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On the regularity of very weak solutions for linear elliptic equations in divergence form

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Abstract. In this paper we consider a linear elliptic equation in divergence form

$$\sum_{i,j} D_j(a_{ij}(x)D_i u) = 0 \quad \text{in } \Omega. \tag{0.1}$$

Assuming the coefficients a_{ij} in $W^{1,n}(\Omega)$ with a modulus of continuity satisfying a certain Dini-type continuity condition, we prove that any very weak solution $u \in L^{n'}_{\text{loc}}(\Omega)$ of (0.1) is actually a weak solution in $W^{1,2}_{\text{loc}}(\Omega)$.

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1. Introduction

Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded open set. In this paper we study regularity properties of very weak solutions to the linear elliptic equation

$$\sum_{i,j} D_j(a_{ij}(x)D_i u) = 0 \quad \text{in } \Omega, \tag{1.1}$$

where the matrix-field $A : \Omega \to \mathbb{R}^{n\times n}$, $A(x) = (a_{ij}(x))_{i,j}$, is elliptic and belongs to $W^{1,n}(\Omega, \mathbb{R}^{n\times n}) \cap L^\infty(\Omega, \mathbb{R}^{n\times n})$, i.e.

$$\sup_{i,j=1,\dots,n} \|a_{ij}\|_{W^{1,n}(\Omega)} \leq M \tag{1.2}$$

and

$$\lambda|\xi|^2 \leq \sum_{i,j} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \quad \forall \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n, \text{ a.e. in } \Omega, \tag{1.3}$$

for some positive constants λ, Λ, and M. Moreover, the matrix A is symmetric, that is $a_{ij} = a_{ji}$ a.e. in Ω for all $i, j \in \{1, ..., n\}$.

Finally we assume that the coefficients $(a_{ij}(x))_{i,j}$ are double-Dini continuous in Ω, i.e. $a_{ij} \in C^0(\Omega)$ and

$$\bar{A}_\Omega(r) := \sum_{i,j} \sup_{\substack{x,y\in\Omega \\ |x-y|\le r}} |a_{ij}(x) - a_{ij}(y)|, \quad r > 0,$$

satisfies

$$\int_0^{diam(\Omega)} \frac{1}{t} \int_0^t \frac{\bar{A}_\Omega(s)}{s} ds\, dt < \infty. \tag{1.4}$$

A common type of double-Dini continuous functions are, of course, $\omega(r) = r^\alpha$, $0 < \alpha \le 1$, thus an example of a matrix-field A satisfying (1.2) and (1.4) is $A \in W^{1,p}(\Omega, \mathbb{R}^{n\times n})$, with $p > n$. On the other hand, condition (1.4) occurs not only for $\omega(r) = r^\alpha$, but more generally for $\omega(r) = \log^\beta\left(\frac{1}{r}\right)$, $\beta < -2$.

Given a measurable matrix $A(x) = (a_{ij}(x))_{i,j}$ satisfying (1.3), a function $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is called a *weak solution* of (1.1) if

$$\sum_{i,j} \int a_{ij}(x) D_i u D_j \varphi\, dx = 0, \quad \forall \varphi \in C_c^\infty(\Omega).$$

The celebrated result by De Giorgi in [5] states that if u is a weak solution of (1.1) then u is locally Hölder continuous.

Subsequently, J. Serrin produced in [14] a famous example, constructing an equation of the form (1.1) which has a solution $u \in W^{1,p}(\Omega)$, with $1 < p < 2$, and $u \notin L^\infty_{\mathrm{loc}}(\Omega)$. Serrin conjectured that if the coefficients a_{ij} are locally Hölder continuous, then any solution (in the sense of distributions) $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ of (1.1) must be a (usual) weak solution, i.e. $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$. Serrin's conjecture was established by Hager and Ross [9], and then in full generality by Brezis [2] (see also [1] for a full proof) starting with $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, or even with $u \in BV_{\mathrm{loc}}(\Omega)$, i.e., $u \in L^1_{\mathrm{loc}}(\Omega)$ and its derivatives (in the sense of distributions) being Radon measures. Let us remark that in Brezis's result the coefficients a_{ij}, satisfying (1.3), are Dini continuous functions in Ω. The Dini continuity of the coefficients is optimal in some sense: for the unit ball B_1 and continuous coefficients, Jin et al. [10] constructed a solution (in the sense of distributions) $u \in W^{1,1}_{\mathrm{loc}}(B_1) \backslash W^{1,p}_{\mathrm{loc}}(B_1)$ for every $p > 1$.

For $A(x) = (a_{ij}(x))_{i,j}$ satisfying (1.2) and (1.3), we will consider a *very weak solution* $u \in L^{n'}_{\mathrm{loc}}(\Omega)$ of (1.1), namely

$$\sum_{i,j} \int u(x) D_i(a_{ij}(x) D_j \varphi(x))\, dx = 0, \quad \forall \varphi \in C_c^\infty(\Omega), \tag{1.5}$$

with $n' = \frac{n}{n-1}$.

Remark 1.1. It is not difficult to prove that the test functions φ in (1.5) can be taken in $W^{2,n}(\Omega) \cap W^{1,\infty}(\Omega)$, with $\mathrm{supp}\, \varphi \Subset \Omega$. Indeed, one can argue by density to show that given a function $\varphi \in W^{2,n}(\Omega) \cap W^{1,\infty}(\Omega)$ with compact support, we may find a sequence $\varphi_k \in C_c^\infty(\Omega)$ such that $\varphi_k \to \varphi$ strongly in $W^{2,n}$ and $\sup_k \|\varphi_k\|_{1,\infty} < \infty$ (so that $D\varphi_k$ converges to $D\varphi$ weakly* in L^∞) and then taking the limit as k goes to infinite in the Eq. (1.5) for φ_k.

The main result of the paper is the following.

Theorem 1.2. *Let u be a very weak solution of (1.1), with $A(x) = (a_{ij}(x))_{i,j}$ satisfying (1.2), (1.3) and (1.4), then u belongs to $W^{1,2}_{\mathrm{loc}}(\Omega)$ and thus it is a weak solution.*

Remark 1.3. It is worth noting that, under hypotheses (1.2) and (1.3), one can consider a very weak solution $u \in L^{n'}_{\mathrm{loc}}(\Omega)$ to (1.1), but when dealing with the regularity properties of u some extra conditions on the coefficients a_{ij} must be considered. The counterexample constructed in [10] provides in fact continuous coefficients a_{ij} which belong also to $W^{1,n}(B_1)$, showing that one can not expect a very weak solution $u \in L^{n'}_{\mathrm{loc}}(\Omega)$ to be a weak solution in $W^{1,2}_{\mathrm{loc}}(\Omega)$ under just conditions (1.2) and (1.3). For the sake of completeness, we will propose the example given in [10] in the "Appendix B", underlining that the constructed coefficients belong also to $W^{1,n}(B_1)$.

On the other hand, in Sect. 4 we propose an alternative to double Dini continuous coefficients which again bypasses the counterexample. In particular, under hypotheses (1.2) and (1.3) we consider a very weak solution in $L^{q}_{\mathrm{loc}}(\Omega)$, with $q > n'$.

Remark 1.4. In [15] Zhang and Bao deal with the case of very weak solutions $u \in L^{1}_{\mathrm{loc}}(\Omega)$ of (1.5), interpreting the coefficients as Lipschitz functions, due to the assumption made on the solutions. Thus our result represents a natural extension from their research.

2. Notation and preliminary results

We collect here the main definitions and notation and some useful results that will be needed in the sequel.

2.1. Notation

In the following, we denote by $B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$ the ball of radius r centered at x.

We indicate by $\{e_1, \dots e_n\}$ the canonical basis of $\mathbb{R}^n$. Given $h \in \mathbb{R} \setminus \{0\}$, for a measurable function $\psi : \mathbb{R}^n \to \mathbb{R}$ and for $\ell = 1, \dots, n$, we introduce the notation

$$\Delta^{\ell}_h \psi := \frac{\psi(x + he_\ell) - \psi(x)}{h}$$

for the incremental quotient in the ℓ-th direction. We recall that for every pair of functions φ, ψ, we have

$$\Delta^{\ell}_h(\varphi\,\psi) = \Delta^{\ell}_h \varphi\,\psi + \varphi(x + he_\ell)\,\Delta^{\ell}_h \psi. \tag{2.1}$$

The following result pertaining to difference quotients of functions in Sobolev spaces is well known t (see [8, Proposition 4.8] for example).

Theorem 2.1. *Let $p > 1$; if $\psi \in W^{1,p}(\Omega)$, then $\Delta_h^\ell \psi \in L^p(\Omega')$ for any $\Omega' \Subset \Omega$ satisfying $h < \frac{dist(\Omega', \partial\Omega)}{2}$, and we have*

$$\|\Delta_h^\ell \psi\|_{L^p(\Omega')} \leq \|D_\ell \psi\|_{L^p(\Omega)}.$$

If $\psi \in L^p(\Omega)$ and there exists $L \geq 0$ such that, for every $h < dist(\Omega', \partial\Omega)$, $\ell = 1, \dots, n$, we have

$$\|\Delta_h^\ell \psi\|_{L^p(\Omega')} \leq L,$$

then $\psi \in W^{1,p}(\Omega')$, $\|D_\ell \psi\|_{L^p(\Omega')} \leq L$ and $\Delta_h^\ell \psi \to D_\ell \psi$ in $L^p(\Omega')$ as $h \to 0$.

Finally, given $p > 1$, we denote by $p' = \frac{p}{p-1}$ the conjugate exponent of p.

2.2. Dini continuous functions

We say that a continuous function f on Ω is Dini continuous if the modulus of continuity $\bar{f}_\Omega : [0, diam(\Omega)] \to \mathbb{R}^+$ defined by

$$\bar{f}_\Omega(r) := \sup_{\substack{x,y \in \Omega \\ |x-y| \leq r}} |f(x) - f(y)|$$

satisfies

$$\int_0^{diam(\Omega)} \frac{\bar{f}_\Omega(t)}{t}\, dt < \infty.$$

We also denote by $C^D(\Omega)$ the space of Dini continuous functions; it turns out to be a Banach space equipped with the following norm:

$$\|f\|_{C^D(\Omega)} := \|f\|_\infty + \int_0^{diam(\Omega)} \frac{\bar{f}_\Omega(t)}{t}\, dt,$$

where $\|\cdot\|_\infty$ is the usual uniform norm.

Let us remark that by the uniform continuity, any function in $C^D(\Omega)$ may be extended up to the boundary of Ω with the same modulus of continuity. Moreover,

$$C^{0,\alpha}(\Omega) \subseteq C^D(\Omega),$$

for any $0 < \alpha \leq 1$, where $C^{0,\alpha}(\Omega)$ denotes the space of Hölder continuous functions.

The space $C_c^D(\Omega)$ will denote the set of functions in $C^D(\Omega)$ with compact support in Ω.

Lemma 2.2. *The space $C_c^\infty(\Omega)$ is dense in $C_c^D(\Omega)$.*

Proof. Let $f \in C_c^D(\Omega)$ that we extend to zero on $\mathbb{R}^n \backslash \Omega$ and set $f_\varepsilon(x) = (\rho_\varepsilon * f)(x)$, where ρ_ε is a standard mollifier. Then, if ε is sufficiently small, $f_\varepsilon \in C_c^\infty(\Omega)$; we will prove that

$$f_\varepsilon \to f \quad \text{in } C^D(\Omega). \tag{2.2}$$

It is easily seen that f_ε uniformly converges to f in Ω, thus in order to prove (2.2) we will just show that

$$\int_0^{diam(\Omega)} \frac{\overline{(f - f_\varepsilon)}_\Omega(t)}{t}\, dt \to 0,$$

as ε tends to 0. Observe that

$$(\overline{f-f_\varepsilon})_\Omega(r) = \sup_{\substack{x,y\in\Omega\\ |x-y|<r}} \{|f_\varepsilon(x)-f(x)-f_\varepsilon(y)+f(y)|\} \le \bar{f}_\Omega(r)+(\bar{f}_\varepsilon)_\Omega(r)$$

and

$$\begin{aligned}(\bar{f}_\varepsilon)_\Omega(r) &= \sup_{\substack{x,y\in\Omega\\ |x-y|<r}} \{|f_\varepsilon(x)-f_\varepsilon(y)|\}\\ &= \sup_{\substack{x,y\in\Omega\\ |x-y|<r}} \left\{\left|\int \rho_\varepsilon(z)\,(f(x-z)-f(y-z))\,dz\right|\right\}\\ &\le \int \rho_\varepsilon(z)\bar{f}_\Omega(r)dz = \bar{f}_\Omega(r),\end{aligned}$$

which together yield

$$(\overline{f_\varepsilon - f})_\Omega(r) \le 2\bar{f}_\Omega(r).$$

On the other hand, since $(\overline{f_\varepsilon - f})_\Omega \to 0$ pointwise, the dominated convergence theorem implies

$$\int_0^{diam(\Omega)} \frac{(\overline{f_\varepsilon - f})_\Omega(t)}{t} \to 0,$$

which concludes the proof of (2.2). □

Remark 2.3. The previous result ensures that $C^D_c(\Omega)$ is a separable space, noting that $C^1_c(\Omega)$ is separable with respect to the usual norm $\|f\|_{1,\infty} := \sum_{|\alpha|\le 1}\|D_\alpha f\|_\infty$, $C^1_c(\Omega)\subseteq C^D_c(\Omega)$ and $\bar{f}_\Omega(r)\le r\|Df\|_\infty$, for every $f\in C^1_c(\Omega)$.

Lemma 2.4. *Let f, f_ε, and g belonging to $C^D(\Omega)$ such that f_ε converges to f in C^D; then gf_ε converges to gf in C^D.*

Proof. As before, it is enough to prove the convergence of the seminorm since the uniform convergence is immediate. Then, writing the definition of the modulus of continuity, we have

$$\begin{aligned}g(f_\varepsilon - f)t_\Omega(r) &= \sup_{\substack{x,y\in\Omega\\ |x-y|<r}} \{|g(x)(f_\varepsilon(x)-f(x)) - g(y)(f_\varepsilon(y)-f(y))|\} \qquad (2.3)\\ &\le \sup_{\substack{x,y\in\Omega\\ |x-y|<r}} \{|g(x)|\,|(f_\varepsilon(x)-f(x))-(f_\varepsilon(y)-f(y))|\}\\ &\quad + \sup_{\substack{x,y\in\Omega\\ |x-y|<r}} \{|g(x)-g(y)||f(y)-f_\varepsilon(y)|\}\\ &\le \|g\|_\infty(\overline{f-f_\varepsilon})_\Omega(r) + \bar{g}_\Omega(r)\|f-f_\varepsilon\|_\infty.\end{aligned}$$

Hence,

$$\begin{aligned}\int_0^{diam(\Omega)} \frac{[\overline{g(f_\varepsilon - f)}]_\Omega(t)}{t}\,dt &\le \|g\|_\infty \int_0^{diam(\Omega)} \frac{(\overline{f-f_\varepsilon})_\Omega(t)}{t}dt\\ &\quad + \|f-f_\varepsilon\|_\infty \int_0^{diam(\Omega)} \frac{\bar{g}_\Omega(t)}{t}dt,\end{aligned}$$

which goes to zero as ε tends to zero. □

2.3. C^1-Dini regularity of solutions to divergence form elliptic equations with Dini-continuous coefficients

For the proof of our result, we will need the following extension of the Schauder regularity theory for elliptic equations in divergence form with Dini continuous coefficients (see [12, Theorem 1.1] and [6, Theorem 1.3], see also [11] which is inclusive of the parabolic case.). For the L^p-regularity theory we refer to [7], where the general case of VMO coefficients is treated (see also [13, Theorem 5.5.3 (a)] or [3, Theorem 2.2. Chapter 10] for the case of continuous coefficients).

Theorem 2.5. *For $\Omega \subset \mathbb{R}^n$, let a_{ij} satisfy (1.3) and (1.4); we consider $f = (f_1, f_2, \dots, f_n)$ with $f_j \in C_c^\infty(\Omega)$ for all $j \in \{1, \dots, n\}$. Assume that $u \in H^1(\Omega)$ is a weak solution of the equation*

$$\sum_{i,j} D_j \left(a_{ij} D_i u\right) = \sum_j D_j f_j \quad \text{in } \Omega \tag{2.4}$$

Then $u \in C^{1,D}(\Omega')$, for any bounded open set Ω', $\Omega' \Subset \Omega$.

Moreover, let Ω a $C^{1,1}$ bounded open subset of $\mathbb{R}^n$, let a_{ij} satisfy (1.2) and (1.3), and let $f_j \in L^p(\Omega)$, for every $j \in \{1, \dots, n\}$, with $1 < p < \infty$, then there exists a unique solution $u \in W_0^{1,p}(\Omega)$ to the problem

$$\sum_{i,j} \int_\Omega a_{ij} D_i u D_j \varphi \, dx = \sum_j \int_\Omega f_j D_j \varphi \, dx \quad \forall \varphi \in W_0^{1,p'}(\Omega),$$

and

$$\|u\|_{W^{1,p}(\Omega)} \le C \sum_j \|f_j\|_{L^p(\Omega)} \tag{2.5}$$

holds, where C depends on $n, \lambda, \Lambda, p, \partial\Omega$, $\|A\|_{W^{1,n}(\Omega, \mathbb{R}^{n\times n})}$.

Remark 2.6. The first conclusion of Theorem 2.5 comes with an estimate of the Dini modulus of continuity of Du involving the Dini modulus of continuity of a_{ij} and f_j. Actually, in [12, Theorem 1.1] and in [6, Theorem 1.3] only the continuity of Du is proved and these results are obtained with a weaker assumption on the coefficients a_{ij}. Assuming (1.4) for the coefficients we are able to prove also the Dini continuity of the gradient of the solution. In "Appendix A" we will resume in broad terms the proof of [12, Theorem 1.1], developing it in order to get the needed Dini continuity result.

2.4. C^2-regularity of solutions to non divergence form elliptic equations with Dini-continuous coefficients

Let us first recall the $W^{2,p}$-solvability of the Dirichlet problem for non divergence elliptic equations with discontinuous coefficients (see [4, Theorem 4.2 and Theorem 4.4]).

Theorem 2.7. *The Dirichlet problem*

$$\begin{cases} \sum_{i,j} a_{ij}(x) D_{ij} u = f & a.e. in\, \Omega \\ u = 0 & on\, \partial\Omega \end{cases} \tag{2.6}$$

where Ω is a $C^{1,1}$ smooth and bounded subset of $\mathbb{R}^n$, $f \in L^p(\Omega)$ with $1 < p < \infty$, and a_{ij} satisfies (1.2) and (1.3), admits a unique solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and

$$||u||_{W^{2,p}(\Omega)} \leq C \left(||u||_{L^p(\Omega)} + ||f||_{L^p(\Omega)} \right), \tag{2.7}$$

where the constant C depends on $n, p, \lambda, \Lambda, \partial\Omega, \|A\|_{W^{1,n}(\Omega, \mathbb{R}^{n\times n})}$.

The next result specifies estimate (2.7); its proof is quite standard but we prefer to write it for the sake of completeness.

Proposition 2.8. *Suppose u is a solution of the elliptic Dirichlet problem (2.6) with a_{ij}, f, p and Ω as above. Then*

$$||u||_{W^{2,p}(\Omega)} \leq C||f||_{L^p(\Omega)}. \tag{2.8}$$

Proof. Let

$$\mathcal{L} = \left\{ L = \sum_{i,j} a_{ij} D_{ij}, \sup_{i,j} ||a_{ij}||_{W^{1,n}(\Omega)} \leq 2M,\ \lambda|\xi|^2 \leq \sum_{i,j} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \right\};$$

having in mind Theorem 2.7, if we prove that for any operator $L \in \mathcal{L}$ and for any $f \in L^p(\Omega)$, the solution u of

$$\begin{cases} Lu = f & \text{a.e. in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

satisfies

$$||u||_{L^p(\Omega)} \leq C||f||_{L^p(\Omega)},$$

we are done. Suppose it is not the case, then this is equivalent to say that for every $N \in \mathbb{N}$, there exists an operator $L_N = \sum_{i,j} a_{ij}^N D_{ij} \in \mathcal{L}$ and a function $f_N \in L^p(\Omega)$ such that the corresponding solution u_N to the Dirichlet problem

$$\begin{cases} L_N u_N = f_N & \text{a.e. in } \Omega \\ u_N = 0 & \text{on } \partial\Omega, \end{cases}$$

satisfies

$$||u_N||_{L^p(\Omega)} > N||f_N||_{L^p(\Omega)}. \tag{2.9}$$

Let us define $v_N = u_N/\|u_N\|_{L^p(\Omega)}$ and $g_N = f_N/\|u_N\|_{L^p(\Omega)}$, so that v_N solves (2.6) with L_N and g_N. By the $W^{2,p}$ estimate (2.7),

$$||v_N||_{W^{2,p}(\Omega)} \leq C \left(||v_N||_{L^p(\Omega)} + ||g_N||_{L^p(\Omega)} \right) < C \left(1 + \frac{1}{N} \right),$$

where C does not depend on N and hence,

$$||v_N||_{W^{2,p}(\Omega)} \leq C. \tag{2.10}$$

Thus v_N is a precompact sequence: up to a non relabeled subsequence, we can suppose $v_N \rightharpoonup u^*$ weakly in $W^{2,p}(\Omega)$, for some $u^* \in W^{2,p}(\Omega)$, moreover $u^* \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. Similarly, we can also say that, for every $i, j = 1, \ldots, n$,

$a_{ij}^N \rightharpoonup a_{ij}^*$ weakly in $W^{1,n}(\Omega)$ and $a_{ij}^N \to a_{ij}^*$ strongly in $L^q(\Omega)$ $\forall\, 1 \le q < \infty$. Thus, the operator $L^* = \sum_{i,j} a_{ij}^* D_{ij}$ belongs to $\mathcal{L}$ and for $\varphi \in L^{p'}(\Omega)$ we have

$$\begin{aligned}
&\left| \int_\Omega (L_N v_N - L^* u^*)\, \varphi \, dx \right| \\
&\quad \le \sum_{i,j=1}^n \left\{ \int_\Omega \left| (a_{ij}^N - a_{ij}^*) \frac{\partial^2 v_N}{\partial x_i \partial x_j} \varphi \right| dx + \left| \int_\Omega a_{ij}^* \varphi \left(\frac{\partial^2 v_N}{\partial x_i \partial x_j} - \frac{\partial^2 u^*}{\partial x_i \partial x_j} \right) dx \right| \right\} \\
&\quad \le C \sum_{i,j=1}^n \| (a_{ij}^N - a_{ij}^*) \varphi \|_{L^{p'}(\Omega)} + \sum_{i,j=1}^n \left\{ \left| \int_\Omega a_{ij}^* \varphi \left(\frac{\partial^2 v_N}{\partial x_i \partial x_j} - \frac{\partial^2 u^*}{\partial x_i \partial x_j} \right) dx \right| \right\}.
\end{aligned}$$

Therefore, $L_N v_N$ converges weakly in $L^p(\Omega)$ to $L^* u^*$. On the other hand, using (2.9), we have

$$||g_N||_{L^p(\Omega)} < \frac{1}{N}.$$

Passing to the limit in the equation satisfied by v_N, we discover that the limit $u^* \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ satisfies $L^* u^* = 0$ a.e. in Ω. By the uniqueness properties of the solutions to (2.6), it follows that $u^* = 0$. Thus v_N converges to zero and the argument becomes contradictory since $\|v_N\|_{L^p(\Omega)} = 1$. □

In [6, Theorem 1.5] it is shown that solutions to elliptic equations in non divergence form with zero Dirichlet boundary conditions are C^2 up to the boundary when the leading coefficients are Dini continuous functions.

Theorem 2.9. *Assume that Ω is a $C^{2,1}$ smooth and bounded open subset of $\mathbb{R}^n$, $f \in C^D(\Omega)$ and a_{ij} satisfies (1.2), (1.3), and (1.4). Let $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ be a solution of the Dirichlet problem*

$$\begin{cases} \sum_{i,j} a_{ij}(x) D_{ij} u = f & \text{a.e. in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.11}$$

then $u \in C^2(\overline{\Omega})$.

Remark 2.10. The assumption in [6] about the coefficients is weaker then (1.4), since they assume that the modulus of continuity

$$\tilde{A}_\Omega(r) := \sum_{i,j} \sup_{x \in \overline{\Omega}} \fint_{B_r(x) \cap \Omega} |a_{ij}(y) - (a_{ij})_{B_r(x) \cap \Omega}|\, dy$$

with $(a_{ij})_{B_r(x)\cap\Omega} = \fint_{B_r(x)\cap\Omega} a_{ij}$, satisfies

$$\int_0 \frac{\tilde{A}_\Omega(r)}{r} dr < \infty.$$

3. Proof of the main theorem

We use a duality argument in conjunction with the regularity properties for elliptic equations in divergence and in non divergence form, stated in Theorems 2.5 and 2.9.

Proof. Let $\Omega' \Subset \Omega$ be an open set and choose a $C^{2,1}$ open set Ω_0 with $\Omega' \Subset \Omega_0 \Subset \Omega$; let $d(\Omega', \partial\Omega_0) = d > 0$. Let $h_0 = d/4$, and $0 < |h| < h_0$.
For the sake of clarity, we divide the proof into two steps.

Step 1 For $\ell = 1, \dots, n$, we claim that $\Delta_h^\ell u$ is bounded in the dual space of Dini continuous functions with compact support $(C_c^D(\Omega'))'$.

Given a Dini continuous function $w \in C_c^D(\Omega')$, according to Theorem 2.9 combined with Theorem 2.7, the solution $v \in W^{2,q}(\Omega_0)$, $\forall q > 1$, to the Dirichlet problem

$$\begin{cases} \sum_{i,j} a_{ij}(x) D_{ij} v = w & \text{a.e in } \Omega_0 \\ v = 0 & \text{on } \partial\Omega_0, \end{cases} \tag{3.1}$$

enjoys the C^2-regularity up to the boundary of Ω_0.

We consider a partition of unity: let $x_1, \dots, x_J \in \Omega'$ and $\eta_1, \dots, \eta_J \in C^\infty(\mathbb{R}^n)$ be such that

$$\Omega' \subset \bar{\Omega}' \subset \bigcup_{k=1}^{J} B_{d/8}(x_k),\ 0 \le \eta_k \le 1,\ \forall k = 1, \dots, J,\ \text{and} \sum_{k=1}^{J} \eta_k = 1 \text{ in } \Omega',$$

and

$$\operatorname{supp} \eta_k \text{ is compact and } \operatorname{supp} \eta_k \subset B_{d/8}(x_k).$$

We fix one of these balls and the related function η_k; we omit to indicate the center x_k and the index k for η_k for simplicity.

In view of Remark 1.1, we can insert $\varphi = \eta \Delta_{-h}^\ell v$ in (1.5), getting

$$\begin{aligned} 0 &= \sum_{i,j} \int u D_i (a_{ij} D_j (\eta \Delta_{-h}^\ell v))\, dx \\ &= \sum_{i,j} \int u D_i a_{ij} D_j (\eta \Delta_{-h}^\ell v)\, dx + \sum_{i,j} \int u\, a_{ij} D_{ij} (\eta \Delta_{-h}^\ell v)\, dx \\ &= \sum_{i,j} \int u D_i a_{ij} D_j \eta \Delta_{-h}^\ell v\, dx + \sum_{i,j} \int u\, \eta D_i a_{ij} D_j (\Delta_{-h}^\ell v)\, dx \\ &\quad + \sum_{i,j} \int \eta\, u\, a_{ij} D_{ij} (\Delta_{-h}^\ell v)\, dx + \sum_{i,j} \int u\, a_{ij} D_j \eta D_i (\Delta_{-h}^\ell v)\, dx \\ &\quad + \sum_{i,j} \int u\, a_{ij} D_i \eta D_j (\Delta_{-h}^\ell v)\, dx + \sum_{i,j} \int u\, a_{ij} D_{ij} \eta \Delta_{-h}^\ell v\, dx. \end{aligned}$$

We can rearrange the previous equation in order to have

$$\begin{aligned}\sum_{i,j}\int \eta\, u\, a_{ij} D_{ij}(\Delta^{\ell}_{-h} v)\,dx = &-\sum_{i,j}\int u D_i a_{ij} D_j \eta \Delta^{\ell}_{-h} v\,dx\\ &-\sum_{i,j}\int u\,\eta D_i a_{ij} D_j(\Delta^{\ell}_{-h} v)\,dx\\ &-\sum_{i,j}\int u\, a_{ij} D_j \eta D_i(\Delta^{\ell}_{-h} v)\,dx\\ &-\sum_{i,j}\int u\, a_{ij} D_i \eta D_j(\Delta^{\ell}_{-h} v)\,dx\\ &-\sum_{i,j}\int u\, a_{ij} D_{ij} \eta \Delta^{\ell}_{-h} v\,dx.\end{aligned}$$

With a simple change of variables, we get

$$\begin{aligned}\sum_{i,j}\int \eta\, u\, a_{ij} D_{ij}(\Delta^{\ell}_{-h} v)\,dx &= \sum_{i,j}\int_{\mathbb{R}^n} \eta\, u\, a_{ij} \Delta^{\ell}_{-h}(D_{ij} v)\,dx\\ &= \sum_{i,j}\int_{\mathbb{R}^n} \Delta^{\ell}_{h}(\eta\, u\, a_{ij}) D_{ij} v\,dx\\ &= \sum_{i,j}\int \Delta^{\ell}_{h} u\,\eta\, a_{ij} D_{ij} v\,dx\\ &\quad+\sum_{i,j}\int u(x+he_\ell)\Delta^{\ell}_{h}(\eta\, a_{ij}) D_{ij} v\,dx,\end{aligned}$$

where we also used (2.1). Thus, we finally have

$$\begin{aligned}\sum_{i,j}\int \eta\Delta^{\ell}_{h} u\, a_{ij} D_{ij} v\,dx = &-\sum_{i,j}\int u D_i a_{ij} D_j \eta \Delta^{\ell}_{-h} v\,dx\\ &-\sum_{i,j}\int u\,\eta D_i a_{ij} D_j(\Delta^{\ell}_{-h} v)\,dx\\ &-\sum_{i,j}\int u\, a_{ij} D_j \eta D_i(\Delta^{\ell}_{-h} v)\,dx\\ &-\sum_{i,j}\int u\, a_{ij} D_i \eta D_j(\Delta^{\ell}_{-h} v)\,dx\\ &-\sum_{i,j}\int u\, a_{ij} D_{ij} \eta \Delta^{\ell}_{-h} v\,dx\\ &-\sum_{i,j}\int u(x+he_\ell)\Delta^{\ell}_{h}(\eta\, a_{ij}) D_{ij} v\,dx\\ =\ &\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4+\mathcal{I}_5+\mathcal{I}_6.\end{aligned}\tag{3.2}$$

Now, we estimate the six terms $\mathcal{I}_m$.

The use of Hölder's inequality gives

$$\begin{aligned}|\mathcal{I}_1| &\le \sum_{i,j}\int_{B_{d/8}} |uD_i a_{ij} D_j \eta \Delta^{\ell}_{-h} v|\,dx\\ &\le ||D\eta||_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)}||A||_{W^{1,n}(\Omega,\mathbb{R}^{n\times n})}||u||_{L^{n'}(\Omega_0)}||\Delta^{\ell}_{-h}v||_{L^\infty(B_{d/8})}\\ &\le C||Dv||_{L^\infty(\Omega_0,\mathbb{R}^n)} \le C\|w\|_{L^\infty(\Omega')},\end{aligned}$$

combined with Sobolev's embedding and Proposition 2.8 in the last inequality. Analogously

$$|\mathcal{I}_2| \le ||A||_{W^{1,n}(\Omega,\mathbb{R}^{n\times n})}||u||_{L^{n'}(\Omega_0)}\|D^2 v\|_{L^\infty(\Omega_0,\mathbb{R}^{n\times n})}.$$

The terms $\mathcal{I}_3$ and $\mathcal{I}_4$ can be treated in the same way. Using Hölder's inequality, Theorem 2.1 and Proposition 2.8, we have

$$|\mathcal{I}_3|, |\mathcal{I}_4| \le \Lambda\|u\|_{L^{n'}(\Omega_0)}\|D\eta\|_{L^\infty(\mathbb{R}^n\mathbb{R}^n)}||v||_{W^{2,n}(\Omega_0)} \le C||w||_{L^n(\Omega')}.$$

Again, for $\mathcal{I}_5$ we have

$$|\mathcal{I}_5| \le \Lambda\|u\|_{L^{n'}(\Omega_0)}\|D^2\eta\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^{n\times n})}\|Dv\|_{L^n(\Omega_0,\mathbb{R}^n)} \le C\|w\|_{L^n(\Omega')}.$$

We finally estimate $\mathcal{I}_6$. From (2.1), we get

$$\begin{aligned}\mathcal{I}_6 = &-\sum_{i,j}\int u(x+he_\ell)\,\eta\Delta^{\ell}_h a_{ij} D_{ij} v\,dx\\ &-\sum_{i,j}\int u(x+he_\ell) a_{ij}(x+he_\ell)\Delta^{\ell}_h\eta\, D_{ij} v\,dx.\end{aligned}$$

The second term can be estimated as $\mathcal{I}_3$ and $\mathcal{I}_4$, thus:

$$\begin{aligned}|\mathcal{I}_6| &\le \sum_{i,j}\int_{B_{d/8}} |u(x+he_\ell)\,\eta\Delta^{\ell}_h a_{ij} D_{ij} v|\,dx + C\|w\|_{L^n(\Omega')}\\ &\le C\|A\|_{W^{1,n}(\Omega,\mathbb{R}^{n\times n})}\|u\|_{L^{n'}(\Omega_0)}\|D^2 v\|_{L^\infty(\Omega_0,\mathbb{R}^{n\times n})} + C\|w\|_{L^n(\Omega')}.\end{aligned} \tag{3.3}$$

Here we have used once more Theorem 2.9.

Finally, combining the estimates found for $\mathcal{I}_m$, $m \in \{1,\dots 6\}$, from (3.2) we get

$$\sum_{i,j}\int_{B_{d/8}} \eta\Delta^{\ell}_h u\, a_{ij} D_{ij} v\,dx \le C,$$

where C depends on $\lambda, \Lambda, \|D\eta\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)}, \|D^2\eta\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^{n\times n})}, \|u\|_{L^{n'}(\Omega_0)}$, $\|A\|_{W^{1,n}(\Omega,\mathbb{R}^{n\times n})}, \|w\|_{L^\infty(\Omega')}$ and $\|D^2 v\|_{L^\infty(\Omega_0,\mathbb{R}^{n\times n})}$, as well as on the modulus of continuity of the coefficients a_{ij} and of the datum w. Summing over $k = 1,\dots,J$, since v is the weak solution to the Dirichlet problem (3.1), we finally have

$$\left|\int_{\Omega'} \eta\, w\Delta^{\ell}_h u\,dx\right| \le C,$$

and we get

$$\left|\int_{\Omega'} w\Delta^{\ell}_h u\,dx\right| \le C,$$

for every $w \in C_c^D(\Omega')$. By the uniform boundedness principle this means that $\{\Delta_h^\ell u\}_h$ is a family of equibounded elements in the dual space of Dini continuous functions $(C_c^D(\Omega'))'$. Since $(C_c^D(\Omega'))'$ is separable, we have that, up to a subsequence,

$$\Delta_h^\ell u \overset{*}{\rightharpoonup} \mu^\ell \in (C_c^D(\Omega'))'.$$

Step 2. We prove that $u \in W^{1,p'}_{\mathrm{loc}}(\Omega)$, with $p > n$.

Using the previous Step we can easily deduce from (1.5) that

$$\sum_{i,j} \langle \mu^i, a_{ij} D_j \varphi \rangle = 0 \quad \forall \varphi \in C_c^\infty(\Omega'), \tag{3.4}$$

where the duality pairing is between $(C_c^D(\Omega'))'$ and $C_c^D(\Omega')$.

For $j \in \{1, \dots, n\}$, let $f = (f_1, \dots, f_n)$ with $f_j \in C_c^\infty(\Omega')$ be such that

$$\sum_j ||f_j||_{L^p(\Omega')} \leq 1,$$

with $p > n$. Introducing as before a regular set Ω_0 between Ω' and Ω we can possibly assume that Ω is a $C^{1,1}$ set. Let $v \in W_0^{1,2}(\Omega)$ be the weak solution of the problem

$$\sum_{i,j} \int a_{ij} D_i v D_j \varphi \, dx = \sum_j \int D_j \varphi f_j \, dx \quad \forall \varphi \in C_c^\infty(\Omega). \tag{3.5}$$

By Theorem 2.5 we have that $v \in W_0^{1,p}(\Omega)$ and

$$||v||_{W^{1,p}(\Omega)} \leq C ||f||_{L^p(\Omega', \mathbb{R}^n)}.$$

Note that, since $p > n$, this means also that the function v is Hölder continuous.

We take $B_{R/2} \subset B_R \subset \Omega'$ a pair of concentric balls centered at $x_0 \in \Omega'$ and we consider $\xi(x) = \xi(|x - x_0|)$ a smooth function such that $\xi(t) = 1$ for $t \in [0, R/2]$ and $\xi(t) = 0$ for $t \geq R$.

We would like to use $\varphi = \xi v$ as test function in (3.4). We first observe that, by Theorem 2.5, the function ξv belongs to $C_c^{1,D}(\Omega')$. Moreover, proving Lemma 2.2, we actually proved that a mollification of a Dini continuous function with compact support strongly converges in C^D to the function itself. Thus, combining this fact with Lemma 2.4, we have that $a_{ij} D_j (\xi v)_\varepsilon$ strongly converges in C^D to $a_{ij} D_j (\xi v)$, where $(\xi v)_\varepsilon(x) = (\rho_\varepsilon * \xi v)(x)$, ρ_ε being a standard mollifier. This in turn implies that the use of $\varphi = \xi v$ as test function in (3.4) is admissible:

$$\sum_{i,j} \langle \mu^i, a_{ij} D_j v \, \xi \rangle + \sum_{i,j} \langle \mu^i, a_{ij} v D_j \xi \rangle = 0. \tag{3.6}$$

Let us come back now to the equation satisfied by v. Let u_ε be a mollification of the solution u, that is $u_\varepsilon = \rho_\varepsilon * u$, with ρ_ε a standard radial mollifier. We use ξu_ε in (3.5):

$$\sum_{i,j} \int a_{ij} \xi D_j u_\varepsilon D_i v \, dx + \sum_{i,j} \int a_{ij} D_j \xi \, u_\varepsilon D_i v \, dx = \sum_j \int \xi f_j D_j u_\varepsilon \, dx + \sum_j \int u_\varepsilon D_j \xi f_j \, dx.$$

Now we claim that this implies, when we pass to the limit as $\varepsilon \to 0$, that

$$\sum_{i,j} \langle \mu^j, a_{ij} \xi D_i v \rangle + \sum_{i,j} \int a_{ij} D_i \xi u D_j v \, dx = \sum_j \langle \mu^j, \xi f_j \rangle + \sum_j \int u D_j \xi f_j \, dx. \tag{3.7}$$

Note that the most delicate terms are the two involving the gradient of u_ε. For a Dini continuous function w (the domain of w is not specified since the function will be multiplied by a function with compact support) we will show that

$$\lim_{\varepsilon \to 0} \lim_{h \to 0} \int \Delta_h^j (u_\varepsilon - u) w \, \xi \, dx = 0,$$

or, in other terms, recalling that μ^j is the limit in the weak* topology of $(C_c^D(\Omega'))'$ of the incremental quotient of u

$$\lim_{\varepsilon \to 0} \int D_j u_\varepsilon w \, \xi \, dx = \langle \mu^j, w \, \xi \rangle.$$

We have:

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \lim_{h \to 0} \int \Delta_h^j u_\varepsilon w \, \xi \, dx \\
&= \lim_{\varepsilon \to 0} \lim_{h \to 0} \int \xi(x) w(x) \int \rho_\varepsilon(x - z) \frac{u(z + he_j) - u(z)}{h} \, dz \, dx \\
&= \lim_{\varepsilon \to 0} \lim_{h \to 0} \int \frac{u(z + he_j) - u(z)}{h} \int \rho_\varepsilon(x - z) \xi(x) w(x) \, dx \, dz \\
&= \lim_{\varepsilon \to 0} \lim_{h \to 0} \int \frac{u(z + he_j) - u(z)}{h} \int \rho_\varepsilon(z - x) \xi(x) w(x) \, dx \, dz \\
&= \lim_{\varepsilon \to 0} \lim_{h \to 0} \int \frac{u(z + he_j) - u(z)}{h} (w \, \xi)_\varepsilon(z) \, dz = \lim_{\varepsilon \to 0} \lim_{h \to 0} \int \Delta_h^j u \, (w \, \xi)_\varepsilon \, dz \\
&= \lim_{\varepsilon \to 0} \langle \mu^j, (w\xi)_\varepsilon \rangle = \langle \mu^j, w \, \xi \rangle,
\end{aligned}$$

where in the last equality we used again that a mollified function of a Dini continuous function with compact support strongly converges in C^D to the function itself. Thus we obtain (3.7).

From it, exploiting the symmetry of a_{ij} and using (3.6) we get

$$\begin{aligned}\sum_j \langle \mu^j, \xi f_j\rangle &= -\sum_{i,j}\langle \mu^i, a_{ij} D_j \xi v\rangle + \sum_{i,j}\int a_{ij} D_i \xi u D_j v\, dx - \sum_j \int u D_j \xi f_j\, dx \\ &= I_1 + I_2 + I_3.\end{aligned} \tag{3.8}$$

We now estimate the three terms I_m, $m = 1, 2, 3$. We have

$$|I_1| \le \sum_{i,j} \|\mu^i\|_{(C_c^D(\Omega'))'} \|a_{ij} v D_i \xi\|_{C^D(\Omega')}.$$

By the definition of the norm in the space of Dini continuous functions we have

$$\|a_{ij} v D_i \xi\|_{C^D(\Omega')} \le \Lambda \|v\|_{L^\infty(\Omega')} \|D_i \xi\|_{L^\infty(B_R)} + \int_0^{diam(\Omega')} \frac{(\overline{a_{ij} v D_i \xi})_{\Omega'}(r)}{r}\, dr.$$

By simple computation we have

$$(\overline{a_{ij} v D_i \xi})_{\Omega'}(r) \le \|v\|_{L^\infty(\Omega')} (\overline{a_{ij} D_i \xi})_{\Omega'}(r) + \|a_{ij} D_i \xi\|_{L^\infty(\Omega')} \overline{v}_{\Omega'}(r),$$

and, using the properties of the solution v (recall that $p > n$), the right hand side can be estimated as

$$\begin{aligned}(\overline{a_{ij} v D_i \xi})_{\Omega'}(r) \le\ & C (\overline{a_{ij} D_j \xi})_{\Omega'}(r) \|f\|_{L^p(\Omega', \mathbb{R}^n)} \\ & + C r^{1-\frac{n}{p}} \|a_{ij} D_j \xi\|_{L^\infty(\Omega')} \|Dv\|_{L^p(\Omega', \mathbb{R}^n)}.\end{aligned}$$

To summarize, we have

$$|I_1| \le C \|f\|_{L^p(\Omega', \mathbb{R}^n)}.$$

The estimate of I_2 and I_3 simply comes by Hölder's inequality and again by the properties of the solution v:

$$\begin{aligned}|I_2| \le \sum_{i,j} \left| \int a_{ij} D_i \xi u D_j v\, dx \right| &\le C \|u\|_{L^{n'}(\Omega')} \|D\xi\|_{L^\infty(B_R, \mathbb{R}^n)} \Lambda \|v\|_{W^{1,n}(\Omega')} \\ &\le C \|f\|_{L^n(\Omega', \mathbb{R}^n)},\end{aligned}$$

and

$$|I_3| \le \sum_j \left| \int u D_j \xi f_j dx \right| \le \|u\|_{L^{n'}(\Omega')} \|D\xi\|_{L^\infty(B_R, \mathbb{R}^n)} \|f\|_{L^n(\Omega', \mathbb{R}^n)}.$$

At the end, the estimates proved for I_1, I_2 and I_3 lead to

$$\sum_j \langle \mu^j, \xi f_j \rangle \leq C \|f\|_{L^p(\Omega', \mathbb{R}^n)},$$

as well

$$\sum_j \langle \mu^j \xi, f_j \rangle \leq C \|f\|_{L^p(\Omega', \mathbb{R}^n)}.$$

Since f is an arbitrary smooth function in $L^p(\Omega', \mathbb{R}^n)$, we conclude

$$\sum_j \|\mu^j \xi\|_{L^{p'}(\Omega')} \leq C,$$

which means, using a finite covering argument, that μ^j is a function in $L^{p'}_{\mathrm{loc}}(\Omega)$ and then $u \in W^{1,p'}_{\mathrm{loc}}(\Omega)$, since, for every $\varphi \in C^\infty_c(\Omega)$ and for h small enough, we have

$$\int \Delta^j_h u \, \varphi \, dx = \int u \Delta^j_{-h} \varphi \, dx;$$

passing to the limit as $h \to 0$, we derive

$$\langle \mu^j, \varphi \rangle = \int \varphi \mu^j \, dx = - \int u D_j \varphi \, dx.$$

Since $u \in W^{1,p'}_{\mathrm{loc}}(\Omega)$, Brezis's result implies that u is a weak solution of the equation (1.5), i.e. our statement. □

4. Sobolev coefficients

As pointed out in the Introduction, very weak solutions in $L^{n'}_{\mathrm{loc}}(\Omega)$ associated to coefficients in $W^{1,n}(\Omega)$ are not weak solutions, due to the counterexample found in [10]. The quoted references on this problem have suggested us to consider Sobolev coefficients with a modulus of continuity satisfying the double Dini condition.

On the other hand, another way to get around the counterexample is to deal with very weak solutions in $L^q_{\mathrm{loc}}(\Omega)$, with $q > n'$. The result is the following.

Theorem 4.1. *Let $u \in L^q_{\mathrm{loc}}(\Omega)$, $q > n'$, be a very weak solution of (1.1), with $A(x) = (a_{ij}(x))_{i,j}$ satisfying (1.2) and (1.3), then u belongs to $W^{1,2}_{\mathrm{loc}}(\Omega)$ and thus it is a weak solution.*

Proof. The proof rests on a duality and a bootstrap argument.

Step 1 We claim that $u \in W^{1,\left(\frac{qn'}{q-n'}\right)'}_{\mathrm{loc}}(\Omega)$.

We proceed as in the Step 1 of the proof of Theorem 1.2 to arrive to (3.2). Now we estimate the six terms $\mathcal{I}_m$. We use Hölder's inequality and Proposition 2.8 to get

$$
\begin{aligned}
|\mathcal{I}_1| &\le \|D\eta\|_{L^\infty(\mathbb{R}^n)}\|A\|_{W^{1,n}(\Omega,\mathbb{R}^{n\times n})}\|u\|_{L^q(\Omega_0)}\|Dv\|_{L^{\frac{qn'}{q-n'}}(\Omega_0,\mathbb{R}^n)} \le C\|w\|_{L^{\frac{qn'}{q-n'}}(\Omega_0)},\\
|\mathcal{I}_2| &\le \|A\|_{W^{1,n}(\Omega,\mathbb{R}^{n\times n})}\|u\|_{L^q(\Omega_0)}\|D^2v\|_{L^{\frac{qn'}{q-n'}}(\Omega_0,\mathbb{R}^{n\times n})} \le C\|w\|_{L^{\frac{qn'}{q-n'}}(\Omega_0)},\\
|\mathcal{I}_3|, |\mathcal{I}_4| &\le \Lambda\|u\|_{L^q(\Omega_0)}\|D\eta\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^n)}\|D^2v\|_{L^{q'}(\Omega_0,\mathbb{R}^{n\times n})} \le C\|w\|_{L^{q'}(\Omega')} \le C\|w\|_{L^{\frac{qn'}{q-n'}}(\Omega')},\\
|\mathcal{I}_5| &\le \Lambda\|u\|_{L^q(\Omega_0)}\|D^2\eta\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^{n\times n})}\|Dv\|_{L^{q'}(\Omega_0,\mathbb{R}^n)} \le C\|w\|_{L^{q'}(\Omega')} \le C\|w\|_{L^{\frac{qn'}{q-n'}}(\Omega')},
\end{aligned}
$$

and finally, as for (3.3),

$$
|\mathcal{I}_6| \le C\|w\|_{L^{\frac{qn'}{q-n'}}(\Omega')} + C\|w\|_{L^{q'}(\Omega')} \le C\|w\|_{L^{\frac{qn'}{q-n'}}(\Omega')}.
$$

So, arguing as in the Step 1 of Theorem 1.2, we deduce

$$
\left|\int_{\Omega'} w\Delta_h^\ell u\,dx\right| \le C\|w\|_{L^{\frac{qn'}{q-n'}}(\Omega')},
$$

which in turn implies, thanks also to Theorem 2.1, that $u \in W^{1,\left(\frac{qn'}{q-n'}\right)'}_{\mathrm{loc}}(\Omega)$. Let us note that thanks to this, the equation satisfied by u may be rewritten as

$$
\sum_{i,j}\int a_{ij}(x)D_iuD_j\varphi\,dx = 0, \tag{4.1}
$$

where the test functions φ can be taken in $W^{1,\frac{qn'}{q-n'}}(\Omega)$ with compact support. On the other hand, the summability of the solution u is not improved by its belonging to this Sobolev space, since $\left(\frac{qn'}{q-n'}\right)' = \frac{qn'}{qn'-q+n'}$ and the Sobolev conjugate of $\frac{qn'}{qn'-q+n'}$ is q.

Step 2 We prove that $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$.

As in the Step 2 of the proof of Theorem 1.2, for $j \in \{1,\dots,n\}$ let $f = (f_1,\dots,f_n)$ with $f_j \in C_c^\infty(\Omega')$ be such that

$$
\sum_j ||f_j||_{L^{q'}(\Omega')} \le 1.
$$

For every $p > 1$, let $v \in W_0^{1,p}(\Omega)$ be the weak solution of the problem

$$
\sum_{i,j}\int a_{ij}D_ivD_j\varphi\,dx = \sum_j\int D_j\varphi f_j\,dx \quad \forall\varphi \in W_0^{1,p'}(\Omega). \tag{4.2}
$$

By Theorem 2.5 we have in particular that

$$
||v||_{W^{1,q'}(\Omega')} \le C||f||_{L^{q'}(\Omega',\mathbb{R}^n)}.
$$

As before, we take $B_{R/2} \subset B_R \subset \Omega'$ a pair of concentric balls centered at $x_0 \in \Omega'$ and we consider $\xi(x) = \xi(|x - x_0|)$ a smooth function such that $\xi(t) = 1$ for $t \in [0, R/2]$ and $\xi(t) = 0$ for $t \geq R$. We can choose $\varphi = v\xi$ in (4.1) and $\varphi = u\xi$ as test function in (4.2), so that

$$\sum_{i,j} \int a_{ij} D_i u D_j v \, \xi \, dx + \sum_{i,j} \int a_{ij} D_i u D_j \xi v \, dx = 0,$$

and

$$\sum_{i,j} \int a_{ij} D_i v D_j u \, \xi \, dx + \sum_{i,j} \int a_{ij} D_i v D_j \xi u \, dx = \sum_j \int f_j D_j u \, \xi \, dx + \sum_j \int f_j D_j \xi u \, dx.$$

Subtracting the two equations and using the symmetry of a_{ij} we get

$$\sum_j \int f_j D_j u \, \xi \, dx = -\sum_j \int f_j D_j \xi u \, dx + \sum_{i,j} \int a_{ij} D_i v D_j \xi u \, dx - \sum_{i,j} \int a_{ij} D_i u D_j \xi v \, dx = I_1 + I_2 + I_3.$$

We estimate the three terms I_m. We have

$$|I_1| \leq \|u\|_{L^q(\Omega')} \|D\xi\|_{L^\infty(B_R, \mathbb{R}^n)} \|f\|_{L^{q'}(\Omega', \mathbb{R}^n)} \leq C \|f\|_{L^{q'}(\Omega', \mathbb{R}^n)},$$
$$|I_2| \leq \Lambda \|D\xi\|_{L^\infty(B_R, \mathbb{R}^n)} \|u\|_{L^q(\Omega')} \|Dv\|_{L^{q'}(\Omega', \mathbb{R}^n)} \leq C \|f\|_{L^{q'}(\Omega', \mathbb{R}^n)},$$

and finally

$$|I_3| \leq \Lambda \|u\|_{W^{1,\left(\frac{qn'}{q-n'}\right)'}(\Omega')} \|v\|_{L^{\frac{qn'}{q-n'}}(\Omega')} \leq C \|v\|_{W^{1,q'}(\Omega')},$$

where the last inequality derives from the fact that the Sobolev conjugate of q' is $\frac{qn'}{q-n'}$. To sum up we have obtained

$$\left| \sum_j \int f_j \xi D_j u \, dx \right| \leq C \|f\|_{L^{q'}(\Omega', \mathbb{R}^n)},$$

as well

$$\|\xi Du\|_{L^q(\Omega', \mathbb{R}^n)} \leq C,$$

and, using a finite covering argument, this implies that $u \in W^{1,q}_{\text{loc}}(\Omega)$. Let us observe that this Sobolev regularity improves the summability of u. In particular, $u \in L^{q^*}_{\text{loc}}(\Omega)$, where q^* is the Sobolev conjugate of q.

Step 3 We claim that if $q > n$ then u is a weak solution.

By the previous step, we deduce that if $q > n$ then the solution u is in $L^\infty_{\text{loc}}(\Omega)$. At this point, it is not difficult to prove, arguing as in Step 1, that $u \in W^{1,n}_{\text{loc}}(\Omega)$.

Step 4 We prove that $u \in L^\infty_{\text{loc}}(\Omega)$.

We just observed that if $q > n$ we are done. Let us consider now $q \leq n$. The solution u is in $W^{1,q}_{\mathrm{loc}}(\Omega)$ and by the Sobolev's embedding $u \in L^{q^*}_{\mathrm{loc}}(\Omega)$, where $q^* = \frac{qn}{n-q}$ if $q < n$ and any number greater then 1 if $q = n$. Arguing exactly as in the Step 2 we derive that $u \in W^{1,q^*}_{\mathrm{loc}}(\Omega)$, which in turn implies that u is in $L^{\infty}_{\mathrm{loc}}(\Omega)$ if $q^* > n$. We already noticed in Step 3 that this gives the desired result. Let us observe that if $q = n$, q^* is any number greater then 1 and so this can be chosen greater then n, while if $q < n$, $q^* > n$ is equivalent to $q > \frac{n}{2}$. We can iterate this procedure. Given $q > n' = \frac{n}{n-1}$ after (at most) $n - 1$ times we deduce that u is locally bounded.

By Step 3 the locally boundedness of the solution gives the desired result. □

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Appendix A: The C^1-Dini regularity of solutions to divergence form elliptic equations with Dini-continuous coefficients

As announced in Remark 2.6, we will specify the modulus of continuity of the gradient of solutions to (2.4) in the proof of [12, Theorem 1.1]. We will consider only the main points of the proof referring for the rest to [12]. The set Ω is supposed to be the ball B_4 centered at 0 and $\Omega' = B_1$. The improvement regards Proposition 1.1 in [12]: for the sake of completeness we will sketch the proof, modifying the original when needed.

Proposition A.1. *For $B_4 \subset \mathbb{R}^n$, $n \geq 1$, let a_{ij}, defined on B_4, satisfy (1.3) and (1.4) and let $f = (f_1, f_2, \dots, f_n)$ with $f_j \in C_c^\infty(B_4)$ for all $j \in \{1, \dots, n\}$. Assume that $u \in H^1(B_4)$ is a weak solution of (2.4), then there exist $a \in \mathbb{R}$ and $b \in \mathbb{R}^n$ such that*

$$\fint_{B_r} |u(x) - (a + b \cdot x)|\, dx \leq r\delta(r)[\|u\|_{L^2(B_2)} + \|f\|_{C^1(B_2)}], \quad \forall r \in (0,1), \tag{A.1}$$

where $\delta(r)$, depending on n, λ, Λ, and on the modulus of continuity of a_{ij} and f, is a monotonically increasing positive function defined on $(0,1)$ satisfying

$$\int_0^1 \frac{\delta(r)}{r}\, dr < +\infty.$$

Remark A.2. As shown in [12, Proposition 1.2], $\delta(r)$ will be the modulus of continuity of Du.

Proof. The proof is carried out for $f = 0$. We use the same notation of [12], denoting by φ the modulus of continuity such that

$$\left(\fint_{B_r} |A - A(0)|^2\right)^{\frac{1}{2}} \leq \varphi(r),$$

where $A = (a_{ij})_{i,j}$. Observe that in our case, assuming (1.4), $\varphi(r)$ has the following form

$$\begin{aligned}\left(\fint_{B_r} |A - A(0)|^2\right)^{\frac{1}{2}} &\leq \left(\fint_{B_r} \bar{A}_{B_4}(|x|)^2\, dx\right)^{\frac{1}{2}} \\ &\leq C\left(\frac{1}{r^n}\int_0^r \bar{A}_{B_4}(\rho)^2 \rho^{n-1} d\rho\right)^{\frac{1}{2}} =: \varphi(r),\end{aligned}$$

which is double-Dini continuous since $\varphi(r) \leq \bar{A}_{B_4}(r)$, and satisfies

$$\max_{r/2 \leq s \leq r} \varphi(s) \leq \mu\varphi(r),$$

with $\mu > 1$. As in [12], by induction, one will find, for $k \geq 0$, $w_k \in H^1(B_{3/4^{k+1}})$ such that

$$\sum_{i,j} D_j(a_{ij}(0) D_i w_k) = 0 \quad \text{in } B_{3/4^{k+1}}, \tag{A.2}$$

$$\|w_k\|_{L^2(B_{2/4^{k+1}})} \leq C4^{-\frac{k(n+2)}{2}}\varphi(4^{-k}),\ \|Dw_k\|_{L^\infty(B_{1/4^{k+1}}, \mathbb{R}^n)} C\varphi(4^{-k}), \tag{A.3}$$

$$\|D^2 w_k\|_{L^\infty(B_{1/4^{k+1}}, \mathbb{R}^{n\times n})} \leq C4^k \varphi(4^{-k}), \tag{A.4}$$

$$\|u - \sum_{j=0}^k w_j\|_{L^2(B_{1/4^{k+1}})} \leq 4^{-\frac{(k+1)(n+2)}{2}}\varphi(4^{-(k+1)}), \tag{A.5}$$

and

$$\|w_k\|_{L^\infty(B_{1/4^{k+1}})} \leq C4^{-k}\varphi(4^{-k}), \tag{A.6}$$

see [12, (14), (15), (16), (17), and (18) of Proposition 1.1]. Here and in the sequel C will denote a universal constant.

For $x \in B_{1/4^{k+1}}$, using (A.3), (A.4), (A.6) and Taylor expansion,

$$\begin{aligned}
&|\sum_{j=0}^{k} w_j(x) - \sum_{j=0}^{\infty} w_j(0) - \sum_{j=0}^{\infty} Dw_j(0)\cdot x| \\
&\quad\le \sum_{j=k+1}^{\infty} (|w_j(0)| + |Dw_j(0)||x|) + \sum_{j=0}^{k} \|D^2 w_j\|_{L^\infty(B_{1/4^{k+1}},\mathbb{R}^{n\times n})}|x|^2 \\
&\quad\le C\sum_{j=k+1}^{\infty} (4^{-j}\varphi(4^{-j}) + \varphi(4^{-j})|x|) + C\sum_{j=0}^{k} 4^j\varphi(4^{-j})|x|^2 \\
&\quad\le C4^{-(k+1)}\int_0^{4^{-k}} \frac{\varphi(r)}{r}\,dr + C|x|^2\int_{\frac{|x|}{2}}^1 \frac{\varphi(r)}{r^2}\,dr.
\end{aligned} \tag{A.7}$$

We then derive from (A.5) and the above, using Hölder's inequality, that

$$\begin{aligned}
&\int_{B_{1/4^{k+1}}} |u(x) - \sum_{j=0}^{\infty} w_j(0) - \sum_{j=0}^{\infty} Dw_j(0)\cdot x|\,dx \\
&\quad\le \|\sum_{j=0}^{k} w_j(x) - \sum_{j=0}^{\infty} w_j(0) - \sum_{j=0}^{\infty} Dw_j(0)\cdot x\|_{L^1(B_{1/4^{k+1}})} \\
&\qquad + \|u - \sum_{j=0}^{k} w_j(x)\|_{L^1(B_{1/4^{k+1}})} \\
&\quad\le C4^{-(k+1)(n+1)}\int_0^{4^{-k}} \frac{\varphi(r)}{r}\,dr + C\int_0^{1/4^{k+1}} \rho^{n+1}\int_{\frac{\rho}{2}}^1 \frac{\varphi(r)}{r^2}\,dr\,d\rho \\
&\qquad + C\,4^{-(k+1)(n+1)}\varphi(1/4^{k+1}).
\end{aligned} \tag{A.8}$$

Proposition A.1 follows from the above with $a = \sum_{j=0}^{\infty} w_j(0)$, $b = \sum_{j=0}^{\infty} Dw_j(0)\cdot x$, and

$$\delta(r) \simeq \int_0^r \frac{\varphi(s)}{s}\,ds + \frac{1}{r^{n+1}}\int_0^r \rho^{n+1}\int_{\frac{\rho}{2}}^1 \frac{\varphi(s)}{s^2}\,ds\,d\rho + \varphi(r),$$

the symbol $\simeq$ standing for $=$ up to a constant. It remains to prove that $\delta(r)$ is a Dini modulus of continuity. Thanks to assumption (1.4), it occurs if we show the Dini continuity of the second term in the previous sum. It yields

$$\frac{1}{r^{n+1}}\int_0^r \rho^{n+1}\int_{\frac{\rho}{2}}^1 \frac{\varphi(s)}{s^2}\,ds\,d\rho \le r\int_{\frac{r}{2}}^1 \frac{\varphi(s)}{s^2}ds + \int_0^r \frac{\varphi(\rho)}{\rho}\,d\rho,$$

so that, integrating by parts,

$$\int_0\int_{\frac{r}{2}}^1 \frac{\varphi(s)}{s^2}ds\,dr = r\int_{\frac{r}{2}}^1 \frac{\varphi(s)}{s^2}ds\Big|_0 + \int_0 \frac{\varphi(\frac{r}{2})}{r/4}dr.$$

It is easy to see that $\lim_{r\to 0} r\int_{\frac{r}{2}}^{1}\frac{\varphi(s)}{s^2}ds = 0$, and thus the thesis follows by the Dini continuity of φ. □

Appendix B: The counterexample

To construct the example, one first considers, for $r \in (0,1)$ and for $\beta > 1$, the function

$$\alpha(r) = \frac{-\beta n}{(n-1)\log\left(\frac{r_0}{r}\right)} + \frac{\beta(\beta+1)}{(n-1)\log^2\left(\frac{r_0}{r}\right)},$$

for some $r_0 > 1$. One takes then $A(x) = (a_{ij}(x))_{i,j}$ defined by

$$a_{ij}(x) = \delta_{ij} + \alpha(|x|)\left(\delta_{ij} - \frac{x_i x_j}{|x|^2}\right),$$

with r_0 large enough so that $\alpha \geq -\frac{1}{2}$, A being then uniformly elliptic.
Let us check now that $A \in W^{1,n}(B_1, \mathbb{R}^{n\times n})$. Simple computation gives

$$\left|\frac{\partial a_{ij}}{\partial x_\ell}\right| \lesssim |\alpha'(|x|)| + |\alpha(|x|)|\frac{1}{|x|},$$

for every $i,j,\ell = 1,\dots,n$ (the symbol $\lesssim$ stand for $\leq$ up to a constant). On the other hand

$$|\alpha'(|x|)| \simeq \frac{1}{|x|\log^2\left(\frac{r_0}{|x|}\right)} + \frac{1}{|x|\log^3\left(\frac{r_0}{|x|}\right)},$$

which in turn implies

$$\left|\frac{\partial a_{ij}}{\partial x_\ell}\right| \lesssim \frac{1}{|x|\log\left(\frac{r_0}{|x|}\right)} + \frac{1}{|x|\log^2\left(\frac{r_0}{|x|}\right)} + \frac{1}{|x|\log^3\left(\frac{r_0}{|x|}\right)} \lesssim \frac{1}{|x|\log\left(\frac{r_0}{|x|}\right)},$$

if r_0 is big enough. Thus, the belonging of A to $W^{1,n}(B_1, \mathbb{R}^{n\times n})$ is provided by the estimate

$$\int_{B_1}\left|\frac{\partial a_{ij}}{\partial x_\ell}\right|^n dx \lesssim \int_{B_1}\frac{1}{|x|^n\log^n\left(\frac{r_0}{|x|}\right)}dx \simeq \int_0^1 \frac{1}{r\log^n\left(\frac{r_0}{r}\right)}dr < +\infty.$$

With such an A, which is continuous but not Dini-continuous, in [10] the authors construct a solution of (1.1) (in the sense of distributions) $u \in W^{1,1}_{\mathrm{loc}}(B_1)\backslash W^{1,p}_{\mathrm{loc}}(B_1)$ for every $p > 1$. In particular, let us observe that such a solution belongs to $L^{n'}_{\mathrm{loc}}(B_1)$.

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