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On the stochastic Dullin–Gottwald–Holm equation: global existence and wave-breaking phenomena

Christian Rohde and Hao Tang

Abstract. We consider a class of stochastic evolution equations that include in particular the stochastic Camassa–Holm equation. For the initial value problem on a torus, we first establish the local existence and uniqueness of pathwise solutions in the Sobolev spaces H^s with $s > 3/2$. Then we show that strong enough nonlinear noise can prevent blow-up almost surely. To analyze the effects of weaker noise, we consider a linearly multiplicative noise with non-autonomous pre-factor. Then, we formulate precise conditions on the initial data that lead to global existence of strong solutions or to blow-up. The blow-up occurs as wave breaking. For blow-up with positive probability, we derive lower bounds for these probabilities. Finally, the blow-up rate of these solutions is precisely analyzed.

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Keywords. Stochastic Dullin–Gottwald–Holm equation, Pathwise solution, Blow-up scenario, Global existence, Wave breaking, Blow-up rate.

Contents

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1. Introduction

The Dullin–Gottwald–Holm (DGH) equation is a third-order dispersive evolution equation given by

$$\begin{aligned} &u_t - \alpha^2 u_{xxt} + c_0 u_x + 3uu_x + \gamma u_{xxx} \\ &\quad = \alpha^2 \left(2u_x u_{xx} + uu_{xxx}\right) \quad \text{in} \quad (0,\infty)\times\mathbb{R}. \end{aligned} \tag{1.1}$$

It was derived by Dullin et al. in [20] as a model governing planar solutions to Euler's equations in the shallow–water regime. The unknown $u = u(t, x)$ in (1.1) stands for the longitudinal velocity component and α^2, γ and c_0 are some physical parameters.

The DGH equation (1.1) embeds two different integrable soliton equations. When $\alpha = 0$, (1.1) reduces to the Korteweg–de–Vries (KdV) equation

$$u_t + c_0 u_x + 3uu_x + \gamma u_{xxx} = 0, \tag{1.2}$$

while (1.1) equals to the following Camassa–Holm (CH) equation for the choices $\gamma = 0$ and $\alpha = 1$,

$$u_t - u_{xxt} + c_0 u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}. \tag{1.3}$$

Both (1.2) and (1.3) have been studied widely in the literature. We notice that the CH equation exhibits two interesting phenomenon, namely (peaked) soliton interaction and wave breaking (the solution remains bounded but its slope becomes unbounded in finite time, cf. [12]), while the KdV equation does not model breaking waves [35] (when $c_0 = 0$, (1.2) admits a smooth soliton). For the CH equation, wave breaking and the necessary and sufficient criterion for the occurrence of breaking waves in the Cauchy problem with smooth initial data have been analyzed [10,12,13,43]. As pointed out in [11,14,15], the essential feature of the CH equation is the occurrence of traveling waves with a peak at their crest, exactly like that the governing equations for water waves admit the so-called Stokes waves of the greatest height. Bressan&Constantin [5,6] developed a new approach to the analysis of the CH equation, and proved the existence of a global conservative and dissipative solutions. Later, Holden&Raynaud [31,32] also obtained global conservative and dissipative solutions using a Lagrangian point of view.

Combining the linear dispersion of the KdV equation with the non-local dispersion of the CH equation, the DGH equation (1.1) preserves its bi-Hamiltonian structure, is completely integrable (via the inverse scattering transform method [20]) and admits also soliton solutions.

Here, we are interested in stochastic variants of the DGH equation to model energy consuming/exchanging mechanisms in (1.1) that are driven by external stochastic influences. Adding multiplicative noise has also been connected to the prevailing hypotheses that the onset of turbulence in fluid models involves randomness, cf. [7,21,39]. Precisely, our stochastic evolution equation is rewritten as

$$\begin{aligned} u_t - \alpha^2 u_{xxt} + c_0 u_x + 3uu_x + \gamma u_{xxx} - \dot{W}(1-\alpha^2\partial_{xx}^2)h(t,u) \\ = \alpha^2\left(2u_x u_{xx} + uu_{xxx}\right), \end{aligned} \tag{1.4}$$

where W is a standard 1-D Brownian motion and $h=(t,u)$ is a typically nonlinear function. We notice that the deterministic counterpart of (1.4) is the weakly dissipative CH equation

$$u_t - u_{xxt} + 3uu_x + \lambda(1-\partial_{xx}^2)h(t,u) = 2u_x u_{xx} + uu_{xxx}, \quad \lambda > 0. \tag{1.5}$$

Equation (1.5) has been introduced and studied for $h(t,u)=u$ in [40,53], In (1.5), the operator $\lambda(1-\partial_{xx}^2)$ is linear and only models the (weak) energy dissipation. In order to model more general random energy exchanges, we consider the possibly nonlinear noise term $-\dot{W}(1-\alpha^2\partial_{xx}^2)h(t,u)$ in (1.4).

To compare our model with deterministic weakly dissipative CH type equations (see [40,52,53] and the references therein), we focus our attention on the case that $\alpha \neq 0$. For convenience, we assume $\alpha = 1$ in this paper. When $\alpha = 1$, applying the operator $(1-\partial_{xx}^2)^{-1}$ to (1.4) gives rise to the following nonlocal equation

$$\mathrm{d}u + \left[(u-\gamma)\,\partial_x u + (1-\partial_{xx}^2)^{-1}\partial_x\left(u^2 + \frac{1}{2}u_x^2 + (c_0+\gamma)\,u\right)\right]\mathrm{d}t = h(t,u)\mathrm{d}W. \tag{1.6}$$

In (1.6), the operator $(1-\partial_{xx}^2)^{-1}$ in torus $\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}$ is understood as

$$\begin{cases} \left[(1-\partial_{xx}^2)^{-1} f\right](x) = [G_{\mathbb{T}} * f](x), \quad \forall\, f \in L^2(\mathbb{T}), \\ G_{\mathbb{T}} = \dfrac{\cosh(x - 2\pi\left[\frac{x}{2\pi}\right] - \pi)}{2\sinh(\pi)}, \end{cases} \tag{1.7}$$

where $[x]$ stands for the integer part of x. Here we remark that for additive noise, (1.6) has been studied in [42]. In this paper we will consider a more general context with noise driven by a cylindrical Wiener process $\mathcal{W}$, rather than a standard Brownian motion W. It is assumed that $\mathcal{W}$ is defined on an auxiliary Hilbert space U which is adapted to a right-continuous filtration $\{\mathcal{F}_t\}_{t\geq 0}$, see Sect. 2 for more details.

With the above notations, the first goal of the present paper is to analyze the existence and uniqueness of pathwise solutions and to determine possible blow-up criterion for the periodic boundary value problem

$$\begin{cases} \mathrm{d}u + [(u-\gamma)\,\partial_x u + F(u)]\ \mathrm{d}t = h(t,u)\,\mathrm{d}\mathcal{W}, \quad x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z},\ t>0, \\ u(\omega,0,x) = u_0(\omega,x), \quad x\in\mathbb{T}, \end{cases} \tag{1.8}$$

where $F(u) = F_1(u) + F_2(u) + F_3(u)$ and

$$\begin{cases} F_1(u) = (1-\partial_{xx}^2)^{-1}\partial_x\left(u^2\right), \\ F_2(u) = (1-\partial_{xx}^2)^{-1}\partial_x\left(\frac{1}{2}u_x^2\right), \\ F_3(u) = (1-\partial_{xx}^2)^{-1}\partial_x\left((c_0+\gamma)\,u\right). \end{cases} \tag{1.9}$$

Under generic assumptions on $h(t,u)$, we will show that (1.8) has a local unique pathwise solution (see Theorem 2.1 below). Here we remark that Chen et al. in [9] have considered the stochastic CH equation with additive noise. For the linear multiplicative noise case, we refer to [48] for the stochastic CH equation, and to [8] for a stochastic modified CH equation.

For stochastic nonlinear evolution equations, the noise effect is a crucial question to study. Can the noise prevent blow-up or does it even drive the formation of singularities? For example, it is known that the well-posedness of linear stochastic transport equations with noise can be established under weaker hypotheses than its deterministic counterpart (cf. [22,24]). For stochastic scalar conservation laws, noise on the flux may bring some regularization effects [27], and it may also trigger the discrete entropy dissipation in the numerical schemes for conservation laws such that the schemes enjoy some stability properties not present in the deterministic case [37]. Moreover, we refer to [29,36,46,48] for the dissipation of energy caused by the linear multiplicative noise.

However, most existing results on the regularization effects by noise for transport type equations are for linear equations or restricted to linear growing noise. Much less is known concerning the cases of nonlinear equations with nonlinear noises. Indeed, the interplay between regularization provided by noise and the nonlinearities of the governing equation is more complicated. For example, singularities can be prevented in some cases (cf. [25]: coalescence of vortices disappears in stochastic 2-D Euler equations). On the other hand, it is known that noise does not prevent shock formation in the Burgers equation, see [23].

Therefore the second goal of this work is to study the case of strong nonlinear noise and consider its effect. As we will see in (2.5) below, for the solution to (1.8), its H^s-norm blows up if and only if its $W^{1,\infty}$-norm blows up. This suggests choosing a noise coefficient involving the $W^{1,\infty}$-norm of u. Therefore in this work we consider the case that $h(t,u)\,\mathrm{d}\mathcal{W} = a\,(1+\|u\|_{W^{1,\infty}})^{\theta}\,u\,\mathrm{d}W$, where $\theta>0$, $a\in\mathbb{R}$ and W is a standard 1-D Brownian motion. We will try to determine the range of θ and a such that the solution to the following problem exists globally in time:

$$\begin{cases} \mathrm{d}u + [(u-\gamma)u_x + F(u)]\,\mathrm{d}t = a\left(1 + \|u\|_{W^{1,\infty}}\right)^{\theta} u\,\mathrm{d}W, & x \in \mathbb{T},\ t > 0, \\ u(\omega, 0, x) = u_0(\omega, x), & x \in \mathbb{T}. \end{cases} \tag{1.10}$$

As is shown in Theorem 2.2 below, if the noise is strong enough (either $\theta > 1/2$, $a \neq 0$ or $\theta = 1/2$, $a^2 \gg 1$), then the global existence holds true for (1.10) almost surely. This result justifies the idea that large nonlinear noise can actually prevent blow-up.

On the other hand, as put forward by e.g. Whitham in [51], the wave breaking phenomenon is one of the most intriguing long-standing problems of water wave theory. For the deterministic CH type equations, the wave breaking phenomenon has been extensively studied, see [12,13,43] for example. Particularly, for equations with dissipation term $\lambda(u - u_{xx})$, we refer to [53] for the phenomenon of wave breaking. When random noise is involved, as far as we know, we can only refer to [17,49] for wave breaking. In [17] the authors proved that temporal stochasticity (in the sense of Stratonovich) in the diffeomorphic flow map for the stochastic CH equation does not prevent the wave breaking process. In [49], wave breaking in the stochastic CH equation with multiplicative Itô noise is considered.

Thus, the third goal of this paper is to consider noise effects associated with the phenomenon of wave breaking. Due to Theorem 2.2, we see that if wave breaking occurs, the noise term does not grow fast. Hence we consider $\theta = 0$ in (1.10) but introduce a non-autonomous pre-factor depending on time t. Precisely, we consider the DGH equation with linear multiplicative noise given by

$$\begin{cases} \mathrm{d}u + [(u-\gamma)\partial_x u + F(u)]\,\mathrm{d}t = b(t)u\,\mathrm{d}W, & x \in \mathbb{T},\ t > 0, \\ u(\omega, 0, x) = u_0(\omega, x), & x \in \mathbb{T}. \end{cases} \tag{1.11}$$

This case can be formally reformulated as the following stochastic evolution equation when $s > 3$

$$u_t - u_{xxt} + c_0 u_x + 3uu_x + \gamma u_{xxx} = 2u_x u_{xx} + uu_{xxx} + b(t)(u - u_{xx})\dot{W}. \tag{1.12}$$

When $c_0 + \gamma = 0$, we give two conditions on the initial data that guarantee the global existence of the solutions. Besides, we also estimate the probability that the solution breaks and describe its breaking rate. See Theorems 2.3-2.7 for the statements.

The precise statements of all the results above can be found in Sect. 2 jointly with the necessary assumptions on the noise coefficient.

2. Definitions, assumptions and main results

We begin by introducing some notations. $L^2(\mathbb{T})$ is the usual space of square–integrable functions on $\mathbb{T}$. For $s \in \mathbb{R}$, $D^s = (1 - \partial_{xx}^2)^{s/2}$ is defined by $\widehat{D^s f}(k) =$

$(1+k^2)^{s/2}\widehat{f}(k)$, where $\widehat{g}$ is the Fourier transform of g. The Sobolev space $H^s(\mathbb{T})$ is defined as

$$H^s(\mathbb{T}) \triangleq \{f \in L^2(\mathbb{T}) : \|f\|^2_{H^s(\mathbb{T})} = \sum_{k\in\mathbb{Z}}(1+k^2)^s|\widehat{f}(k)|^2 < \infty\},$$

and the inner product $(f,g)_{H^s}$ is $(f,g)_{H^s} := \sum_{k\in\mathbb{Z}}(1+k^2)^s\widehat{f}(k)\cdot\overline{\widehat{g}}(k) = (D^s f, D^s g)_{L^2}$. When the function space refers to $\mathbb{T}$, we will drop $\mathbb{T}$ if there is no ambiguity. We will use $\lesssim$ to denote estimates that hold up to some universal *deterministic* constant which may change from line to line but whose meaning is clear from the context. For linear operators A and B, we denote $[A,B] = AB - BA$.

We briefly recall some aspects of the theory of infinite dimensional stochastic analysis which we will use below. We refer the readers to [18, 26, 33] for an extended treatment of this subject.

We call $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{W})$ a stochastic basis, where $\{\mathcal{F}_t\}_{t\geq 0}$ is a right-continuous filtration on $(\Omega,\mathcal{F})$ such that $\{\mathcal{F}_0\}$ contains all the $\mathbb{P}$-negligible subsets and $\mathcal{W}(t) = \mathcal{W}(\omega,t)(\omega\in\Omega)$ is a cylindrical Wiener process adapted to $\{\mathcal{F}_t\}_{t\geq 0}$. More precisely, we consider a separable Hilbert space U as well as a larger Hilbert space U_0 such that the embedding $U \hookrightarrow U_0$ is Hilbert–Schmidt. Therefore we define

$$\mathcal{W} = \sum_{k=1}^{\infty} W_k e_k \in C([0,\infty); U_0) \ \ \mathbb{P} - a.s.,$$

where $\{W_k\}_{k\geq 1}$ is a sequence of mutually independent 1-D Brownian motions and $\{e_k\}_{k\in\mathbb{N}}$ is a complete orthonormal basis of U.

For a predictable stochastic process G taking values in the space of Hilbert–Schmidt operators from U to H^s, denoted by $\mathcal{L}_2(U;H^s)$, the Itô stochastic integral

$$\int_0^\tau G\mathrm{d}\mathcal{W} = \sum_{k=1}^{\infty}\int_0^\tau Ge_k\mathrm{d}W_k$$

is well defined (see [18, 44] for example). Remember that

$$G \in \mathcal{L}_2(U;H^s) \Longleftrightarrow \|G\|^2_{\mathcal{L}_2(U;H^s)} = \sum_{k=1}^{\infty}\|Ge_k\|^2_{H^s} < \infty.$$

The stochastic integral $\int_0^t G\,\mathrm{d}\mathcal{W}$ is an H^s-valued square–integrable martingale. In our case we have the Burkholder-Davis-Gundy inequality

$$\mathbb{E}\left(\sup_{t\in[0,T]}\left\|\int_0^t G\,\mathrm{d}\mathcal{W}\right\|^p_{H^s}\right) \leq C(p,s)\mathbb{E}\left(\int_0^T \|G\|^2_{\mathcal{L}_2(U;H^s)}\,\mathrm{d}t\right)^{\frac{p}{2}}, \quad p\geq 1. \tag{2.1}$$

2.1. Definitions of the solutions

We now precise the notion of pathwise solutions to (1.8).

Definition 2.1. [Pathwise solutions] Let $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{W})$ be a fixed stochastic basis. Let $s > 3/2$ and u_0 be an H^s-valued $\mathcal{F}_0$ measurable random variable.

(1) A local pathwise solution to (1.8) is a pair (u, τ), where τ is a stopping time satisfying $\mathbb{P}\{\tau > 0\} = 1$ and $u : \Omega \times [0, \infty) \to H^s$ is an $\mathcal{F}_t$ predictable H^s-valued process satisfying

$$u(\cdot \wedge \tau) \in C([0, \infty); H^s) \ \ \mathbb{P} - a.s.,$$

and for all $t > 0$,

$$u(t \wedge \tau) - u(0) + \int_0^{t\wedge\tau} [(u - \gamma)\partial_x u + F(u)]\, \mathrm{d}t' = \int_0^{t\wedge\tau} h(t', u)\mathrm{d}\mathcal{W} \ \ \mathbb{P} - a.s.$$

(2) Local pathwise solutions are said to be pathwise unique, if given any two pairs of local pathwise solutions (u_1, τ_1) and (u_2, τ_2) with $\mathbb{P}\{u_1(0) = u_2(0)\} = 1$, we have

$$\mathbb{P}\{u_1(t, x) = u_2(t, x), \quad \forall\, (t, x) \in [0, \tau_1 \wedge \tau_2] \times \mathbb{T}\} = 1.$$

(3) Additionally, (u, τ^*) is called a maximal solution to (1.8) if $\tau^* > 0$ almost surely and if there is an increasing sequence $\tau_n \to \tau^*$ such that for any $n \in \mathbb{N}$, (u, τ_n) is a pathwise solution to (1.8) and on the set $\{\tau^* < \infty\}$,

$$\sup_{t\in[0,\tau_n]} \|u\|_{H^s} \geq n.$$

(4) If $\tau^* = \infty$ almost surely, then we say that the pathwise solution exists globally.

2.2. Assumptions

Next, we prescribe some conditions on the noise coefficient h in (1.8) and on b in (1.12).

Assumption 2.1. Let $s > 1/2$. We assume that $h : [0, \infty) \times H^s \ni (t, u) \mapsto h(t, u) \in \mathcal{L}_2(U; H^s)$ is continuous in (t, u). Moreover, we assume the following:

- There is a non-decreasing locally bounded function $f(\cdot) : [0, \infty) \to [0, \infty)$ such that for any $t > 0$,

$$\|h(t, u)\|_{\mathcal{L}_2(U;H^s)} \leq f(\|u\|_{W^{1,\infty}})\|u\|_{H^s}. \tag{2.2}$$

  Particularly, in the additive noise case, we assume $h : [0, \infty) \times \mathbb{T} \ni (t, x) \mapsto h(t, x) \in \mathcal{L}_2(U; H^s)$ is continuous meaning that (2.2) reduces to $\|h(t, x)\|_{\mathcal{L}_2(U;H^s)} \leq C$ for some $C > 0$.
- There is a non-decreasing locally bounded function $q(\cdot) : [0, \infty) \to [0, \infty)$, such that for any $t > 0$,

$$\sup_{\|u\|_{H^s}, \|v\|_{H^s} \leq N} \left\{ \mathbf{1}_{\{u\neq v\}} \frac{\|h(t, u) - h(t, v)\|_{\mathcal{L}_2(U;H^s)}}{\|u - v\|_{H^s}} \right\} \leq q(N), \quad N \geq 1. \tag{2.3}$$

After the regularization effect of strong noise is established in Theorem 2.2, to analyze the effect of noise on the regularity of pathwise solutions, we restrict ourselves to the linear-noise case (1.12) imposing the following bounds on the coefficient b.

Assumption 2.2. When considering (1.12) with non-autonomous linear noise $b(t)u\,\mathrm{d}W$, we assume that there are constants $b_*, b^* > 0$ such that $0 < b_* \le b^2(t) \le b^*$ for all t.

Remark 2.1. Let us give some brief explanations for the assumptions.

- The function $h : [0,\infty) \times H^s \ni (t,u) \mapsto h(t,u) \in \mathcal{L}_2(U; H^s)$ is required to be continuous in (t,u). This will be essential to pass to the limit when establishing the existence of a martingale solution as an intermediate step, cf. [3,48,49].
- The uniform-in-time assumption (2.2) bounds the growth of the $\mathcal{L}_2(U; H^s)$-norm of the noise coefficient in terms of a product of a nonlinear function of the $W^{1,\infty}$-norm and the H^s-norm. This allows us to control the $W^{1,\infty}$-norm by some cut-off later.
- Formula (2.3) ensures local Lipschitz continuity in H^s, which will be used to obtain (local) existence and uniqueness.
- Let us outline that we will use a Girsanov-type transformation to study (1.11) (see Remark 2.6 and Sect. 6). The assumption $b^2(t) \le b^*$ is used to guarantee that such transformation is well-defined and the condition $b(t) \ne 0, t \ge 0$ is needed to establish certain estimates for the Girsanov-type process $\mathrm{e}^{\int_0^t b(t')\mathrm{d}W_{t'} - \int_0^t \frac{b^2(t')}{2}\mathrm{d}t'}$ (see Lemma 3.7). In Theorem 2.3, the condition $0 < b_* \le b^2(0)$ is used to bound the initial data, cf. (2.7).

2.3. Main results and remarks

Now we present our results. For the general case (1.8), we have the following local existence result which moreover relates the possible blow-up in the H^s-norm to simultaneous blow up in the $W^{1,\infty}$-norm.

Theorem 2.1. (Maximal solutions) *Let $s > 3/2$, $c_0, \gamma \in \mathbb{R}$ and let $h(t,u)$ satisfy Assumption 2.1. For a given stochastic basis $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\ge0}, \mathcal{W})$ and an H^s-valued $\mathcal{F}_0$ measurable random variable u_0 satisfying $\mathbb{E}\|u_0\|^2_{H^s} < \infty$, the initial value problem (1.8) admits a local unique pathwise solution (u,τ) in the sense of Definition 2.1 with*

$$u(\cdot \wedge \tau) \in L^2\left(\Omega; C\left([0,\infty); H^s\right)\right). \tag{2.4}$$

Besides, (u,τ) can be extended to a unique maximal solution (u,τ^*) in the sense of Definition 2.1 and the following blow-up criterion holds true:

$$\mathbf{1}_{\{\limsup_{t\to\tau^*}\|u(t)\|_{H^s}=\infty\}} = \mathbf{1}_{\left\{\limsup_{t\to\tau^*}\|u(t)\|_{W^{1,\infty}}=\infty\right\}} \quad \mathbb{P}-a.s. \tag{2.5}$$

Remark 2.2. For the proof of Theorem 2.1 one can follow the ideas in e.g. [2–4,16,19,29,48] by constructing a sequence of approximations for a problem with cut-off for the $W^{1,\infty}$-norm. Such a cut-off implies at-most linear growth of u and guarantees the global existence of an approximate solution. Otherwise we have to find a positive lower bound for the lifespan of the approximate solutions, which is a priori not clear. Besides, with the cut-off, one can close the a priori $L^2(\Omega; H^s)$ estimate by splitting $\mathbb{E}(\|u\|^2_{H^s}\|u\|_{W^{1,\infty}})$.

Turning to noise-driven regularization effects, the blow-up criterion (2.5) suggests relating the noise coefficient to the $W^{1,\infty}$-norm of u. Therefore we consider (1.10) with scalable noise impact, i.e., we assume $h(t,u) = a(1+\|u\|)^{\theta}_{W^{1,\infty}}u$ for some $\theta > 0$ and $a \in \mathbb{R}$. When a and θ satisfy certain strength-conditions, the noise term counteracts the formation of singularities and we have

Theorem 2.2. (Global existence for strong nonlinear noise) *Let $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\ge 0}, W)$ be a fixed stochastic basis. Let $s > \frac{5}{2}$, $c_0, \gamma \in \mathbb{R}$ and $u_0 \in H^s$ be an H^s-valued $\mathcal{F}_0$-measurable random variable with $\mathbb{E}\|u_0\|^2_{H^s} < \infty$. Assume that θ and a satisfy*

$$\text{either} \quad a \in \mathbb{R}\backslash\{0\},\ \theta > \frac{1}{2} \quad \text{or} \quad a^2 > 2Q, \theta = \frac{1}{2}, \tag{2.6}$$

where $Q = Q(s, c_0, \gamma)$ is a constant that will be specified in Lemma 3.5. Then the corresponding maximal solution (u, τ^*) to (1.10) satisfies

$$\mathbb{P}\{\tau^* = \infty\} = 1.$$

Theorem 2.2 implies that blow-up of pathwise solutions might only be observed if the noise is weak. To detect such noise, we analyze the simpler ansatz $h(t,u) = b(t)u$ as in (1.11). Even in this linear noise case the situation is quite subtle allowing for global existence as well as blow-up of solutions. For global existence, we can identify two cases.

Theorem 2.3. (Global existence for weak noise I) *Let $s > 3/2$. Assume $c_0 + \gamma = 0$. Let $b(t)$ satisfy Assumption 2.2 and $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\ge 0}, W)$ be a fixed stochastic basis. Assume u_0 is an H^s-valued $\mathcal{F}_0$ measurable random variable satisfying $\mathbb{E}\|u_0\|^2_{H^s} < \infty$. Let $K = K(s) > 0$ be a constant such that the embedding $\|\cdot\|_{W^{1,\infty}} < K\|\cdot\|_{H^s}$ holds. Then there is a $C = C(s) > 1$ such that for any $R > 1$ and $\lambda_1 > 2$, if*

$$\|u_0\|_{H^s} < \frac{b_*}{CK\lambda_1 R} \quad \mathbb{P} - a.s., \tag{2.7}$$

then (1.11) has a maximal solution (u, τ^*) satisfying for any $\lambda_2 > \frac{2\lambda_1}{\lambda_1 - 2}$ the estimate

$$\mathbb{P}\left\{\|u(t)\|_{H^s} < \frac{b_*}{CK\lambda_1} \mathrm{e}^{-\frac{((\lambda_1-2)\lambda_2 - 2\lambda_1)}{2\lambda_1\lambda_2}\int_0^t b^2(t')\mathrm{d}t'} \text{ for all } t > 0\right\} \ge 1 - \left(\frac{1}{R}\right)^{2/\lambda_2}. \tag{2.8}$$

Theorem 2.4. (Global existence for weak noise II) *Let $c_0 + \gamma = 0$ and $s > 3$. Let $b(t)$ satisfy Assumption 2.2 and $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\ge 0}, W)$ be a fixed stochastic basis. Assume u_0 is an H^s-valued $\mathcal{F}_0$ measurable random variable satisfying $\mathbb{E}\|u_0\|^2_{H^s} < \infty$. If u_0 satisfies*

$$\mathbb{P}\left\{(1 - \partial^2_{xx})u_0(x) > 0, \quad \forall\, x \in \mathbb{T}\right\} = p,$$
$$\mathbb{P}\left\{(1 - \partial^2_{xx})u_0(x) < 0, \quad \forall\, x \in \mathbb{T}\right\} = q,$$

for some $p, q \in [0,1]$, then the corresponding maximal solution (u, τ^*) to (1.11) satisfies

$$\mathbb{P}\{\tau^* = \infty\} \ge p + q.$$

Remark 2.3. Theorem 2.3 provides a global existence result for initial data with bounded H^s-norm depending on the strength of the noise. This result can *not* be observed in the deterministic case because $0 < b_* \leq b^2(t)$ is required (see Assumption 2.2). On the other hand, since the proof of Theorem 2.4 relies on the analysis of a PDE with random coefficient (see (6.2) below), the deterministic case can be included by formally letting the random coefficient be 1. Therefore, in this sense, Theorem 2.4 covers the corresponding deterministic result, cf. [13,41]. Indeed, by letting $\beta \equiv 1$ in (6.2) and taking $(p, q) = (1, 0)$ or $(p, q) = (0, 1)$ in Theorem 2.4, we obtain the global existence for the deterministic DGH equation.

According to (2.5) in Theorem 2.1, a blow-up comes along with an explosion of the $W^{1,\infty}$-norm. For the special noise in (1.11) we can improve the result by showing that a blow-up is related to the first spatial derivative only and corresponds to the wave-breaking phenomenon with exploding negative slope.

Theorem 2.5. (Blow-up scenario) *Let $c_0 + \gamma = 0$, $s > 3$ and Assumption 2.2 be satisfied. Let $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, W)$ be fixed in advance. Let (u, τ^*) be the unique maximal solution to (1.11) starting from an $\mathcal{F}_0$ measurable random variable $u_0 \in L^2(\Omega; H^s)$. Then the singularities can arise only in the form of wave breaking, i.e.,*

$$\mathbb{P}\left\{\|u(t)\|_{L^\infty} \lesssim A\|u_0\|_{H^1} < \infty, \quad \forall\, t > 0\right\} = 1, \tag{2.9}$$

where $A = A(\omega) = \sup_{t>0} \mathrm{e}^{\int_0^t b(t')\ \mathrm{d}W_{t'} - \int_0^t \frac{b^2(t')}{2}\ \mathrm{d}t'} < \infty\ \mathbb{P}-a.s.$, and

$$\mathbf{1}_{\{\limsup_{t\to\tau^*} \|u(t)\|_{H^s}=\infty\}} = \mathbf{1}_{\{\liminf_{t\to\tau^*} [\min_{x\in\mathbb{T}} u_x(t,x)]=-\infty\}} \quad \mathbb{P}-a.s. \tag{2.10}$$

Still we have not identified initial data for (1.11) that lead to a blow-up. A precise condition in terms of probability is given in the next theorem. To formulate it, we introduce the number $\lambda > 0$ such that for any $f \in H^3$, the estimate

$$\max_{x\in\mathbb{T}} f^2(x) \leq \lambda\|f\|^2_{H^1} \tag{2.11}$$

holds.

Theorem 2.6. (Wave breaking and its probability) *Let $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, W)$ be a fixed stochastic basis, $c_0 + \gamma = 0$ and $s > 3$. Let Assumption 2.2 be verified and let $u_0 \in L^2(\Omega; H^s)$ be $\mathcal{F}_0$ measurable. If for some $c \in (0, 1)$,*

$$\min_{x\in\mathbb{T}} \partial_x u_0(x) < -\frac{1}{2}\sqrt{\frac{(b^*)^2}{c^2} + 4\lambda\|u_0\|^2_{H^1}} - \frac{b^*}{2c} \quad \mathbb{P}-a.s.,$$

where b^* is given in Assumption 2.2 and λ is given in (2.11), then the maximal solution (u, τ^*) to (1.11) (or (1.12), equivalently) satisfies

$$\mathbb{P}\left\{\tau^* < \infty\right\} \geq \mathbb{P}\left\{\mathrm{e}^{\int_0^t b(t')\,\mathrm{d}W_{t'}} > c, \quad \forall\, t > 0\right\} > 0.$$

By Theorem 2.5, we have $\mathbb{P}\left\{u \text{ breaks in finite time}\right\} \geq \mathbb{P}\left\{\mathrm{e}^{\int_0^t b(t')\,\mathrm{d}W_{t'}} > c,\ \forall\, t > 0\right\} > 0.$

Remark 2.4. Whereas Theorem 2.3 provides a global existence result, Theorem 2.6 detects the formation of singularities in finite time under certain conditions on the initial data. We stress that these two conditions are mutually exclusive. In Theorem 2.3 we suppose $\|u_0\|_{H^s} \le \frac{b_*}{CK\lambda_1 R}$ with $C > 1, \lambda_1 > 2$ and $R > 1$ almost surely. Then u satisfies (2.8). In Theorem 2.6 we suppose for some $c \in (0,1)$ that $\min_{x\in\mathbb{T}} \partial_x u_0(x) < -\frac{1}{2}\sqrt{\frac{(b^*)^2}{c^2} + 4\lambda\|u_0\|^2_{H^1}} - \frac{b^*}{2c} < -\frac{b^*}{2c}$ almost surely holds. But this means $\|u_0\|_{H^s} > \frac{1}{K}\|u_0\|_{W^{1,\infty}} \ge \frac{1}{K}|\min_{x\in\mathbb{T}} \partial_x u_0(x)| > \frac{b^*}{2cK} > \frac{b_*}{CK\lambda_1 R}$.

We conclude this section with a result refining Theorem 2.5. It is possible to quantify the blow-up rate.

Theorem 2.7. (Wave breaking rate) *Let the conditions in Theorem 2.5 hold true. Then*

$$\lim_{t\to\tau^*}\left(\min_{x\in\mathbb{T}}[u_x(t,x)]\int_t^{\tau^*}\beta(t')\,\mathrm{d}t'\right) = -2\beta(\tau^*) \quad \text{a.e. on} \quad \{\tau^* < \infty\}, \tag{2.12}$$

where

$$\beta(\omega,t) = \mathrm{e}^{\int_0^t b(t')\mathrm{d}W_{t'} - \int_0^t \frac{b^2(t')}{2}\mathrm{d}t'}.$$

Remark 2.5. As a corollary of Theorems 2.5 and 2.7, we have that as long as singularities occurs, they can arise only in the form of wave breaking and the breaking rate is given by (2.12). This result is optimal in the sense that it is consistent with the result for the corresponding deterministic case. Indeed, for the deterministic DGH equation (cf. [41, Theorem 4.2]), the blow-up rate is

$$\lim_{t\to\tau^*}\left(\min_{x\in\mathbb{T}}[u_x(t,x)](\tau^*-t)\right) = -2.$$

Formally, since the deterministic DGH equation can be viewed as (6.2) with $\beta \equiv 1$, we see that the blow-up estimate (2.12) coincides with the above deterministic result when $\beta \equiv 1$.

Remark 2.6. Let us make a comment on the idea for the subsequent analysis of (1.11), which is motivated by [29,46,48]. By introducing the Girsanov-type transformation

$$v = \frac{1}{\beta(\omega,t)}u, \quad \beta(\omega,t) = \mathrm{e}^{\int_0^t b(t')\mathrm{d}W_{t'} - \int_0^t \frac{b^2(t')}{2}\mathrm{d}t'},$$

we obtain an equation for v (see Sect. 6 for the detailed calculation), namely

$$v_t + \beta v v_x - \gamma v_x + \beta(1-\partial^2_{xx})^{-1}\partial_x\left(v^2 + \frac{1}{2}v_x^2\right) + (c_0+\gamma)(1-\partial^2_{xx})^{-1}\partial_x v = 0.$$

Although the above equation for v does not depend on a stochastic integral on v itself, to extend the deterministic results to the stochastic setting, we need to overcome a few technical difficulties since the system is not only random but also non-autonomous (see e.g., (6.6), (6.8) and (6.10)). With the help of certain estimates and asymptotic limits of Girsanov-type processes (see Lemma 3.7), we are able to apply the energy estimate pathwisely (for a.e. $\omega \in \Omega$) to study the global existence and possible blow-up of solutions.

We outline the rest of the paper. In the next section, we briefly recall some relevant preliminaries. In Sect. 4, we prove Theorem 2.1. For the large noise case, we prove Theorem 2.2 in Sect. 5. For the non-autonomous linear multiplicative noise case, we consider the global existence, decay, wave breaking and the blow-up rate of the pathwise solutions and prove Theorems 2.3, 2.4, 2.6 and 2.7 in Sect. 6.

3. Preliminary results

We summarize some auxiliary results, which will be used to prove our main results from Sect. 2. Define the regularizing operator T_ε on $\mathbb{T}$ as

$$T_\varepsilon f(x) := (1-\varepsilon^2\Delta)^{-1} f(x) = \sum_{k\in\mathbb{Z}} \left(1+\varepsilon^2|k|^2\right)^{-1} \widehat{f}(k)\, \mathrm{e}^{\mathrm{i}xk}, \quad \varepsilon \in (0,1). \tag{3.1}$$

Since T_ε can be characterized by its Fourier multipliers, it is easy to see

$$[D^s, T_\varepsilon] = 0, \tag{3.2}$$

$$(T_\varepsilon f, g)_{L^2} = (f, T_\varepsilon g)_{L^2}, \tag{3.3}$$

$$\|T_\varepsilon u\|_{H^s} \le \|u\|_{H^s}. \tag{3.4}$$

Furthermore, we have

Lemma 3.1. ([49]) *Let $f, g : \mathbb{T} \to \mathbb{R}$ such that $g \in W^{1,\infty}$ and $f \in L^2$. Then for some $C > 0$,*

$$\|[T_\varepsilon, g] f_x\|_{L^2} \le C\|g\|_{W^{1,\infty}} \|f\|_{L^2}.$$

The following estimates are classical for Sobolev spaces.

Lemma 3.2. ([34]) *Let $s > 1$. There is a $C_s > 0$ such that for all $f \in H^s \cap W^{1,\infty}$, $g \in H^{s-1} \cap L^\infty$ we have*

$$\|\, [D^s, f]\, g\|_{L^2} \le C_s\left(\|D^s f\|_{L^2}\|g\|_{L^\infty} + \|\partial_x f\|_{L^\infty}\|D^{s-1} g\|_{L^2}\right).$$

Lemma 3.3. ([34]) *Let $s > 0$, then there is a $C_s > 0$ such that we have for all $f, g \in H^s \cap L^\infty$ the estimate*

$$\|fg\|_{H^s} \le C_s\left(\|f\|_{H^s}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|g\|_{H^s}\right).$$

Specifically, for our problem (1.8), we have introduced the nonlocal term $F(\cdot)$ in (1.9). Using the Moser estimate from Lemma 3.3, we can obtain the next statement on $F(\cdot)$ (see [50]).

Lemma 3.4. *For $F(\cdot)$ defined in (1.9) and for any $v, v_1, v_2 \in H^s$ with $s > 1/2$, we have*

$$\|F(v)\|_{H^s} \lesssim \left(\|v\|_{L^\infty} + \|\partial_x v\|_{L^\infty} + (c_0+\gamma)\right)\|v\|_{H^s}, \quad s > 3/2$$

$$\|F(v_1) - F(v_2)\|_{H^s} \lesssim \left(\|v_1\|_{H^s} + \|v_2\|_{H^s} + (c_0+\gamma)\right)\|v_1 - v_2\|_{H^s}, \quad s > 3/2$$

$$\|F(v_1) - F(v_2)\|_{H^s} \lesssim \left(\|v_1\|_{H^{s+1}} + \|v_2\|_{H^{s+1}} + (c_0+\gamma)\right)\|v_1 - v_2\|_{H^s}, \quad 3/2 \ge s > 1/2.$$

The following estimate will be used in the proof of the blow-up criterion (2.5) and of Theorem 2.2.

Lemma 3.5. *Let $s > 3/2$, $c_0, \gamma \in \mathbb{R}$. Let $F(\cdot)$ and T_ε be given in* (1.9) *and* (3.1)*, respectively. There is a constant $Q = Q(s, c_0, \gamma) > 0$ such that for all $\varepsilon > 0$,*

$$|(T_\varepsilon [(u-\gamma)u_x], T_\varepsilon u)_{H^s}| + |(T_\varepsilon F(u), T_\varepsilon u)_{H^s}| \le Q\,(1 + \|u\|_{W^{1,\infty}})\,\|u\|_{H^s}^2.$$

Proof. We first notice that

$$\begin{aligned}(T_\varepsilon [(u-\gamma)u_x], T_\varepsilon u)_{H^s} &= \int_{\mathbb{T}} D^s T_\varepsilon [(u-\gamma)u_x] \cdot D^s T_\varepsilon u \,\mathrm{d}x \\ &= \int_{\mathbb{T}} D^s T_\varepsilon [u u_x] \cdot D^s T_\varepsilon u \,\mathrm{d}x.\end{aligned}$$

Due to (3.2) and (3.3), we commute the operator to derive

$$\begin{aligned}&(D^s T_\varepsilon [u u_x], D^s T_\varepsilon u)_{L^2} \\ &\quad = \left([D^s, u]\, u_x, D^s T_\varepsilon^2 u\right)_{L^2} + ([T_\varepsilon, u] D^s u_x, D^s T_\varepsilon u)_{L^2} + (u D^s T_\varepsilon u_x, D^s T_\varepsilon u)_{L^2}.\end{aligned}$$

Then it follows from Lemmas 3.1 and 3.2, integration by parts, (3.4) and $H^s \hookrightarrow W^{1,\infty}$ that

$$|(T_\varepsilon [(u-\gamma)u_x], T_\varepsilon u)_{H^s}| \lesssim \|u\|_{W^{1,\infty}} \|u\|_{H^s}^2.$$

Using Lemma 3.4 and (3.4) directly, we have

$$|(T_\varepsilon F(u), T_\varepsilon u)_{H^s}| \lesssim (\|u\|_{W^{1,\infty}} + (c_0 + \gamma))\,\|u\|_{H^s}^2.$$

Combining the above two inequalities gives rise to the desired estimate of the lemma. □

The following lemma has been established for the real-line case in [12] and [10], respectively. They hold likewise for $x \in \mathbb{T}$, using the periodicity on $\mathbb{T}$.

Lemma 3.6. *Let $T > 0$ and $u \in C^1([0,T); H^2(\mathbb{T}))$. Then given any $t \in [0,T)$, there is at least one point $z(t)$ with*

$$M(t) := \min_{x \in \mathbb{T}} [u_x(t,x)] = u_x(t, z(t)).$$

Moreover, the function $M = M(t)$ is almost everywhere differentiable on $(0,T)$ with

$$\frac{\mathrm{d}}{\mathrm{d}t} M(t) = u_{tx}(t, z(t)) \quad \text{a.e. on } (0,T).$$

We conclude this preparatory section with some results from [47], which are needed to establish the theorems on global existence.

Lemma 3.7. *Let Assumption 2.2 hold true and assume that $a(t) \in C([0,\infty))$ is a bounded function. For*

$$X = \mathrm{e}^{\int_0^t b(t')\,\mathrm{d}W_{t'} + \int_0^t a(t') - \frac{b^2(t')}{2}\,\mathrm{d}t'}$$

the following properties hold true.

(i) *Let* $\phi(t) := \int_0^t b^2(t')\,\mathrm{d}t'$ *with inverse* $\phi^{-1}(t)$*. If*

$$\limsup_{t\to\infty} \frac{1}{\sqrt{2t\log\log t}}\left(\int_0^{\phi^{-1}(t)} a(t')\,\mathrm{d}t' - \frac{t}{2}\right) < -1,$$

then

$$\lim_{t\to\infty} X(t) = 0 \quad \mathbb{P}-a.s.$$

If

$$\liminf_{t\to\infty} \frac{1}{\sqrt{2t\log\log t}}\left(\int_0^{\phi^{-1}(t)} a(t')\,\mathrm{d}t' - \frac{t}{2}\right) > 1,$$

then

$$\lim_{t\to\infty} X(t) = \infty \quad \mathbb{P}-a.s.$$

(ii) *Let* $a(t) = \lambda b^2(t)$ *with* $\lambda < \frac{1}{2}$ *and* $\tau_R = \inf\{t \geq 0 : X(t) > R\}$ *with* $R > 1$*, then*

$$\mathbb{P}(\tau_R = \infty) \geq 1 - \left(\frac{1}{R}\right)^{1-2\lambda}.$$

4. Proof of Theorem 2.1

We consider the initial value problem (1.8). The proof of existence and uniqueness of pathwise solutions can be carried out by standard procedures used in many works, see [2,3,28,29,47–49] for more details. Therefore we only give a sketch.

(1) Firstly, one constructs a suitable approximation scheme using a cut-off function to control the $W^{1,\infty}$-norm (arising from $(u-\gamma)u_x$, (2.2) and Lemma 3.4). With such cut-off, both the drift and diffusion coefficients in the problem become locally Lipschitz continuous and grow linearly in u (cf. [47–49]). Thus the approximation solutions exist globally. Besides, such cut-off enables us to close the a priori $L^2(\Omega; H^s)$ estimate by splitting $\mathbb{E}(\|u\|^2_{H^s}\|u\|_{W^{1,\infty}})$. Therefore by using Lemma 3.2 and (2.2), uniform estimates for the approximation solutions can be established. We refer the readers to [48,49] for some closely related models;
(2) Secondly, by the uniform estimates, one obtains the tightness of the distributions of the approximation solution in $\mathscr{P}\left(C([0,T]; H^{s-1})\right)$, where $\mathscr{P}\left(C([0,T]; H^{s-1})\right)$ is the collection of Borel probability measures on $C([0,T]; H^{s-1})$. We refer to [45,48,49] for example. Applying the probabilistic compactness arguments, i.e., the Prokhorov theorem and the Skorokhod theorem, and using some technical convergence results as in [1–3,19], one verifies the existence of a martingale solution in H^s with $s > 3$. In this step $s > 3$ is an intermediate requirement because the convergence is in H^{s-1} and we need to control the $W^{1,\infty}$-norm by the embedding $H^{s-1} \hookrightarrow W^{1,\infty}$;

(3) Thirdly, by Lemma 3.4 and (2.3), one can show that pathwise uniqueness holds. Then the Gyöngy–Krylov characterization of the convergence in probability (see [30]) can be applied to show the existence and uniqueness of a pathwise solution in H^s with $s > 3$, cf. [3,29,49];

(4) Finally, mollifying initial data, analyzing the convergence and employing the argument as in [28,29,48,49] lead to a local pathwise solution (u,τ) to (1.6) with $u(\cdot\wedge\tau) \in L^2(\Omega; C([0,\infty); H^s))$ for $u_0 \in L^2(\Omega; H^s)$ with $s > 3/2$.

To finish the proof of Theorem 2.1, we only need to verify the blow-up criterion (2.5). Motivated by [16,47], we first consider, in next lemma, the relationship between the explosion time of $\|u(t)\|_{H^s}$ and the explosion time of $\|u(t)\|_{W^{1,\infty}}$ for (1.8). The results of the lemma will not only immediately imply the blow-up criterion (2.5) but also be used in the next sections.

Lemma 4.1. *Let (u,τ^*) be the unique maximal solution to (1.8). Then the real-valued stochastic process $\|u\|_{W^{1,\infty}}$ is also $\mathcal{F}_t$-adapted. Besides, for any $m,n \in \mathbb{N}$, define*

$$\tau_{1,m} = \inf\left\{t \geq 0 : \|u(t)\|_{H^s} \geq m\right\}, \quad \tau_{2,n} = \inf\left\{t \geq 0 : \|u(t)\|_{W^{1,\infty}} \geq n\right\}.$$

For $\tau_1 = \lim_{m\to\infty} \tau_{1,m}$ and $\tau_2 = \lim_{n\to\infty} \tau_{2,n}$, we have then

$$\tau_1 = \tau_2 \quad \mathbb{P} - a.s.$$

Proof. To begin with, since $u(\cdot\wedge\tau) \in C([0,\infty); H^s)$ almost surely, we see that for any $t \in [0,\tau]$,

$$[u(t)]^{-1}(Y) = [u(t)]^{-1}(H^s \cap Y), \quad \forall\, Y \in \mathcal{B}(W^{1,\infty}).$$

Therefore $u(t)$, as a $W^{1,\infty}$-valued process, is also $\mathcal{F}_t$-adapted. Moreover, the embedding $H^s \hookrightarrow W^{1,\infty}$ for $s > 3/2$ means that there is a $K = K(s) > 0$ such that $\|\cdot\|_{W^{1,\infty}} < K\|\cdot\|_{H^s}$. Then for every $m \in \mathbb{N}$,

$$\sup_{t\in[0,\tau_{1,m}]} \|u(t)\|_{W^{1,\infty}} \leq K \sup_{t\in[0,\tau_{1,m}]} \|u(t)\|_{H^s} \leq ([K]+1)m,$$

where $[K]$ means the integer part of K. Consequently, $\tau_{1,m} \leq \tau_{2,([K]+1)m} \leq \tau_2$ almost surely, which means that $\tau_1 \leq \tau_2$ $\mathbb{P} - a.s.$ Now we only need to prove the contrary inequality. Let $n,k \in \mathbb{N}$, one has

$$\left\{\sup_{t\in[0,\tau_{2,n}\wedge k]} \|u(t)\|_{H^s} < \infty\right\} = \bigcup_{m\in\mathbb{N}} \left\{\sup_{t\in[0,\tau_{2,n}\wedge k]} \|u(t)\|_{H^s} < m\right\}$$
$$\subset \bigcup_{m\in\mathbb{N}} \left\{\tau_{2,n}\wedge k \leq \tau_{1,m}\right\}.$$

Notice that

$$\bigcup_{m\in\mathbb{N}} \left\{\tau_{2,n}\wedge k \leq \tau_{1,m}\right\} \subset \left\{\tau_{2,n}\wedge k \leq \tau_1\right\}.$$

If $\mathbb{P}\{\tau_{2,n}\wedge k\le\tau_1\}=1$ for all $n,k\in\mathbb{N}$, then we have

$$\mathbb{P}\{\tau_2\le\tau_1\}=\mathbb{P}\left\{\bigcap_{n\in\mathbb{N}}\{\tau_{2,n}\le\tau_1\}\right\}=\mathbb{P}\left\{\bigcap_{n,k\in\mathbb{N}}\{\tau_{2,n}\wedge k\le\tau_1\}\right\}=1. \quad (4.1)$$

To this end, we only need to prove

$$\mathbb{P}\left\{\sup_{t\in[0,\tau_{2,n}\wedge k]}\|u(t)\|_{H^s}<\infty\right\}=1,\quad \forall\, n,k\in\mathbb{N}. \quad (4.2)$$

Consider first $\mathbb{E}\sup_{t\in[0,\tau_{2,n}\wedge k]}\|u(t)\|^2_{H^s}$. We cannot estimate this expectation using the Itô formula directly. Indeed, the Itô formula in a Hilbert space ([18, Theorem 4.32] or [26, Theorem 2.10]) requires $((u-\gamma)\,u_x,u)_{H^s}$ to be well-defined and the Itô formula under a Gelfand triplet ([38, Theorem I.3.1] or [44, Theorem 4.2.5]) requires the dual product ${}_{H^{s-1}}\langle(u-\gamma)\,u_x,u\rangle_{H^{s+1}}$ to be well-defined. In our case we only have $u\in H^s$ and $(u-\gamma)\,u_x\in H^{s-1}$ such that neither requirement is fulfilled. Therefore we utilize the mollifier operator T_ε defined in (3.1). We first apply T_ε to (1.8), and then use the Itô formula for $\|T_\varepsilon u\|^2_{H^s}=\|D^sT_\varepsilon u\|^2_{L^2}$ to deduce that for any $n,k>1$ and $t\in[0,\tau_{2,n}\wedge k]$,

$$\begin{aligned}\|T_\varepsilon u(t)\|^2_{H^s}-\|T_\varepsilon u(0)\|^2_{H^s}&=2\sum_{k=1}^{\infty}\int_0^t\left(D^sT_\varepsilon h(t',u)e_k,D^sT_\varepsilon u\right)_{L^2}\mathrm{d}W_k\\&\quad-2\int_0^t\left(D^sT_\varepsilon\left[(u-\gamma)\partial_x u\right],D^sT_\varepsilon u\right)_{L^2}\,\mathrm{d}t'\\&\quad-2\int_0^t\left(D^sT_\varepsilon F(u),D^sT_\varepsilon u\right)_{L^2}\,\mathrm{d}t'\\&\quad+\int_0^t\sum_{k=1}^{\infty}\|D^sT_\varepsilon h(t',u)e_k\|^2_{L^2}\,\mathrm{d}t'\\&=:\sum_{k=1}^{\infty}\int_0^t L_{1,k}\,\mathrm{d}W_k+\sum_{i=2}^{4}\int_0^t L_i\,\mathrm{d}t'.\end{aligned}$$

On account of the Burkholder-Davis-Gundy inequality (2.1), for the expectation of the H^s-norm of $T_\varepsilon u$, we arrive at

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,\tau_{2,n}\wedge k]}\|T_\varepsilon u(t)\|^2_{H^s}&\le\mathbb{E}\|T_\varepsilon u_0\|^2_{H^s}+C\mathbb{E}\left(\int_0^{\tau_{2,n}\wedge k}\sum_{k=1}^{\infty}|L_{1,k}|^2\mathrm{d}t\right)^{\frac{1}{2}}\\&\quad+\sum_{i=2}^{4}\mathbb{E}\int_0^{\tau_{2,n}\wedge k}|L_i|\,\mathrm{d}t.\end{aligned}$$

We can infer from (3.4) and Assumption 2.1 that

$$\mathbb{E}\left(\int_0^{\tau_{2,n}\wedge k}\sum_{k=1}^{\infty}|L_{1,k}|^2\mathrm{d}t\right)^{\frac{1}{2}}$$
$$\leq \frac{1}{2}\mathbb{E}\sup_{t\in[0,\tau_{2,n}\wedge k]}\|T_\varepsilon u\|^2_{H^s}+Cf^2(n)\int_0^k\left(1+\mathbb{E}\|u\|^2_{H^s}\right)\,\mathrm{d}t.$$

For L_2 and L_3, we use Lemma 3.5 to find

$$\mathbb{E}\int_0^{\tau_{2,n}\wedge k}|L_2|+|L_3|\,\mathrm{d}t\leq C(1+n)\int_0^k\left(1+\mathbb{E}\|u\|^2_{H^s}\right)\mathrm{d}t.$$

Similarly, it follows from the assumption (2.2) that

$$\mathbb{E}\int_0^{\tau_{2,n}\wedge k}|L_4|\,\mathrm{d}t\leq Cf^2(n)\int_0^k\left(1+\mathbb{E}\|u\|^2_{H^s}\right)\mathrm{d}t.$$

If we combine the above estimates and use (3.4), we are led for some constant $C=C_n>0$ depending on n to

$$\mathbb{E}\sup_{t\in[0,\tau_{2,n}\wedge k]}\|T_\varepsilon u(t)\|^2_{H^s}\leq 2\mathbb{E}\|u_0\|^2_{H^s}+C_n\int_0^k\left(1+\mathbb{E}\sup_{t'\in[0,t\wedge\tau_{2,n}]}\|u(t')\|^2_{H^s}\right)\mathrm{d}t.$$

Since the right hand side of the last estimate does not depend on ε, and $T_\varepsilon u$ tends to u in $C\left([0,T];H^s\right)$ for any $T>0$ almost surely as $\varepsilon\to 0$, one can send $\varepsilon\to 0$ to obtain

$$\mathbb{E}\sup_{t\in[0,\tau_{2,n}\wedge k]}\|u(t)\|^2_{H^s}\leq 2\mathbb{E}\|u_0\|^2_{H^s}+C_n\int_0^k\left(1+\mathbb{E}\sup_{t'\in[0,t\wedge\tau_{2,n}]}\|u(t')\|^2_{H^s}\right)\mathrm{d}t.$$

Then Grönwall's inequality shows that for each $n,k\in\mathbb{N}$, there is a constant $C=C(n,k,u_0)>0$ such that

$$\mathbb{E}\sup_{t\in[0,\tau_{2,n}\wedge k]}\|u(t)\|^2_{H^s}<C(n,k,u_0),$$

which gives (4.2). □

We finish the section with the proof of the blow-up criterion in Theorem (2.1).

Proof of (2.5). Let $\tau_{1,m}$, $\tau_{2,n}$, τ_1 and τ_2 be given in Lemma 4.1. If u is the unique pathwise solution with maximal existence time τ^*, for fixed $m,n>0$, even if $\mathbb{P}\{\tau_{1,m}=0\}$ or $\mathbb{P}\{\tau_{2,n}=0\}$ is larger than 0, for a.e. $\omega\in\Omega$, there is $m>0$ or $n>0$ such that $\tau_{1,m},\tau_{2,n}>0$. By continuity of $\|u(t)\|_{H^s}$ and the uniqueness of u, it is easy to check that $\tau_1=\tau_2=\tau^*$. Consequently, we obtain the desired blow-up criterion.

5. Proof of Theorem 2.2: strong nonlinear noise

To begin with, we note the following algebraic inequality.

Lemma 5.1. *Let* $c, M > 0$. *Assume*

$$\textit{either} \quad \eta > 1,\ a, b > 0 \quad \textit{or} \quad \eta = 1,\ b > a > 0.$$

There is a $C > 0$ *such that for all* $0 \le x \le My < \infty$,

$$\frac{a(1+x)y^2 + b(1+x)^\eta y^2}{1+y^2} - \frac{2b(1+x)^\eta y^4}{(1+y^2)^2} + \frac{c(1+x)^\eta y^4}{(1+y^2)^2(1+\log(1+y^2))} \le C.$$

Proof. Since $My \ge x$, we have

$$\frac{a(1+x)y^2 + b(1+x)^\eta y^2}{1+y^2} - \frac{2b(1+x)^\eta y^4}{(1+y^2)^2} + \frac{c(1+x)^\eta y^4}{(1+y^2)^2(1+\log(1+y^2))}$$

$$\le a(1+x) + b(1+x)^\eta - 2b(1+x)^\eta \frac{\left(\frac{x}{M}\right)^4}{\left(1+\left(\frac{x}{M}\right)^2\right)^2} + \frac{c(1+x)^\eta}{\left(1+\log\left(1+\left(\frac{x}{M}\right)^2\right)\right)}.$$

When $\eta > 1$ and $a, b > 0$ or $\eta = 1$ and $b > a > 0$, the latter expression tends to $-\infty$ for $x \to +\infty$, which implies the statement of the lemma. □

We are now ready to prove Theorem 2.2 following [45] to large extent.

Proof of Theorem 2.2. Assume $s > 5/2$ and let u_0 be an H^s-valued $\mathcal{F}_0$-measurable random variable with $\mathbb{E}\|u_0\|^2_{H^s} < \infty$. Let $h(t,u) = h(u) = a\left(1 + \|u\|_{W^{1,\infty}}\right)^\theta u$ with $\theta \ge 1/2$ and $a \ne 0$.

For $r > 3/2$, the embedding $H^r \hookrightarrow W^{1,\infty}$ implies that we have for any $u, v \in H^r$ the estimate

$$\sup_{\|u\|_{H^r}, \|v\|_{H^r} \le N} \left\{ \mathbf{1}_{\{u \ne v\}} \frac{\|h(u) - h(v)\|_{H^r}}{\|u - v\|_{H^r}} \right\} \le q(N), \quad N \ge 1.$$

This means that one can establish the pathwise uniqueness for (1.10) in H^r with $r > 3/2$. Hence, in the same way as proving Theorem 2.1, one can show that (1.10) admits a unique pathwise solution u in H^s with $s > 5/2$ and maximal existence time τ^*. We recall the definition of the mollifier T_ε from Sect. 3 and define

$$\tau_m = \inf\left\{t \ge 0 : \|u(t)\|_{H^s} \ge m\right\}.$$

Applying the Itô formula to $\|T_\varepsilon u(t)\|^2_{H^s}$ gives

$$\mathrm{d}\|T_\varepsilon u\|^2_{H^s} = 2a\left(1 + \|u\|_{W^{1,\infty}}\right)^\theta \left(T_\varepsilon u, T_\varepsilon u\right)_{H^s} \mathrm{d}W - 2\left(T_\varepsilon\left[(u-\gamma)u_x\right], T_\varepsilon u\right)_{H^s} \mathrm{d}t$$
$$- 2\left(T_\varepsilon F(u), T_\varepsilon u\right)_{H^s} \mathrm{d}t + a^2\left(1 + \|u\|_{W^{1,\infty}}\right)^{2\theta} \|T_\varepsilon u\|^2_{H^s} \mathrm{d}t.$$

Again, using Itô formula to $\log(1 + \|T_\varepsilon u\|^2_{H^s})$ yields

$$\mathrm{d}\log(1 + \|T_\varepsilon u\|^2_{H^s})$$
$$= \frac{2a\left(1 + \|u\|_{W^{1,\infty}}\right)^\theta}{1 + \|T_\varepsilon u\|^2_{H^s}} \left(T_\varepsilon u, T_\varepsilon u\right)_{H^s} \mathrm{d}W$$
$$- \frac{1}{1 + \|T_\varepsilon u\|^2_{H^s}} \left\{2\left(T_\varepsilon\left[(u-\gamma)u_x\right], T_\varepsilon u\right)_{H^s} + 2\left(T_\varepsilon F(u), T_\varepsilon u\right)_{H^s}\right\} \mathrm{d}t$$
$$+ \frac{a^2\left(1 + \|u\|_{W^{1,\infty}}\right)^{2\theta}}{1 + \|T_\varepsilon u\|^2_{H^s}} \|T_\varepsilon u\|^2_{H^s} \mathrm{d}t - 2\frac{a^2\left(1 + \|u\|_{W^{1,\infty}}\right)^{2\theta}}{(1 + \|T_\varepsilon u\|^2_{H^s})^2} \|T_\varepsilon u\|^4_{H^s} \mathrm{d}t.$$

Lemma 3.5 and (3.4) imply that there is a $Q = Q(s, c_0, \gamma) > 0$ such that for any $t > 0$ we have

$$
\begin{aligned}
&\mathbb{E}\log\left(1+\|T_\varepsilon u(t\wedge\tau_m)\|_{H^s}^2\right)-\mathbb{E}\log\left(1+\|T_\varepsilon u_0\|_{H^s}^2\right)\\
&\quad= \mathbb{E}\int_0^{t\wedge\tau_m}\frac{1}{1+\|T_\varepsilon u\|_{H^s}^2}\left\{-2\left(T_\varepsilon[(u-\gamma)u_x],T_\varepsilon u\right)_{H^s}\right.\\
&\qquad\left.-2\left(T_\varepsilon F(u),T_\varepsilon u\right)_{H^s}\right\}\mathrm{d}t'\\
&\qquad+\mathbb{E}\int_0^{t\wedge\tau_m}\frac{1}{1+\|T_\varepsilon u\|_{H^s}^2}a^2\left(1+\|u\|_{W^{1,\infty}}\right)^{2\theta}\|T_\varepsilon u\|_{H^s}^2\,\mathrm{d}t'\\
&\qquad-\mathbb{E}\int_0^{t\wedge\tau_m}\frac{2}{(1+\|T_\varepsilon u\|_{H^s}^2)^2}a^2\left(1+\|u\|_{W^{1,\infty}}\right)^{2\theta}\|T_\varepsilon u\|_{H^s}^4\,\mathrm{d}t'\\
&\quad\le \mathbb{E}\int_0^{t\wedge\tau_m}\left[\frac{1}{1+\|T_\varepsilon u\|_{H^s}^2}\left\{2Q\left(1+\|u\|_{W^{1,\infty}}\right)\|u\|_{H^s}^2+a^2\left(1+\|u\|_{W^{1,\infty}}\right)^{2\theta}\right.\right.\\
&\qquad\left.\left.\|T_\varepsilon u\|_{H^s}^2\right\}\right]\mathrm{d}t'\\
&\quad-\mathbb{E}\int_0^{t\wedge\tau_m}\frac{1}{(1+\|T_\varepsilon u\|_{H^s}^2)^2}2a^2\left(1+\|u\|_{W^{1,\infty}}\right)^{2\theta}\|T_\varepsilon u\|_{H^s}^4\,\mathrm{d}t'.
\end{aligned}
$$

Notice that for any $T > 0$, $T_\varepsilon u$ tends to u in $C([0,T];H^s)$ almost surely as $\varepsilon \to 0$. Then, by (3.4) and the dominated convergence theorem, the last estimate leads to

$$
\begin{aligned}
&\mathbb{E}\log\left(1+\|u(t\wedge\tau_m)\|_{H^s}^2\right)-\mathbb{E}\log(1+\|u_0\|_{H^s}^2)\\
&\quad= \lim_{\varepsilon\to0}\left(\mathbb{E}\log\left(\left(1+\|T_\varepsilon u(t\wedge\tau_m)\|_{H^s}^2\right)\right)-\mathbb{E}\log\left(\left(1+\|T_\varepsilon u_0\|_{H^s}^2\right)\right)\right)\\
&\quad\le \lim_{\varepsilon\to0}\mathbb{E}\int_0^{t\wedge\tau_m}\frac{1}{1+\|T_\varepsilon u\|_{H^s}^2}\left\{2Q\left(1+\|u\|_{W^{1,\infty}}\right)\|u\|_{H^s}^2\right.\\
&\qquad\qquad\left.+a^2\left(1+\|u\|_{W^{1,\infty}}\right)^{2\theta}\|T_\varepsilon u\|_{H^s}^2\right\}\mathrm{d}t'\\
&\qquad-\lim_{\varepsilon\to0}\mathbb{E}\int_0^{t\wedge\tau_m}\frac{1}{(1+\|T_\varepsilon u\|_{H^s}^2)^2}2a^2\left(1+\|u\|_{W^{1,\infty}}\right)^{2\theta}\|T_\varepsilon u\|_{H^s}^4\,\mathrm{d}t'\\
&\quad= \mathbb{E}\int_0^{t\wedge\tau_m}\frac{2Q\left(1+\|u\|_{W^{1,\infty}}\right)\|u\|_{H^s}^2+a^2\left(1+\|u\|_{W^{1,\infty}}\right)^{2\theta}\|u\|_{H^s}^2}{1+\|u\|_{H^s}^2}\,\mathrm{d}t'\\
&\qquad\qquad-\mathbb{E}\int_0^{t\wedge\tau_m}\frac{2a^2\left(1+\|u\|_{W^{1,\infty}}\right)^{2\theta}\|u\|_{H^s}^4}{(1+\|u\|_{H^s}^2)^2}\,\mathrm{d}t'.
\end{aligned}
$$

Since we have assumed (2.6), Lemma 5.1 immediately shows that there are constants $K_1, K_2 > 0$ such that

$$
\begin{aligned}
&\mathbb{E}\log\left(1+\|u(t\wedge\tau_m)\|_{H^s}^2\right)-\mathbb{E}\log\left(1+\|u_0\|_{H^s}^2\right)\\
&\quad\le \mathbb{E}\int_0^{t\wedge\tau_m}K_1-K_2\frac{a^2\left(1+\|u\|_{W^{1,\infty}}\right)^{2\theta}\|u\|_{H^s}^4}{(1+\|u\|_{H^s}^2)^2\left(1+\log(1+\|u\|_{H^s}^2)\right)}\,\mathrm{d}t',
\end{aligned}
$$

which means that for some $C(u_0, K_1, K_2, t) > 0$,

$$\mathbb{E}\int_0^{t\wedge\tau_m} \frac{a^2\left(1+\|u\|_{W^{1,\infty}}\right)^{2\theta}\|u\|_{H^s}^4}{(1+\|u\|_{H^s}^2)^2\,(1+\log(1+\|u\|_{H^s}^2))}\,\mathrm{d}t' \leq C(u_0, K_1, K_2, t) < \infty, \tag{5.1}$$

and

$$\mathbb{E}\int_0^{t\wedge\tau_m} \left| K_1 - K_2 \frac{a^2\left(1+\|u\|_{W^{1,\infty}}\right)^{2\theta}\|u\|_{H^s}^4}{(1+\|u\|_{H^s}^2)^2\,(1+\log(1+\|u\|_{H^s}^2))}\right| \mathrm{d}t' \\ \leq C(u_0, K_1, K_2, t) < \infty. \tag{5.2}$$

Next, we notice that there is a function $\delta : [0,\infty) \to [0,\infty)$ with $\delta(\varepsilon) \to 0$ when $\varepsilon \to 0$ such that

$$\begin{aligned}
&\frac{2Q\left(1+\|u\|_{W^{1,\infty}}\right)\|u\|_{H^s}^2 + a^2\left(1+\|u\|_{W^{1,\infty}}\right)^{2\theta}\|T_\varepsilon u\|_{H^s}^2}{1+\|T_\varepsilon u\|_{H^s}^2} \\
&\quad - \frac{2a^2\left(1+\|u\|_{W^{1,\infty}}\right)^{2\theta}\|T_\varepsilon u\|_{H^s}^4}{(1+\|T_\varepsilon u\|_{H^s}^2)^2} \\
&\leq \frac{2Q\left(1+\|u\|_{W^{1,\infty}}\right)\|u\|_{H^s}^2 + a^2\left(1+\|u\|_{W^{1,\infty}}\right)^{2\theta}\|u\|_{H^s}^2}{1+\|u\|_{H^s}^2} \\
&\quad - \frac{2a^2\left(1+\|u\|_{W^{1,\infty}}\right)^{2\theta}\|u\|_{H^s}^4}{(1+\|u\|_{H^s}^2)^2} + \delta(\varepsilon)
\end{aligned}$$

holds. Therefore, for any $T > 0$, by using Lemma 5.1, the Burkholder-Davis-Gundy inequality (2.1) and (5.2), we find that

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T\wedge\tau_m]} \log\left(1+\|T_\varepsilon u\|_{H^s}^2\right) - \mathbb{E}\log\left(1+\|T_\varepsilon u_0\|_{H^s}^2\right) \\
&\leq C\mathbb{E}\left(\int_0^{T\wedge\tau_m} \frac{a^2\left(1+\|u\|_{W^{1,\infty}}\right)^{2\theta}\|T_\varepsilon u\|_{H^s}^4}{\left(1+\|T_\varepsilon u\|_{H^s}^2\right)^2}\,\mathrm{d}t\right)^{\frac{1}{2}} \\
&\quad + \mathbb{E}\int_0^{T\wedge\tau_m} \left| K_1 - K_2 \frac{a^2\left(1+\|u\|_{W^{1,\infty}}\right)^{2\theta}\|u\|_{H^s}^4}{(1+\|u\|_{H^s}^2)^2\,(1+\log(1+\|u\|_{H^s}^2))} + \delta(\varepsilon)\right| \mathrm{d}t \\
&\leq \frac{1}{2}\mathbb{E}\sup_{t\in[0,T\wedge\tau_m]}\left(1+\log(1+\|T_\varepsilon u\|_{H^s}^2)\right) \\
&\quad + C\mathbb{E}\int_0^{T\wedge\tau_m} \frac{a^2\left(1+\|u\|_{W^{1,\infty}}\right)^{2\theta}\|T_\varepsilon u\|_{H^s}^4}{\left(1+\|T_\varepsilon u\|_{H^s}^2\right)^2(1+\log(1+\|T_\varepsilon u\|_{H^s}^2))}\,\mathrm{d}t \\
&\quad + K_1 T + \mathbb{E}\int_0^{T\wedge\tau_m} K_2 \frac{a^2\left(1+\|u\|_{W^{1,\infty}}\right)^{2\theta}\|u\|_{H^s}^4}{(1+\|u\|_{H^s}^2)^2\,(1+\log(1+\|u\|_{H^s}^2))}\,\mathrm{d}t + \delta(\varepsilon)T \\
&\leq \frac{1}{2}\mathbb{E}\sup_{t\in[0,T\wedge\tau_m]}\left(1+\log(1+\|T_\varepsilon u\|_{H^s}^2)\right) \\
&\quad + C\mathbb{E}\int_0^{T\wedge\tau_m} \frac{a^2\left(1+\|u\|_{W^{1,\infty}}\right)^{2\theta}\|T_\varepsilon u\|_{H^s}^4}{\left(1+\|T_\varepsilon u\|_{H^s}^2\right)^2(1+\log(1+\|T_\varepsilon u\|_{H^s}^2))}\,\mathrm{d}t
\end{aligned}$$

$$+ C(u_0, K_1, K_2, T) + \delta(\varepsilon)T.$$

Thus, we use the dominated convergence theorem, Fatou's lemma and (5.1) to obtain finally

$$\mathbb{E} \sup_{t\in[0,T\wedge\tau_m]} \log\left(1+\|u\|_{H^s}^2\right) \le C(u_0, K_1, K_2, T).$$

Since $\log(1+x)$ is continuous and increasing for $x>0$, we have that for any $m \ge 1$,

$$\mathbb{P}\{\tau^* < T\} \le \mathbb{P}\{\tau_m < T\} \le \mathbb{P}\left\{ \sup_{t\in[0,T]} \log(1+\|u\|_{H^s}^2) \ge \log(1+m^2)\right\} \le \frac{C(u_0, K_1, K_2, T)}{\log(1+m^2)}.$$

Letting $m\to\infty$ forces $\mathbb{P}\{\tau^* < T\} = 0$ for any $T>0$, which means $\mathbb{P}\{\tau^* = \infty\} = 1$. □

6. Proofs of Theorems 2.3–2.7: non-autonomous linear noise case

In this section, we study (1.12) with linear noise. Depending on the strength of the noise in (1.12), we provide either the global existence of pathewise solutions or the precise blow-up scenarios for the maximal pathwise solution. As discussed in Remark 2.6, we rely on the Girsanov-type transform

$$v = \frac{1}{\beta(\omega,t)}u, \quad \beta(\omega,t) = \mathrm{e}^{\int_0^t b(t')\mathrm{d}W_{t'} - \int_0^t \frac{b^2(t')}{2}\mathrm{d}t'}. \tag{6.1}$$

We first collect some properties of v.

Proposition 6.1. *Let $s>3/2$, $\alpha=1$ and $h(t,u)=b(t)u$ such that $b(t)$ satisfies Assumption 2.2. Let $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\ge0}, W)$ be fixed in advance. If $u_0(\omega,x)$ is an H^s-valued $\mathcal{F}_0$ measurable random variable with $\mathbb{E}\|u_0\|_{H^s}^2 < \infty$ and (u,τ^*) is the corresponding unique maximal solution to (1.11), then for any $c_0, \gamma \in \mathbb{R}$ and for $t\in[0,\tau^*)$, the process v defined by (6.1) solves the following problem on $\mathbb{T}$ almost surely,*

$$\begin{cases} v_t + \beta v v_x - \gamma v_x + \beta(1-\partial_{xx}^2)^{-1}\partial_x\left(v^2 + \frac{1}{2}v_x^2\right) + (c_0+\gamma)(1-\partial_{xx}^2)^{-1}\partial_x v = 0, \\ v(\omega,0,x) = u_0(\omega,x). \end{cases} \tag{6.2}$$

Moreover, we have $v\in C\left([0,\tau^*);H^s\right)\cap C^1([0,\tau^*);H^{s-1})$ $\mathbb{P}-a.s.$ and, if $s>3$, then it holds

$$\mathbb{P}\big\{\|v(t)\|_{H^1} = \|u_0\|_{H^1} \quad \textit{for all}\, t\ge 0\big\} = 1. \tag{6.3}$$

Proof. Since $b(t)$ satisfies Assumption 2.2, $h(t,u)=b(t)u$ satisfies Assumption 2.1. Consequently, Theorem 2.1 implies that (1.11) (that is (1.8) with $h(t,u)=b(t)u$) has a unique maximal solution (u,τ^*).

A direct computation with the Itô formula yields

$$\mathrm{d}\frac{1}{\beta}=-b(t)\frac{1}{\beta}\mathrm{d}W+b^2(t)\frac{1}{\beta}\mathrm{d}t.$$

Therefore we arrive at

$$\begin{aligned}
\mathrm{d}v=&\ \frac{1}{\beta}\left[-\left[(u-\gamma)\,\partial_x u+F(u)\right]\mathrm{d}t+b(t)u\,\mathrm{d}W\right]\\
&\quad+u\left[-b(t)\frac{1}{\beta}\mathrm{d}W+b^2(t)\frac{1}{\beta}\mathrm{d}t\right]-b^2(t)\frac{1}{\beta}u\,\mathrm{d}t\\
=&\ \frac{1}{\beta}\left[-\left((u-\gamma)\,\partial_x u+F(u)\right)\mathrm{d}t\right]\\
=&\ \left\{-\beta vv_x+\gamma v_x-\beta(1-\partial_{xx}^2)^{-1}\partial_x\left(v^2+\frac{1}{2}v_x^2\right)-(c_0+\gamma)(1-\partial_{xx}^2)^{-1}v_x\right\}\mathrm{d}t,
\end{aligned}\tag{6.4}$$

which is $(6.2)_1$. Since $v(0)=u_0(\omega,x)$, we see that v satisfies (6.2). Moreover, Theorem 2.1 implies $u\in C\left([0,\tau^*);H^s\right)\ \mathbb{P}-a.s.$, so is v. Besides, from Lemma 3.4 and $(6.2)_1$, we see that for a.e. $\omega\in\Omega$, $v_t=\gamma v_x-\beta vv_x-\beta(F_1(v)+F_2(v))-F_3(v)\in C([0,\tau^*);H^{s-1})$. Hence $v\in C^1\left([0,\tau^*);H^{s-1}\right)\ \mathbb{P}-a.s.$

Notice that if $s>3$, $(6.2)_1$ is equivalent to

$$v_t-v_{xxt}+c_0v_x+\gamma v_{xxx}+3\beta vv_x=2\beta v_xv_{xx}+\beta vv_{xxx}. \tag{6.5}$$

Multiplying both sides of (6.5) by v and then integrating the resulting equation on $x\in\mathbb{T}$, we see that for a.e. $\omega\in\Omega$ and for all $t>0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}}\left(v^2+v_x^2\right)\,\mathrm{d}x=0,$$

which implies (6.3). □

6.1. Theorem 2.3: global existence for weak noise I

Now we prove the first global existence result, which is motivated by [29,46–48].

Proof of Theorem 2.3. To begin with, we apply the operator D^s to (6.4), multiply both sides of the resulting equation by D^sv and integrate over $\mathbb{T}$ to obtain for a.e. $\omega\in\Omega$

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_{H^s}^2=&\ \gamma\int_{\mathbb{T}}D^sv\cdot D^sv_x\,\mathrm{d}x-\beta(\omega,t)\int_{\mathbb{T}}D^sv\cdot D^s\left[vv_x\right]\,\mathrm{d}x\\
&-\beta(\omega,t)\int_{\mathbb{T}}D^sv\cdot D^sF(v)\,\mathrm{d}x\\
=&\ -\beta(\omega,t)\int_{\mathbb{T}}D^sv\cdot D^s\left[vv_x\right]\,\mathrm{d}x\\
&-\beta(\omega,t)\int_{\mathbb{T}}D^sv\cdot D^sF(v)\,\mathrm{d}x.
\end{aligned}$$

Using Lemma 3.2, integration by parts and Lemma 3.4, we conclude that there is a $C = C(s) > 1$ such that for a.e. $\omega \in \Omega$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_{H^s}^2 \le C\beta(t)\ \|v\|_{W^{1,\infty}}\|v\|_{H^s}^2,$$

where β is given in (6.1) (If necessary, T_ε can be used as in Lemma 4.1). Then $w = \mathrm{e}^{-\int_0^t b(t')\mathrm{d}W_{t'}} u = \mathrm{e}^{-\int_0^t \frac{b^2(t')}{2}\mathrm{d}t'} v$ satisfies

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|_{H^s} + \frac{b^2(t)}{2}\|w(t)\|_{H^s}\\ &\quad \le C\alpha(\omega,t)\|w(t)\|_{W^{1,\infty}}\|w(t)\|_{H^s}, \quad \alpha(\omega,t) = \mathrm{e}^{\int_0^t b(t')\mathrm{d}W_{t'}}.\end{aligned}$$

Let $R > 1$ and $\lambda_1 > 2$. Assume $\|u_0\|_{H^s} < \frac{b_*}{CK\lambda_1 R} < \frac{b_*}{CK\lambda_1}$ almost surely and define

$$\tau_1 = \inf\left\{t > 0 : \alpha(\omega,t)\|w\|_{W^{1,\infty}} = \|u\|_{W^{1,\infty}} > \frac{b^2(t)}{C\lambda_1}\right\}. \tag{6.6}$$

Then it follows from the embedding $\|u(0)\|_{W^{1,\infty}} \le K\|u(0)\|_{H^s} < \frac{b_*}{C\lambda_1}$ that $\mathbb{P}\{\tau_1 > 0\} = 1$, and for $t \in [0,\tau_1)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|_{H^s} + \frac{(\lambda_1-2)b^2(t)}{2\lambda_1}\|w(t)\|_{H^s} \le 0.$$

The above inequality and $w = \mathrm{e}^{-\int_0^t b(t')\mathrm{d}W_{t'}} u$ imply that for a.e. $\omega \in \Omega$, for any $\lambda_2 > \frac{2\lambda_1}{\lambda_1-2}$ and for $t \in [0,\tau_1)$,

$$\begin{aligned}\|u(t)\|_{H^s} &\le \|w_0\|_{H^s}\mathrm{e}^{\int_0^t b(t')\mathrm{d}W_{t'} - \int_0^t \frac{(\lambda_1-2)b^2(t')}{2\lambda_1}\,\mathrm{d}t'}\\ &= \|u_0\|_{H^s}\mathrm{e}^{\int_0^t b(t')\,\mathrm{d}W_{t'} - \int_0^t \frac{b^2(t')}{\lambda_2}\mathrm{d}t'}\mathrm{e}^{-\frac{((\lambda_1-2)\lambda_2-2\lambda_1)}{2\lambda_1\lambda_2}\int_0^t b^2(t')\,\mathrm{d}t'}.\end{aligned} \tag{6.7}$$

Define the stopping time

$$\tau_2 = \inf\left\{t > 0 : \mathrm{e}^{\int_0^t b(t')\mathrm{d}W_{t'} - \int_0^t \frac{b^2(t')}{\lambda_2}\mathrm{d}t'} > R\right\}. \tag{6.8}$$

Notice that $\mathbb{P}\{\tau_2 > 0\} = 1$. From (6.7), we have that almost surely

$$\begin{aligned}\|u(t)\|_{H^s} &< \frac{b_*}{CK\lambda_1 R} \times R \times \mathrm{e}^{-\frac{((\lambda_1-2)\lambda_2-2\lambda_1)}{2\lambda_1\lambda_2}\int_0^t b^2(t')\,\mathrm{d}t'}\\ &= \frac{b_*}{CK\lambda_1}\mathrm{e}^{-\frac{((\lambda_1-2)\lambda_2-2\lambda_1)}{2\lambda_1\lambda_2}\int_0^t b^2(t')\,\mathrm{d}t'} \le \frac{b_*}{CK\lambda_1}, \quad t \in [0,\tau_1\wedge\tau_2).\end{aligned} \tag{6.9}$$

By Assumption 2.2, (6.9) and (6.6), we find that on $[0,\tau_1\wedge\tau_2)$,

$$\|u(t)\|_{W^{1,\infty}} \le K\|u(t)\|_{H^s} \le \frac{b_*}{C\lambda_1} \le \frac{b^2(t)}{C\lambda_1}\ \ \mathbb{P}-a.s.,$$

which means

$$\mathbb{P}\{\tau_1 \ge \tau_2\} = 1. \tag{6.10}$$

Therefore it follows from (6.9) that

$$\mathbb{P}\left\{\|u(t)\|_{H^s} < \frac{b_*}{CK\lambda_1}\mathrm{e}^{-\frac{((\lambda_1-2)\lambda_2-2\lambda_1)}{2\lambda_1\lambda_2}\int_0^t b^2(t')\,\mathrm{d}t'}\ \text{for all } t > 0\right\} \ge \mathbb{P}\{\tau_2 = \infty\}.$$

We apply (ii) in Lemma 3.7 to find that

$$\mathbb{P}\{\tau_2 = \infty\} > 1 - \left(\frac{1}{R}\right)^{2/\lambda_2},$$

which completes the proof. □

6.2. Theorem 2.4: global existence for weak noise II

Let $\beta(\omega, t)$ be given as in (6.1). With Proposition 6.1 at hand, we can proceed to prove Theorem 2.4. We see that for a.e. $\omega \in \Omega$, the transform $v(\omega, t, x)$ solves (6.2) on $[0, \tau^*)$. Moreover, since $H^s \hookrightarrow C^2$ for $s > 3$, we have $v, v_x \in C^1([0, \tau^*) \times \mathbb{T})$. Then for a.e. $\omega \in \Omega$, for any $x \in \mathbb{T}$ and $c_0, \gamma \in \mathbb{R}$, the problem

$$\begin{cases} \dfrac{\mathrm{d}q(\omega,t,x)}{\mathrm{d}t} = \beta(\omega,t)v(\omega,t,q(\omega,t,x)) - \gamma, & t \in [0,\tau^*), \\ q(\omega,0,x) = x, \quad x \in \mathbb{T}, \end{cases} \tag{6.11}$$

has a unique solution $q(\omega, t, x)$ such that $q(\omega, t, x) \in C^1([0, \tau^*) \times \mathbb{T})$ for a.e $\omega \in \Omega$. Moreover, differentiating (6.11) with respect to x yields that for a.e. $\omega \in \Omega$,

$$\begin{cases} \dfrac{dq_x(\omega,t,x)}{dt} = \beta(\omega,t)v_x(\omega,t,q)q_x, & t \in [0,\tau^*), \\ q_x(\omega,0,x) = 1, \quad x \in \mathbb{T}. \end{cases}$$

For a.e. $\omega \in \Omega$, we solve the above equation to obtain

$$q_x(\omega,t,x) = \exp\left(\int_0^t \beta(\omega,t')v_x(\omega,t',q(\omega,t',x))\,\mathrm{d}t'\right).$$

Thus for a.e. $\omega \in \Omega$, $q_x > 0$, $(t, x) \in [0, \tau^*) \times \mathbb{T}$. On the other hand, if v solves (6.2) (or equivalently (6.5)) $\mathbb{P}-a.s.$, then the momentum variable $V = v - v_{xx}$ satisfies

$$V_t + c_0 v_x + \beta v V_x + 2\beta V v_x + \gamma v_{xxx} = 0 \ \ \mathbb{P}-a.s. \tag{6.12}$$

Particularly, if $c_0 + \gamma = 0$, (6.12) becomes

$$V_t + c_0 v_x + \beta v V_x + 2\beta V v_x + \gamma v_{xxx} = V_t - \gamma V_x + \beta v V_x + 2\beta V v_x = 0 \ \ \mathbb{P}-a.s.,$$

which means

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[V(\omega,t,q(\omega,t,x))q_x^2(\omega,t,x)\right] = q_x^2\left[V_t + \beta v V_x - \gamma V_x + 2\beta V v_x\right] = 0 \ \ \mathbb{P}-a.s.$$

This, and $q_x(\omega, 0, x) = 1$ imply that

$$V(\omega,t,q(\omega,t,x))q_x^2(\omega,t,x) = V_0(\omega,x).$$

Consequently, we have $\mathrm{sign}(V) = \mathrm{sign}(V_0)$. Besides, since $v = G_{\mathbb{T}} * V$ with $G_{\mathbb{T}} > 0$ given in (1.7), we have $\mathrm{sign}(v) = \mathrm{sign}(V)$. Summarizing the above analysis, we have the following result:

Lemma 6.1. *Assume $c_0 + \gamma = 0$ and $s > 3$. Let $V_0(\omega, x) = (1 - \partial_{xx}^2)u_0(\omega, x)$ and $V(\omega, t, x) = v(\omega, t, x) - v_{xx}(\omega, t, x)$, where $v(\omega, t, x)$ solves (6.2) on $[0, \tau^*)$ $\mathbb{P}-a.s.$ Then for a.e. $\omega \in \Omega$,*

$$\mathrm{sign}(v) = \mathrm{sign}(V) = \mathrm{sign}(V_0), \quad (t,x) \in [0,\tau^*) \times \mathbb{T}.$$

The next step is to control $\|u(\omega,t)\|_{W^{1,\infty}}$. In combination with (2.5), we will then directly verify Theorem 2.4.

Lemma 6.2. *Let all the conditions as in the statement of Proposition 6.1 hold true. Let V and V_0 be defined in Lemma 6.1. If additionally we have $c_0+\gamma=0$ and*

$$\mathbb{P}\{V_0(\omega,x)>0,\quad \forall\, x\in\mathbb{T}\}=p,\quad \mathbb{P}\{V_0(\omega,x)<0,\quad \forall\, x\in\mathbb{T}\}=q,$$

for some $p,q\in[0,1]$, then the maximal solution (u,τ^*) of (1.12) satisfies

$$\mathbb{P}\Big\{\|u_x(\omega,t)\|_{L^\infty}\le\|u(\omega,t)\|_{L^\infty}\lesssim\beta(\omega,t)\|u_0\|_{H^1},\quad \forall\, t\in[0,\tau^*)\Big\}\ge p+q.$$

Proof. Using (1.7), one can derive (see [48]) that for a.e. $\omega\in\Omega$, and for all $(t,x)\in[0,\tau^*)\times\mathbb{T}$,

$$[v+v_x](\omega,t,x)=\frac{1}{2\sinh(\pi)}\int_0^{2\pi}\mathrm{e}^{(x-y-2\pi[\frac{x-y}{2\pi}]-\pi)}V(\omega,t,y)\,\mathrm{d}y,\quad (6.13)$$

$$[v-v_x](\omega,t,x)=\frac{1}{2\sinh(\pi)}\int_0^{2\pi}\mathrm{e}^{(y-x+2\pi[\frac{x-y}{2\pi}]+\pi)}V(\omega,t,y)\,\mathrm{d}y.\quad (6.14)$$

Then one can employ (6.13), (6.14) and Lemma 6.1 to obtain that for a.e. $\omega\in\Omega$ and for all $(t,x)\in[0,\tau^*)\times\mathbb{T}$,

$$\begin{cases}-v(\omega,t,x)\le v_x(\omega,t,x)\le v(\omega,t,x), & \text{if } V_0(\omega,x)=(1-\partial^2_{xx})u_0(\omega,x)>0,\\ v(\omega,t,x)\le v_x(\omega,t,x)\le -v(\omega,t,x), & \text{if } V_0(\omega,x)=(1-\partial^2_{xx})u_0(\omega,x)<0.\end{cases}\quad (6.15)$$

Notice that

$$\{V_0(\omega,x)>0\}\cap\{V_0(\omega,x)<0\}=\emptyset.\quad (6.16)$$

Combining (6.15) and (6.16) yields

$$\mathbb{P}\Big\{|v_x(\omega,t,x)|\le|v(\omega,t,x)|,\quad \forall\,(t,x)\in[0,\tau^*)\times\mathbb{T}\Big\}\ge p+q.\quad (6.17)$$

In view of $H^1\hookrightarrow L^\infty$, (6.3) and (6.17), we arrive at

$$\mathbb{P}\Big\{\|v_x(\omega,t)\|_{L^\infty}\le\|v(\omega,t)\|_{L^\infty}\lesssim\|v(\omega,t)\|_{H^1}=\|u_0\|_{H^1},\quad \forall\, t\in[0,\tau^*)\Big\}\ge p+q.$$

Via (6.1), we obtain the desired estimate. □

Proof of Theorem 2.4. Let (u,τ^*) be the maximal solution to (1.12). Then Lemma 6.2 implies that

$$\mathbb{P}\Big\{\|u\|_{W^{1,\infty}}\lesssim 2\beta(\omega,t)\|u_0\|_{H^1},\quad \forall\, t\in[0,\tau^*)\Big\}\ge p+q.$$

It follows from (i) in Lemma 3.7 that $\sup_{t>0}\beta(\omega,t)<\infty\ \mathbb{P}-a.s.$ Then we can infer from (2.5) that $\mathbb{P}\{\tau^*=\infty\}\ge p+q$. That is to say, $\mathbb{P}\{u \text{ exists globally}\}\ge p+q$. □

6.3. Theorem 2.5: blow-up scenario

Proof of Theorem 2.5. Recall (6.1). By (i) in Lemma 3.7, $A = A(\omega) = \sup_{t>0}\beta(\omega,t) < \infty$ $\mathbb{P}-a.s.$ Then we can first infer from $H^1 \hookrightarrow L^\infty$ and (6.3) that for all $t>0$,

$$\sup_{t>0}\|u\|_{L^\infty} \lesssim A\|u_0\|_{H^1} < \infty \ \ \mathbb{P}-a.s.,$$

which is (2.9). Now we prove (2.10). Let

$$\Omega_1 = \left\{\limsup_{t\to\tau^*}\|u(t)\|_{H^s} = \infty\right\} \quad \text{and} \quad \Omega_2 = \left\{\liminf_{t\to\tau^*}\left[\min_{x\in\mathbb{T}} u_x(t,x)\right] = -\infty\right\}.$$

By the previously proven blow-up criterion in Theorem 2.1, we have that for a.e. $\omega\in\Omega_2$, $\omega\in\Omega_1$. Now we prove that for a.e. $\omega\in\Omega_1$, $\omega\in\Omega_2$. Suppose not. Then there is a positive random variable $K=K(\omega)<\infty$ almost surely such that

$$u_x(\omega,t,x) > -K, \quad (t,x)\in[0,\tau^*(\omega))\times\mathbb{T} \ \ \mathbb{P}-a.s.$$

Using (6.12), (6.1) and integration by parts, we find that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} V^2\,\mathrm{d}x &= 2\int_{\mathbb{T}} V[-\beta v V_x - 2\beta V v_x + \gamma V_x]\,\mathrm{d}x\\ &= -4\beta\int_{\mathbb{T}} V^2 v_x\,\mathrm{d}x - 2\beta\int_{\mathbb{T}} V V_x v\,\mathrm{d}x\\ &= -3\beta\int_{\mathbb{T}} V^2 v_x\,\mathrm{d}x \le 3K\int_{\mathbb{T}} V^2\,\mathrm{d}x, \quad t\in[0,\tau^*) \ \ \mathbb{P}-a.s.,\end{aligned}$$

which yields that

$$\|V\|_{L^2} \lesssim \mathrm{e}^{3Kt}\|V(0)\|_{L^2} < \infty, \quad t\in[0,\tau^*) \quad \mathbb{P}-a.s.$$

Combining the above estimate, (6.1) and $A(\omega)=\sup_{t>0}\beta(\omega,t)<\infty$ $\mathbb{P}-a.s.$ (cf. (i) in Lemma 3.7), we have that

$$\|u(t)\|_{H^2} \lesssim \beta(t)\mathrm{e}^{Kt}\|u(0)\|_{H^2} < \infty, \quad t\in[0,\tau^*) \ \ \mathbb{P}-a.s.$$

By the embedding $H^2 \hookrightarrow W^{1,\infty}$ and the blow-up criterion in Theorem 2.1, almost surely we have that $\|u(t)\|_{H^s}$ can be extended beyond τ^*. Therefore we obtain a contradiction and hence $\omega\in\Omega_2$. Therefore we obtain (2.10). □

6.4. Theorem 2.6: wave breaking and its probability

The proof of Theorem 2.6 relies on certain properties of the solution v to the problem (6.2).

Proposition 6.2. *Let $\mathcal{S}=(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{t\ge0},W)$ be a fixed stochastic basis, let $b(t)$ satisfy Assumption 2.2, $c_0+\gamma=0$, $s>3$ and $u_0=u_0(x)\in H^s$ be an H^s-valued $\mathcal{F}_0$ measurable random variable with $\mathbb{E}\|u_0\|^2_{H^s}<\infty$. Let (u,τ^*) be the maximal solution to (1.11) with initial random variable u_0. Recall the process β given in (6.1) and the constant λ as in Eq. (2.11). Let $N=\frac{\lambda}{2}\|u_0\|^2_{H^1}<\infty$. Then for v, defined by (6.1), we have that*

$$M(\omega,t) := \min_{x\in\mathbb{T}}[v_x(\omega,t,x)] \tag{6.18}$$

satisfies the following estimate almost surely:

$$\frac{\mathrm{d}}{\mathrm{d}t}M(t) \leq \beta N - \beta\frac{1}{2}M^2(t) \quad \text{a.e. on } (0,\tau^*). \tag{6.19}$$

Moreover, if $M(0) < -\sqrt{2N}$ almost surely, then

$$M(t) \leq -\sqrt{2N}, \quad \forall\, t \in [0,\tau^*) \quad \mathbb{P}-a.s., \tag{6.20}$$

and M is non-increasing on $[0,\tau^*)$ $\mathbb{P}-a.s.$

Proof. For any $v \in H^1$, it is easy to see that, cf. [10],

$$G_{\mathbb{T}} * \left(v^2 + \frac{1}{2}v_x^2\right)(x) \geq \frac{1}{2}v^2. \tag{6.21}$$

Using (1.9), (1.7), (6.4) and (6.1), we find for $c_0 + \gamma = 0$ that

$$v_{tx} - \gamma v_{xx} + \beta v v_{xx} = \beta v^2 - \beta\frac{1}{2}v_x^2 - \beta G_{\mathbb{T}} * \left(v^2 + \frac{1}{2}v_x^2\right), \quad t \in [0,\tau^*) \quad \mathbb{P}-a.s. \tag{6.22}$$

By Proposition 6.1, $v(\omega,t,x) \in C^1([0,\tau^*);H^{s-1})$ with $s > 3$ almost surely. To apply Lemma 3.6 for each path, we recall (6.18) and let $z(\omega,t)$ be a point where the infimum of v_x is attained as in Lemma 3.6. Then for a.e. $\omega \in \Omega$, $v_{xx}(t,z(\omega,t)) = 0$. Moreover, Lemma 3.6 also implies that for a.e. $\omega \in \Omega$, the path of $M(\omega,t)$ is locally Lipschitz.

Then for almost all $t \in [0,\tau^*)$, evaluating (6.22) in $(t,z(t))$ with using Lemma 3.6 yields for a.e. $\omega \in \Omega$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}M(t) &= \beta v^2(t,z(t)) \\ &\quad -\beta\frac{1}{2}M^2(t) - \beta G_{\mathbb{T}} * \left(v^2 + \frac{1}{2}v_x^2\right)(t,z(t)) \quad \text{a.e. on } (0,\tau^*).\end{aligned} \tag{6.23}$$

Since $\mathbb{E}\|u_0\|_{H^s} < \infty$, $N = \frac{\lambda}{2}\|u_0\|_{H^1}^2 < \infty$ $\mathbb{P}-a.s.$ Applying (6.21), (2.11) and (6.3) in the above equation gives that for a.e. $\omega \in \Omega$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}M(t) &\leq \beta\frac{1}{2}v^2(t,z(t)) - \beta\frac{1}{2}M^2(t) \\ &\leq \beta\frac{\lambda}{2}\|v(t)\|_{H^1}^2 - \beta\frac{1}{2}M^2(t) \\ &= \beta N - \beta\frac{1}{2}M^2(t) \quad \text{a.e. on } (0,\tau^*),\end{aligned}$$

which is (6.19). In order to show (6.20), we define τ as

$$\tau(\omega) := \inf\left\{t > 0 : M(\omega,t) > -\sqrt{2N}\right\} \wedge \tau^*.$$

If $M(0) < -\sqrt{2N}$, then $\mathbb{P}\{\tau > 0\} = 1$. Now we only need to show that

$$\mathbb{P}\{\tau(\omega) = \tau^*(\omega)\} = 1. \tag{6.24}$$

Actually, failure of (6.24) would ensure the existence of a set $\Omega' \subseteq \Omega$ such that $\mathbb{P}\{\Omega'\} > 0$ and $0 < \tau(\omega') < \tau^*(\omega')$ for a.e. $\omega' \in \Omega'$. In view of the time continuity of M (recall Lemma 3.6), we find that $M(\omega',\tau(\omega')) = -\sqrt{2N}$. From

(6.19) we have that $M(\omega', t)$ is non-increasing for $t \in [0, \tau(\omega'))$. Hence by the continuity of the path of $M(\omega', t)$ again, we see that $M(\omega', \tau(\omega')) \le M(0) < -\sqrt{2N}$, which is a contradiction. Hence (6.24) is true and so is (6.20). $\square$

Proposition 6.3. *Let all the conditions as in Proposition 6.2 hold true. Let* $0 < c < 1$ *and*

$$\Omega^* = \left\{\omega : \beta(t) \ge c\mathrm{e}^{-\frac{b^*}{2}t} \quad \text{for all } t\right\}.$$

If $M(0) < -\frac{1}{2}\sqrt{\frac{(b^*)^2}{c^2} + 8N} - \frac{b^*}{2c}$ *almost surely, then for a.e.* $\omega \in \Omega^*$,

$$\tau^*(\omega) < \infty.$$

Proof. We rewrite (6.19) as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}M(t) \le -\frac{\beta}{2}\left(1 - \frac{2N}{M^2(0)}\right)M^2(t)\\ - \frac{\beta N}{M^2(0)}M^2(t) + \beta N \quad \text{a.e. on } (0, \tau^*) \quad \mathbb{P} - a.s.\end{aligned}$$

Due to Proposition 6.2, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}M(t) &\le -\frac{\beta(t)}{2}\left(1 - \frac{2N}{M^2(0)}\right)M^2(t) - \left(\frac{M^2(t)}{M^2(0)} - 1\right)\beta(t)N\\ &\le -\frac{\beta(t)}{2}\left(1 - \frac{2N}{M^2(0)}\right)M^2(t) \quad \text{a.e. on } (0, \tau^*) \quad \mathbb{P} - a.s.\end{aligned}$$

Since $M(t)$ is locally Lipschitz continuous in t and satisfies (6.20), $\frac{1}{M(t)}$ is also locally Lipschitz continuous in t almost surely. Therefore an integration leads to

$$\frac{1}{M(t)} - \frac{1}{M(0)} \ge \left(1 - \frac{2N}{M^2(0)}\right)\int_0^t \frac{\beta(t')}{2}\,\mathrm{d}t', \quad t \in (0, \tau^*) \;\; \mathbb{P} - a.s.,$$

which together with (6.20) means that for a.e. $\omega \in \Omega^*$,

$$-\frac{1}{M(0)} \ge \left(\frac{1}{2} - \frac{N}{M^2(0)}\right)\int_0^{\tau^*}\beta(t)\,\mathrm{d}t \ge \left(\frac{1}{2} - \frac{N}{M^2(0)}\right)\left(\frac{2c}{b^*} - \frac{2c}{b^*}\mathrm{e}^{-\frac{b^*}{2}\tau^*}\right).$$

Recall that $M(0) < -\frac{1}{2}\sqrt{\frac{(b^*)^2}{c^2} + 8N} - \frac{b^*}{2c}$ almost surely. We finally arrive at

$$\left(\frac{1}{2} - \frac{N}{M^2(0)}\right)\frac{2c}{b^*}\mathrm{e}^{-\frac{b^*}{2}\tau^*} \ge \frac{2c}{b^*}\left(\frac{1}{2} - \frac{N}{M^2(0)}\right) + \frac{1}{M(0)} > 0 \quad \text{a.e. on } \Omega^*.$$

Therefore we have $\tau^* < \infty$ a.e. on Ω^*. $\square$

Proof of Theorem 2.6. Proposition 6.3 implies that

$$\mathbb{P}\{\tau^* < \infty\} \ge \mathbb{P}\left\{\beta(t) \ge c\mathrm{e}^{-\frac{b^*}{2}t} \quad \text{for all } t\right\}.$$

Since $b^2(t) < b^*$ for all $t > 0$, we have

$$\left\{\mathrm{e}^{\int_0^t b(t')\mathrm{d}W_{t'}} > c \quad \text{for all } t\right\} \subseteq \left\{\beta(t) \ge c\mathrm{e}^{-\frac{b^*}{2}t} \quad \text{for all } t\right\}.$$

Therefore we arrive at

$$\mathbb{P}\{\tau^* < \infty\} \geq \mathbb{P}\left\{e^{\int_0^t b(t')\mathrm{d}W_{t'}} > c \quad \text{for all } t\right\} > 0,$$

which gives the desired estimate in Theorem 2.6. □

6.5. Theorem 2.7: wave breaking rate

As the last contribution of the paper, we prove Theorem 2.7, which provides a precise bound on the wave breaking rate.

Proof of Theorem 2.7. Recalling (6.23), we have that almost surely

$$-\beta\left\|G_{\mathbb{T}} * \left(v^2 + \frac{1}{2}v_x^2\right)\right\|_{L^\infty} \leq \frac{\mathrm{d}}{\mathrm{d}t}M(t) + \beta\frac{1}{2}M^2(t) \leq \beta\|v\|_{L^\infty}^2 \quad \text{a.e. on } (0,\tau^*).$$

Using $\|G_{\mathbb{T}}\|_{L^\infty} < \infty$ and (6.3), we have

$$\left\|G_{\mathbb{T}} * \left(v^2 + \frac{1}{2}v_x^2\right)\right\|_{L^\infty} \lesssim \left\|v^2 + \frac{1}{2}v_x^2\right\|_{L^1} \lesssim \|v\|_{H^1}^2 = \|u_0\|_{H^1}^2.$$

Therefore there is a constant $C > 0$ such that

$$-C\beta\|u_0\|_{H^1}^2 \leq \frac{\mathrm{d}}{\mathrm{d}t}M(t) + \beta\frac{1}{2}M^2(t) \leq C\beta\|u_0\|_{H^1}^2 \quad \text{a.e. on } (0,\tau^*). \tag{6.25}$$

Let $\varepsilon \in (0, \frac{1}{2})$ and $K = C\|u_0\|_{H^1}^2$. Since $\liminf_{t\to\tau^*} M(t) = -\infty$ a.e. on $\{\tau^* < \infty\}$ and $K < \infty$ almost surely (cf. Theorem 2.5), for a.e. $\omega \in \Omega$, there is some $t_0 = t_0(\omega,\varepsilon) \in (0,\tau^*)$ with $M(t_0) < 0$ and $M^2(t_0) > \frac{K}{\varepsilon}$. Similar to the proof of (6.20), we have that for a.e. $\omega \in \{\tau^* < \infty\}$,

$$M^2(t) > \frac{K}{\varepsilon}, \quad t \in [t_0,\tau^*). \tag{6.26}$$

A combination of (6.25) and (6.26) enables us to infer that for a.e. $\omega \in \{\tau^* < \infty\}$,

$$\beta\frac{K}{M^2(t)} + \frac{\beta}{2} > -\frac{\frac{\mathrm{d}}{\mathrm{d}t}M(t)}{M^2(t)} > -\beta\frac{K}{M^2(t)} + \frac{\beta}{2} \quad \text{a.e. on } (t_0,\tau^*).$$

Since for a.e. $\omega \in \{\tau^* < \infty\}$, the path of M is locally Lipschitz in t and satisfies (6.26), $\frac{1}{M}$ is also locally Lipschitz.

Then we integrate the above estimate on (t,τ^*) to derive that for a.e. $\omega \in \{\tau^* < \infty\}$,

$$\left(\frac{1}{2}+\varepsilon\right)\int_t^{\tau^*}\beta(t')\,\mathrm{d}t' \geq -\frac{1}{M(t)} \geq \left(\frac{1}{2}-\varepsilon\right)\int_t^{\tau^*}\beta(t')\,\mathrm{d}t', \quad t_0 < t < \tau^*.$$

Therefore we can infer from (6.1) and (6.18) that for a.e. $\omega \in \{\tau^* < \infty\}$,

$$\frac{1}{\frac{1}{2}+\varepsilon} \leq -\min_{x\in\mathbb{T}}[u_x(\omega,t,x)]\beta^{-1}(t)\int_t^{\tau^*}\beta(t')\,\mathrm{d}t' \leq \frac{1}{\frac{1}{2}-\varepsilon}, \quad t_0 < t < \tau^*.$$

Since $\varepsilon \in (0, \frac{1}{2})$ is arbitrary, we obtain that for $\beta(\omega,t) = e^{\int_0^t b(t')\,\mathrm{d}W_{t'} - \int_0^t \frac{b^2(t')}{2}\,\mathrm{d}t'}$,

$$\lim_{t\to\tau^*}\left(\min_{x\in\mathbb{T}}[u_x(t,x)]\int_t^{\tau^*}\beta(t')\,\mathrm{d}t'\right) = -2\beta(\tau^*) \quad \text{a.e. on} \quad \{\tau^* < \infty\},$$

which completes the proof. □

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Christian Rohde and Hao Tang
Institut für Angewandte Analysis und Numerische Simulation
Universität Stuttgart
Pfaffenwaldring 57
70569 Stuttgart
Germany
e-mail: Hao.Tang@mathematik.uni-stuttgart.de

Christian Rohde
e-mail: christian.rohde@mathematik.uni-stuttgart.de

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