Concave power solutions of the dominative p -Laplace equation

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Abstract. In this paper, we study properties of solutions of the Dominative p -Laplace equation with homogeneous Dirichlet boundary conditions in a bounded convex domain Ω . For the equation $-\mathcal{D}_p u = 1$, we show that $\sqrt{u}$ is concave, and for the eigenvalue problem $\mathcal{D}_p u + \lambda u = 0$, we show that $\log u$ is concave.

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1. Introduction

The *Dominative* p -Laplace operator was defined as

$$\mathcal{D}_p u := \Delta u + (p-2)\lambda_{\max}(D^2 u), \quad p \geq 2,$$

by Brustad in [3] and later studied in [4]. See also [6] for a stochastic interpretation and a game-theoretic approach of the equation. Here, $\lambda_{\max}$ denotes the largest eigenvalue of the Hessian matrix

$$D^2 u = \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right)_{ij}.$$

We shall study the two equations

$$-\mathcal{D}_p u = 1 \quad \text{and} \quad \mathcal{D}_p u + \lambda u = 0$$

in a bounded convex domain $\Omega \subset \mathbb{R}^n$. The positive solutions with zero boundary values have the property for $-\mathcal{D}_p u = 1$ that $\sqrt{u}$ is concave, see Theorem 1.1 below. In Theorem 1.2 we show that for $\mathcal{D}_p u + \lambda u = 0$, $\log u$ is concave. Problems related to concave solutions have been studied for p -Laplace type equations, and we give a quick review of the results. The operator is closely related to the *normalized* p -Laplace operator,

$$\Delta_p^N u = |\nabla u|^{2-p} \operatorname{div} (|\nabla u|^{p-2} \nabla u),$$

which describes a Tug-of-war game with noise, see [18]. Due to this, the operator has been studied extensively over the last 15 years, and we refer to [2, 10, 11] for an introduction and some regularity results. The solutions are weak and appear in the form of *viscosity solutions* and we refer to [8] for an introduction of viscosity solutions. If u is a solution of the problem

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

in a bounded convex domain $\Omega \subset \mathbb{R}^n$, one can show that $\sqrt{u}$ is concave. This problem, including more complex right-hand sides, was studied in the 1970's and 1980's by [13–15, 17]. For $n = 1$ and $n = 2$, a brute force calculation shows that $\sqrt{u}$ is concave. For $n \geq 3$ the proofs are more complicated. For the ordinary p -Laplacian, [19] showed that $u^{\frac{p-1}{p}}$ is concave. One should note that simply setting $p = 2$ does not simplify the proof. Thus, the papers [14, 15] are still of great value. For the infinity Laplacian, $\Delta_\infty u = \langle D^2 u \nabla u, \nabla u \rangle$, [7] showed that $u^{\frac{3}{4}}$ is concave. Our result for the Dominative p -Laplace equation can be formulated in the following theorem. We say that Ω satisfies the interior sphere condition if for all $y \in \partial\Omega$ there is an $x \in \Omega$ and an open ball $B_r(x)$ such that $B_r(x) \subset \Omega$ and $y \in \partial B_r(x)$.

Theorem 1.1. *Let $u \in C(\bar{\Omega})$ be a viscosity solution of*

$$\begin{cases} -\mathcal{D}_p u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

in a bounded convex domain $\Omega \subset \mathbb{R}^n$ which satisfies the interior sphere condition. Then $\sqrt{u}$ is concave.

Further, we study the eigenvalue problem and give the following result.

Theorem 1.2. *Let $u \in C(\bar{\Omega})$ be a positive viscosity solution of*

$$\begin{cases} -\mathcal{D}_p u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $\lambda > 0$ in a bounded convex domain $\Omega \subset \mathbb{R}^n$. Then $\log u$ is concave.

Remark 1.3. We give a remark on what happens when $p \rightarrow \infty$ in Theorem 1.1. After dividing the equation by p and letting p approach infinity, the following equation is obtained

$$\begin{cases} -\lambda_{\max}(D^2 u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

This equation has the solution $u = 0$, which is obviously already concave. This is better than the square root being concave, so for $p = \infty$ a stronger result is obtained. (For a less trivial result, another normalization with p is needed.)

For the Helmholtz equation $\Delta u + \lambda u = 0$, the problem related to concave logarithmic solutions has been studied in [5, 9, 15]. The nonlinear eigenvalue problem associated with the p -Laplace equation has been studied for example in [16, 19]. In [19], Sakaguchi showed that $\log u$ is a concave function.

2. Preliminaries and notation

The gradient of a function $f : \Omega_T \rightarrow \mathbb{R}$ is

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$$

and its Hessian matrix is

$$(D^2 f)_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

We will use the operator

$$\mathcal{D}_p u = \Delta u + (p-2)\lambda_{\max}(D^2 u)$$

and if applied to a matrix $X \in S^n$, we use

$$D_p X = \operatorname{tr}(X) + (p-2)\lambda_{\max}(X).$$

Also, the *normalized* p -Laplace operator is referred to,

$$\Delta_p^N u = \Delta u + (p-2) \cdot \frac{1}{|\nabla u|^2} \sum_{i,j=1}^N u_{x_i} u_{x_j} u_{x_i x_j}.$$

Viscosity solutions The Dominative p -Laplace operator is uniformly elliptic. Therefore, it is convenient to use viscosity solutions as a notion of weak solutions. Throughout the text, we always keep $p \geq 2$. In the definition below, g is assumed to be continuous in all variables.

Definition 2.1. A function $u \in USC(\bar{\Omega})$ is a viscosity subsolution to $-\mathcal{D}_p u = g(x, u, \nabla u)$ if, for all $\phi \in C^2(\Omega)$,

$$-\mathcal{D}_p \phi(x) \leq g(x, u, \nabla \phi).$$

at any point $x \in \Omega$ where $u - \phi$ attains a local maximum. A function $u \in LSC(\bar{\Omega})$ is a viscosity supersolution to $-\mathcal{D}_p u = g(x, u, \nabla u)$ if, for all $\phi \in C^2(\Omega)$,

$$-\mathcal{D}_p \phi(x) \geq g(x, u, \nabla \phi).$$

at any point $x \in \Omega$ where $u - \phi$ attains a local minimum.

A function $u \in C(\bar{\Omega})$ is a viscosity solution of

$$\begin{cases} -\mathcal{D}_p u = g(x, u, \nabla u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

if it is a viscosity sub- and supersolution of $-\mathcal{D}_p u = g(x, u, \nabla u)$ and $u = 0$ on $\partial\Omega$.

When defining viscosity solutions to $\Delta_p^N u = g(x, u, \nabla u)$, one has to be careful at points where the gradient vanishes.

Definition 2.2. A function $u \in USC(\bar{\Omega})$ is a viscosity subsolution of $-\Delta_p^N u = 1$ if, for all $\phi \in C^2(\Omega)$,

$$\begin{cases} -\Delta_p^N \phi(x) \leq 1, & \text{if } \nabla \phi(x) \neq 0 \\ -\mathcal{D}_p \phi(x) \leq 1, & \text{if } \nabla \phi(x) = 0. \end{cases}$$

at any point $x \in \Omega$ where $u - \phi$ attains a local minimum. A function $u \in LSC(\bar{\Omega})$ is a viscosity supersolution of $-\Delta_p^N u = 1$ if, for all $\phi \in C^2(\Omega)$,

$$\begin{cases} -\Delta_p^N \phi(x) \geq 1, & \text{if } \nabla \phi(x) \neq 0 \\ \mathcal{D}_p(-\phi(x)) \geq 1, & \text{if } \nabla \phi(x) = 0. \end{cases}$$

at any point $x \in \Omega$ where $u - \phi$ attains a local minimum. A function $u \in C(\bar{\Omega})$ is a viscosity solution of

$$\begin{cases} -\Delta_p^N u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

if it is a viscosity sub- and supersolution of $-\Delta_p^N u = 1$ and $u = 0$ on $\partial\Omega$.

We also need an equivalent definition of viscosity solutions using the *sub*- and *superjets*. For functions $u : \Omega \rightarrow \mathbb{R}^n$ they are given by

$$\begin{aligned} J^{2,+}u(x) = \{ & (q, X) \in \mathbb{R}^n \times S^n : u(y) \leq u(x) + \langle q, y - x \rangle \\ & + \frac{1}{2} \langle X(y - x), y - x \rangle + o(|y - x|^2) \text{ as } y \rightarrow x \} \end{aligned}$$

and

$$\begin{aligned} J^{2,-}u(x) = \{ & (q, X) \in \mathbb{R}^n \times S^n : u(y) \geq u(x) + \langle q, y - x \rangle \\ & + \frac{1}{2} \langle X(y - x), y - x \rangle + o(|y - x|^2) \text{ as } y \rightarrow x \}. \end{aligned}$$

Definition 2.3. A function $u \in USC(\bar{\Omega})$ is a viscosity subsolution to $-\mathcal{D}_p u = g(x, u, \nabla u)$ if $(q, X) \in J^{2,+}u(x)$ implies

$$-D_p X \leq g(x, u, q).$$

A function $u \in LSC(\bar{\Omega})$ is a viscosity supersolution of $-\mathcal{D}_p u = g(x, u, \nabla u)$ if $(q, X) \in J^{2,-}u(x)$ implies

$$-D_p X \geq g(x, u, q).$$

A function $u \in C(\bar{\Omega})$ is a viscosity solution of

$$\begin{cases} -\mathcal{D}_p u = g(x, u, \nabla u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

if it is a viscosity sub- and supersolution of $-\mathcal{D}_p u = g(x, u, \nabla u)$ and $u = 0$ on $\partial\Omega$.

We mention some results in [12] obtained for the normalized p -Laplace equation, which we will use together with the relationship between the normalized p -Laplace equation and the Dominative p -Laplace equation.

Lemma 2.4. *A function $u \in USC(\bar{\Omega})$ is a positive viscosity subsolution of $-\Delta_p^N u = 1$ with $u = 0$ on $\partial\Omega$ if and only if $v = -\sqrt{u} \in LSC(\bar{\Omega})$ is a negative viscosity supersolution of*

$$-\Delta_p^N v = \frac{1}{v} \left((p-1)|\nabla v|^2 + \frac{1}{2} \right).$$

Lemma 2.5. *Let $\lambda > 0$. A function $u \in USC(\bar{\Omega})$ is a positive viscosity subsolution of $-\Delta_p^N u = \lambda u$ if and only if $v = -\ln u \in LSC(\bar{\Omega})$ is a negative viscosity supersolution of*

$$-\Delta_p^N v = -(p-1)|\nabla v|^2 - \lambda.$$

Properties of the operator We give some properties of viscosity solutions of the Dominative p -Laplace equation.

- Comparison principle: let $u \in USC(\bar{\Omega})$ be a viscosity subsolution of $-\mathcal{D}_p u = 1$ and let $v \in LSC(\bar{\Omega})$ be a viscosity supersolution of $-\mathcal{D}_p v = 1$. Then $u \leq v$ on $\partial\Omega$ implies $u \leq v$ in Ω . For a proof, see [Theorem 3.3, [8]].
- Positive supersolutions: if $u \in LSC(\bar{\Omega})$ is a viscosity supersolution of $-\mathcal{D}_p u = 1$ with $u = 0$ on $\partial\Omega$, then $u > 0$ in Ω . To see this, note that $w = 0$ is a viscosity subsolution, and $u \geq w$ by the comparison principle. This inequality must be strict. If $u(x_0) = 0$, then x_0 is a minimum for u . Let $\phi(x) = u(x_0)$ be a test function. Then $u - \phi$ has a local minimum at x_0 . But $-\mathcal{D}_p \phi = 0 < 1$, which contradicts u being a supersolution.

The Dominative p -Laplace operator has many of the same properties that the normalized p -Laplace operator possess. Here, we give some connections for viscosity solutions.

Lemma 2.6. *If $u \in LSC(\bar{\Omega})$ is a viscosity supersolution of*

$$-\mathcal{D}_p u = g(x, u, \nabla u),$$

then u is a viscosity supersolution of

$$-\Delta_p^N u = g(x, u, \nabla u).$$

Here, g is assumed to be continuous in all variables. Similarly, if $u \in USC(\bar{\Omega})$ is a viscosity subsolution of $-\Delta_p^N u = g(x, u, \nabla u)$, then u is a viscosity supersolution of $-\mathcal{D}_p u = g(x, u, \nabla u)$.

Proof. Assume u is a viscosity supersolution of $-\mathcal{D}_p u = g(x, u, \nabla u)$. If $u - \phi$ obtains a minimum at $x \in \Omega$, we have, provided $\nabla\phi(x) \neq 0$,

$$-\Delta_p^N \phi \geq -\mathcal{D}_p \phi \geq g(x, u, \nabla \phi).$$

If $\nabla\phi(x) = 0$,

$$-\Delta \phi - (p-2)\lambda_{\min}(D^2\phi) \geq -\mathcal{D}_p \phi \geq g(x, u, \nabla \phi).$$

Hence, u is a viscosity supersolution of $-\Delta_p^N u = g(x, u, \nabla u)$. If u is a viscosity subsolution of $-\Delta_p^N u = g(x, u, \nabla u)$ and $u - \phi$ obtains a maximum at $x \in \Omega$,

$$-\mathcal{D}_p \phi \leq -\Delta_p^N \phi \leq g(x, u, \nabla \phi), \quad \text{provided } \nabla\phi(x) \neq 0.$$

On the other hand, if $\nabla\phi(x) = 0$, $-\mathcal{D}_p\phi \leq g(x, u, 0)$ by definition. Hence, u is a viscosity supersolution of $-\mathcal{D}_p u = g(x, u, \nabla u)$. $\square$

The following Lemma will be applied in the proof of the concavity, and it relies on the fact that the mapping $(q, A) \rightarrow \langle q, A^{-1}q \rangle$ is convex in S^+ for each $q \in \mathbb{R}^n$. Here, S^+ consists of the symmetric positive definite matrices.

Lemma 2.7. *Let $X_i \in S^+$, $\nu_i \in [0, 1]$, $i = 1, \dots, k$, with $\sum_{i=1}^k \nu_i = 1$. Then*

$$\frac{1}{D_p \left(\sum_{i=1}^k \nu_i X_i \right)^{-1}} \geq \sum_{i=1}^k \frac{\nu_i}{D_p X_i^{-1}}.$$

Proof. In the appendix of [1] it was shown that $(q, A) \rightarrow \langle q, A^{-1}q \rangle$ is convex,

$$\langle q, (\mu A_1 + (1 - \mu) A_2)^{-1} q \rangle \leq \mu \langle q, A_1^{-1} q \rangle + (1 - \mu) \langle q, A_2^{-1} q \rangle,$$

for $q \in \mathbb{R}^n$, $A_1, A_2 \in S^+$ and $\mu \in [0, 1]$. Consequently,

$$D_p (\mu A_1 + (1 - \mu) A_2)^{-1} \leq \mu D_p A_1^{-1} + (1 - \mu) D_p A_2^{-1}. \quad (2.1)$$

We label $c_1 = D_p(X_1^{-1})$, $c_2 = D_p(X_2^{-1})$ and choose

$$A_1 = \frac{X_1}{c_2}, \quad A_2 = \frac{X_2}{c_1}, \quad \mu = \frac{\nu c_2}{\nu c_2 + (1 - \nu) c_1}.$$

With these choices,

$$D_p (\nu X_1 + (1 - \nu) X_2)^{-1} = \frac{D_p (\mu A_1 + (1 - \mu) A_2)^{-1}}{\nu c_2 + (1 - \nu) c_1}.$$

Using inequality (2.1) we find

$$\begin{aligned} \frac{1}{D_p (\nu X_1 + (1 - \nu) X_2)^{-1}} &= \frac{\nu c_2 + (1 - \nu) c_1}{D_p (\mu A_1 + (1 - \mu) A_2)^{-1}} \\ &\geq \frac{\nu c_2 + (1 - \nu) c_1}{\mu D_p(A_1^{-1}) + (1 - \mu) D_p(A_2^{-1})} \\ &= \frac{\nu c_2 + (1 - \nu) c_1}{\mu c_1 c_2 + (1 - \mu) c_1 c_2} \\ &= \frac{\nu}{c_1} + \frac{1 - \nu}{c_2} \\ &= \frac{\nu}{D_p(X_1^{-1})} + \frac{1 - \nu}{D_p(X_2^{-1})}. \end{aligned}$$

By induction, the inequality in Lemma 2.7 holds. $\square$

Convex envelope

The *convex envelope* of a function $u : \Omega \rightarrow \mathbb{R}^n$ is defined as

$$u_{**}(x) = \inf \left\{ \sum_{i=1}^k \mu_i u(x_i) : x_i \in \Omega, \sum \mu_i x_i = x, \sum \mu_i = 1, k \leq n+1, \mu_i \geq 0 \right\}.$$

We are interested in the convex envelope of the square root, $v = -\sqrt{u}$, and we have the following result on what happens near the boundary of Ω .

Lemma 2.8. *Let u be a viscosity solution to $-\mathcal{D}_p u = 1$ in a convex domain Ω that satisfies the interior sphere condition. Further let $x \in \Omega$, $x_1, \dots, x_k \in \bar{\Omega}$, $\sum_{i=1}^k \mu_i = 1$ with*

$$x = \sum_{i=1}^k \mu_i x_i, \quad u_{**}(x) = \sum_{i=1}^k \mu_i u(x_i).$$

Then $x_1, \dots, x_k \in \Omega$.

Proof. Since u is, in particular a viscosity supersolution to $-\Delta_p^N u = 1$, Lemma 3.2 in [12] gives the result. $\square$

3. Concave square-root solutions

First, we examine which equation $v = -\sqrt{u}$ solves in the viscosity sense.

Lemma 3.1. *A function $u \in USC(\bar{\Omega})$ is a positive viscosity subsolution of $-\mathcal{D}_p u = 1$ with $u = 0$ on $\partial\Omega$ if and only if $v = -\sqrt{u} \in LSC(\bar{\Omega})$, with $v = 0$ on $\partial\Omega$, is a negative viscosity supersolution of*

$$-\mathcal{D}_p v = \frac{1}{v} \left((p-1)|\nabla v|^2 + \frac{1}{2} \right).$$

Proof. Let u be a viscosity subsolution of $-\mathcal{D}_p u = 1$. Take $\phi \in C^2(\Omega)$ such that for some $r > 0$,

$$0 = (v - \phi)(x_0) < (v - \phi)(x), \quad \text{for all } x \in B_r(x_0),$$

so that $v - \phi$ has a strict local minimum point at $x_0 \in \Omega$. Let $\psi(x) = \phi(x)^2$. Then, since $v(x), \phi(x) < 0$,

$$\begin{aligned} (u - \psi)(x_0) &= (v(x_0) - \phi(x_0))(v(x_0) + \phi(x_0)) = 0 \\ (u - \psi)(x) &= (v(x) - \phi(x))(v(x) + \phi(x)) < 0. \end{aligned}$$

Hence, $u - \psi$ has a strict local *maximum* at x_0 . We see that $\psi_{x_i} = 2\phi\phi_{x_i}$, $\psi_{x_i x_j} = 2\phi_{x_i}\phi_{x_j} + 2\phi\phi_{x_i x_j}$. Since u is a viscosity subsolution we have at x_0 ,

$$\begin{aligned} 1 &\geq -\mathcal{D}_p \psi \\ &= -2\text{tr}(\nabla\phi \otimes \nabla\phi + \phi D^2\phi) - 2(p-2)\lambda_{\max}(\nabla\phi \otimes \nabla\phi + \phi D^2\phi) \\ &\geq -2|\nabla\phi|^2 - 2\phi\Delta\phi - 2(p-2)\lambda_{\max}(\nabla\phi \otimes \nabla\phi) - 2\phi(p-2)\lambda_{\max}(D^2\phi) \\ &= -2(p-1)|\nabla\phi|^2 - 2\phi\mathcal{D}_p\phi. \end{aligned}$$

Dividing by $\frac{1}{2\phi(x_0)}$ gives $-\mathcal{D}_p \phi(x_0) \geq \frac{1}{2\phi(x_0)} \left((p-1)|\nabla\phi(x_0)|^2 + \frac{1}{2} \right)$, which shows that v is a viscosity supersolution of

$$-\mathcal{D}_p v = \frac{1}{v} \left((p-1)|\nabla v|^2 + \frac{1}{2} \right).$$

On the other hand, suppose $v \in LSC(\Omega)$ is a negative viscosity supersolution of $-\mathcal{D}_p v = \frac{1}{v}((p-1)|\nabla v|^2 + \frac{1}{2})$. By Lemma 2.6, v is a viscosity supersolution of

$$-\Delta_p^N v \geq \frac{1}{v} \left((p-1)|\nabla v|^2 + \frac{1}{2} \right).$$

Applying Lemma 2.4 we see that $u = v^2$ is a positive viscosity subsolution of

$$-\Delta_p^N u = 1.$$

A second application of Lemma 2.6 shows that u is a viscosity subsolution of

$$-\mathcal{D}_p u = 1.$$

□

We now focus our attention on the convex envelope, v_{**} . It turns out that v_{**} is a viscosity supersolution to the same equation as v .

Lemma 3.2. *Let $u \in USC(\bar{\Omega})$ be a positive viscosity subsolution to $-\mathcal{D}_p u = 1$ with $u = 0$ on $\partial\Omega$ in a convex domain Ω that satisfies the interior sphere condition. If $v = -\sqrt{u}$, then v_{**} is a negative viscosity supersolution to*

$$-\mathcal{D}_p v_{**} = \frac{1}{v_{**}} \left((p-1)|\nabla v_{**}|^2 + \frac{1}{2} \right)$$

with $v_{**} = 0$ on $\partial\Omega$.

Proof. According to ([1], Lemma 4) we have $v_{**} = v = 0$ on $\partial\Omega$ so we only have to show that v_{**} is a viscosity supersolution. To this end, let $(q, A) \in J^{2,-}v_{**}(x)$. By Lemma 2.8 we can decompose x in a convex combination of interior points,

$$x = \sum_{i=1}^k \mu_i x_i, \quad v_{**}(x) = \sum_{i=1}^k \mu_i v(x_i), \quad \sum_{i=1}^k \mu_i = 1,$$

with $x_1, \dots, x_k \in \Omega$. By Proposition 1 in [1] there are $A_1, \dots, A_k \in S^+$ such that $(q, A_i) \in \bar{J}^{2,-}v(x_i)$ and

$$A - \epsilon A^2 \leq (\mu_1 A_1^{-1} + \dots + \mu_k A_k^{-1})^{-1}$$

for all $\epsilon > 0$ small enough. Since v is a viscosity supersolution,

$$-D_p(A_i) \geq \frac{1}{v(x_i)} \left((p-1)|q|^2 + \frac{1}{2} \right)$$

Multiplying both sides with $\mu_i v(x_i)$ and a summation $i = 1, \dots, k$ yields

$$-v_{**}(x) \leq \left((p-1)|q|^2 + \frac{1}{2} \right) \sum_{i=1}^k \frac{\mu_i}{D_p(A_i)}.$$

Using this inequality we find

$$\begin{aligned} & -D_p(A - \epsilon A^2) - \frac{1}{v_{**}(x)} \left((p-1)|q|^2 + \frac{1}{2} \right) \\ & \geq -D_p(A - \epsilon A^2) + \left(\sum_{i=1}^k \frac{\mu_i}{D_p(A_i)} \right)^{-1}. \end{aligned}$$

Lemma 2.7 then gives

$$\begin{aligned} & -D_p(A - \epsilon A^2) - \frac{1}{v_{**}(x)} \left((p-1)|q|^2 + \frac{1}{2} \right) \\ & \geq -D_p(A - \epsilon A^2) + D_p \left(\sum_{i=1}^k \mu_i X_i^{-1} \right)^{-1} \geq 0 \end{aligned}$$

since $A - \epsilon A^2 \leq \left(\sum_{i=1}^k \mu_i X_i^{-1} \right)^{-1}$. Letting $\epsilon \rightarrow 0$ we see that

$$-D_p(A) \geq \frac{1}{v_{**}(x)} \left((p-1)|q|^2 + \frac{1}{2} \right)$$

which shows that v_{**} is a viscosity supersolution to

$$-\mathcal{D}_p v_{**} = \frac{1}{v_{**}} \left((p-1)|\nabla v_{**}|^2 + \frac{1}{2} \right).$$

□

Proof of Theorem 1.1. We have to show that $v = -\sqrt{u}$ is convex, making $\sqrt{u}$ concave, if u is a viscosity solution of

$$\begin{aligned} -\mathcal{D}_p u &= 1 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.1}$$

Since u is, in particular, a supersolution, it is positive. By Lemma 3.2, v_{**} is a negative supersolution of

$$-\mathcal{D}_p v_{**} = \frac{1}{v_{**}} \left((p-1)|\nabla v_{**}|^2 + \frac{1}{2} \right).$$

By Lemma 3.1

$$-\mathcal{D}_p (v_{**})^2 \leq 1.$$

We have found a subsolution of equation (3.1). The comparison principle allows us to conclude that

$$v_{**}^2 \leq u = v^2, \quad \text{in } \Omega.$$

But $v_{**} \leq v < 0$. Thus we must have $v_{**} = v$, showing that v is convex. □

4. Log-concavity for the eigenvalue problem

We proceed in the same manner as in section 4. The proofs of the following two Lemmas are similar to the proofs of Lemma 3.1 and 3.2. We note that the interior sphere condition is not needed here, since $v = -\ln u$ converges to infinity on the boundary. This makes a similar version of Lemma 2.8 redundant.

Lemma 4.1. *Assume that Ω is a convex domain in $\mathbb{R}^n$ and let $\lambda > 0$. A function $u \in USC(\bar{\Omega})$ is a positive viscosity subsolution to $-\mathcal{D}_p u = \lambda u$ with $u = 0$ on $\partial\Omega$ if and only if $v = -\ln u \in LSC(\bar{\Omega})$ is a negative viscosity supersolution to*

$$-\mathcal{D}_p v = -(p-1)|\nabla v|^2 - \lambda.$$

Lemma 4.2. *Assume that Ω is a convex domain in $\mathbb{R}^n$ and let $\lambda > 0$. Let $u \in USC(\bar{\Omega})$ be a positive viscosity subsolution to $-\mathcal{D}_p u = \lambda u$ with $u = 0$ on $\partial\Omega$. If $v = -\ln u$, then v_{**} is a viscosity supersolution to*

$$-\mathcal{D}_p v_{**} = -(p-1)|\nabla v_{**}|^2 - \lambda.$$

Proof of Theorem 1.2. Let u be a positive viscosity solution of $-\mathcal{D}_p u = \lambda u$. Denoting $v = -\ln u$, Lemma 4.2 gives that v_{**} is a viscosity supersolution to

$$-\mathcal{D}_p v_{**} = -(p-1)|\nabla v_{**}|^2 - \lambda.$$

Lemma 4.1 gives

$$-\mathcal{D}_p e^{v_{**}} \leq \lambda e^{-v_{**}}$$

in the viscosity sense. By the comparison principle,

$$e^{-v_{**}} \leq u = e^{-v}, \quad \text{in } \Omega.$$

This together with the fact that $v_{**} \leq v$ shows that $v_{**} = v$, making v a convex function and $\log u$ a concave function. $\square$

5. Conclusion and further problems

In this paper, we showed certain concavity properties of power functions for solutions of the homogeneous Dirichlet problem for the Dominative p -Laplace equation. This was due to the structure of the equation and its relation to the normalized p -Laplace operator. An interesting question is whether the parabolic version, $u_t = \mathcal{D}_p u$ has similar concavity properties and in what way it depends on the initial data. Further, for $n = 2$, the equation can be explicitly written out, and it would be interesting to see a simple proof of the same result.

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