Approaching optimality in blow-up results for Keller–Segel systems with logistic-type dampening

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Abstract. Nonnegative solutions of the Neumann initial-boundary value problem for the chemotaxis system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) + \lambda u - \mu u^\kappa, \\ 0 = \Delta v - \overline{m}(t) + u, \quad \overline{m}(t) = \frac{1}{|\Omega|} \int_{\Omega} u(\cdot, t) \end{cases} \quad (\star)$$

in smooth bounded domains $\Omega \subset \mathbb{R}^n$, $n \geq 1$, are known to be global in time if $\lambda \geq 0$, $\mu > 0$ and $\kappa > 2$. In the present work, we show that the exponent $\kappa = 2$ is actually critical in the four- and higher dimensional setting. More precisely, if

$$\begin{array}{lll} n \geq 4, & \kappa \in (1, 2) & \text{and } \mu > 0 \\ \text{or } n \geq 5, & \kappa = 2 & \text{and } \mu \in \left(0, \frac{n-4}{n}\right), \end{array}$$

for balls $\Omega \subset \mathbb{R}^n$ and parameters $\lambda \geq 0$, $m_0 > 0$, we construct a nonnegative initial datum $u_0 \in C^0(\overline{\Omega})$ with $\int_{\Omega} u_0 = m_0$ for which the corresponding solution (u, v) of $(\star)$ blows up in finite time. Moreover, in 3D, we obtain finite-time blow-up for $\kappa \in (1, \frac{3}{2})$ (and $\lambda \geq 0$, $\mu > 0$). As the corner stone of our analysis, for certain initial data, we prove that the mass accumulation function $w(s, t) = \int_0^{s_0} \rho^{n-1} u(\rho, t) d\rho$ fulfills the estimate $w_s \leq \frac{w}{s}$. Using this information, we then obtain finite-time blow-up of u by showing that for suitably chosen initial data, s_0 and γ , the function $\phi(t) = \int_0^{s_0} s^{-\gamma} (s_0 - s) w(s, t)$ cannot exist globally.

Keywords. Chemotaxis, Finite-time blow-up, Logistic source.

1. Introduction

A considerable amount of the literature on chemotaxis systems deals with detecting critical parameters distinguishing between global existence and finite-time blow-up. Such a dichotomy is already present in the *minimal Keller–Segel system*

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), \\ v_t = \Delta v - v + u \end{cases} \quad (1.1)$$

proposed by Keller and Segel to model chemotactic behavior of bacteria attracted by a chemical substance they produce themselves [10]. Considered in two-dimensional balls, the mass of u_0 is critical: If the initial datum u_0 is sufficiently regular, radially symmetric and satisfies $\int_{\Omega} u_0 < 8\pi$, then the corresponding solutions are global in time and bounded [18] while for any $m_0 > 8\pi$, there exists $u_0 \in C^0(\bar{\Omega})$ with $\int_{\Omega} u_0 = m_0$ leading to finite-time blow-up [5, 15]. (See also [17] for corresponding results in a parabolic–elliptic simplification of (1.1).) Let us note that this specific critical mass phenomenon is limited to the two-dimensional setting: While solutions to (1.1) are always global in time and bounded if considered in one-dimensional domains [20], in the spatially higher dimensional cases, finite-time blow-up has been detected even for arbitrary positive initial masses [30].

Other dichotomies between boundedness and blow-up include critical exponents both for nonlinear diffusion as well as nonlinear sensitivity [7] and nonlinear signal production [33]. Moreover, for a chemotaxis system with indirect signal production, another critical mass phenomenon has been detected in [25], this time distinguishing between boundedness and blow-up in infinite time. Instead of presenting these findings in detail here, we refer to the surveys [1, 13] for a broader overview of chemotaxis systems and related results.

Aiming to further enhance our understanding of the exact strength of the destabilising taxis term, in this article, we present another critical parameter distinguishing between global existence and finite-time blow-up, namely the exponent $\kappa = 2$ in Keller–Segel systems with logistic-type degradation.

Before stating our main result, let us introduce systems featuring such dampening terms and recall some of the corresponding results. That is, we will first consider the *Keller–Segel system with logistic source*

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) + \lambda u - \mu u^2, \\ \tau v_t = \Delta v - v + u \end{cases} \quad (1.2)$$

in smooth, bounded domains $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and given parameters $\lambda \in \mathbb{R}$, $\mu > 0$ and $\tau \geq 0$. (We note that in view of the global existence result for $\lambda = \mu = 0$ in one-dimensional domains mentioned above, at least for the question whether finite-time blow-up occurs, we may confine ourselves to the assumption $n \geq 2$.) The system (1.2) and variations thereof describe several biological processes such as population dynamics [6, 23], pattern formation [38] or embryogenesis [21] (see also [6] for an overview).

Already in 2007, Tello and Winkler showed that for $\tau = 0$, any $\lambda \in \mathbb{R}$, $\mu > \frac{n-2}{n}$ and any reasonably smooth initial data, the system (1.2) possesses global, bounded classical solutions [26]. Moreover, for $n \geq 3$ and $\mu = \frac{n-2}{n}$ (and again $\tau = 0$ and at least $\lambda \geq 0$) solutions to (1.2) are global in time [9], but to the best of our knowledge it is unknown whether these are also always bounded. For the parabolic–parabolic case, that is, for $\tau > 0$, the situation is similar: In the two-dimensional setting, assuming merely $\mu > 0$ suffices to guarantee global existence of classical solutions [19], even for dampening terms growing slightly slower than quadratically [41]. Moreover, for higher dimensional convex domains, global classical solutions have been constructed for $\mu > \mu_0$ for some $\mu_0 > 0$ in [28], where explicit upper bounds of μ_0 then have been derived in [16, 40] and the convexity assumption has been removed in [39] at the cost of worsening the condition on μ . In all these settings, however, the known upper bounds for μ_0 are larger than $\frac{n-2}{n}$.

However, if one resorts to more general solution concepts, further existence results are available. Under rather mild conditions, global weak solutions have been constructed in [12, 26] for the cases $\tau = 0$ and $\tau = 1$, respectively. Moreover, if the degradation term $-\mu u^2$ in (1.2) is replaced by a weaker but still sufficiently strong superlinear dampening term, global generalized solutions have been obtained, again both for the parabolic–elliptic [27] and the fully parabolic case [35, 37, 42].

On the other hand, it has been observed that despite the presence of quadratic dampening terms, structures may form on intermediate time scales which even surpass so-called population thresholds to an arbitrary high extent (cf. [9, 11, 31] for the parabolic–elliptic and [32] for the parabolic–parabolic case).

While these findings already show that the aggregating effect of the chemotaxis term is strongly countered although not completely nullified by quadratic degradation terms, the question arises whether the most drastic form of spatial aggregation—finite-time blow-up—still occurs in Keller–Segel systems with superlinear degradation terms. A first partial (and affirmative) answer has been given in [29]: There, the compared to (1.2) with $\tau = 0$ slightly simplified system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) + \lambda u - \mu u^\kappa, \\ 0 = \Delta v - \overline{m}(t) + u, \quad \overline{m}(t) := \frac{1}{|\Omega|} \int_{\Omega} u(\cdot, t) \end{cases} \quad (1.3)$$

is considered in balls $\Omega \subset \mathbb{R}^n$, $n \geq 5$ and, for any $\lambda \geq 0$ and $\kappa \in (1, \frac{3}{2} + \frac{1}{2(n-1)})$, initial data leading to finite-time blow-up are constructed. The second important finding in this direction transfers this result to physically meaningful space dimensions. More concretely, [34] detects finite-time blow-up even in the system (1.2) with $\tau = 0$ (inter alia) for balls $\Omega \in \mathbb{R}^n$, $n \in \{3, 4\}$, $\lambda \geq 0$ and $\kappa \in (1, \frac{7}{6})$.

Recently, the regime of exponents allowing for finite-time blow-up in (1.3) has been further widened to $\kappa > \frac{4}{3}$ and $\kappa > \frac{3}{2}$ in the three- and four-dimensional settings, respectively [2]. Moreover, in planar domains, chemotactic collapse can be obtained if one replaces the term $-u^\kappa$ in (1.3) with certain heterogeneous dampening terms such as $-|x|^2 u^2$ [4]. (Let us additionally note that similar finite-time blow-up results are also available for systems with non-linear diffusion [2, 14] or sublinear taxis sensitivity [24].)

Main results At least in the four- and higher dimensional settings, the journey of detecting finite-time blow-up in (1.3) for ever increasing values of κ comes to an end with the present article; we obtain the corresponding result up to (and for $n \geq 5$ even including) the optimal exponent $\kappa = 2$.

More precisely, our main result reads

Theorem 1.1. *Suppose*

$$n \geq 3, \quad \kappa \in \left(1, \min\left\{2, \frac{n}{2}\right\}\right) \quad \text{and} \quad \mu > 0 \quad (1.4a)$$

$$\text{or} \quad n \geq 5, \quad \kappa = 2 \quad \text{and} \quad \mu \in \left(0, \frac{n-4}{n}\right). \quad (1.4b)$$

Moreover, let $R > 0$, $\Omega := B_R(0) \subset \mathbb{R}^n$, $m_0 > 0$, $m_1 \in (0, m_0)$ and $\lambda \geq 0$. Then there exists $r_1 \in (0, R)$ such that whenever

$$u_0 \in C^1(\overline{\Omega}) \quad \text{is positive, radially symmetric as well as radially decreasing} \quad (1.5)$$

and fulfills

$$\int_{\Omega} u_0 = m_0 \quad \text{as well as} \quad \int_{B_{r_1}(0)} u_0 \geq m_1, \quad (1.6)$$

the following holds: There exist $T_{\max} < \infty$ and a classical solution

$$(u, v) \in (C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})))^2 \quad (1.7)$$

of

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) + \lambda u - \mu u^\kappa & \text{in } \Omega \times (0, T_{\max}), \\ 0 = \Delta v - \overline{m}(t) + u, \quad \overline{m}(t) := \frac{1}{|\Omega|} \int_{\Omega} u(\cdot, t) & \text{in } \Omega \times (0, T_{\max}), \\ \partial_\nu u = \partial_\nu v = 0 & \text{on } \partial\Omega \times (0, T_{\max}), \\ u(\cdot, 0) = u_0 & \text{in } \Omega, \end{cases} \quad (1.8)$$

which blows up at $T_{\max}$ in the sense that $\lim_{t \nearrow T_{\max}} u(0, t) = \infty$.

Main ideas Following Jäger and Luckhaus [8], we rely on the mass accumulation function given by $w(s, t) := \int_0^s \rho^{n-1} u(\rho, t) d\rho$, which transforms (1.8) to a scalar equation, see Lemma 3.1. The predecessors [2, 34] of this article, which deal with (variations of) the system (1.3), then proceed to show that the function ϕ defined by

$$\phi(s_0, t) := \int_0^{s_0} s^{-\gamma} (s_0 - s) w(s, t) ds \quad (1.9)$$

cannot, at least not for certain initial data, $s_0 \in (0, R^n)$ and $\gamma \in (0, 1)$, exist globally in time, implying that u must blow up in finite time. One of the most challenging terms to estimate arises from the degradation term; one essentially has to control the integral $\int_0^{s_0} w_s^\kappa(s, t) ds$. At this point, pointwise estimates for w_s come in handy, which due to the identity $w_s(s, t) = u(s^{\frac{1}{n}}, t)$ are available once pointwise estimates for u are known. These in turn can for instance be obtained by analyzing general parabolic equations in divergence form [3, 36] or by arguments similar to Lemma 3.3 below. In fact, one of the main points in [2] is to discuss how pointwise upper estimates for u of the form $u(x) \leq C|x|^{-p}$ influence the possibility to detect finite-time blow-up.

However, a natural limitation of this approach is the exponent $p = n$ since for fixed $C > 0$ and $p < n$, nonnegative functions $u_0 \in C^0(\bar{\Omega})$ with $u_0(x) \leq C|x|^{-p}$ cannot have their mass concentrated arbitrary close to the origin; that is, depending on the value of C and p , none of these functions may fulfill (1.6). However, as seen in [2], even the choice $p = n$ ‘only’ yields finite-time blow-up for the system (1.3) for certain $\kappa < \frac{3}{2}$.

Thus, in the present article, where we handle exponents $\kappa \leq 2$, we choose a slightly different path. At the basis of our analysis stands Lemma 3.3: There, we derive the key estimate $w_s \leq \frac{w}{s}$, which due to $w(0, \cdot) \equiv 0$ actually improves on $w_s \leq \frac{C}{s}$. Its proof is surprisingly simple: As already observed in similar contexts (cf. [2, 4, 33]), for radially decreasing initial data, $w_s(\cdot, t)$ is decreasing for all times t , see Lemma 3.2. The desired estimate is then just a consequence of the mean value theorem.

Another major difference of our methods compared to [2, 34] is that we do not limit our analysis of (1.9) to $\gamma \in (0, 1)$ but also allow for parameters γ being larger than 1. In the five- and higher dimensional settings, this will then allow us to obtain finite-time blow-up even for $\kappa = 2$. (In 3D and 4D, the term stemming from the diffusion forces γ to be smaller than 1 and hence we cannot employ the same method as in higher dimensions.) We also note that the realization of the idea of taking $\gamma > 1$ is made possible by the new crucial estimate $w_s \leq \frac{w}{s}$.

The rest of the article is organized as follows: After stating some preliminary results in Sect. 2, in Sect. 3 we derive $w_s \leq \frac{w}{s}$ in Lemma 3.3. Section 4 then starts with the definition of the function ϕ and a calculation of its derivative, see Lemma 4.1. Next, in the Lemma 4.2, we suitably estimate the term originating in the logistic source, before dealing with the remaining terms and the initial datum of ϕ in the subsequent lemmata. In Lemma 4.6, we then finally prove finiteness of the maximal existence time $T_{\max}$.

2. Preliminaries

In the sequel, we fix $n \geq 3$, $R > 0$, $\Omega := B_R(0) \subset \mathbb{R}^n$, $\kappa \in (1, 2]$, $\lambda \geq 0$ and $\mu > 0$.

Lemma 2.1. *Suppose u_0 complies with (1.5). There exists $T_{\max} \in (0, \infty]$ and a unique pair (u, v) of regularity (1.7) which solves (1.8) classically and is such*

that if $T_{\max} < \infty$, then $\lim_{t \nearrow T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty$. Moreover, both u and v are radially symmetric and u is positive in $\overline{\Omega} \times [0, T_{\max})$.

Proof. This is contained in [29, Lemma 1.1]. $\square$

Given u_0 as in (1.5), we denote the solution given in Lemma 2.1 by (u, v) and its maximal existence time by $T_{\max}$. Moreover, we always set $\overline{m}(t) := \frac{1}{|\Omega|} \int_{\Omega} u(\cdot, t)$ for $t \in [0, T_{\max})$.

Since the zeroth order term in the first equation in (1.8), $\lambda u - \mu u^\kappa$, grows at most linearly in u , we can easily control the mass of the first solution component.

Lemma 2.2. *Suppose that u_0 satisfies (1.5). Then*

$$\int_{\Omega} u(\cdot, t) \leq e^{\lambda t} \int_{\Omega} u_0 \quad \text{for all } t \in (0, T_{\max}).$$

Proof. This immediately follows from integrating the first equation in (1.8) and using that $\mu > 0$. $\square$

As used multiple times in the sequel, let us also state the following elementary

Lemma 2.3. *Given $a > -1$, there is $B \in (0, \infty)$ such that for any $s_0 > 0$, the identity*

$$\int_0^{s_0} s^a (s_0 - s) \, ds = B s_0^{a+2}$$

holds.

Proof. We substitute $s \mapsto s_0 s$ and take $B := \int_0^1 s^a (1 - s) \, ds \in (0, \infty)$. $\square$

3. The mass accumulation function w

Given u_0 as in (1.5) (and thus (u, v) as in Lemma 2.1), we denote the mass accumulation function by

$$w(s, t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} u(\rho, t) \, d\rho, \quad (s, t) \in [0, R^n] \times [0, T_{\max}), \quad (3.1)$$

which has been introduced in the context of chemotaxis systems in [8]. In this section, we prove some of its properties, most notably the crucial estimate $w_s \leq \frac{w}{s}$ in Lemma 3.3.

Lemma 3.1. *For every u_0 satisfying (1.5), the function w given by (3.1) belongs to $C^0([0, R^n] \times [0, T_{\max})) \cap C^{2,1}([0, R^n] \times (0, T_{\max}))$ and fulfills*

$$w_s(s, t) = \frac{u(s^{\frac{1}{n}}, t)}{n} \quad \text{for all } (s, t) \in [0, R^n] \times [0, T_{\max}) \quad (3.2)$$

as well as

$$\begin{aligned} w_t &= n^2 s^{2-\frac{2}{n}} w_{ss} + n w w_s - n \overline{m}(t) s w_s \\ &\quad + \lambda w - n^{\kappa-1} \mu \int_0^s w_s^\kappa(\sigma, t) \, d\sigma \quad \text{in } (0, R^n) \times (0, T_{\max}). \end{aligned} \quad (3.3)$$

Proof. This can be seen by a direct calculation. In fact, the asserted regularity is a consequence of Lemma 2.1, the identity (3.2) follows from the chain rule, and [29, equation (1.4)] asserts that (3.3) holds. $\square$

Next, as a major step towards proving $w_s \leq \frac{w}{s}$, we show that for initial data fulfilling (1.5), the first solution component is radially decreasing throughout evolution.

Lemma 3.2. *Suppose u_0 complies with (1.5). Then $u_r \leq 0$ in $(0, R) \times (0, T_{\max})$.*

Proof. This can be shown as in [2, Lemma 5.1] or [4, Lemma 3.7] (which in turn both follow [33, Lemma 2.2]). However, due to the importance of this lemma for showing the crucial estimate $w_s \leq \frac{w}{s}$ in the succeeding lemma, we choose to at least sketch the proof here. First, by an approximation argument as in [33, Lemma 2.2], we may assume $u_r \in C^0([0, R] \times [0, T_{\max})) \cap C^{2,1}((0, R) \times (0, T_{\max}))$.

Furthermore, the second equation in (1.8) asserts

$$r^{1-n}(r^{n-1}uv_r)_r = u_r v_r + ur^{1-n}(r^{n-1}v_r)_r = u_r v_r - u^2 + \overline{m}(t)u$$

$$\text{in } (0, R) \times (0, T_{\max})$$

and hence, with $f(z) := \lambda z - \mu z^\kappa$ for $z \geq 0$,

$$\begin{aligned} u_{rt} &= (r^{1-n}(r^{n-1}(u_r - uv_r))_r + f(u))_r \\ &= u_{rrr} + \frac{n-1}{r}u_{rr} - \frac{n-1}{r^2}u_r - u_{rr}v_r - u_r v_{rr} + 2uu_r - \overline{m}(t)u_r + f'(u)u_r \\ &= u_{rrr} + a(r, t)u_{rr} + b(r, t)u_r \quad \text{in } (0, R) \times (0, T_{\max}), \end{aligned}$$

where

$$\begin{aligned} a(r, t) &:= \frac{n-1}{r} - v_r(r, t) \quad \text{and} \\ b(r, t) &:= -\frac{n-1}{r^2} - v_{rr}(r, t) + 2u(r, t) - \overline{m}(t) + f'(u(r, t)) \end{aligned}$$

for $(r, t) \in (0, R) \times (0, T_{\max})$.

As can be rapidly seen by writing the second equation in (1.8) in radial coordinates (and has been argued in more detail in [4, Lemma 3.6], for instance), $-v_{rr} \leq u$ holds throughout $(0, R) \times (0, T_{\max})$, so that for fixed $T \in (0, T_{\max})$, we can estimate

$$\sup_{r \in (0, R), t \in (0, T)} b(r, t) \leq 3\|u\|_{L^\infty((0, R) \times (0, T))} + \|f'\|_{L^\infty(0, \|u\|_{L^\infty((0, R) \times (0, T))})} < \infty.$$

An application of the maximum principle (cf. [22, Proposition 52.4]) then gives $u_r \leq 0$ in $(0, R) \times (0, T)$, which upon taking $T \nearrow T_{\max}$ implies the statement. $\square$

As already advertised multiple times, this lemma now allows us to rapidly obtain the important estimate $w_s \leq \frac{w}{s}$.

Lemma 3.3. *Assume that u_0 satisfies (1.5). For all $s \in [0, R^n]$ and $t \in [0, T_{\max})$,*

$$w_s(s, t) \leq \frac{w(s, t)}{s} \leq w_s(0, t) \quad (3.4)$$

holds. In particular, for all $t_0 \in (0, T_{\max})$ there is $C > 0$ such that

$$\frac{s}{C} \leq w(s, t) \leq Cs \quad \text{for } s \in [0, R^n] \text{ and } t \in [0, t_0]. \quad (3.5)$$

Proof. For fixed $t \in [0, T_{\max})$ and $s \in [0, R^n]$, the mean value theorem provides us with $\xi \in (0, s)$ such that $w(s, t) = sw_s(\xi, t)$, which already proves (3.4) since w_s is decreasing by Lemma 3.2 and (3.2).

Moreover, a consequence thereof is (3.5), since w_s is positive and bounded in $[0, R^n] \times [0, t_0]$ for any $t_0 \in (0, T_{\max})$ by Lemma 2.1 and (3.2). $\square$

4. A supersolution to a superlinear ODE: finite-time blow-up

We will construct initial data leading to finite-time blow-up and hence prove Theorem 1.1 in this section. As already mentioned in the introduction, our argument is based on constructing a function ϕ which cannot exist globally, implying that the solution of (1.8) also can only exist on a finite time interval. In fact, we define ϕ as in [2] or [34]; that is, for given u_0 as in (1.5) and $\gamma \in (0, 2)$, we set

$$\phi(s_0, t) := \int_0^{s_0} s^{-\gamma}(s_0 - s)w(s, t) \, ds, \quad s_0 \in (0, R^n), t \in (0, T_{\max}). \quad (4.1)$$

However, as the parameter γ herein may be larger than 1 (unlike as in [2] or [34]), some care is needed for calculating the time derivative of ϕ . This is done in the following

Lemma 4.1. *Suppose that u_0 complies with (1.5). Let $\gamma \in (0, 2)$ and ϕ be as in (4.1). For every $s_0 \in (0, R^n)$, $\phi(s_0, \cdot)$ belongs to $C^0([0, T_{\max})) \cap C^1((0, T_{\max}))$ and fulfills*

$$\begin{aligned} \phi_t(s_0, t) &\geq n^2 \int_0^{s_0} s^{2-\frac{2}{n}-\gamma}(s_0 - s)w_{ss} \, ds \\ &\quad + n \int_0^{s_0} s^{-\gamma}(s_0 - s)ww_s \, ds \\ &\quad - n\overline{m}(t) \int_0^{s_0} s^{1-\gamma}(s_0 - s)w_s \, ds \\ &\quad - n^{\kappa-1}\mu \int_0^{s_0} s^{-\gamma}(s_0 - s) \int_0^s w_s^\kappa(\sigma, t) \, d\sigma \, ds \\ &=: I_1(s_0, t) + I_2(s_0, t) + I_3(s_0, t) + I_4(s_0, t) \quad \text{for all } t \in (0, T_{\max}). \end{aligned} \quad (4.2)$$

Proof. We first fix $s_0 \in (0, R^n)$ and note that $\phi(s_0, \cdot) \in C^0([0, T_{\max}))$ because of (3.5) and $1 - \gamma > -1$. Letting $0 < t_0 < t_1 < T_{\max}$, we then make use of Lemmata 3.1 and 3.3 to obtain $c_1, c_2, c_3, c_4 > 0$ such that

$$\begin{aligned} w(s, t) &\leq c_1 s, \quad w_s(s, t) \leq c_2, \quad |w_{ss}(s, t)| \leq c_3 \quad \text{and} \quad \overline{m}(t) \leq c_4 \\ &\text{for } (s, t) \in [0, s_0] \times [t_0, t_1]. \end{aligned}$$

Recalling (3.3), we obtain

$$\begin{aligned} & \frac{d}{dt} (s^{-\gamma}(s_0 - s)w) \\ &= \left(n^2 s^{2-\frac{2}{n}} w_{ss} + n w w_s - n \overline{m}(t) s w_s + \lambda w - n^{\kappa-1} \mu \int_0^s w_s^\kappa(\sigma, t) d\sigma \right) s^{-\gamma}(s_0 - s) \end{aligned}$$

for $s \in (0, s_0)$ and $t \in (0, T_{\max})$, so that

$$\begin{aligned} & \left| \frac{d}{dt} (s^{-\gamma}(s_0 - s)w(s, t)) \right| \\ & \leq \left(n^2 c_3 s_0^{1-\frac{2}{n}} + n c_1 c_2 + n c_2 c_4 + \lambda c_1 + n^{\kappa-1} \mu c_2^\kappa \right) s^{1-\gamma}(s_0 - s) \end{aligned}$$

for all $s \in (0, s_0)$ and $t \in (t_0, t_1)$. Again due to $1 - \gamma > -1$, we therefore have $\phi(s_0, \cdot) \in C^1((0, T_{\max}))$ and

$$\phi_t(s_0, t) = I_1(s_0, t) + I_2(s_0, t) + I_3(s_0, t) + \lambda \int_0^{s_0} s^{-\gamma}(s_0 - s)w ds + I_4(s_0, t)$$

for all $t \in (0, T_{\max})$, which due to $\lambda \geq 0$ implies (4.2). $\square$

Aiming to derive that ϕ is a supersolution to a superlinear ODE, we now estimate the terms $I_1, \dots, I_4$ in (4.2) and begin with I_4 , the term stemming from the logistic source. In the following proof, we will crucially make use of the estimate (3.4) to improve on corresponding results obtained by the predecessors [2] and [34].

Lemma 4.2. *Let I_2 and I_4 be as in (4.2).*

(i) *If $\kappa = 2$, $\gamma > 1$ and u_0 fulfills (1.5), then*

$$I_4(s_0, t) \geq -\frac{\mu}{\gamma-1} I_2(s_0, t) \quad \text{for all } s_0 \in (0, R^n) \text{ and } t \in (0, T_{\max}). \quad (4.3)$$

(ii) *Let $\kappa \in (1, 2)$ and $\gamma \in (\frac{2(\kappa-1)}{\kappa}, 1)$. There exists $C_4 > 0$ such that whenever u_0 fulfills (1.5), then*

$$I_4(s_0, t) \geq C_4 s_0^{\frac{2-\kappa}{2}} I_2^{\frac{\kappa}{2}}(s_0, t) \quad \text{for all } s_0 \in (0, \min\{1, R^n\}) \text{ and } t \in (0, T_{\max}). \quad (4.4)$$

Proof. We let $\gamma \in (0, \infty) \setminus \{1\}$ and also fix u_0 as in (1.5) but will make sure that C_4 can be taken independently of u_0 .

By Fubini's theorem, we first observe that

$$\begin{aligned} I_4(s_0, t) &= -n^{\kappa-1} \mu \int_0^{s_0} s^{-\gamma}(s_0 - s) \int_0^s w_s^\kappa(\sigma, t) d\sigma ds \\ &= -n^{\kappa-1} \mu \int_0^{s_0} \left(\int_\sigma^{s_0} s^{-\gamma}(s_0 - s) ds \right) w_s^\kappa(\sigma, t) d\sigma \\ &\geq -n^{\kappa-1} \mu \int_0^{s_0} \left(\int_\sigma^{s_0} s^{-\gamma} ds \right) (s_0 - \sigma) w_s^\kappa(\sigma, t) d\sigma \\ &= -\frac{n^{\kappa-1} \mu}{1-\gamma} \int_0^{s_0} \left(s_0^{1-\gamma} - s^{1-\gamma} \right) (s_0 - s) w_s^\kappa(s, t) ds \end{aligned} \quad (4.5)$$

for all $s_0 \in (0, R^n)$ and $t \in (0, T_{\max})$.

In the case of $\gamma > 1$ and $\kappa = 2$, we drop a positive term and employ (3.4) in calculating

$$\begin{aligned} I_4(s_0, t) &\geq -\frac{n\mu}{\gamma-1} \int_0^{s_0} s^{1-\gamma}(s_0-s)w_s^2 ds \\ &\geq -\frac{n\mu}{\gamma-1} \int_0^{s_0} s^{-\gamma}(s_0-s)ww_s ds \\ &= -\frac{\mu}{\gamma-1} I_2(s_0, t) \quad \text{for all } s_0 \in (0, R^n) \text{ and } t \in (0, T_{\max}), \end{aligned}$$

which already implies (4.3).

If on the other hand $\gamma \in (0, 1)$ and $\kappa \in (0, 1)$, going back to (4.5) and making use of (3.4), we see that

$$\begin{aligned} I_4(s_0, t) &\geq -\frac{n^{\kappa-1}\mu}{1-\gamma} s_0^{1-\gamma} \int_0^{s_0} (s_0-s)w_s^\kappa(s, t) ds \\ &\geq -\frac{n^{\kappa-1}\mu}{1-\gamma} R^{n(1-\gamma)} \int_0^{s_0} s^{-\frac{\kappa}{2}}(s_0-s)(ww_s)^{\frac{\kappa}{2}} ds \end{aligned} \quad (4.6)$$

for all $s_0 \in (0, R^n)$ and $t \in (0, T_{\max})$. By Hölder's inequality (with exponents $\frac{2}{2-\kappa}, \frac{2}{\kappa}$), we have therein

$$\begin{aligned} \int_0^{s_0} s^{-\frac{\kappa}{2}}(s_0-s)(ww_s)^{\frac{\kappa}{2}} ds &= \int_0^{s_0} s^{-\frac{(1-\gamma)\kappa}{2}}(s_0-s)(s^{-\gamma}ww_s)^{\frac{\kappa}{2}} ds \\ &\leq \left(\int_0^{s_0} s^{-\frac{(1-\gamma)\kappa}{2-\kappa}}(s_0-s) ds \right)^{\frac{2-\kappa}{2}} \left(\int_\Omega s^{-\gamma}(s_0-s)ww_s ds \right)^{\frac{\kappa}{2}} \end{aligned} \quad (4.7)$$

for all $s_0 \in (0, R^n)$ and $t \in (0, T_{\max})$. We assume now moreover that $\gamma > \frac{2(\kappa-1)}{\kappa}$ and hence $\gamma-1 > \frac{\kappa-2}{\kappa}$ as well as $a := \frac{(\gamma-1)\kappa}{2-\kappa} > -1$, so that applying Lemma 2.3 (with B as in that lemma) gives

$$\int_0^{s_0} s^{-\frac{(1-\gamma)\kappa}{2-\kappa}}(s_0-s) ds = B s_0^{a+2} \leq B s_0 \quad \text{for all } s_0 \in (0, \min\{1, R^n\}). \quad (4.8)$$

Finally, combining (4.6)–(4.8) and the definition of I_2 yields (4.4) for some $C_4 > 0$ independent of u_0 . $\square$

The remaining integrals in (4.2) can be estimated as in [34] or [2]. However, at least for the statement concerning I_1 , we would like to give a full proof here in order to show the basis of the restriction on κ in Theorem 1.1. Indeed, while in Lemma 4.2 above, γ has to be taken sufficiently large, for estimating I_1 , we need γ to be suitably small. We will obtain finite-time blow-up precisely in the cases where the set of admissible γ for both these lemmata is nonempty. Moreover, compared to [34], the proof below makes use of the estimate (3.4) and is hence somewhat shorter.

Lemma 4.3. *Let $\gamma \in (0, 2 - \frac{4}{n})$. There is $C_1 > 0$ such that whenever u_0 satisfies (1.5) and I_1, I_2 are as in (4.2), then*

$$I_1(s_0, t) \geq -C_1 s_0^{\frac{3-\gamma}{2} - \frac{2}{n}} I_2^{\frac{1}{2}}(s_0, t) \quad \text{for all } s_0 \in (0, R^n) \text{ and } t \in (0, T_{\max}).$$

Proof. For convenience, we fix u_0 as in (1.5), albeit we emphasize that the constants below do not depend on u_0 . An integration by parts gives

$$\begin{aligned} & \int_0^{s_0} s^{2-\frac{2}{n}-\gamma}(s_0-s)w_{ss} \, ds \\ &= -\left(2-\frac{2}{n}-\gamma\right) \int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0-s)w_s \, ds \\ & \quad + \int_0^{s_0} s^{2-\frac{2}{n}-\gamma}w_s \, ds + \left[s^{2-\frac{2}{n}-\gamma}(s_0-s)w_s\right]_0^{s_0} \quad \text{in } (0, T_{\max}). \end{aligned}$$

Herein, the second term on the right hand side is positive and the last one is zero because of $\gamma < 2 - \frac{4}{n} < 2 - \frac{2}{n}$.

Setting $c_1 := 2 - \frac{2}{n} - \gamma > 0$, we hence infer from (3.4) and Hölder's inequality that

$$\begin{aligned} & \int_0^{s_0} s^{2-\frac{2}{n}-\gamma}(s_0-s)w_{ss} \, ds \\ & \geq -\left(2-\frac{2}{n}-\gamma\right) \int_0^{s_0} s^{1-\frac{2}{n}-\gamma}(s_0-s)w_s \, ds \\ & \geq -c_1 \int_0^{s_0} s^{\frac{1}{2}-\frac{2}{n}-\gamma}(s_0-s)(ww_s)^{\frac{1}{2}} \, ds \\ & \geq -c_1 \left(\int_0^{s_0} s^{1-\gamma-\frac{4}{n}}(s_0-s) \, ds\right)^{\frac{1}{2}} \left(\int_0^{s_0} s^{-\gamma}(s_0-s)ww_s \, ds\right)^{\frac{1}{2}} \quad \text{holds in } (0, T_{\max}). \end{aligned}$$

Since $\gamma < 2 - \frac{4}{n}$ and hence $a := 1 - \gamma - \frac{4}{n} > -1$, Lemma 2.3 asserts that (with B as in that lemma)

$$\left(\int_0^{s_0} s^{1-\gamma-\frac{4}{n}}(s_0-s) \, ds\right)^{\frac{1}{2}} = B^{\frac{1}{2}} s_0^{\frac{3-\gamma}{2}-\frac{2}{n}} \quad \text{for all } s_0 \in (0, R^n),$$

so that the statement follows by the definitions of I_1 and I_2 . $\square$

Next, for estimating the integrals I_2 and I_3 in (4.2), we basically recall the corresponding results from [34].

Lemma 4.4. *There exist $C_2, C_3 > 0$ such that for u_0 satisfying (1.5), we have*

$$I_2(s_0, t) \geq C_2 s_0^{-(3-\gamma)} \phi^2(s_0, t) \quad \text{for all } s_0 \in (0, R^n) \text{ and } t \in (0, T_{\max}) \quad (4.9)$$

and

$$I_3(s_0, t) \geq -C_3 \left(\int_{\Omega} u_0\right) s_0^{\frac{3-\gamma}{2}} I_2^{\frac{1}{2}}(s_0, t) \quad \text{for all } s_0 \in (0, R^n) \text{ and } t \in (0, \min\{1, T_{\max}\}), \quad (4.10)$$

where ϕ is in (4.1) and I_2, I_3 are defined in (4.2).

Proof. Arguing as in [34, Lemma 4.4], we obtain

$$\phi(s_0, t) \leq c_1 s_0^{\frac{3-\gamma}{2}} I_2^{\frac{1}{2}}(s_0, t) \quad \text{for all } s_0 \in (0, R^n) \text{ and } t \in (0, T_{\max}) \quad (4.11)$$

for some $c_1 > 0$ independent of u_0 . Taking both the left and the right hand side therein to the power 2 already yields (4.9).

Moreover, as

$$I_3(s_0, t) = -n\overline{m}(t) \int_0^{s_0} s^{1-\gamma}(s_0 - s)w_s(s, t) \, ds \geq -\frac{n}{|\Omega|} \left(\int_{\Omega} u_0 \right) e^{\lambda t} \phi(s_0, t)$$

for $(s_0, t) \in (0, R^n) \times (0, T_{\max})$ by Lemma 2.2 and (3.4), another consequence of (4.11) is (4.10). $\square$

As a final preparation, we note that, under certain circumstances, $\phi(s_0, 0)$ can be shown to be sufficiently large.

Lemma 4.5. *For every $m_1 > 0$, there exists $C_0 > 0$ with the following property: Let $s_0 \in (0, R^n)$, set $s_1 := \frac{s_0}{4}$ as well as $r_1 := s_1^{\frac{1}{n}}$ and suppose that u_0 fulfills (1.5) as well as $\int_{B_{r_1}(0)} u_0 \geq m_1$. Then $\phi(s_0, 0) \geq C_0 s_0^{2-\gamma}$.*

Proof. See [34, estimate (5.5)]; the main idea is to use the monotonicity of w_0 which in turn is implied by nonnegativity of u_0 . $\square$

A combination of the results obtained above now reveals that for initial data whose mass is sufficiently concentrated near the origin, the corresponding solution cannot exist globally in time. Again, the argument is not too different from [29] or [34], but we choose to give it nonetheless in order to show that s_0 and u_0 can be chosen in such a way that ϕ would blow up in finite time if (u, v) were a global solution.

Lemma 4.6. *Let $m_0 > m_1 > 0$ and suppose that (1.4) holds. There exists $r_1 \in (0, R)$ such that whenever u_0 fulfills (1.5) and (1.6), then $T_{\max} \leq \frac{1}{2}$.*

Proof. Let us begin by fixing some parameters. If (1.4a) holds, then $\kappa \in (1, \frac{n}{2})$ and hence

$$\frac{2(\kappa - 1)}{\kappa} - \left(2 - \frac{4}{n} \right) < \frac{2 \cdot \frac{n-2}{2}}{\frac{n}{2}} - \frac{2(n-2)}{n} = 0.$$

As additionally $\kappa < 2$, we may hence choose $\gamma \in (\frac{2(\kappa-1)}{\kappa}, \min\{2 - \frac{4}{n}, 1\})$. We moreover fix an arbitrary $\varepsilon > 0$ and apply Lemma 4.2 (ii) as well as Young's inequality (with exponents $\frac{2}{2-\kappa}, \frac{2}{\kappa}$) to obtain $C'_4 > 0$ with

$$I_4(s_0, t) \geq -\frac{\mu}{\mu + \varepsilon} I_2(s_0, t) - C'_4 s_0 \quad \text{for all } s_0 \in (0, \min\{1, R^n\}) \text{ and } t \in (0, T_{\max}), \quad (4.12)$$

whenever u_0 satisfies (1.5) and where I_2 and I_4 are as in (4.2).

Suppose now that on the other hand (1.4b) holds. Because of $\mu \in (0, \frac{n-4}{n})$, we may then choose $\gamma \in (1 + \mu, 2 - \frac{4}{n})$. Setting moreover $\varepsilon := \gamma - 1 - \mu > 0$, an application of Lemma 4.2 (i) reveals that (4.12) holds also in this case (with $C'_4 := 0$ and for all u_0 complying with (1.5)).

In both cases, the definition of γ entails $0 < \gamma < 2 - \frac{4}{n}$, hence by Lemma 4.1, Lemma 4.3, (4.10), (4.12), Young's inequality and (4.9), there are $c_1, c_2 > 0$ such that

$$\begin{aligned} \phi_t(s_0, t) &\geq I_1(s_0, t) + I_2(s_0, t) + I_3(s_0, t) + I_4(s_0, t) \\ &\geq \frac{\varepsilon}{\mu + \varepsilon} I_2(s_0, t) - \left(C_1 s_0^{\frac{3-\gamma}{2} - \frac{2}{n}} + C_3 m_0 s_0^{\frac{3-\gamma}{2}} \right) I_2^{\frac{1}{2}}(s_0, t) - C'_4 s_0 \\ &\geq c_1 I_2(s_0, t) - c_2 s_0^{\min\{3-\gamma-\frac{4}{n}, 3-\gamma, 1\}} \\ &\geq C_2 c_1 s_0^{-(3-\gamma)} \phi^2(s_0, t) - c_2 s_0 \end{aligned} \quad (4.13)$$

for all $s_0 \in (0, \min\{1, R^n\})$, $t \in (0, \min\{1, T_{\max}\})$ and u_0 satisfying (1.5) as well as $\int_{\Omega} u_0 = m_0$, where $\phi, I_1, \dots, I_4$ are as in Lemma 4.1, C_1 is as in Lemma 4.3 and C_2, C_3 are as in Lemma 4.4.

For $s_0 > 0$, we set $c_3 := C_2 c_1$,

$$\begin{aligned} d_1(s_0) &:= c_3 s_0^{-(3-\gamma)}, \quad d_2(s_0) := c_2 s_0, \quad d_3(s_0) := \left(\frac{d_2(s_0)}{d_1(s_0)} \right)^{\frac{1}{2}} \\ \text{and } \phi_0(s_0) &:= C_0 s_0^{2-\gamma}, \end{aligned}$$

where C_0 is as in Lemma 4.5. We observe that $d_1(s_0) \rightarrow \infty$ for $s_0 \searrow 0$ since $3 - \gamma > 1 > 0$. Therefore, noting further that

$$\frac{1}{2}(1 + 3 - \gamma) = 2 - \frac{\gamma}{2} > 2 - \gamma,$$

we may also fix $s_0 \in (0, \min\{1, R^n\})$ so small that

$$\phi_0(s_0) \geq d_3(s_0) + \frac{2}{d_1(s_0)}. \quad (4.14)$$

Moreover, we now fix u_0 not only complying with (1.5) but also with (1.6) for $r_1 := (\frac{s_0}{4})^{\frac{1}{n}}$ and will show that the corresponding solution given by Lemma 2.1 blows up in finite time. From (4.13) and Lemma 4.5, we infer that $\phi(s_0, \cdot)$ satisfies

$$\begin{cases} \phi_t(s_0, t) \geq d_1(s_0) \phi^2(s_0, t) - d_2(s_0) & \text{for all } t \in (0, \min\{1, T_{\max}\}), \\ \phi(s_0, 0) \geq \phi_0(s_0). \end{cases} \quad (4.15)$$

Since (4.14) implies $\phi_0(s_0) \geq d_3(s_0)$ and because of $d_1(s_0)d_3(s_0)^2 - d_2(s_0) = 0$, the comparison principle and (4.15) assert $\phi(s_0, t) \geq d_3(s_0)$ for all $t \in (0, \min\{1, T_{\max}\})$, so that by (4.15) we have

$$\begin{aligned} \phi_t(s_0, t) &\geq d_1(s_0) (\phi^2(s_0, t) - d_3(s_0)^2) \\ &\geq d_1(s_0) (\phi(s_0, t) - d_3(s_0))^2 \quad \text{for all } t \in (0, \min\{1, T_{\max}\}). \end{aligned}$$

Dividing by the right hand side therein yields upon an integration in time

$$t = \int_0^t 1 \, ds \leq \int_{\phi(s_0, 0)}^{\phi(s_0, t)} \frac{d\sigma}{d_1(s_0)(\sigma - d_3(s_0))^2} \leq \left[-\frac{1}{d_1(s_0)(\sigma - d_3(s_0))} \right]_{\phi_0(s_0)}^{\infty} \leq \frac{1}{2}$$

for all $t \in (0, \min\{1, T_{\max}\})$, implying $T_{\max} \leq \frac{1}{2}$. $\square$

Finally, we conclude that Theorem 1.1 is now merely a direct consequence of the lemmata above.

Proof of Theorem 1.1. Lemma 4.6 asserts that there is $r_1 \in (0, R)$ such that under the conditions of Theorem 1.1, the maximal existence time $T_{\max}$ is finite. By Lemma 3.2 and Lemma 2.1, this then implies $u(0, t) = \|u(\cdot, t)\|_{L^\infty(\Omega)} \rightarrow \infty$ as $t \nearrow T_{\max}$. $\square$

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