Lambda^{-1} e E(t) u_0^n\|_{\mathcal{H}^1(\mathbb{R}^n)}.
 \end{aligned}$$

At this stage, using Proposition 3.3 and the boundedness of the Riesz transforms R_j on the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ for $j = 1, \dots, n$, we have

$$\|\bar{u}^n\|_{\mathcal{H}^1(\mathbb{R}^n)} \lesssim t^{-\frac{\alpha+\beta}{2}} \int_{\mathbb{R}_+^n} |y'|^\alpha y_n^\beta |u_0(y', y_n)| dy.$$

Similarly,

$$\|\bar{u}'\|_{\mathcal{H}^1(\mathbb{R}^n)} \lesssim t^{-\frac{\alpha+\beta}{2}} \int_{\mathbb{R}_+^n} |y'|^\alpha y_n^\beta |u_0(y', y_n)| dy.$$

This completes our proof.

□

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