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Asymptotic profile of solutions for semilinear wave equations with structural damping

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Abstract. This paper is concerned with the initial value problem for semilinear wave equation with structural damping $u_{tt}+(-\Delta)^{\sigma}u_t-\Delta u = f(u)$, where $\sigma \in (0, \frac{1}{2})$ and $f(u) \sim |u|^p$ or $u|u|^{p-1}$ with $p > 1 + 2/(n - 2\sigma)$. We first show the global existence for initial data small in some weighted Sobolev spaces on $\mathbb{R}^n$ $(n \geq 2)$. Next, we show that the asymptotic profile of the solution above is given by a constant multiple of the fundamental solution of the corresponding parabolic equation, provided the initial data belong to weighted L^1 spaces.

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1. Introduction

In this paper, we consider the unique global existence of solutions and diffusion phenomina for the Cauchy problem of the semilinear wave equation with structural damping (damping term depends on the frequency) for $\sigma \in (0, \frac{1}{2})$:

$$\begin{cases} u_{tt} - \Delta u + (-\Delta)^{\sigma}u_t = f(u), & t \geq 0, \ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

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where $f \in C^{[\bar{s}],1}(\mathbb{R})$ $(1 \le \bar{s}, \ [\bar{s}] < p)$ satisfies

$$
\begin{cases}
|\dfrac{d^j}{du^j} f(u)| \le C|u|^{p-j} & (0 \le j \le [\bar{s}]), \\
|\dfrac{d^j}{du^j}(f(u) - f(v))| \le C|u - v|(|u| + |v|)^{p-[\bar{s}]} & (j = [\bar{s}]),
\end{cases}
\tag{1.2}
$$

for a positive constant C. Here, $[\bar{s}]$ denotes the integer part of $\bar{s}$.

For linear wave equations with structural damping:

$$
\begin{cases}
u_{tt} + (-\Delta)^\sigma u_t - \Delta u = 0, & t \ge 0, \ x \in \mathbb{R}^n, \\
u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), & x \in \mathbb{R}^n,
\end{cases}
\tag{1.3}
$$

with $\sigma \in (0, \frac{1}{2})$, Narazaki and Reissig [20] gave some $L^p - L^q$ $(1 \le p \le q \le \infty)$ estimates of the solutions. D'Abbicco and Ebert [2] showed the diffusion phenomena, by giving the $L^p - L^q$ decay estimates of the difference between the low frequency part of the solution of (1.3) and that of the corresponding parabolic equation

$$
v_t + (-\Delta)^{1-\sigma} v = 0, \quad t \ge 0, \ x \in \mathbb{R}^n,
\tag{1.4}
$$

with initial data $(-\Delta)^\sigma u_0 + u_1$. Ikehata and Takeda [11] showed that a constant multiple of the fundamental solution of the parabolic equation (1.4) gives the asymptotic profile of the solutions of (1.3) with $(u_0, u_1) \in (L^1 \cap H^1) \times (L^1 \cap L^2)$ (see Remark 3).

For semilinear structural damped wave equation (1.1) with $\sigma \in (0, \frac{1}{2})$, D'Abbicco and Reissig [5] first showed global existence and decay estimates of the solution of (1.1) with small initial data for space dimension $1 \le n \le 4$ and $p \in [2, n/[n-2]_+]$ such that

$$
p > p_\sigma := 1 + \frac{2}{n - 2\sigma}.
\tag{1.5}
$$

They showed the results by using $(L^1 \cap L^2) - L^2$ estimates of solutions of the linear wave equation with structural damping (1.3). In [5], they considered also for $\sigma \in [\frac{1}{2}, 1]$ and showed that p_σ is critical in a particular case $u_{tt} + 2(-\Delta)^\sigma u_t - \Delta u = 0$. Using the $L^p - L^q$ decay estimate $(1 \le p \le q \le \infty)$ of solutions of the linear wave equations with structural damping (1.3) by [2] for low frequency part, D'Abbicco and Ebert [4] (see also [3]) showed the unique existence of solutions of (1.1) for small initial data in some Sobolev spaces and gave the decay estimates of the solutions, in the following two cases:

$$
p_\sigma < p, \quad n < 1 + 2\max\left\{m \in \mathbb{N}; m < \frac{1 + 2\sigma}{1 - 2\sigma}\right\},
\tag{1.6}
$$

or

$$
p_\sigma < p < 1 + \frac{2(1 + 2\sigma)}{[n - 2(1 + 2\sigma)]_+}, \quad \left[\frac{n}{2}\right]\left[\frac{2}{p} - 1\right]_+ (1 - 2\sigma) < 1 + 2\sigma.
\tag{1.7}
$$

In [4], they also treated the case where $-\Delta u$ is replaced by $(-\Delta)^\delta u$ with $\delta > 0$.

The assumption (1.6) and (1.7) for $p < 2$ restrict the space dimension from above. The first purpose of this paper is to remove restriction of the space dimension n from above for every $\sigma \in (0, \frac{1}{2})$.

The second purpose is to give the asymptotic profile of the solutions of (1.1) as $t \to \infty$, if small initial data belongs to some weighted L^1 spaces. We show that a constant multiple of the fundamental solution of the parabolic equation (1.4) gives the asymptotic profile of (1.1) (Theorem 3). As as far as the author knows, there seems to be no results on the asymptotic profile for semilinear wave equation with structural damping (1.1) for $\sigma \in (0, \frac{1}{2})$.

In the case $\sigma = 0$, the asymptotic profile for semilinear damped wave equation is investigated. Since we treat nonlinear term not necessarily absorbing, we only refer to the results for non-absorbing type nonlinear term. Then if $1 < p \leq p_0$ where

$$p_0 := 1 + \frac{2}{n} \ : \text{Fujita Exponent},$$

then the solution of the semilinear damped wave equation blows up when $f(u) = |u|^p$ and the integrals of initial data on $\mathbb{R}^n$ are positive (see [10, 15, 23, 24]). On the other hand, in the case $p > p_0$, small data global existence is widely studied, (see [7–9, 12, 16–19, 21, 23], for example, and the references therein). The asymptotic profiles of the solutions are obtained as follows. Galley and Raugel [6] $(n = 1)$, Hosono and Ogawa [8] $(n = 2)$, showed that the asymptotic profile of the solutions is given by a constant multiple of the heat kernel $G(t, x)$, provided the initial data belong to some Sobolev spaces. (See also Kawakami and Takeda [14] for higher order asymptotic expansion in the case $n \leq 3$.) For general space dimensions, Hayashi, Kaikina and Naumkin [7] proved the unique existence of global solution $u \in C([0, \infty); H^{\bar{s}} \cap H^{0,\delta})$ for small initial data belonging to some weighted L^1 spaces, and showed that a constant multiple of the heat kernel gives the asymptotic profile of the solutions (see Remark 9).

We consider the equation in weighted Sobolev spaces as in [7]. The high frequency part of the structural damped wave equation has a good regularizing property. However, unlike the damped wave equation $(\sigma = 0)$, the Fourier transform of the kernel of the linear structural damped wave equation is singular at the origin. This fact causes the difficulty when we treat the equation in weighted Sobolev spaces. To get around this difficulty, we estimate the low frequency part in a new way employing Lorentz spaces (Lemma 1). For the estimate of nonlinear term, we use the method of [7,9].

This paper is organized as follows.

- In Sect. 2, we list some notations and state main results.
- In Sect. 3, we list known preliminary lemmas.
- In Sect. 4, we estimate kernels.
- In Sect. 5, we prove Theorem 1.
- In Sect. 6, we estimate a nonlinear term.
- In Sect. 7, we estimate a convolution term.

- In Sect. 8, we prove Proposition 1 and Theorems 2 and 3. That is, we prove the global existence of the solution of semilinear wave equation with structural damping, and give the asymptotic profile of the solutions.

2. Main results

Before stating our results, we list some notations.

Notation 1. *We write $\varphi(x) \lesssim \psi(x)$ on I if there exists a positive constant C such that*

$$\varphi(x) \leq C\psi(x) \text{ for every } x \in I.$$

We write $\varphi(x) \sim \psi(x)$ on I, if $\varphi(x) \lesssim \psi(x)$ and $\psi(x) \lesssim \varphi(x)$ on I.

Notation 2. *For $a \in \mathbb{R}$, $[a]_+ := \max\{a, 0\}$.*

Notation 3. *For every $q \in [1, \infty]$, we abbreviate $\mathbb{R}^n$ in $L^q(\mathbb{R}^n)$, and L^q norm is denoted by $\| \cdot \|_q$.*

Notation 4. *Let $H^{s,\delta} = H^{s,\delta}(\mathbb{R}^n)$ denote the weighted Sobolev space equipped with the norm*

$$\|u\|_{H^{s,\delta}} = \|\langle x \rangle^\delta (1 - \Delta)^{s/2} u\|_{L^2}.$$

$H^{s,0}$ equals H^s. Let $\dot{H}^s = \dot{H}^s(\mathbb{R}^n)$ denote the homogeneous Sobolev space equipped with the norm

$$\|u\|_{\dot{H}^s} = \|(-\Delta)^{s/2} u\|_{L^2}.$$

Notation 5. *(see [1, section 1.3], for example) Let $q \in (1, \infty)$ and $r \in [1, \infty]$. Let μ be the Lebesgue measure on $\mathbb{R}^n$. The distribution function $m(\tau, \varphi)$ is defined by*

$$m(\tau, \varphi) := \mu(\{x; |\varphi(x)| > \tau\}).$$

The Lorentz space $L_{q,r}$ consists of all locally integrable function φ on $\mathbb{R}^n$ such that

$$\|\varphi\|_{q,r} := \left(\int_0^\infty (t^{1/q} \varphi^*(t))^r \frac{dt}{t} \right)^{1/r} < \infty \quad \text{when} \quad r < \infty,$$

$$\|\varphi\|_{q,\infty} := \sup_t t^{1/q} \varphi^*(t) = \sup_\tau \tau m(\tau, \varphi)^{1/q} < \infty,$$

where $\varphi^*(t) = \inf\{\tau; m(\tau, \varphi) \leq t\}$ (the rearrangement of φ).

Notation 6. *For $\kappa \in (0, n)$, Riesz potential is the operator*

$$I_\kappa f(x) := \frac{1}{|x|^{n-\kappa}} * f = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\kappa}} dy = C_{n,\kappa} \mathcal{F}^{-1}(|\xi|^{-\kappa} \hat{f}(\rho)).$$

First we give the asymptotic profile of the solutions to linear wave equation with structural damping.

Theorem 1. Let $n \geq 1$. Let $(u_0, u_1) \in (L^1 \cap L^2) \times (L^1 \cap H^{-2\sigma})$ such that $|\cdot|^{\theta_j} u_j \in L^1$ with $\theta_j \in [0,1]$ $(j = 0, 1)$. Let $u \in C([0, \infty); H^1) \cap C^1((0, \infty); L^2)$ be a unique global solution of (1.3). Then the following holds.

$$\|u(t, \cdot) - \vartheta_0 H_\sigma(t, \cdot) - \vartheta_1 G_\sigma(t, \cdot)\|_2$$

$$\lesssim \langle t \rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{4}+2\sigma-1\right)} \|u_0\|_1 + \langle t \rangle^{\max\{\frac{1}{1-\sigma}\left(-\frac{n}{4}+3\sigma-1\right), \frac{1}{\sigma}\left(-\frac{n}{4}+\sigma\right)\}} \|u_1\|_1$$

$$+ e^{-\varepsilon_\sigma t}(\|u_0\|_2 + \|u_1\|_{H^{-2\sigma}})$$

$$+ t^{\frac{1}{1-\sigma}\left(-\frac{n}{4}-\frac{\theta_0}{2}\right)} \||\cdot|^{\theta_0} u_0\|_1 + t^{\frac{1}{1-\sigma}\left(-\frac{n}{4}+\sigma-\frac{\theta_1}{2}\right)} \||\cdot|^{\theta_1} u_1\|_1, \qquad (2.1)$$

where

$$H_\sigma(t, x) := \mathcal{F}^{-1}[e^{-|\xi|^{2(1-\sigma)}t}](x),$$

$$G_\sigma(t, x) := \mathcal{F}^{-1}[|\xi|^{-2\sigma} e^{-|\xi|^{2(1-\sigma)}t}](x) = C_{n,2\sigma}^{-1} I_{2\sigma} H_\sigma(t, x), \qquad (2.2)$$

$$\vartheta_0 := \int_{\mathbb{R}^n} u_0(y) dy, \quad \vartheta_1 := \int_{\mathbb{R}^n} u_1(y) dy. \qquad (2.3)$$

Remark 1.

$$\max\left\{ \frac{1}{1-\sigma}\left(-\frac{n}{4}+3\sigma-1\right), \frac{1}{\sigma}\left(-\frac{n}{4}+\sigma\right)\right\} = \begin{cases} \frac{1}{1-\sigma}\left(-\frac{n}{4}+3\sigma-1\right) & \text{if } 8\sigma \leq n, \\ \frac{1}{\sigma}\left(-\frac{n}{4}+\sigma\right) & \text{if } 8\sigma \geq n. \end{cases}$$

Remark 2. The function $H_\sigma(t, x)$ is the fundamental solution of the parabolic equation (1.4). We easily see that

$$\|G_\sigma(t, \cdot)\|_2 = \|\widehat{G_\sigma}(t, \cdot)\|_2 \sim t^{\frac{1}{1-\sigma}\left(-\frac{n}{4}+\sigma\right)}, \quad \|H_\sigma(t, \cdot)\| = \|\widehat{H_\sigma}(t, \cdot)\|_2 \sim t^{-\frac{n}{4(1-\sigma)}}, \qquad (2.4)$$

(see (5.8) and (5.16)). Putting $\theta_0 = [\theta - 2\sigma]_+$ and $\theta_1 = \theta$ $(\theta \in (0, 1])$ in (2.1), and taking (2.4) and the assumption $\sigma \in (0, \frac{1}{2})$ into consideration, we obtain

$$\|u(t, \cdot) - \vartheta_1 G_\sigma(t, \cdot)\|_2$$

$$\lesssim t^{\max\{\frac{1}{1-\sigma}\left(-\frac{n}{4}+\sigma-\min\{1-2\sigma, \frac{\theta}{2}\}\right), \frac{1}{\sigma}\left(-\frac{n}{4}+\sigma\right)\}}$$

$$\times (\|u_0\|_2 + \|u_1\|_{H^{-2\sigma}} + \|\langle\cdot\rangle^{[\theta-2\sigma]_+} u_0\|_1 + \|\langle\cdot\rangle^\theta u_1\|_1).$$

Thus, the decay order of $\|u(t, \cdot) - \vartheta_1 G_\sigma(t, \cdot)\|_2$ is larger than that of $\|G_\sigma(t, \cdot)\|_2$ itself, and therefore, $\vartheta_1 G_\sigma(t, x)$ gives the asymptotic profile of the solution if $\vartheta_1 \neq 0$.

If $u_1 = 0$, then (2.1) implies

$$\|u(t, \cdot) - \vartheta_0 H_\sigma(t, \cdot)\|_2 \lesssim t^{\frac{1}{1-\sigma}\left(-\frac{n}{4}-\min\{1-2\sigma, \frac{\theta_0}{2}\}\right)}(\|u_0\|_2 + \|\langle\cdot\rangle^{\theta_0} u_0\|_1). \qquad (2.5)$$

Thus, the decay order of $\|u(t, \cdot) - \vartheta_0 H_\sigma(t, \cdot)\|_2$ is larger than that of $\|H_\sigma(t, \cdot)\|_2$ itself if $\theta_0 > 0$, and therefore, $\vartheta_0 H_\sigma(t, x)$ gives the asymptotic profile if $\vartheta_0 \neq 0$.

Remark 3. Ikehata and Takeda [11, Theorem 1.2] showed

$$\|u(t, \cdot) - \vartheta_1 G_\sigma(t, \cdot)\|_2 = o(t^{\frac{1}{1-\sigma}\left(-\frac{n}{4}+\sigma\right)})$$

as $t \to \infty$ for initial data in $(u_0, u_1) \in (L^1 \cap H^1) \times (L^1 \cap L^2)$.

If $u_1 = 0$, Karch [13, Corollary 4.1] showed

$$\|u(t, \cdot) - \vartheta_0 H_\sigma(t, \cdot)\|_2 = o(t^{-\frac{n}{4(1-\sigma)}})$$

as $t \to \infty$ for $u_0 \in L^1$.

Theorem 2. (Global existence of the solution) *Let $n \geq 2$, and*

$$p > p_\sigma := 1 + \frac{2}{n - 2\sigma}. \tag{2.6}$$

Assume that $\bar{s} \geq 1$ and $[\bar{s}] < p$. If $2\bar{s} < n$, assume moreover that

$$p \leq 1 + \frac{2}{n - 2\bar{s}}. \tag{2.7}$$

Assume that $f \in C^{[\bar{s}],1}(\mathbb{R})$ satisfies (1.2). *Let q_j $(j = 0, 1)$ be numbers such that*

$$\begin{cases} q_0 = \dfrac{2n}{n + 2 - 4\sigma}, \quad q_1 = \dfrac{2n}{n + 2} \qquad \text{if} \quad p_\sigma < p < 1 + \dfrac{4}{n + 2 - 4\sigma}, \\[2mm] 1 < q_0 < \dfrac{n(p-1)}{2}, \quad 1 < q_1 < \dfrac{n(p-1)}{2(1 + (p-1)\sigma)} \\[2mm] \qquad\qquad\qquad\qquad \text{if} \quad 1 + \dfrac{4}{n + 2 - 4\sigma} < p \leq 1 + \dfrac{4}{n}, \\[2mm] 1 < q_0 < 2, \qquad\qquad 1 < q_1 < \dfrac{2n}{n + 4\sigma} \qquad \text{if} \quad 1 + \dfrac{4}{n} < p. \end{cases} \tag{2.8}$$

(Case 1). *In the case $p_\sigma < p \leq 1 + \frac{4}{n+2-4\sigma}$, let δ be a number satisfying*

$$2\left(\frac{1}{p-1} + \sigma\right) - \frac{n}{2} - 1 < \delta \leq \frac{2}{p-1} - \frac{n}{2}. \tag{2.9}$$

Then there exists a positive number ε such that if initial data

$$u_0 \in H^{\bar{s}} \cap H^{0,\delta}, \langle \cdot \rangle^\delta u_0 \in L^{q_0,2}, \quad u_1 \in \dot{H}^{\bar{s}-1}, \langle \cdot \rangle^\delta u_1 \in L^{q_1,2} \tag{2.10}$$

satisfy

$$\|\langle \cdot \rangle^\delta u_0\|_{q_0,2} + \|\langle \cdot \rangle^\delta u_0\|_2 + \|u_0\|_{H^{\bar{s}}} + \|\langle \cdot \rangle^\delta u_1\|_{q_1,2} + \|u_1\|_{\dot{H}^{\bar{s}-1}} \leq \varepsilon, \tag{2.11}$$

then initial value problem (1.1) *has a unique global solution*
$$u \in C([0, \infty); H^{\bar{s}} \cap H^{0,\delta}) \cap C^1((0, \infty); H^{\bar{s}-1}).$$

(Case 2). *In the case $p > 1 + \frac{4}{n+2-4\sigma}$, there exists a positive number ε such that if initial data*

$$u_0 \in H^{\bar{s}} \cap L^{q_0,2}, \quad u_1 \in H^{\bar{s}-1} \cap L^{q_1,2},$$

satisfy

$$\|u_0\|_{q_0,2} + \|u_0\|_{H^{\bar{s}}} + \|u_1\|_{q_1,2} + \|u_1\|_{H^{\bar{s}-1}} \leq \varepsilon, \tag{2.12}$$

then initial value problem (1.1) *has a unique global solution*
$$u \in C([0, \infty); H^{\bar{s}}) \cap C^1((0, \infty); H^{\bar{s}-1}).$$

Remark 4. We note that $L^{q_j} = L^{q_j, q_j} \subset L^{q_j, 2}$ by Lemma A given later, since $q_j \leq 2$.

Remark 5. If the space dimension $n = 2$, then $1 + \frac{4}{n+2-4\sigma} \leq p_\sigma$, and therefore, (Case 1) does not occur.

We prove Theorem 2 by using the following proposition.

Proposition 1. (Global existence of the solution) *Let* $n \geq 2$ *and* $r \in [1, \frac{2n}{n+4\sigma})$. *Let*

$$p > p_{\sigma,r} := 1 + \frac{2r}{n - 2r\sigma}. \tag{2.13}$$

Assume that $\bar{s} \geq 1$ *and* $[\bar{s}] < p$. *If* $2\bar{s} < n$, *assume moreover* (2.7). *Assume that* $f \in C^{[\bar{s}],1}(\mathbb{R})$ *satisfies* (1.2). *Let* δ *be a non-negative constant satisfying*

$$n\left(\frac{1}{r} - \frac{1}{2}\right) - 1 \leq \delta < n\left(\frac{1}{r} - \frac{1}{2}\right) - 2\sigma. \tag{2.14}$$

and

$$\begin{cases} n\left(\frac{1}{p} - \frac{1}{2}\right) < \delta, & if \quad r = 1, \\[2mm] n\left(\frac{1}{pr} - \frac{1}{2}\right) \leq \delta & if \quad r \in \left(1, \frac{2n}{n+4\sigma}\right). \end{cases} \tag{2.15}$$

Let

$$\hat{q}_0 = \frac{nr}{n - r(\delta + 2\sigma)}, \quad \hat{q}_1 = \frac{nr}{n - r\delta}. \tag{2.16}$$

Then there exists a positive number ε *such that if initial data*

$$u_0 \in H^{\bar{s}} \cap H^{0,\delta}, \langle \cdot \rangle^\delta u_0 \in L^{\hat{q}_0,2}, \quad u_1 \in \dot{H}^{\bar{s}-1}, \langle \cdot \rangle^\delta u_1 \in L^{\hat{q}_1,2} \tag{2.17}$$

satisfy

$$\begin{aligned} \|\langle \cdot \rangle^\delta u_0\|_{\hat{q}_0,2} + \|\langle \cdot \rangle^\delta u_0\|_2 + \|u_0\|_{H^{\bar{s}}} \\ + \|u_1\|_1 + \|\langle \cdot \rangle^\delta u_1\|_{\hat{q}_1,2} + \|(-\Delta)^{\frac{\bar{s}}{2}}(1-\Delta)^{-\frac{1}{2}} u_1\|_2 \leq \varepsilon, \end{aligned} \tag{2.18}$$

in the case $r = 1$, *and*

$$\begin{aligned} \|\langle \cdot \rangle^\delta u_0\|_{\hat{q}_0,2} + \|\langle \cdot \rangle^\delta u_0\|_2 + \|u_0\|_{H^{\bar{s}}} \\ + \|\langle \cdot \rangle^\delta u_1\|_{\hat{q}_1,2} + \|(-\Delta)^{\frac{\bar{s}}{2}}(1-\Delta)^{-\frac{1}{2}} u_1\|_2 \leq \varepsilon, \end{aligned} \tag{2.19}$$

in the case $r \in (1,2]$, *then initial value problem* (1.1) *has a unique global solution* $u \in C([0,\infty); H^{\bar{s}} \cap H^{0,\delta})) \cap C^1((0,\infty); H^{\bar{s}-1})$.

Furthermore, the solution satisfies estimate:

$$\sup_{t>0} \left(\langle t \rangle^{\frac{1}{1-\sigma}\left(\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\sigma\right)} \|u(t,\cdot)\|_2 + \langle t \rangle^{\frac{1}{1-\sigma}\left(\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{\delta}{2}-\sigma\right)} \||\cdot|^\delta u(t,\cdot)\|_2 \right.$$

$$\left. + \langle t \rangle^{\frac{1}{1-\sigma}\left(\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\sigma+\frac{\bar{s}}{2}\right)} \|(-\Delta)^{\frac{\bar{s}}{2}} u(t,\cdot)\|_2 \right) < \infty. \tag{2.20}$$

Remark 6. The assumption $r < \frac{2n}{n+4\sigma}$ implies that $n(\frac{1}{r} - \frac{1}{2}) - 2\sigma > 0$. The inequality

$$n\left(\frac{1}{pr} - \frac{1}{2}\right) < n\left(\frac{1}{r} - \frac{1}{2}\right) - 2\sigma$$

is equivalent to

$$p > 1 + \frac{2r\sigma}{n - 2r\sigma},$$

which holds by (2.13) since $\sigma < 1$. Hence, we can take a non-negative number δ satisfying assumptions (2.14) and (2.15).

If initial data belong to weighted L^1 space, the asymptotic profile of the solution is given by a constant multiple of the fundamental solution of the parabolic equation (1.4).

Theorem 3. (Asymptotic profile) *Assume the assumption of Proposition* 1 *with* $r = 1$. *Let* ε *be a positive constant given by Proposition* 1 *for* $r = 1$, *and let* $\theta \in [0, 1]$. *Assume that initial data satisfy* (2.17) *and* (2.18) *and that* $\langle \cdot \rangle^{[\theta - 2\sigma]_+} u_0, \langle \cdot \rangle^\theta u_1 \in L^1$. *Let* ν *be an arbitrary number satisfying*

$$0 < \nu < \min\left\{ \frac{n}{4}(p-2) + \frac{1}{2}p\delta, \delta \right\}. \tag{2.21}$$

Assume moreover that

$$\nu < \frac{\delta}{2\bar{s}} \left(n - \frac{p}{2}(n - 2\bar{s}) \right) \qquad if \quad \bar{s} < \frac{n}{2}. \tag{2.22}$$

Then there is a constant C depending on

$$\|\langle \cdot \rangle^{[\theta - 2\sigma]_+} u_0\|_1 + \|\langle \cdot \rangle^\delta u_0\|_{\frac{n}{n-(\delta+2\sigma)}} + \|\langle \cdot \rangle^\delta u_0\|_2 + \|u_0\|_{H^{\bar{s}}}$$

$$+ \|\langle \cdot \rangle^\theta u_1\|_1 + \|\langle \cdot \rangle^\delta u_1\|_{\frac{n}{n-\delta}} + \|(-\Delta)^{\frac{\bar{s}}{2}}(1-\Delta)^{-\frac{1}{2}} u_1\|_2$$

such that the solution $u \in C([0,\infty); H^{\bar{s}} \cap H^{0,\delta}) \cap C^1((0,\infty); H^{\bar{s}-1})$ of (1.1), *which is given by Proposition* 1, *satisfies the following:*

$$\left\| u(t,\cdot) - \Theta G_\sigma(t,\cdot) \right\|_2$$
$$\le C t^{\max\left\{ \frac{1}{1-\sigma}\left(-\frac{n}{4} + \sigma - \min\{(p-1)\left(\frac{n}{2}-\sigma\right)-1, 1-2\sigma, \nu, \frac{\theta}{2}\}\right), \frac{1}{\sigma}\left(-\frac{n}{4}+\sigma\right)\right\}}, \tag{2.23}$$

where G_σ is defined by (2.2) *and*

$$\Theta := \int_{\mathbb{R}^n} u_1(y) dy + \int_0^\infty \int_{\mathbb{R}^n} f(u(\tau, y)) dy d\tau. \tag{2.24}$$

Remark 7. The right-hand sides of (2.21) and (2.22) are positive. In fact, assumtion (2.15) implies $\frac{n}{2}(p-2) + p\delta > 0$, and (2.7) implies $n - \frac{p}{2}(n - 2\bar{s}) > 0$. Hence we can take ν satisfying (2.21) and (2.22).

Remark 8. Since (2.4) holds, (2.23) implies that G_σ gives the asymptotic profile of the solution if $\Theta \neq 0$.

Remark 9. In the case $\sigma = 0$, Hayashi, Kaikina and Naumkin [7] showed the existence of global solution $u \in C([0,\infty); H^{\bar{s}} \cap H^{0,\delta})$ of the semilinear damped wave (1.1) with $\sigma = 0$ for small initial data $u_0 \in H^{\bar{s}} \cap H^{0,\delta}, u_1 \in H^{\bar{s}-1} \cap H^{0,\delta}$ with $\delta > \frac{n}{2}$, and showed

$$\left\| u(t,\cdot) - \tilde{\Theta} G_0(t,\cdot) \right\|_q \le C t^{-\frac{n}{2}(1-\frac{1}{q}) - \min\{\frac{n}{2}(p-1)-1, \frac{\delta}{2} - \frac{n}{4}, \nu\}},$$

for $2 \leq q \leq \frac{2n}{n-2\bar{s}}$, where $\tilde{\Theta} = \int_{\mathbb{R}^n}(u_0(y) + u_1(y))dy + \int_0^\infty \int_{\mathbb{R}^n} f(u(\tau, y))dyd\tau$, G_0 is the heat kernel ((2.2) with $\sigma = 0$) and $0 < \nu < 1$.

3. Preliminary lemmas

We list some properties for weak L^p and Lorentz spaces which are used in this paper (see [1, section 1.3], [22], for example).

Lemma A. *Let $q \in (0, \infty)$. Then*

$$L^{q,q} = L^q, \quad L^{q,\infty} = L^*_q,$$
$$L^{q,\rho_1} \subset L^{q,\rho_2} \quad if \quad 1 \leq \rho_1 \leq \rho_2 \leq \infty.$$

Lemma B. *Assume that $\mu, \rho, \nu \in (1, \infty)$ and $\tilde{\mu}, \tilde{\rho}, \tilde{\nu} \in [1, \infty]$ satisfy*

$$\frac{1}{\mu} = \frac{1}{\rho} + \frac{1}{\nu}, \quad \frac{1}{\tilde{\mu}} = \frac{1}{\tilde{\rho}} + \frac{1}{\tilde{\nu}}.$$

Then

$$\|fg\|_{\mu,\tilde{\mu}} \lesssim \|f\|_{\rho,\tilde{\rho}}\|g\|_{\nu,\tilde{\nu}},$$

provided the right-hand side is finite.

The next corollary immediately follows from Lemma B.

Corollary A. *Let $\omega > 0$, $\mu, \nu \in (1, \infty)$ and $\tilde{\mu} \in [1, \infty]$. If*

$$\frac{1}{\mu} = \frac{\omega}{n} + \frac{1}{\nu},$$

then the following hold.

$$\left\||x|^{-\omega} f\right\|_{\mu,\tilde{\mu}} \lesssim \left\||x|^{-\omega}\right\|_{\frac{n}{\omega},\infty}\|f\|_{\nu,\tilde{\mu}} \lesssim \|f\|_{\nu,\tilde{\mu}}$$

Lemma C. *Let $q \in (2, \infty)$, and let q' be the dual exponent of q, that is, $\frac{1}{q} + \frac{1}{q'} = 1$. Let $\nu \in [1, \infty]$. Then*

$$\|\mathcal{F}[\varphi]\|_{q,\nu} \lesssim \|\varphi\|_{q',\nu}.$$

Lemma D. *(Young's inequality) Let $q, \rho \in (1, 2]$ such that $\frac{1}{q} + \frac{1}{\rho} = \frac{3}{2}$. Let $s, t \in [2, \infty)$ such that $\frac{1}{s} + \frac{1}{t} = \frac{1}{2}$. Then*

$$\|\varphi * \psi\|_2 \lesssim \|\varphi\|_{\rho,s}\|\psi\|_{q,t}.$$

Lemma E. *(sharp Sobolev embedding theorem) Let $q \in [2, \infty)$ and $s \geq 0$. If*

$$\frac{n}{2} - s \leq \frac{n}{q},$$

then

$$H^s(\mathbb{R}^n) \subset L^{q,2}(\mathbb{R}^n).$$

4. Decay estimate for the kernels

In this section, we estimate the kernel of the following linear wave equation with structural damping (1.3).

By Fourier transform, the equation (1.3) is transformed to

$$\hat{u}_{tt} + |\xi|^{2\sigma}\hat{u}_t + |\xi|^2\hat{u} = 0 \quad (t > 0), \qquad \hat{u}(0) = \hat{u}_0, \quad \hat{u}_t(0) = \hat{u}_1.$$

Hence the solution u of (1.3) is expressed as

$$u(t,x) = (K_0(t,\cdot) * u_0)(x) + (K_1(t,\cdot) * u_1)(x), \tag{4.1}$$

where

$$\widehat{K_0}(t,\xi) = \frac{1}{\lambda_+(|\xi|) - \lambda_-(|\xi|)}(\lambda_+(|\xi|)e^{\lambda_-(|\xi|)t} - \lambda_-(|\xi|)e^{\lambda_+(|\xi|)t}), \tag{4.2}$$

$$\widehat{K_1}(t,\xi) = \frac{1}{\lambda_+(|\xi|) - \lambda_-(|\xi|)}(e^{\lambda_+(|\xi|)t} - e^{\lambda_-(|\xi|)t}), \tag{4.3}$$

$$\lambda_\pm(|\xi|) = \frac{1}{2}\left(-|\xi|^{2\sigma} \pm \sqrt{|\xi|^{4\sigma} - 4|\xi|^2}\right) \tag{4.4}$$

$$= \begin{cases} \frac{1}{2}|\xi|^{2\sigma}\left(-1 \pm \sqrt{1 - 4|\xi|^{2(1-2\sigma)}}\right) & \text{if} \quad |\xi|^{1-2\sigma} < \frac{1}{2}, \\ \frac{1}{2}|\xi|^{2\sigma}\left(-1 \pm i\sqrt{4|\xi|^{2(1-2\sigma)} - 1}\right) & \text{if} \quad |\xi|^{1-2\sigma} > \frac{1}{2}. \end{cases} \tag{4.5}$$

We divide K_0 and K_1 into

$$\widehat{K_1^\pm}(t,\xi) := \pm\frac{e^{\lambda_\pm(|\xi|)t}}{\lambda_+(|\xi|) - \lambda_-(|\xi|)}, \tag{4.6}$$

$$\widehat{K_0^\pm}(t,\xi) := \mp\frac{\lambda_\mp(|\xi|)e^{\lambda_\pm(|\xi|)t}}{\lambda_+(|\xi|) - \lambda_-(|\xi|)} = -\lambda_\mp(\xi)\widehat{K_1^\pm}(t,\xi). \tag{4.7}$$

Let $\chi_{low}(\xi) \in C^\infty(\mathbb{R}^n)$ be a function such that $\chi_{low}(\xi) = 1$ for $|\xi| \leq 2^{-\frac{3}{1-2\sigma}}$ and $\chi_{low}(\xi) = 0$ for $|\xi| \geq 2^{-\frac{2}{1-2\sigma}}$. Let $\chi_{high}(\xi) \in C^\infty(\mathbb{R}^n)$ be a function such that $\chi_{high}(\xi) = 1$ for $|\xi| \geq 2$ and $\chi_{high}(\xi) = 0$ for $|\xi| \leq 1$.

We put

$$\chi_{mid}(\xi) : = 1 - \chi_{low}(\xi) - \chi_{high}(\xi),$$
$$\chi_{mh}(\xi) := 1 - \chi_{low}(\xi) = \chi_{mid}(\xi) + \chi_{high}(\xi).$$

Here we note that

$$\text{supp}\,\chi_{mid} \subset \{\xi; |\xi| \in [2^{-\frac{3}{1-2\sigma}}, 2]\}, \quad \text{supp}\,\chi_{hm} \subset \{\xi; |\xi| \in [2^{-\frac{3}{1-2\sigma}}, \infty)\}. \tag{4.8}$$

We put

$$K_{j,low}(t,x) := \mathcal{F}^{-1}[\hat{K}_j(t,\cdot)\chi_{low}(\cdot)],$$
$$K_{j,mid}(t,x) := \mathcal{F}^{-1}[\hat{K}_j(t,\cdot)\chi_{mid}(\cdot)],$$
$$K_{j,high}(t,x) := \mathcal{F}^{-1}[\hat{K}_j(t,\cdot)\chi_{high}(\cdot)],$$
$$K_{j,mh}(t,x) := K_{j,mid}(t,x) + K_{j,high}(t,x),$$

for $j = 0, 1$. Dividing the kernel into

$$K_j = K_{j,low} + K_{j,mid} + K_{j,high} = K_{j,low} + K_{j,mh}$$

for $j = 0, 1$, we estimate each part.

4.1. Estimate of the kernels for low frequency part

In this subsection, we consider low frequency region: $|\xi| \leq 2^{-\frac{2}{1-2\sigma}}$.

Lemma 1. *Let $\alpha > -\frac{n}{2}$ and $\beta > 0$. Let $a > 2^{-\frac{2}{1-2\sigma}}$. Let $g(t, \rho)$ be a smooth function on $[0, \infty) \times (0, a)$ satisfying*

$$\left| \frac{\partial^k}{\partial \rho^k} g(t, \rho) \right| \lesssim \rho^{\alpha-k} e^{-\frac{1}{2}\rho^\beta t} \tag{4.9}$$

on $[0, \infty) \times (0, a)$ for every $k = 0, 1, \cdots$. Put

$$K(t, x) := \mathcal{F}^{-1}[g(t, |\xi|)\chi_{low}](x).$$

Then for every $q_j \in [1, 2)$ $(j = 0, 1)$ and $\vartheta \in [0, \frac{n}{2} + \alpha)$ satisfying

$$\frac{1}{q_1} \geq \frac{1}{2} + \frac{\vartheta - \alpha}{n}, \quad \frac{1}{q_2} \geq \frac{1}{2} - \frac{\alpha}{n}, \tag{4.10}$$

the following holds.

$$\left\| |x|^\vartheta \left(K(t, \cdot) * \varphi(\cdot) \right) \right\|_2 \lesssim \langle t \rangle^{\frac{1}{\beta}\left(-n\left(\frac{1}{q_1}-\frac{1}{2}\right)+\vartheta-\alpha\right)} \|\varphi\|'_{q_1}$$
$$+ \langle t \rangle^{\frac{1}{\beta}\left(-n\left(\frac{1}{q_2}-\frac{1}{2}\right)-\alpha\right)} \||x|^\vartheta \varphi\|'_{q_2}, \tag{4.11}$$

where $\| \cdot \|'_q$ denote

$$\| \cdot \|'_q = \begin{cases} \| \cdot \|_1 & \text{if } q = 1 \\ \| \cdot \|_{q,2} & \text{if } q \in (1, 2]. \end{cases} \tag{4.12}$$

Before proving Lemma 1, we state two corollaries:

Corollary 1. *Let $\alpha > -\frac{n}{2}$ and $\beta > 0$. Let $a > 2^{-\frac{2}{1-2\sigma}}$. Let v and λ be smooth functions on some interval $(0, a)$ such that*

$$|v^{(j)}(\rho)| \lesssim \rho^{\alpha-j}, \tag{4.13}$$

$$|\lambda^{(j)}(\rho)| \lesssim \rho^{\beta-j}, \quad -\lambda(\rho) \sim \rho^\beta \tag{4.14}$$

on $(0, a)$ for every $j = 0, 1, \cdots$. Put

$$K(t, x) := \mathcal{F}^{-1}[v(|\xi|)e^{\lambda(|\xi|)t}(x)\chi_{low}](x). \tag{4.15}$$

Then the conclusion of Lemma 1 holds.

In fact, we easily see that

$$\left| \frac{\partial^k}{\partial \rho^k} \left(v(\rho)e^{\lambda(\rho)t} \right) \right| \lesssim \rho^{\alpha-k} \left(\sum_{j=0}^k (\rho^\beta t)^j \right) e^{-\rho^\beta t} \lesssim \rho^{\alpha-k} e^{-\frac{1}{2}\rho^\beta t} \tag{4.16}$$

on $(0, a)$ for every $k = 0, 1, \ldots$. Hence, $g(t, \rho) = v(\rho)e^{\lambda(\rho)t}$ satisfies the assumption (4.9) of Lemma 1, and thus the conclusion holds.

Corollary 2. *Let α, β, γ be numbers such that $\alpha - \beta + \gamma > -\frac{n}{2}$, $\beta > 0$ and $\gamma > 0$. Let $a > 2^{-\frac{2}{1-2\sigma}}$. Let υ and λ be smooth functions on $(0, a)$ such that*

$$|\upsilon^{(j)}(\rho)| \lesssim \rho^{\alpha-j}, \tag{4.17}$$

$$|\lambda^{(j)}(\rho)| \lesssim \rho^{\beta-j}, \quad -\lambda(\rho) \sim \rho^{\beta} \tag{4.18}$$

$$|\mu^{(j)}(\rho)| \lesssim \rho^{\gamma-j}, \quad -\mu(\rho) \sim \rho^{\gamma} \tag{4.19}$$

on $(0, a)$ for every $j = 1, 2, \cdots$. Put

$$K(t, x) := \mathcal{F}^{-1}[\upsilon(|\xi|)e^{\lambda(|\xi|)t}(1 - e^{\mu(|\xi|)t})\chi_{low}](x).$$

Then for every $q_j \in [1, 2)$ $(j = 1, 2)$ and $\vartheta \in [0, \frac{n}{2} + \alpha - \beta + \gamma)$ satisfying

$$\frac{1}{q_1} \geq \frac{1}{2} + \frac{\vartheta - \alpha + \beta - \gamma}{n}, \quad \frac{1}{q_2} \geq \frac{1}{2} + \frac{-\alpha + \beta - \gamma}{n}, \tag{4.20}$$

the following holds.

$$\left\| |x|^{\vartheta} \left(K(t, \cdot) * \varphi(\cdot) \right) \right\|_2 \lesssim \langle t \rangle^{\frac{1}{\beta}\left(-n(\frac{1}{q_1} - \frac{1}{2}) + \vartheta - \alpha + \beta - \gamma\right)} \|\varphi\|'_{q_1}$$
$$+ \langle t \rangle^{\frac{1}{\beta}\left(-n(\frac{1}{q_2} - \frac{1}{2}) - \alpha + \beta - \gamma\right)} \||x|^{\vartheta}\varphi\|'_{q_2}. \tag{4.21}$$

Remark 10. D'Abbicco and Ebert [2] considered the kernels:

$$K(t, x) = \mathcal{F}^{-1}[\upsilon(|\xi|)e^{\lambda(|\xi|)t}\chi_{low}](x),$$

where υ and λ satisfy the assumptions (4.13) and (4.14) for $\alpha > -1$ (see [2, Lemma 3.1]), and

$$K(t, x) = \mathcal{F}^{-1}\left[\upsilon(|\xi|)e^{\lambda(|\xi|)t}\frac{1 - e^{\mu(|\xi|)t}}{\mu(|\xi|)t}\chi_{low}\right](x),$$

where υ, λ and μ satisfy (4.17), (4.18) and (4.19) for $\alpha > -1, \beta > 0, \gamma > 0$ (see [2, Lemma 3.2]), and showed $L^p - L^q$ estimates of $\varphi \mapsto K(t, \cdot) * \varphi$ for $1 \leq p \leq q \leq \infty$ such that

(i) $p \neq q$ if $\alpha = 0$ and υ is not a constant,
(ii) $\frac{1}{p} - \frac{1}{q} \geq -\frac{\alpha}{n}$ if $\alpha \in (-1, 0)$,

by using the description of kernels by Bessel functions.

In this paper, we show weighted L^2 estimates of $K(t, \cdot) * \varphi$ in a way different from [2] by employing Lorentz spaces.

Proof of Corollary 2. By the Leibniz rule, we have

$$\frac{\partial^k}{\partial\rho^k}\left(\upsilon(\rho)e^{\lambda(\rho)t}(1 - e^{\mu(\rho)t})\right) = \sum_{j=0}^{k} C_{k,j}\frac{\partial^j}{\partial\rho^j}(\upsilon(\rho)e^{\lambda(\rho)t})\frac{\partial^{k-j}}{\partial\rho^{k-j}}(1 - e^{\mu(\rho)t}). \tag{4.22}$$

By assumtion (4.19), we have

$$\left|\frac{\partial^{k-j}}{\partial\rho^{k-j}}(1-e^{\mu(\rho)t})\right| = \left|\frac{\partial^{k-j}}{\partial\rho^{k-j}}e^{\mu(\rho)t}\right|$$

$$\lesssim \rho^{-(k-j)}\left(\sum_{i=1}^{k-j}(\rho^\gamma t)^i\right)e^{-\rho^\gamma t} = \rho^{-k+j}\rho^\gamma t\sum_{i=0}^{k-j-1}(\rho^\gamma t)^i e^{-\rho^\gamma t}$$

$$\lesssim \rho^{-k+j+\gamma}te^{-\frac{\rho^\gamma t}{2}} \leq \rho^{-k+j+\gamma}t, \qquad (4.23)$$

if $j \leq k-1$, and

$$\left|\frac{\partial^{k-j}}{\partial\rho^{k-j}}(1-e^{\mu(\rho)t})\right| = |1-e^{\mu(\rho)t}| = |\mu(\rho)te^{\theta\mu(\rho)t}| \lesssim \rho^\gamma t, \qquad (4.24)$$

with $\theta \in (0,1)$ if $j = k$. From (4.22), (4.16) with k replaced by j, (4.23) and (4.24), it follows that

$$\left|\frac{\partial^k}{\partial\rho^k}\left(\upsilon(\rho)e^{\lambda(\rho)t}(1-e^{\mu(\rho)t})\right)\right| \lesssim \rho^{\alpha+\gamma-k}te^{-\frac{1}{2}\rho^\beta t} \lesssim \rho^{\alpha+\gamma-\beta-k}e^{-\frac{1}{4}\rho^\beta t}$$

on $(0,a)$. Hence, $g(t,\rho) = \upsilon(\rho)e^{\lambda(\rho)t}(1-e^{\mu(\rho)t})$ satisfies the assumption (4.9) with α replaced by $\alpha - \beta + \gamma$, and therefore, Lemma 1 implies the assertion. $\square$

Now we prove Lemma 1.

Proof of Lemma 1. (Step 1) Let k be a non-negative integer and $\nu \in (0,\infty)$. We show that

$$\left\|(-\Delta)^{\frac{k}{2}}\hat{K}(t,\cdot)\right\|_\nu \lesssim \langle t\rangle^{\frac{1}{\beta}(-\alpha+k-\frac{n}{\nu})} \qquad (4.25)$$

for every $t \geq 0$ if $(-\alpha+k)\nu < n$, and

$$\left\|(-\Delta)^{\frac{k}{2}}\hat{K}(t,\cdot)\right\|_{\nu,\infty} \lesssim \langle t\rangle^{\frac{1}{\beta}(-\alpha+k-\frac{n}{\nu})} = 1, \qquad (4.26)$$

for every $t \geq 0$ if $(-\alpha+k)\nu = n$.

First, we assume that $(-\alpha+k)\nu < n$. Using assumtion (4.9) and changing variables by $t^{1/\beta}\rho = r$, we have

$$\left\|(-\Delta)^{\frac{k}{2}}\hat{K}(t,\cdot)\right\|_\nu^\nu \lesssim \left\|\sum_{|\gamma|=k}\partial_\xi^\gamma\hat{K}(t,\xi)\right\|_\nu^\nu$$

$$\lesssim \int_0^a \rho^{(\alpha-k)\nu}e^{-\frac{1}{2}\nu\rho^\beta t}\rho^{n-1}d\rho \qquad (4.27)$$

$$= t^{\frac{1}{\beta}((-\alpha+k)\nu-n)}\int_0^{ta}r^{-(-\alpha+k)\nu}e^{-\frac{1}{2}\nu r^\beta}r^{n-1}dr$$

$$\lesssim t^{\frac{1}{\beta}((-\alpha+k)\nu-n)}. \qquad (4.28)$$

By (4.27), we have

$$\left\|(-\Delta)^{\frac{k}{2}}\hat{K}(t,\cdot)\right\|_\nu^\nu \lesssim \int_0^a \rho^{(\alpha-k)\nu}\rho^{n-1}d\rho < \infty,$$

for $0 < t \le 1$, which together with (4.28) yields (4.25).

Next we assume that $(-\alpha + k)\nu = n$. By (4.9), we have

$$\left|\partial_\xi^\gamma \hat{K}(t,\xi)\right| \lesssim |\xi|^{\alpha - |\gamma|}$$

for every $t \ge 0$. Hence,

$$s\mu(\{\xi; |\partial_\xi^\gamma \hat{K}(t,\xi)| > s\})^{\frac{1}{\nu}} \lesssim s^{1 - \frac{n}{(-\alpha+k)\nu}} = 1,$$

if $|\gamma| = k$, and therefore,

$$\left\|(-\Delta)^{\frac{k}{2}} \hat{K}(t,\cdot)\right\|_{\nu,\infty} \lesssim \sum_{|\gamma|=k} \left\|\partial_\xi^\gamma \hat{K}(t,\xi)\right\|_{\nu,\infty} \lesssim 1,$$

for every $t \ge 0$, that is, (4.26) holds in the case $(-\alpha + k)\nu = n$.

(Step 2) Let $\vartheta \in [0,\delta]$ and $\kappa \in (1,2]$. We prove that

$$\left\||\cdot|^\vartheta K(t,\cdot)\right\|_{\kappa,2} \lesssim \langle t \rangle^{\frac{1}{\beta}(\vartheta - \alpha - n(1-\frac{1}{\kappa}))}, \tag{4.29}$$

for every $t > 0$ if $-\alpha + \vartheta < n(1 - \frac{1}{\kappa})$, and

$$\left\||\cdot|^\vartheta K(t,\cdot)\right\|_{\kappa,\infty} \lesssim \langle t \rangle^{\frac{1}{\beta}(\vartheta - \alpha - n(1-\frac{1}{\kappa}))} = 1, \tag{4.30}$$

for every $t > 0$ if $-\alpha + \vartheta = n(1 - \frac{1}{\kappa})$.

Let ω be a non-negative number such that $\vartheta + \omega$ becomes an integer and that

$$n\left(\frac{1}{\kappa} - \frac{1}{2}\right) \le \omega < \frac{n}{\kappa}. \tag{4.31}$$

Since $n \ge 2$, we can take ω satisfying above conditions. Let ν and its dual exponent ν' be the numbers defined by

$$\frac{1}{\kappa} = \frac{\omega}{n} + 1 - \frac{1}{\nu} = \frac{\omega}{n} + \frac{1}{\nu'}. \tag{4.32}$$

Since $0 < 1/\nu' = 1/\kappa - \omega/n \le 1/2$ by assumtion (4.31), we have

$$1 < \nu \le 2 \le \nu' < \infty. \tag{4.33}$$

Now we prove (4.29) under the assumption $-\alpha + \vartheta < n(1 - \frac{1}{\kappa})$. By Corollary A and Lemmas A and C together with the relation (4.33), we have

$$\left\||\cdot|^\vartheta K(t,\cdot)\right\|_{\kappa,2} = \left\||\cdot|^{-\omega}|\cdot|^{\omega+\vartheta} K(t,\cdot)\right\|_{\kappa,2} \lesssim \left\||\cdot|^{-\omega}\right\|_{\frac{n}{\omega},\infty} \left\||\cdot|^{\vartheta+\omega} K(t,\cdot)\right\|_{\nu',2}$$

$$\lesssim \left\||\cdot|^{\vartheta+\omega} K(t,\cdot)\right\|_{\nu',2} = \left\|\mathcal{F}^{-1}[(-\Delta)^{\frac{\vartheta+\omega}{2}} \hat{K}(t,\cdot)]\right\|_{\nu',2}$$

$$\lesssim \left\|(-\Delta)^{\frac{\vartheta+\omega}{2}} \hat{K}(t,\cdot)\right\|_{\nu,2}$$

$$\lesssim \left\|(-\Delta)^{\frac{\vartheta+\omega}{2}} \hat{K}(t,\cdot)\right\|_{\nu,\nu} = \left\|(-\Delta)^{\frac{\vartheta+\omega}{2}} \hat{K}(t,\cdot)\right\|_\nu. \tag{4.34}$$

Since the assumption $-\alpha + \vartheta < n(1-\frac{1}{\kappa})$ and (4.32) imply that $-\alpha + \vartheta + \omega < \frac{n}{\nu}$, we can take $k = \vartheta + \omega$ in (4.25). Substituting the inequality into (4.34), we obtain (4.29).

Next we prove (4.30) under the assumption $-\alpha + \vartheta = n(1 - \frac{1}{\kappa})$. By Corollary A and Lemma C together with (4.33), we have

$$
\begin{aligned}
\left\| |\cdot|^{\vartheta} K(t, \cdot) \right\|_{\kappa,\infty} &= \left\| |\cdot|^{-\omega} \cdot |\cdot|^{\omega+\vartheta} K(t, \cdot) \right\|_{\kappa,\infty} \\
&\lesssim \left\| |\cdot|^{\vartheta+\omega} K(t, \cdot) \right\|_{\nu',\infty} = \left\| \mathcal{F}^{-1} [(-\Delta)^{\frac{\vartheta+\omega}{2}} \hat{K}(t, \cdot)] \right\|_{\nu',\infty} \\
&\lesssim \left\| (-\Delta)^{\frac{\vartheta+\omega}{2}} \hat{K}(t, \cdot) \right\|_{\nu,\infty}.
\end{aligned}
\tag{4.35}
$$

Since the assumption $-\alpha + \vartheta = n(1 - \frac{1}{\kappa})$ implies $-\alpha + \vartheta + \omega = \frac{n}{\nu}$, we can take $k = \vartheta + \omega$ in (4.26). Substituting the inequality into (4.35), we obtain (4.30).

(Step 3) We define $r_j \in (1, 2]$ by $\frac{1}{q_j} + \frac{1}{r_j} = \frac{3}{2}$ $(j = 1, 2)$. We estimate each term of the right-hand side of

$$
\left\| |\cdot|^{\vartheta} (K(t, \cdot) * \varphi) \right\|_2 \lesssim \left\| (|\cdot|^{\vartheta} K(t, \cdot)) * \varphi \right\|_2 + \left\| K(t, \cdot) * (|\cdot|^{\vartheta} \varphi) \right\|_2. \tag{4.36}
$$

If $q_1 \in (1, 2)$, Lemma D yields

$$
\left\| (|\cdot|^{\vartheta} K(t, \cdot)) * \varphi \right\|_2 \lesssim \left\| |\cdot|^{\vartheta} K(t, \cdot) \right\|_{r_1,\infty} \|\varphi\|_{q_1,2}. \tag{4.37}
$$

The assumption (4.10) implies $-\alpha + \vartheta \leq n(\frac{1}{q_1} - \frac{1}{2}) = n(1 - \frac{1}{r_1})$. Hence, noting that $\left\| |\cdot|^{\vartheta} K(t, \cdot) \right\|_{r_1,\infty} \leq \left\| |\cdot|^{\vartheta} K(t, \cdot) \right\|_{r_1,2}$ and substituting (4.29) or (4.30) with $(\vartheta, \kappa) = (\vartheta, r_1)$ into (4.37), we obtain

$$
\left\| (|\cdot|^{\vartheta} K(t, \cdot)) * \varphi \right\|_2 \lesssim \langle t \rangle^{\frac{1}{\beta} \left(-n(\frac{1}{q_1} - \frac{1}{2}) + \vartheta - \alpha \right)} \|\varphi\|_{q_1}'. \tag{4.38}
$$

In the case $q_1 = 1$, Young's inequality yields

$$
\left\| (|\cdot|^{\vartheta} K(t, \cdot)) * \varphi \right\|_2 \lesssim \left\| |\cdot|^{\vartheta} K(t, \cdot) \right\|_2 \|\varphi\|_1 = \left\| |\cdot|^{\vartheta} K(t, \cdot) \right\|_{2,2} \|\varphi\|_1. \tag{4.39}
$$

The assumption $\vartheta < \frac{n}{2} + \alpha$ implies $-\alpha + \vartheta < \frac{n}{2} = n(1 - \frac{1}{2})$. Hence, substituting (4.29) with $(\vartheta, \kappa) = (\vartheta, 2)$ into (4.39), we see that (4.38) holds also for $q_1 = 1$.

We can estimate the second term of (4.36) in the same way: If $q_2 \in (1, 2)$, Lemma D yields

$$
\left\| K(t, \cdot) * (|\cdot|^{\vartheta} \varphi) \right\|_2 \lesssim \| K(t, \cdot) \|_{r_2,\infty} \left\| |\cdot|^{\vartheta} \varphi \right\|_{q_2,2}. \tag{4.40}
$$

The assumption (4.10) implies $-\alpha \leq n(\frac{1}{q_2} - \frac{1}{2}) = n(1 - \frac{1}{r_2})$. Hence, substituting (4.29) or (4.30) with $(\vartheta, \kappa) = (0, r_2)$ into (4.40), we obtain

$$
\left\| K * (|\cdot|^{\vartheta} \varphi) \right\|_2 \lesssim \langle t \rangle^{\frac{1}{\beta} \left(-n(\frac{1}{q_2} - \frac{1}{2}) - \alpha \right)} \| |x|^{\vartheta} \varphi \|_{q_2}'. \tag{4.41}
$$

In the case $q_2 = 1$, Young's inequality yields

$$
\left\| K(t, \cdot) * (|\cdot|^{\vartheta} \varphi) \right\|_2 \lesssim \| K(t, \cdot) \|_2 \left\| |\cdot|^{\vartheta} \varphi \right\|_1 = \| K(t, \cdot) \|_{2,2} \left\| |\cdot|^{\vartheta} \varphi \right\|_1. \tag{4.42}
$$

Since $-\alpha < \frac{n}{2} = n(1 - \frac{1}{2})$, we have (4.29) with $(\vartheta, \kappa) = (0, 2)$, which together with (4.42) yields (4.41) with $q_2 = 1$.

Hence, (4.38) and (4.41) hold for every case. Substituting (4.38) and (4.41) into (4.36), we obtain (4.11). $\qquad \square$

Lemma 2. *Assume that $0 \leq s_2 \leq s_1$ and $\vartheta \geq 0$ satisfy $\vartheta - s_1 + s_2 < \frac{n}{2} - 2\sigma$. If $q_j \in [1, 2)$ $(j = 1, 2, 3, 4)$ satisfy*

$$\frac{1}{q_1} \geq \frac{1}{2} + \frac{2\sigma + \vartheta - s_1 + s_2}{n}, \quad \frac{1}{q_2} \geq \frac{1}{2} + \frac{2\sigma - s_1 + s_2}{n}, \quad \frac{1}{q_3} \geq \frac{1}{2} + \frac{\vartheta - s_1 + s_2}{n},$$
$$(4.43)$$

then the following hold provided the right-hand sides are finite:

$$\left\| |\cdot|^{\vartheta} (-\Delta)^{\frac{s_1}{2}} \left(\mathcal{F}^{-1}[\widehat{K_1^+}(t, \cdot)\chi_{low}] * \varphi \right) \right\|_2$$
$$\lesssim \langle t \rangle^{\frac{1}{1-\sigma} \left(-\frac{n}{2}(\frac{1}{q_1} - \frac{1}{2}) + \frac{\vartheta - s_1 + s_2}{2} + \sigma \right)} \|(-\Delta)^{\frac{s_2}{2}} \varphi\|'_{q_1}$$
$$+ \langle t \rangle^{\frac{1}{1-\sigma} \left(-\frac{n}{2}(\frac{1}{q_2} - \frac{1}{2}) - \frac{s}{2} + \sigma \right)} \| |\cdot|^{\vartheta} (-\Delta)^{\frac{s_2}{2}} \varphi\|'_{q_2}, \quad (4.44)$$

$$\left\| |\cdot|^{\vartheta} (-\Delta)^{\frac{s_1}{2}} \left(\mathcal{F}^{-1}[\widehat{K_1^-}(t, \cdot)\chi_{low}] * \varphi \right) \right\|_2$$
$$\lesssim \langle t \rangle^{\frac{1}{\sigma} \left(-\frac{n}{2}(\frac{1}{q_1} - \frac{1}{2}) + \frac{\vartheta - s_1 + s_2}{2} + \sigma \right)} \|(-\Delta)^{\frac{s_2}{2}} \varphi\|'_{q_1}$$
$$+ \langle t \rangle^{\frac{1}{\sigma} \left(-\frac{n}{2}(\frac{1}{q_2} - \frac{1}{2}) - \frac{s}{2} + \sigma \right)} \| |\cdot|^{\vartheta} (-\Delta)^{\frac{s_2}{2}} \varphi\|'_{q_2}, \quad (4.45)$$

$$\left\| |\cdot|^{\vartheta} (-\Delta)^{\frac{s_1}{2}} \left(K_{1,low}(t, \cdot) * \varphi \right) \right\|_2$$
$$\lesssim \langle t \rangle^{\frac{1}{1-\sigma} \left(-\frac{n}{2}(\frac{1}{q_1} - \frac{1}{2}) + \frac{\vartheta - s_1 + s_2}{2} + \sigma \right)} \|(-\Delta)^{\frac{s_2}{2}} \varphi\|'_{q_1}$$
$$+ \langle t \rangle^{\frac{1}{1-\sigma} \left(-\frac{n}{2}(\frac{1}{q_2} - \frac{1}{2}) - \frac{s}{2} + \sigma \right)} \| |\cdot|^{\vartheta} (-\Delta)^{\frac{s_2}{2}} \varphi\|'_{q_2}, \quad (4.46)$$

$$\left\| |\cdot|^{\vartheta} (-\Delta)^{\frac{s_1}{2}} \left(\mathcal{F}^{-1}[\widehat{K_0^+}(t, \cdot)\chi_{low}] * \varphi \right) \right\|_2$$
$$\lesssim \langle t \rangle^{\frac{1}{1-\sigma} \left(-\frac{n}{2}(\frac{1}{q_3} - \frac{1}{2}) + \frac{\vartheta - s_1 + s_2}{2} \right)} \|(-\Delta)^{\frac{s_2}{2}} \varphi\|'_{q_3}$$
$$+ \langle t \rangle^{\frac{1}{1-\sigma} \left(-\frac{n}{2}(\frac{1}{q_4} - \frac{1}{2}) - \frac{s}{2} \right)} \| |\cdot|^{\vartheta} (-\Delta)^{\frac{s_2}{2}} \varphi\|'_{q_4}, \quad (4.47)$$

$$\left\| |\cdot|^{\vartheta} (-\Delta)^{\frac{s_1}{2}} \left(\mathcal{F}^{-1}[\widehat{K_0^-}(t, \cdot)\chi_{low}] * \varphi \right) \right\|_2$$
$$\lesssim \langle t \rangle^{\frac{1}{\sigma} \left(-\frac{n}{2}(\frac{1}{q_3} - \frac{1}{2}) + \frac{\vartheta - s_1 + s_2}{2} + 2\sigma - 1 \right)} \|(-\Delta)^{\frac{s_2}{2}} \varphi\|'_{q_3}$$
$$+ \langle t \rangle^{\frac{1}{\sigma} \left(-\frac{n}{2}(\frac{1}{q_4} - \frac{1}{2}) - \frac{s}{2} + 2\sigma - 1) \right)} \| |\cdot|^{\vartheta} (-\Delta)^{\frac{s_2}{2}} \varphi\|'_{q_4}, \quad (4.48)$$

$$\left\| |\cdot|^{\vartheta} (-\Delta)^{\frac{s_1}{2}} \left(K_{0,low}(t, \cdot) * \varphi \right) \right\|_2$$
$$\lesssim \langle t \rangle^{\frac{1}{1-\sigma} \left(-\frac{n}{2}(\frac{1}{q_3} - \frac{1}{2}) + \frac{\vartheta - s_1 + s_2}{2} \right)} \|(-\Delta)^{\frac{s_2}{2}} \varphi\|'_{q_3}$$
$$+ \langle t \rangle^{\frac{1}{1-\sigma} \left(-\frac{n}{2}(\frac{1}{q_4} - \frac{1}{2}) - \frac{s}{2} \right)} \| |\cdot|^{\vartheta} (-\Delta)^{\frac{s_2}{2}} \varphi\|'_{q_4}, \quad (4.49)$$

where $\| \cdot \|'_q$ is defined by (4.12).

Proof. Since

$$\left\| |\cdot|^{\vartheta} (-\Delta)^{\frac{s_1}{2}} \left(\mathcal{F}^{-1}[\hat{K}(t,\cdot)\chi_{low}] * \varphi \right) \right\|_2$$

$$= \left\| |\cdot|^{\vartheta} \mathcal{F}^{-1}[|\xi|^{s_1-s_2} \hat{K}(t,\xi)\chi_{low}|\xi|^{s_2}\hat{\varphi}(\xi)] \right\|_2$$

$$= \left\| |\cdot|^{\vartheta} (-\Delta)^{\frac{s_1-s_2}{2}} \mathcal{F}^{-1}[\hat{K}(t,\cdot)\chi_{low}] * (-\Delta)^{\frac{s_2}{2}} \varphi \right\|_2,$$

the conclusion reduces to the case $s_2 = 0$ by taking $s_1 - s_2$ and $(-\Delta)^{s_2}\varphi$ as s_1 and φ respectively.

Let $\lambda_{\pm}$ be the functions defined by (4.4). From (4.5), it follows that

$$-2\rho^{2-2\sigma} \le \lambda_+(\rho) = -\frac{2\rho^{2-2\sigma}}{1 + \sqrt{1 - 4\rho^{2(1-2\sigma)}}} \le -\rho^{2-2\sigma}, \qquad (4.50)$$

$$-\rho^{2\sigma} \le \lambda_-(\rho) \le -\frac{1}{2}\rho^{2\sigma}, \qquad (4.51)$$

$$\lambda_+(\rho) - \lambda_-(\rho) = \rho^{2\sigma}\sqrt{1 - 4\rho^{2(1-2\sigma)}} \sim \rho^{2\sigma}, \qquad (4.52)$$

on the support of χ_{low}.

We first prove (4.44). It is written that

$$(-\Delta)^{\frac{s_1}{2}} \left(\mathcal{F}^{-1}[\widehat{K_1^+}(t,\cdot)\chi_{low}] * \varphi \right) = \mathcal{F}^{-1}[|\xi|^{s_1}\widehat{K_1^+}(t,\cdot)\chi_{low}] * \varphi,$$

where K_1^+ is defined by (4.6). By definition, $K(t,x) = \mathcal{F}^{-1}[|\xi|^{s_1}\widehat{K_1^+}(t,\cdot)\chi_{low}]$ has the form (4.15) with

$$\upsilon(\rho) = \frac{\rho^{s_1}}{\lambda_+(\rho) - \lambda_-(\rho)}\chi_{low}, \quad \lambda(\rho) = \lambda_+(\rho).$$

By using (4.50) and (4.52), we easily see that υ and λ above satisfy the assumption of Corollary 1 with $\alpha = s_1 - 2\sigma$, $\beta = 2(1 - \sigma)$. The assumption $n \ge 2$ and $0 < 2\sigma < 1$ implies $\alpha > -\frac{n}{2}$ and $\beta > 0$, that is, α and β satisfy the assumption of Corollary 1. Definition of α and (4.43) imply (4.10) (here we note that we assume $s_2 = 0$). Hence, applying Corollary 1, we obtain (4.44).

$K(t,x) = \mathcal{F}^{-1}[|\xi|^{s_1}\widehat{K_1^-}(t,\cdot)\chi_{low}]$ (K_1^- is defined by (4.6)) has the form (4.15) with

$$\upsilon(\rho) = \frac{-\rho^{s_1}}{\lambda_+(\rho) - \lambda_-(\rho)}\chi_{low}, \quad \lambda(\rho) = \lambda_-(\rho).$$

By using (4.51) and (4.52), we easily see that υ and λ above satisfy the assumption of Corollary 1 for $\alpha = s_1 - 2\sigma(> -\frac{n}{2})$, $\beta = 2\sigma(> 0)$ and therefore, (4.45) holds in the same way as in the proof of (4.44).

Since $\sigma < 1 - \sigma$ by the assumption that $\sigma \in (0, 1/2)$, the estimate (4.46) follows from (4.44) and (4.45).

$K(t,x) = \mathcal{F}^{-1}[|\xi|^{s_1}\widehat{K_0^+}(t,\cdot)\chi_{low}]$ (K_0^+ is defined by (4.7)) has the form (4.15) with

$$\upsilon(\rho) = \frac{-\rho^{s_1}\lambda_-(\rho)}{\lambda_+(\rho) - \lambda_-(\rho)}\chi_{low}, \quad \lambda(\rho) = \lambda_+(\rho),$$

By using (4.50)–(4.52), we easily see that υ and λ satisfy the assumption of Corollary 1 with $\alpha = s_1, \beta = 2(1-\sigma)$. Definition of α and (4.43) imply assumtion (4.10) of q_1 for $q_1 = q_3$. Since $q_4 < 2$, the assumption on q_2 of (4.10) holds for $q_2 = q_4$. Hence, we can apply Corollary 1 to obtain (4.47).

Kernel $K = \mathcal{F}^{-1}[|\xi|^{s_1}\widehat{K_0^-}(t,\cdot)\chi_{low}]$ (K_0^- is defined by (4.7)) has the form (4.15) with

$$\upsilon(\rho) = \frac{\rho^{s_1}\lambda_+(\rho)}{\lambda_+(\rho) - \lambda_-(\rho)}\chi_{low}, \quad \lambda(\rho) = \lambda_-(\rho),$$

By using (4.50)–(4.52), we easily see that υ and λ above satisfy the assumption of Corollary 1 with $\alpha = s_1 + 2(1-2\sigma)(> 0), \beta = 2\sigma(> 0)$. Definition of α and (4.43) imply the assumption on q_1 of (4.10) for $q_1 = q_3$. The assumption on q_2 of (4.10) holds for $q_2 = q_4$ in the same reason as above. Hence, (4.48) holds by (4.11).

Since $\sigma \in (0, 1/2)$, inequality (4.49) follows from (4.47) and (4.48).   □

Lemma 3. *Let $\vartheta \geq 0$ and $0 \leq s_2 \leq s_1$ such that $\vartheta - s_1 + s_2 < \frac{n}{2} - 2\sigma$. Assume that $q_j \in [1, 2)$ $(j = 1, 2, 3, 4)$ satisfy*

$$\frac{1}{q_1} \geq \frac{1}{2} + \frac{2\sigma + \vartheta - s_1 + s_2}{n}, \quad \frac{1}{q_2} \geq \frac{1}{2} + \frac{2\sigma - s_1 + s_2}{n}, \quad \frac{1}{q_3} \geq \frac{1}{2} + \frac{\vartheta - s_1 + s_2}{n}.$$
$$(4.53)$$

Then the following hold provided the right-hand sides are finite:

$$\left\| |\cdot|^{\vartheta}(-\Delta)^{\frac{s_1}{2}}\left(\mathcal{F}^{-1}[(\widehat{K_1^+}(t,\cdot) - |\xi|^{-2\sigma}e^{-|\xi|^{2(1-\sigma)}t})\chi_{low}] * \varphi\right)\right\|_2$$
$$\lesssim \langle t\rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}(\frac{1}{q_1}-\frac{1}{2})+\frac{\vartheta-s_1+s_2}{2}+3\sigma-1\right)}\|(-\Delta)^{\frac{s_2}{2}}\varphi\|_{q_1}'$$
$$+ \langle t\rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}(\frac{1}{q_2}-\frac{1}{2})+3\sigma-1\right)}\||\cdot|^{\vartheta}(-\Delta)^{\frac{s_2}{2}}\varphi\|_{q_2}', \qquad (4.54)$$

$$\left\| |\cdot|^{\vartheta}(-\Delta)^{\frac{s_1}{2}}\left(\mathcal{F}^{-1}[(\widehat{K_0^+}(t,\xi) - e^{-|\xi|^{2(1-\sigma)}t})\chi_{low}] * \varphi\right)\right\|_2$$
$$\lesssim \langle t\rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}(\frac{1}{q_1}-\frac{1}{2})+\frac{\vartheta-s_1+s_2}{2}+2\sigma-1\right)}\|(-\Delta)^{\frac{s_2}{2}}\varphi\|_{q_1}'$$
$$+ \langle t\rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}(\frac{1}{q_2}-\frac{1}{2})+2\sigma-1\right)}\||\cdot|^{\vartheta}(-\Delta)^{\frac{s_2}{2}}\varphi\|_{q_2}', \qquad (4.55)$$

where $\|\cdot\|_q'$ is defined by (4.12).

Proof. We first prove (4.54). Let $\lambda_\pm$ be the functions defined by (4.5). By the same reason as in the proof of Lemma 2, we may assume that $s_2 = 0$. It follows from the definition that

$$|\xi|^{s_1}\widehat{K_1^+}(t,\xi) - |\xi|^{s_1-2\sigma}e^{-|\xi|^{2(1-\sigma)}t}$$
$$= |\xi|^{s_1-2\sigma}\left(\frac{1}{\sqrt{1-4|\xi|^{2(1-2\sigma)}}}\exp\left(\frac{-2|\xi|^{2(1-\sigma)}t}{1+\sqrt{1-4|\xi|^{2(1-2\sigma)}}}\right) - e^{-|\xi|^{2(1-\sigma)}t}\right)$$

$$= \frac{|\xi|^{s_1-2\sigma} e^{-|\xi|^{2(1-\sigma)}t}}{\sqrt{1-4|\xi|^{2(1-2\sigma)}}} \left(\exp\left(\frac{-4|\xi|^{2(2-3\sigma)}t}{(1+\sqrt{1-4|\xi|^{2(1-2\sigma)}})^2} \right) - 1 \right)$$

$$- \frac{4|\xi|^{s_1+2(1-3\sigma)} e^{-|\xi|^{2(1-\sigma)}t}}{(1+\sqrt{1-4|\xi|^{2(1-2\sigma)}})\sqrt{1-4|\xi|^{2(1-2\sigma)}}}.$$

$$:= M_{1,1} + M_{1,2} \quad \text{(we put).} \tag{4.56}$$

We easily see that

$$\upsilon(\rho) = \frac{-\rho^{s_1-2\sigma}}{\sqrt{1-4\rho^{2(1-2\sigma)}}}, \quad \lambda(\rho) = -\rho^{2(1-\sigma)}, \quad \mu(\rho) = \frac{4\rho^{2(2-3\sigma)}}{(1+\sqrt{1-4\rho^{2(1-2\sigma)}})^2}$$

satisfy the assumption (4.17)–(4.19) of Corollary 2 with $\alpha = s_1 - 2\sigma, \beta = 2(1-\sigma), \gamma = 2(2-3\sigma)$. Then the assumption $n \geq 2$ and $2\sigma < 1$ implies $\alpha - \beta + \gamma = s_1 + 2(1-3\sigma) > -\frac{n}{2}$ and $\beta > 0$, that is, α, β and γ satisfy the assumption of Corollary 2. The assumption (4.53) and the definition of α, β, γ above imply

$$\frac{1}{q_1} \geq \frac{1}{2} + \frac{2\sigma + \vartheta - s_1}{n} \geq \frac{1}{2} + \frac{\vartheta - \alpha + \beta - \gamma}{n},$$

$$\frac{1}{q_2} \geq \frac{1}{2} + \frac{2\sigma - s_1}{n} \geq \frac{1}{2} + \frac{-\alpha + \beta - \gamma}{n},$$

that is, (4.20) holds. Hence, we can apply Corollary 2 for the above choice to obtain

$$|M_{1,1}| \lesssim \langle t \rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}(\frac{1}{q_1}-\frac{1}{2})+\frac{\vartheta-s_1}{2}+3\sigma-1\right)} \|\varphi\|'_{q_1}$$

$$+ \langle t \rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}(\frac{1}{q_2}-\frac{1}{2})-\frac{s}{2}+3\sigma-1\right)} \||\cdot|^{\vartheta}\varphi\|'_{q_2}. \tag{4.57}$$

We also see that

$$\upsilon(\rho) = \frac{4\rho^{s_1+2(1-3\sigma)}}{(1+\sqrt{1-4\rho^{2(1-2\sigma)}})\sqrt{1-4\rho^{2(1-2\sigma)}}}, \quad \lambda(\rho) = -\rho^{2(1-\sigma)}$$

satisfy the assumption of Corollary 1 with $\alpha = s_1 + 2(1-3\sigma)(> -\frac{n}{2})$ and $\beta = 2(1-\sigma)(> 0)$. The assumption (4.10) is satisfied by (4.53) together with the definition of α. Hence, we can apply Corollary 1 to obtain

$$|M_{1,2}| \lesssim \langle t \rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}(\frac{1}{q_1}-\frac{1}{2})+\frac{\vartheta-s_1}{2}+3\sigma-1\right)} \|\varphi\|'_{q_1}$$

$$+ \langle t \rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}(\frac{1}{q_2}-\frac{1}{2})-\frac{s_1}{2}+3\sigma-1\right)} \||\cdot|^{\vartheta}\varphi\|'_{q_2}. \tag{4.58}$$

Inequality (4.54) follows from (4.56), (4.57) and (4.58).

Next we prove (4.55). It follows from (4.5), (4.50) and (4.52) that

$$|\xi|^{s_1} \widehat{K_0^+}(t,\xi) - |\xi|^{s_1} e^{-|\xi|^{2(1-\sigma)}t}$$

$$= |\xi|^{s_1} \left(\frac{-\lambda_-(|\xi|)}{\lambda_+(|\xi|) - \lambda_-(|\xi|)} - 1 \right) e^{\lambda_+(|\xi|)t} + |\xi|^{s_1} \left(e^{\lambda_+(|\xi|)t} - e^{-|\xi|^{2(1-\sigma)}t} \right)$$

$$= -\frac{2|\xi|^{s_1+2(1-2\sigma)}}{(1+\sqrt{1-4|\xi|^{2(1-2\sigma)}})\sqrt{1-4|\xi|^{2(1-2\sigma)}}} \exp(\lambda_+(|\xi|)t)$$

$$+ |\xi|^{s_1} \exp(-|\xi|^{2(1-\sigma)}t) \left(\exp\left(\frac{-4|\xi|^{2(2-3\sigma)}t}{(1 + \sqrt{1 - 4|\xi|^{2(1-2\sigma)}})^2} \right) - 1 \right) \quad (4.59)$$

$$=: M_{2,1} + M_{2,2} \quad \text{(we put)}. \tag{4.60}$$

We easily see that

$$\upsilon(\rho) = \frac{2\rho^{s_1 + 2(1-2\sigma)}}{(1 + \sqrt{1 - 4\rho^{2(1-2\sigma)}})\sqrt{1 - 4\rho^{2(1-2\sigma)}}},$$

$$\lambda(\rho) = \lambda_+(\rho) = \frac{-2\rho^{2-2\sigma}t}{1 + \sqrt{1 - 4\rho^{2(1-2\sigma)}}}$$

satisfy the assumption of Corollary 1 with $\alpha = s_1 + 2(1 - 2\sigma), \beta = 2(1 - \sigma)$. The assumption (4.53) and the definition of α above yield

$$\frac{1}{q_3} \geq \frac{1}{2} + \frac{\vartheta - s_1}{n} \geq \frac{1}{2} + \frac{\vartheta - \alpha}{n},$$

$$\frac{1}{q_4} > \frac{1}{2} \geq \frac{1}{2} + \frac{-\alpha}{n},$$

that is, (4.10) holds for $q_1 = q_3$ and $q_2 = q_4$. Hence, we can apply Corollary 1 to obtain

$$|M_{2,1}| \lesssim \langle t \rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}(\frac{1}{q_3} - \frac{1}{2}) + \frac{\vartheta - s_1}{2} + 2\sigma - 1\right)} \|\varphi\|'_{q_3}$$

$$+ \langle t \rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}(\frac{1}{q_4} - \frac{1}{2}) - \frac{s_1}{2} + 2\sigma - 1\right)} \||\cdot|^{\vartheta}\varphi\|'_{q_4}. \tag{4.61}$$

We also see that

$$\upsilon(\rho) = \rho^{s_1}, \quad \lambda(\rho) = -\rho^{2(1-\sigma)}, \quad \mu(\rho) = \frac{4\rho^{2(2-3\sigma)}}{(1 + \sqrt{1 - 4\rho^{2(1-2\sigma)}})^2}$$

satisfy the assumption of Corollary 2 with $\alpha = s_1, \beta = 2(1 - \sigma), \gamma = 2(2 - 3\sigma)$. The assumption (4.53) and the definition of α, β, γ above yield

$$\frac{1}{q_3} \geq \frac{1}{2} + \frac{\vartheta - s_1}{n} > \frac{1}{2} + \frac{\vartheta - s_1 - 2(1 - 2\sigma)}{n} = \frac{1}{2} + \frac{\vartheta - \alpha + \beta - \gamma}{n},$$

$$\frac{1}{q_4} > \frac{1}{2} \geq \frac{1}{2} + \frac{-s_1 - 2(1 - 2\sigma)}{n} = \frac{1}{2} + \frac{-\alpha + \beta - \gamma}{n},$$

that is, (4.20) holds for $q_1 = q_3, q_2 = q_4$. Hence, we can apply Corollary 2 to obtain

$$|M_{2,2}| \lesssim \langle t \rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}(\frac{1}{q_3} - \frac{1}{2}) + \frac{\vartheta - s_1}{2} + 2\sigma - 1\right)} \|\varphi\|'_{q_3}$$

$$+ \langle t \rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}(\frac{1}{q_4} - \frac{1}{2}) - \frac{s_1}{2} + 2\sigma - 1\right)} \||\cdot|^{\vartheta}\varphi\|'_{q_4}. \tag{4.62}$$

Inequality (4.55) follows from (4.60), (4.61) and (4.62). $\qquad\qquad \square$

4.2. Estimate of the kernels for high frequency part($|\xi| \geq 1$)

In this subsection, we consider high frequency region: $|\xi| \geq 1$.

Lemma 4. *For every $s \geq 0, \delta \geq 0$, the following hold.*

$$\left\|(-\Delta)^{\frac{s}{2}}\left(K_{1,high}(t,\cdot) * \varphi\right)\right\|_2 \lesssim e^{-\frac{t}{2}}\left\|(-\Delta)^{\frac{s}{2}}(1-\Delta)^{-\frac{1}{2}}\varphi\right\|_2, \tag{4.63}$$

$$\left\|\langle\cdot\rangle^{\delta}\left(K_{1,high}(t,\cdot) * \varphi\right)\right\|_2 \lesssim e^{-\frac{t}{2}}\left\|(1-\Delta)^{-\frac{1}{2}}\langle\cdot\rangle^{\delta}\varphi\right\|_2, \tag{4.64}$$

$$\left\|(-\Delta)^{\frac{s}{2}}\left(K_{0,high}(t,\cdot) * \varphi\right)\right\|_2 \lesssim e^{-\frac{t}{2}}\left\|(-\Delta)^{\frac{s}{2}}\varphi\right\|_2, \tag{4.65}$$

$$\left\|\langle\cdot\rangle^{\delta}\left(K_{0,high}(t,\cdot) * \varphi\right)\right\|_2 \lesssim e^{-\frac{t}{2}}\left\|\langle\cdot\rangle^{\delta}\varphi\right\|_2, \tag{4.66}$$

provided the right-hand sides are finite.

Proof. We easily see that

$$\langle\xi\rangle|\widehat{K_1^{\pm}}(t,\xi)| \lesssim e^{-\frac{|\xi|^{2\sigma}}{2}t} \leq e^{-\frac{t}{2}}, \tag{4.67}$$

on the support of χ_{high}. Hence,

$$\left\|(-\Delta)^{\frac{s}{2}}\left(K_{1,high}(t,\cdot) * \varphi\right)\right\|_2 = \left\||\xi|^s \widehat{K_1}(t,\xi)\chi_{high}(\xi)\hat{\varphi}(\xi)\right\|_2$$

$$\leq \left\|\widehat{K_1}(t,\xi)\chi_{high}(\xi)\langle\xi\rangle\right\|_{\infty}\left\||\xi|^s\langle\xi\rangle^{-1}\hat{\varphi}(\xi)\right\|_2 \lesssim e^{-\frac{t}{2}}\left\|(-\Delta)^{\frac{s}{2}}(1-\Delta)^{-\frac{1}{2}}\varphi\right\|_2,$$

that is, (4.63) holds.

In the proof of [9, p. 10] (see also [7, p. 643]), the following Leibniz rule is shown:

$$\|(1-\Delta)^{\frac{\vartheta}{2}}(\varphi\psi)\|_2 \leq \sum_{j=1}^{n}\sum_{k=1}^{[\vartheta]+1}\|\partial_j^k\varphi\|_{\infty}\|(1-\Delta)^{\frac{\vartheta}{2}}\psi\|_2. \tag{4.68}$$

Since $|\partial_j^k(\widehat{K_1}(t,\xi)\chi_{high}(\xi)\langle\xi\rangle)| \leq C_k e^{-\frac{t}{2}}$ for every nonnegative integer k, we have

$$\left\||\cdot|^{\vartheta}\left(K_{1,high} * \varphi\right)\right\|_2 = \left\||(-\Delta)^{\frac{\vartheta}{2}}\left((\widehat{K_1}(t,\xi)\chi_{high}(\xi)\langle\xi\rangle)(\langle\xi\rangle^{-1}\hat{\varphi}(\xi))\right)\right\|_2$$

$$\lesssim e^{-\frac{t}{2}}\left\|\langle\xi\rangle^{-1}(1-\Delta)^{\vartheta/2}\hat{\varphi}(\xi)\right\|_2 \sim e^{-\frac{t}{2}}\|(1-\Delta)^{-\frac{1}{2}}\langle\cdot\rangle^{\vartheta}\varphi\|_2.$$

Taking $\vartheta = 0$ and δ in this inequality, we obtain (4.64).

We can prove (4.65) and (4.66) in the same way. $\qquad\square$

4.3. Estimate of the kernels for middle frequency part

In this subsection, we consider the region: $|\xi| \in [2^{-\frac{3}{1-2\sigma}}, 2]$.

Lemma 5. *There is a constant $\varepsilon_{\sigma} \in (0, \frac{1}{2})$ such that the following hold for every $s \geq 0, \delta \geq 0$:*

$$\left\|(-\Delta)^{\frac{s}{2}}\left(K_{1,mid}(t,\cdot) * \varphi\right)\right\|_2 \lesssim e^{-\varepsilon_{\sigma}t}\|(1-\Delta)^{-\frac{1}{2}}\varphi\|_2, \tag{4.69}$$

$$\left\|\langle\cdot\rangle^{\delta}\left(K_{1,mid}(t,\cdot) * \varphi\right)\right\|_2 \lesssim e^{-\varepsilon_{\sigma}t}\|(1-\Delta)^{-\frac{1}{2}}\langle\cdot\rangle^{\delta}\varphi\|_2, \tag{4.70}$$

$$\left\|(-\Delta)^{\frac{s}{2}}\left(K_{0,mid}(t,\cdot) * \varphi\right)\right\|_2 \lesssim e^{-\varepsilon_{\sigma}t}\|\varphi\|_2, \tag{4.71}$$

$$\left\|\langle\cdot\rangle^{\delta}\left(K_{0,mid}(t,\cdot) * \varphi\right)\right\|_2 \lesssim e^{-\varepsilon_{\sigma}t}\|\langle\cdot\rangle^{\delta}\varphi\|_2. \tag{4.72}$$

provided the right-hand sides are finite.

Proof. By definitions (4.3), (4.4) and (4.5), we have

$$|\widehat{K_1}(t,\xi)| = \left| \frac{e^{\lambda_+(|\xi|)t} - e^{\lambda_-(|\xi|)t}}{\lambda_+(|\xi|) - \lambda_-(|\xi|)} \right| = t \left| e^{(\theta\lambda_+(|\xi|)+(1-\theta)\lambda_-(|\xi|))t} \right|$$
$$= t \left| e^{\frac{t}{2}(-|\xi|^{2\sigma}+(2\theta-1)|\xi|^{2\sigma}\sqrt{1-4|\xi|^{2(1-2\sigma)}})} \right| \tag{4.73}$$

for some $\theta \in (0,1)$. Hence,

$$|\widehat{K_1}(t,\xi)| = te^{-\frac{1}{2}|\xi|^{2\sigma}t} \le te^{-2^{-\frac{1}{1-2\sigma}}t} \lesssim e^{-2\varepsilon_\sigma t} \tag{4.74}$$

in the case $2|\xi|^{1-2\sigma} \ge 1$, where $\varepsilon_\sigma = 2^{-\frac{6}{1-2\sigma}-1}$. Next, we consider the case $2|\xi|^{1-2\sigma} \le 1$. Then

$$\frac{1}{2}|\xi|^{2\sigma}(-1+(2\theta-1)\sqrt{1-4|\xi|^{2(1-2\sigma)}}) \le \frac{1}{2}|\xi|^{2\sigma}(-1+\sqrt{1-4|\xi|^{2(1-2\sigma)}})$$
$$= \frac{-2|\xi|^{2-2\sigma}}{1+\sqrt{1-4|\xi|^{2(1-2\sigma)}}} \le -|\xi|^{2-2\sigma} \le -2^{-\frac{6(1-\sigma)}{1-2\sigma}} \le -2\varepsilon_\sigma,$$

and thus,

$$|\widehat{K_1}(t,\xi)| \le te^{-2\varepsilon_\sigma t} \lesssim e^{-\varepsilon_\sigma t}$$

on $[2^{-\frac{3}{1-2\sigma}}, 2^{-\frac{1}{1-2\sigma}}]$, which together with (4.74) yields

$$\langle\xi\rangle|\widehat{K_1}(t,\xi)\chi_{mid}(\xi)| \lesssim e^{-\varepsilon_\sigma t}. \tag{4.75}$$

Calculating in the same way as in the proof of (4.63) by using (4.75) instead of (4.67), and noting that $-\Delta$ is bounded operator on the support χ_{mid}, we obtain (4.69).

In the same way as in the proof of (4.75), we see that

$$\|(1-\Delta)^{\frac{k}{2}}\widehat{K_1}(t,\cdot)\chi_{mid}\|_\infty \le C_k e^{-\varepsilon_\sigma t}, \tag{4.76}$$

for every $k \in \mathbb{N} \cup \{0\}$. Then by the same calculation as in the proof of (4.64), we obtain (4.70).

We can estimate

$$\widehat{K_0} = \frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} = \frac{e^{\lambda_+} - e^{\lambda_-}}{\lambda_+ - \lambda_-}\lambda_+ + \lambda_-,$$

in the same way, and obtain the assertion for $K_{0,mid}$. □

5. Asymptotic profile of the solutions of linear equation

In this section, we prove Theorem 1.

Since the solution u of (1.3) is written as

$$u(t,x) = (K_0(t,\cdot) * u_0)(x) + (K_1(t,\cdot) * u_1)(x),$$

the conclusion of Theorem 1 follows from the following lemma.

Lemma 6. *Let $u_j \in L^1 \cap L^2$ for $j = 0, 1$. Then the following hold.*

$$\left\| K_1(t, \cdot) * u_1 - G_\sigma(t, x) \int_{\mathbb{R}^n} u_1(y) dy \right\|_2$$
$$\lesssim \langle t \rangle^{\max\{\frac{1}{1-\sigma}(-\frac{n}{4}+3\sigma-1), \frac{1}{\sigma}(-\frac{n}{4}+\sigma)\}} \|u_1\|_1 + e^{-\varepsilon_\sigma t} \|u_1\|_{H^{-2\sigma}}$$
$$+ t^{\frac{1}{1-\sigma}(-\frac{n}{4}+\sigma-\frac{\theta}{2})} \left\| |\cdot|^\theta u_1 \right\|_1, \tag{5.1}$$

$$\left\| K_0(t, \cdot) * u_0 - H_\sigma(t, x) \int_{\mathbb{R}^n} u_0(y) dy \right\|_2$$
$$\lesssim \langle t \rangle^{\frac{1}{1-\sigma}(-\frac{n}{4}+2\sigma-1)} \|u_0\|_1 + e^{-\varepsilon_\sigma t} \|u_0\|_2 + t^{\frac{1}{1-\sigma}(-\frac{n}{4}-\frac{\theta}{2})} \left\| |\cdot|^\theta u_0 \right\|_1. \tag{5.2}$$

Proof. First we prove (5.1). We have

$$\left\| K_1(t, \cdot) * u_1 - G_\sigma(t, \cdot) \int_{\mathbb{R}^n} u_1(y) dy \right\|$$
$$\leq \left\| \mathcal{F}^{-1}[\widehat{K_1^+}(t, \xi)\chi_{low}] * u_1 - \mathcal{F}^{-1}[|\xi|^{-2\sigma} e^{-|\xi|^{2(1-\sigma)}t}\chi_{low}] * u_1 \right\|_2$$
$$+ \left\| \mathcal{F}^{-1}[\widehat{K_1^-}(t, \xi)\chi_{low}] * u_1 \right\|_2 + \|K_{1,mh}(t, \cdot) * u_1\|_2$$
$$+ \left\| \mathcal{F}^{-1}\left[\widehat{G_\sigma}(t, \xi)\chi_{mh} \right] * u_1 \right\|_2 + \|G_\sigma(t, \cdot) * u_1 - G_\sigma(t, \cdot) \int_{\mathbb{R}^n} u_1(y) dy\|_2$$
$$=: I_{1,1} + I_{1,2} + I_{1,3} + I_{1,4} + I_{1,5}. \tag{5.3}$$

By (4.54) and (4.45) for $q_1 = q_2 = 1$ and $\vartheta = s_1 = s_2 = 0$, we have

$$I_{1,1} + I_{1,2} \lesssim \langle t \rangle^{\frac{1}{1-\sigma}(-\frac{n}{4}+3\sigma-1)} \|u_1\|_1 + \langle t \rangle^{\frac{1}{\sigma}(-\frac{n}{4}+\sigma)} \|u_1\|_1. \tag{5.4}$$

By (4.63) and (4.69), we have

$$I_{1,3} \lesssim e^{-\varepsilon_\sigma t} \|u_1\|_{H^{-1}}. \tag{5.5}$$

Since the support of χ_{mh} is included in $[2^{-\frac{3}{1-2\sigma}}, \infty)$ and $2^{-\frac{6(1-\sigma)}{1-2\sigma}} > \varepsilon_\sigma$, we have

$$I_{1,4} = \left\| \widehat{G_\sigma}(t, \cdot)\chi_{mh}\hat{u}_1 \right\|_2 = \left\| |\xi|^{-2\sigma} e^{-|\xi|^{2(1-\sigma)}t}\chi_{mh}\hat{u}_1 \right\|_2 \lesssim e^{-\varepsilon_\sigma t} \|u_1\|_{H^{-2\sigma}}. \tag{5.6}$$

It is written that

$$I_{1,5} = \|\widehat{G_\sigma}(t, \cdot)(\hat{u}_1(\cdot) - \hat{u}_1(0))\|_2. \tag{5.7}$$

Since $\widehat{G_\sigma}(t, \xi) = |\xi|^{-2\sigma} e^{-|\xi|^{2(1-\sigma)}t}$, we have by the transformation $t^{\frac{1}{2(1-\sigma)}} r = \rho$ that

$$\|\widehat{G_\sigma}(t, \cdot)| \cdot |^\theta\|_2^2 = \int_0^\infty r^{2(\theta-2\sigma)+n-1} e^{-2r^{2(1-\sigma)}t} dr$$
$$= t^{-\frac{n+2(\theta-2\sigma)}{2(1-\sigma)}} \int_0^\infty \rho^{2(\theta-2\sigma)+n-1} e^{-2\rho^{2(1-\sigma)}} d\rho \sim t^{-\frac{n+2(\theta-2\sigma)}{2(1-\sigma)}},$$

that is,

$$\|\widehat{G_\sigma}(t, \cdot)| \cdot |^\theta\|_2 \sim t^{\frac{1}{1-\sigma}(-\frac{n}{4}-\frac{\theta}{2}+\sigma)}. \tag{5.8}$$

On the other hand, since $\theta \in [0,1]$, we have

$$
\begin{aligned}
|\hat{u}_1(\xi) - \hat{u}_1(0)| &\leq \int_{\mathbb{R}^n} \left|(e^{ix\cdot\xi} - 1)u_1(x)\right| dx = \int_{\mathbb{R}^n} \left|(e^{\frac{ix\cdot\xi}{2}} - e^{\frac{-ix\cdot\xi}{2}})u_1(x)\right| dx \\
&= 2\int_{\mathbb{R}^n} \left|\sin\left(\frac{x\cdot\xi}{2}\right)u_1(x)\right| dx \\
&\leq 2\int_{\mathbb{R}^n} \left(\frac{|x\cdot\xi|}{2}\right)^\theta |u_1(x)|dx = 2^{1-\theta}|\xi|^\theta \left\||\cdot|^\theta u_1\right\|_1 \quad (5.9)
\end{aligned}
$$

for every $\xi \in \mathbb{R}^n$. From (5.7), (5.8) and (5.9), it follows that

$$
\begin{aligned}
I_{1,5} &\leq 2\left\|\widehat{G_\sigma}(t,\cdot)|\cdot|^\theta \left\||\cdot|^\theta u_1\right\|_1\right\|_2 = 2\|\widehat{G_\sigma}(t,\cdot)|\cdot|^\theta\|_2 \left\||\cdot|^\theta u_1\right\|_1 \\
&\lesssim t^{\frac{1}{1-\sigma}\left(-\frac{n}{4}-\frac{\theta}{2}+\sigma\right)} \left\||\cdot|^\theta u_1\right\|_1. \quad (5.10)
\end{aligned}
$$

Substituting (5.4)–(5.6) and (5.10) into (5.3), we obtain (5.1).

Next we prove (5.2). We have

$$
\begin{aligned}
\left\|K_0(t,\cdot)\right. &\left.* u_0 - H_\sigma(t,x)\int_{\mathbb{R}^n} u_0(y)dy\right\|_2 \\
&\leq \left\|\mathcal{F}^{-1}[\widehat{K_0^+}(t,\cdot)\chi_{low}] * u_0 - \mathcal{F}^{-1}[\widehat{H_\sigma}(t,\cdot)\chi_{low}] * u_0\right\|_2 \\
&\quad + \left\|\mathcal{F}^{-1}[\widehat{K_0^-}(t,\cdot)\chi_{low}] * u_0\right\|_2 + \|K_{0,mh}(t,\cdot) * u_0\|_2 \\
&\quad + \left\|\mathcal{F}^{-1}\left[\widehat{H_\sigma}(t,\cdot)\chi_{mh}\right] * u_0\right\|_2 + \|H_\sigma(t,\cdot) * u_0 - H_\sigma(t,\cdot)\int_{\mathbb{R}^n} u_0(y)dy\|_2 \\
&=: I_{0,1} + I_{0,2} + I_{0,3} + I_{0,4} + I_{0,5}. \quad (5.11)
\end{aligned}
$$

By (4.55) with $q_j = 1, s_j = 0$ ($j = 1,2$) and $\vartheta = 0$, we have

$$
\begin{aligned}
I_{0,1} &= \left\|\mathcal{F}^{-1}\left[\left(\widehat{K_0^+}(t,\xi) - e^{-|\xi|^{2(1-\sigma)}t}\right)\chi_{low}\right] * u_0\right\|_2 \\
&\lesssim \langle t\rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{4}+2\sigma-1\right)}\|u_0\|_1. \quad (5.12)
\end{aligned}
$$

Inequality (4.48) implies

$$
I_{0,2} \lesssim \langle t\rangle^{\frac{1}{\sigma}\left(-\frac{n}{4}+2\sigma-1\right)}\|u_0\|_1, \quad (5.13)
$$

and inequalities (4.65) and (4.71) imply

$$
I_{0,3} \lesssim e^{-\varepsilon_\sigma t}\|u_0\|_2. \quad (5.14)
$$

Since the support of χ_{mh} is included in $[2^{-\frac{3}{1-2\sigma}}, \infty)$ and $2^{-\frac{6(1-\sigma)}{1-2\sigma}} > \varepsilon_\sigma$, we have

$$
I_{0,4} = \left\|e^{-|\xi|^{2(1-\sigma)}t}\chi_{mh}\hat{u}_0\right\|_2 \lesssim e^{-\varepsilon_\sigma t}\|u_0\|_2. \quad (5.15)
$$

By (5.8) with θ replaced by $2\sigma + \theta$, we have

$$
\|\widehat{H_\sigma}(t,\cdot)|\cdot|^\theta\|_2 = \|\widehat{G_\sigma}(t,\cdot)|\cdot|^{2\sigma+\theta}\|_2 \sim t^{\frac{1}{1-\sigma}\left(-\frac{n}{4}-\frac{\theta}{2}\right)}. \quad (5.16)
$$

Then, in the same way as in the proof of (5.10), by using (5.16) instead of (5.8), we have

$$I_{0,5} \lesssim t^{\frac{1}{1-\sigma}\left(-\frac{n}{4}-\frac{\theta}{2}\right)} \left\| |\cdot|^{\theta} u_0 \right\|_1. \tag{5.17}$$

Since $\sigma < 1 - \sigma$, (5.2) follows from (5.11)–(5.15) and (5.17). $\qquad\square$

6. Estimate of the nonlinear term

Throughout this section, we suppose assumtion (1.2), and estimate nonlinear terms by using the argument of [7] and [9]. For $r \in [1,2)$, $\delta \in [0, \frac{n}{2} - 2\sigma)$ and $\bar{s} \geq 1$, we define

$$X_{r,\delta,\bar{s}} := \{ u \in C((0,\infty); H^{\bar{s}} \cap H^{0,\delta}); \|u\|_{X_{r,\delta,\bar{s}}} < \infty \}, \tag{6.1}$$

where

$$\begin{aligned}
\|\varphi\|_{X_{r,\delta,\bar{s}}} &:= \sup_{t>0} \left(\langle t \rangle^{\frac{1}{1-\sigma}\left(\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\sigma+\frac{\bar{s}}{2}\right)} \left\| (-\Delta)^{\bar{s}/2} \varphi(t) \right\|_2 \right. \\
&\quad \left. + \langle t \rangle^{\frac{1}{1-\sigma}\left(\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{\delta}{2}-\sigma\right)} \left\| \langle \cdot \rangle^{\delta} \varphi(t) \right\|_2 \right).
\end{aligned} \tag{6.2}$$

For $\vartheta \in [0, \frac{n}{2} - 2\sigma)$, we put

$$\begin{aligned}
\zeta_{r,\vartheta} &:= \frac{1}{1-\sigma} \left(-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right) + \sigma + \frac{\vartheta}{2} - (p-1)\left(\frac{n}{2r}-\sigma\right) + \frac{1}{2} \right) \\
&= \frac{1}{1-\sigma} \left(-\left(\frac{n}{2r}-\sigma\right)p + \frac{n}{4} + \frac{\vartheta}{2} + \frac{1}{2} \right).
\end{aligned} \tag{6.3}$$

For $s \geq 0$, we define

$$\tilde{q}_s := \frac{2n}{n+2+2[s]-2s}, \quad \text{that is,} \quad \frac{1}{\tilde{q}_s} = \frac{1}{2} + \frac{1+[s]-s}{n}. \tag{6.4}$$

Lemma 7. *Let $r \in [1,2)$, $\delta \in [0, \frac{n}{2} - 2\sigma)$ and $\bar{s} > 2\sigma$. Let $X = X_{r,\delta,\bar{s}}$. Then the following holds for every $\vartheta \in [0,\delta]$, $s \in [0,\bar{s}]$ and $u \in X$:*

(i) We have

$$\left\| (-\Delta)^{\frac{s}{2}} u(t,\cdot) \right\|_2 \lesssim \langle t \rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\sigma-\frac{s}{2}\right)} \|u\|_X, \tag{6.5}$$

$$\left\| |\cdot|^{\vartheta} u(t,\cdot) \right\|_2 \lesssim \langle t \rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{\vartheta}{2}+\sigma\right)} \|u\|_X. \tag{6.6}$$

(ii) We have

$$\|u(t,\cdot)\|_{q,2} \lesssim \langle t \rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{q}\right)+\sigma\right)} \|u\|_X, \quad \text{if} \quad q = \frac{2n}{n+2\delta}, \tag{6.7}$$

$$\|u(t,\cdot)\|_q \lesssim \langle t \rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{q}\right)+\sigma\right)} \|u\|_X$$

$$\text{if} \quad q \in \begin{cases} \left(\frac{2n}{n+2\delta}, \frac{2n}{n-2\bar{s}}\right] & (2\bar{s} < n), \\ \left(\frac{2n}{n+2\delta}, \infty\right) & (2\bar{s} \geq n). \end{cases} \tag{6.8}$$

(iii) We have

$$\left\|(-\Delta)^{\frac{[s]}{2}} f(u(t,\cdot))\right\|_{\tilde{q}_s} \lesssim \langle t \rangle^{\frac{1}{1-\sigma}\left((-\frac{n}{2r}+\sigma)p+\frac{n}{4}+\frac{1}{2}-\frac{s}{2}\right)} \|u\|_X^p, \tag{6.9}$$

$$\left\|(-\Delta)^{\frac{s}{2}}(1-\Delta)^{-\frac{1}{2}} f(u(t,\cdot))\right\|_2 \lesssim \langle t \rangle^{\frac{1}{1-\sigma}\left((-\frac{n}{2r}+\sigma)p+\frac{n}{4}+\frac{1}{2}-\frac{s}{2}\right)} \|u\|_X^p \tag{6.10}$$

$$\left\|\langle\cdot\rangle^{\vartheta} f(u(t,\cdot))\right\|_{2n/(n+2)} \lesssim \langle t \rangle^{\zeta_{r,\vartheta}} \|u\|_X^p. \tag{6.11}$$

Proof. Except use of the weak L^p estimate, we follow the argument of [9, Lemmas 2.3 and 2.5], which is originated in [7, Lemma 2.1, 2.3 and 2.5].

(i) By Plancherel's theroem and Hölder's inequality, we have

$$\left\|(-\Delta)^{\frac{s}{2}} u(t,\cdot)\right\|_2 = \||\cdot|^s \hat{u}(t,\cdot)\|_2$$

$$\leq \||\cdot|^{\bar{s}} \hat{u}(t,\cdot)\|_2^{\frac{s}{\bar{s}}} \|\hat{u}(t,\cdot)\|_2^{1-\frac{s}{\bar{s}}} = \|(-\Delta)^{\frac{\bar{s}}{2}} u(t,\cdot)\|_2^{\frac{s}{\bar{s}}} \|u(t,\cdot)\|_2^{1-\frac{s}{\bar{s}}},$$

which together with the definition of $\|\cdot\|_X$ implies (6.5). In the same way, we see that (6.6) holds by Hölder's inequality.

(ii) We first consider the case $\frac{2n}{n+2\delta} = q$, that is, $n(\frac{1}{q} - \frac{1}{2}) = \delta$. Then

$$\|u(t,\cdot)\|_{q,2} \lesssim \||\cdot|^{-\delta}\|_{\frac{n}{\delta},\infty} \||\cdot|^{\delta} u(t,\cdot)\|_2 \lesssim \||\cdot|^{\delta} u\|_2, \tag{6.12}$$

which together with the definition of $\|\cdot\|_X$ implies (6.7).

Next we consider the case $\frac{2n}{n+2\delta} < q < 2$, that is, $0 < n\left(\frac{1}{q} - \frac{1}{2}\right) < \delta$. In [9, (2.12)] (see also [7, (2.5)]), the following is shown

$$\|u\|_q \lesssim \|u\|_2^{1-\frac{n}{\delta}\frac{2-q}{2q}} \||\cdot|^{\delta} u\|_2^{\frac{n}{\delta}\frac{2-q}{2q}}, \tag{6.13}$$

when $0 < n\left(\frac{1}{q} - \frac{1}{2}\right) < \delta$. This together with the definition of $\|\cdot\|_X$ implies (6.8).

We consider the case $2 \leq q \leq \frac{2n}{n-2\bar{s}}$ and $\bar{s} < 2n$. Let $\tilde{s} = \frac{n}{2} - \frac{n}{q}(\leq \bar{s})$. Then Sobolev's embedding theorem together with (6.5) implies

$$\|u\|_q \leq \|u\|_{\dot{H}^{\tilde{s}}} \leq \langle t \rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\sigma-\frac{s}{2}\right)} \|u\|_X = \langle t \rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{q}\right)+\sigma\right)} \|u\|_X,$$

that is, (6.8) holds. In the same way, (6.8) holds also in the case $2 \leq q < \infty$ and $\bar{s} \leq 2n$.

(iii) We put

$$\kappa := \frac{n}{2} - \frac{1}{p-1}. \tag{6.14}$$

By the Leibniz rule together with assumtion (1.2), we have

$$\|\nabla^{[s]} f(u(t,\cdot))\|_{\tilde{q}_s} \leq \left\| u(t,\cdot)^{p-[s]} \sum_{\sum_{j=1}^{[s]} |\nu_j|=[s]} \prod_{j=1}^{[s]} |D_x^{\nu_j} u(t,\cdot)| \right\|_{\tilde{q}_s}, \tag{6.15}$$

where ν_j is a multi index. Put $k_j = |\nu_j|$. Then, as in the proof of [9, Lemma 2.5], we can choose $s_j \in [0, k_j - \frac{1}{p-1})$ such that q_j $(j = 1, \dots, [s])$ defined by

$$\frac{1}{q_0} = \left(\frac{1}{2} - \frac{\kappa}{n}\right)(p - [s]), \tag{6.16}$$

$$\frac{1}{q_j} = \frac{1}{2} - \frac{\kappa + s_j - k_j}{n} \quad (j = 1, \ldots, [s]) \tag{6.17}$$

satisfies

$$\sum_{j=0}^{[s]} \frac{1}{q_j} = \frac{1}{\tilde{q}_s}, \tag{6.18}$$

$$(p - [s])q_0 \in [2, \infty) \text{ and } q_j \in [2, \infty) \text{ for } j = 1, \ldots, [s]. \tag{6.19}$$

Since

$$\frac{1}{\tilde{q}_s} - \frac{1}{q_0} = \frac{1}{2} + \frac{1}{n} + \frac{[s] - s}{n} - \left(\frac{1}{2} - \frac{\kappa}{n} \right)(p - [s]),$$

$$\sum_{j=1}^{[s]} \frac{1}{q_j} = \sum_{j=1}^{[s]} \left(\frac{1}{2} - \frac{\kappa + s_j - k_j}{n} \right) = \left(\frac{1}{2} - \frac{\kappa}{n} \right)[s] - \frac{1}{n} \sum_{j=1}^{[s]} s_j + \frac{[s]}{n},$$

the condition (6.18) is equivalent to

$$\sum_{j=1}^{[s]} s_j = s - \kappa, \tag{6.20}$$

and thus, $\kappa + s_j \leq s$. Taking (6.16)–(6.19) into account, we apply Hölder's inequality and Sobolev's embedding theorem to (6.15). Then we obtain

$$\|\nabla^{[s]} f(u(t, \cdot))\|_{\tilde{q}_s} \lesssim \|u^{p-[s]}\|_{q_0} \sum_{k_j \geq 0, \sum_{j=1}^{[s]} k_j = [s]} \prod_{j=1}^{[s]} \||\nabla|^{k_j} u(t, \cdot)\|_{q_j}$$
$$\lesssim \|\nabla^\kappa u\|_2^{p-[s]} \sum_{k_j \geq 0, \sum_{j=1}^{[s]} k_j = [s]} \prod_{j=1}^{[s]} \||\nabla|^{\kappa + s_j} u(t, \cdot)\|_2, \tag{6.21}$$

where $|\nabla| := (-\Delta)^{1/2}$. Then estimating the right-hand side of (6.21) by the definition of $\|\cdot\|_X$, and using (6.20) and (6.14), we obtain

$$\|\nabla^{[s]} f(u(t, \cdot))\|_{\tilde{q}_s} \lesssim \langle t \rangle^{\frac{1}{1-\sigma}\left(\left(-\frac{n}{2}\left(\frac{1}{r} - \frac{1}{2}\right) + \sigma \right)p - \frac{\kappa}{2}(p-[s]) - \frac{1}{2}\sum_{j=1}^{[s]}(\kappa + s_j) \right)} \|u\|_X^p$$
$$= \langle t \rangle^{\frac{1}{1-\sigma}\left(\left(-\frac{n}{2}\left(\frac{1}{r} - \frac{1}{2}\right) + \sigma - \frac{\kappa}{2} \right)p - \frac{1}{2}\sum_{j=1}^{[s]} s_j \right)} \|u\|_X^p$$
$$= \langle t \rangle^{\frac{1}{1-\sigma}\left(\left(-\frac{n}{2r} + \sigma \right)p + \frac{n}{4} + \frac{1}{2} - \frac{s}{2} \right)} \|u\|_X^p,$$

that is, (6.9) holds.

Sobolev's embedding theorem together with inequality (6.9) implies (6.10).

By Hölder's inequality and assumtion (1.2), we have

$$\|\langle \cdot \rangle^\vartheta f(u(t, \cdot))\|_{2n/(n+2)} \lesssim \|\langle \cdot \rangle^\vartheta u(t, \cdot)\|_2 \|u(t, \cdot)\|_{n(p-1)}^{p-1}. \tag{6.22}$$

The assumption (2.13) implies

$$n(p-1) \geq \frac{2rn}{n - 2r\sigma} > \frac{2n}{n + 2\delta},$$

and (2.7) implies

$$n(p-1) \leq \frac{2n}{n-2\bar{s}}$$

if $2\bar{s} < n$. Thus, we can apply (6.6) and (6.8) with $q = n(p-1)$, which together with (6.22) yields (6.11). □

7. Estimates of a convolution term

Throughout this section, we suppose assumtion (1.2).

7.1. Decay estimates

Throughout this subsection, we suppose the assumption of Proposition 1.

Lemma 8. *Let $\vartheta \in [0, \delta]$. For every $u \in X = X_{r,\delta,\bar{s}}$, we have*

$$\int_0^t \||\cdot|^\vartheta \mathcal{F}^{-1}\left[\widehat{K_1^\pm}\chi_{mh}\right](t-\tau, \cdot) * f(u(\tau, \cdot))\|_2 d\tau \lesssim \langle t \rangle^{\zeta_{r,\vartheta}} \|u\|_X^p, \qquad (7.1)$$

where $\zeta_{r,\vartheta}$ is the number defined by (6.3).

Proof. By (4.64) and (4.70), Sobolev's embedding theorem and (6.11), we have

$$\left\||\cdot|^\vartheta \mathcal{F}^{-1}\left[\widehat{K_1^\pm}\chi_{mh}\right](t-\tau, \cdot) * f(u(\tau, \cdot))\right\|_2$$
$$\lesssim e^{-\varepsilon_\sigma(t-\tau)}\|(1-\Delta)^{-\frac{1}{2}}\langle\cdot\rangle^\vartheta f(u(\tau, \cdot))\|_2$$
$$\lesssim e^{-\varepsilon_\sigma(t-\tau)}\|\langle\cdot\rangle^\vartheta f(u(\tau, \cdot))\|_{\frac{2n}{n+2}} \lesssim e^{-\varepsilon_\sigma(t-\tau)}\langle\tau\rangle^{\zeta_{r,\vartheta}}\|u\|_X^p,$$

which yields (7.1). □

Lemma 9. *For every $u, v \in X = X_{r,\delta,\bar{s}}$, we have*

$$\int_0^t \||\cdot|^\delta \left(K_1(t-\tau, \cdot) * f(u(\tau, \cdot))\right)\|_2 d\tau \lesssim \langle t \rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}(\frac{1}{r}-\frac{1}{2})+\frac{\delta}{2}+\sigma\right)}\|u\|_X^p. \quad (7.2)$$

$$\int_0^t \||\cdot|^\delta \left(K_1(t-\tau, \cdot) * (f(u(\tau, \cdot)) - f(v(\tau, \cdot)))\right)\|_2 d\tau$$
$$\lesssim \langle t \rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}(\frac{1}{r}-\frac{1}{2})+\frac{\delta}{2}+\sigma\right)}(\|u\|_X + \|v\|_X)^{p-1}\|u-v\|_X. \qquad (7.3)$$

Proof. First, we estimate the low frequency part. By (4.46) with $q_1 = r$, $q_2 = \frac{2n}{n+2}$ and $\vartheta = \delta$, $s_1 = s_2 = 0$, we have

$$\int_0^t \||\cdot|^\delta \left(K_{1,low}(t-\tau, \cdot) * f(u(\tau, \cdot))\right)\|_2 d\tau \leq I_1 + I_2, \qquad (7.4)$$

where

$$I_1 := \begin{cases} \int_0^t \langle t-\tau \rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}(\frac{1}{r}-\frac{1}{2})+\frac{\delta}{2}+\sigma\right)}\|f(u(\tau, \cdot))\|_r d\tau & \text{if} \quad r \geq 1, \\ \int_0^t \langle t-\tau \rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}(\frac{1}{r}-\frac{1}{2})+\frac{\delta}{2}+\sigma\right)}\|f(u(\tau, \cdot))\|_{r,2} d\tau & \text{if} \quad r > 1, \end{cases}$$

$$I_2 := \int_0^t \langle t-\tau \rangle^{\frac{1}{1-\sigma}\left(-\frac{1}{2}+\sigma\right)}\||\cdot|^\delta f(u(\tau, \cdot))\|_{2n/n+2} d\tau.$$

Since $\bar{s} \geq 1$, assumtion (2.7) implies that $pr \leq 2p \leq \frac{2n}{n-2\bar{s}}$ if $2s_o < n$. From (2.15), it follows that $\frac{2n}{n+2\delta} < pr$ in the case $r = 1$, and $\frac{2n}{n+2\delta} \leq pr$ in the case $r > 1$. In the case $\frac{2n}{n+2\delta} < pr$, we can apply (6.8) with $q = pr$ to obtain

$$\|f(u(\tau,\cdot))\|_r \lesssim \|u(\tau,\cdot)\|_{pr}^p \lesssim \langle\tau\rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}\left(\frac{p-1}{r}\right)+p\sigma\right)}\|u\|_X^p. \qquad (7.5)$$

In the case $pr = \frac{2n}{n+2\delta}$, making use of the equality $(|u|^p)^* = (u^*)^p$, we have

$$\|f(u(\tau,\cdot))\|_{r,2} \lesssim \||u(\tau,\cdot)|^p\|_{r,2} = \|u(\tau,\cdot)\|_{pr,2p}^p \lesssim \|u(\tau,\cdot)\|_{pr,2}^p.$$

Hence, by (6.7) with $q = pr$, we obtain

$$\|f(u(\tau,\cdot))\|_{r,2} \lesssim \langle\tau\rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}\left(\frac{p-1}{r}\right)+p\sigma\right)}\|u\|_X^p. \qquad (7.6)$$

Substituting (7.5) or (7.6) into I_1, we obtain

$$I_1 \lesssim \int_0^t \langle t-\tau\rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{\delta}{2}+\sigma\right)}\langle\tau\rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}\left(\frac{p}{r}-\frac{1}{r}\right)+p\sigma\right)}d\tau\|u\|_X^p.$$

The following inequality is commonly used to estimate the nonlinear term.

$$\int_0^t \langle t-\tau\rangle^\rho s^\eta d\tau \lesssim \begin{cases} \langle t\rangle^{\max\{\rho,\eta\}} & \text{if} \quad \min\{\rho,\eta\} < -1, \\ \langle t\rangle^{\max\{\rho,\eta\}}\log(2+t) & \text{if} \quad \min\{\rho,\eta\} = -1, \quad (7.7) \\ \langle t\rangle^{1+\rho+\eta} & \text{if} \quad \min\{\rho,\eta\} > -1. \end{cases}$$

The assumption that $\delta \geq n(\frac{1}{r}-\frac{1}{2}) - 1$ implies

$$\frac{1}{1-\sigma}\left(-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{\delta}{2}+\sigma\right) \geq -1. \qquad (7.8)$$

The assumption (2.13) is equivalent to $p\left(\frac{n}{2r}-\sigma\right) > \frac{n}{2r} - \sigma + 1$, which is equivalent to

$$\frac{1}{1-\sigma}\left(-\frac{n}{2}\left(\frac{p-1}{r}\right)+p\sigma\right) < -1. \qquad (7.9)$$

Hence, by using (7.7), we obtain

$$I_1 \lesssim \langle t\rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{\delta}{2}+\sigma\right)}\|u\|_X^p. \qquad (7.10)$$

Since $\frac{1}{1-\sigma}\left(-\frac{1}{2}+\sigma\right) > -1$, it follows from (6.11) and (7.7) that

$$I_2 \lesssim \int_0^t \langle t-\tau\rangle^{\frac{1}{1-\sigma}\left(-\frac{1}{2}+\sigma\right)}\langle\tau\rangle^{\zeta_{r,\delta}}d\tau\|u\|_X^p$$

$$\lesssim \begin{cases} \langle t\rangle^{\frac{1}{1-\sigma}\left(-\frac{1}{2}+\sigma\right)}\|u\|_X^p & \text{if } \zeta_{r,\delta} < -1 \\ \langle t\rangle^{\frac{1}{1-\sigma}\left(-\frac{1}{2}+\sigma\right)+\zeta_{r,\delta}+1}\log(t+2)\|u\|_X^p & \text{if } \zeta_{r,\delta} \geq -1. \end{cases}$$

The assumption that $\delta \geq n(\frac{1}{r}-\frac{1}{2}) - 1$ implies $-\frac{1}{2}+\sigma \leq -\frac{n}{2}(\frac{1}{r}-\frac{1}{2})+\frac{\delta}{2}+\sigma$. Hence, we have

$$I_2 \lesssim \langle t\rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{\delta}{2}+\sigma\right)}\|u\|_X^p \qquad (7.11)$$

in the case $\zeta_{r,\delta} < -1$. By definition (6.3) and assumption (2.13), we have

$$\frac{1}{1-\sigma}\left(-\frac{1}{2}+\sigma\right)+\zeta_{r,\delta}+1$$

$$=\frac{1}{1-\sigma}\left(-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{\delta}{2}+\sigma+1-(p-1)\left(\frac{n}{2r}-\sigma\right)\right)$$

$$<\frac{1}{1-\sigma}\left(-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{\delta}{2}+\sigma\right).$$

Hence, (7.11) holds also in the case $\zeta_{r,\delta}\geq -1$. Substituting (7.10) and (7.11) into (7.4), we obtain

$$\int_0^t\left\||\cdot|^\delta\left(K_{1,low}(t-\tau,\cdot)*f(u(\tau,\cdot))\right)\right\|_2 d\tau\lesssim\langle t\rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{\delta}{2}+\sigma\right)}\|u\|_X^p,$$

which together with (7.1) yields (7.2).

The assumption (1.2) implies

$$|f(u(\tau,x))-f(v(\tau,x))|\lesssim(|u(\tau,x)|+|v(\tau,x)|)^{p-1}|u(\tau,x)-v(\tau,x)|,$$

and we can prove (7.3) in the same way. $\qquad\square$

Lemma 10. *Let $s\in[0,\bar{s}]$. For every $u,v\in X=X_{r,\delta,\bar{s}}$, we have*

$$\int_0^t\left\|(-\Delta)^{\frac{s}{2}}\left(K_1^+(t-\tau,\cdot)*f(u(\tau,\cdot))\right)\right\|_2 d\tau\lesssim\langle t\rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{s}{2}+\sigma\right)}\|u\|_X^p.$$
$$(7.12)$$

$$\int_0^t\left\|(-\Delta)^{\frac{s}{2}}\left(K_1^-(t-\tau,\cdot)*f(u(\tau,\cdot))\right)\right\|_2 d\tau$$
$$\lesssim\langle t\rangle^{\max\{\frac{1}{\sigma}\left(-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{s}{2}+\sigma\right),\frac{1}{1-\sigma}\left(-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{s}{2}+\sigma-(p-1)\left(\frac{n}{2r}-\sigma\right)+1\right)\}}\|u\|_X^p$$
$$(7.13)$$

$$\int_0^t\left\|(-\Delta)^{\frac{s}{2}}\left(K_1(t-\tau,\cdot)*f(u(\tau,\cdot))\right)\right\|_2 d\tau\lesssim\langle t\rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{s}{2}+\sigma\right)}\|u\|_X^p.$$
$$(7.14)$$

$$\int_0^t\left\|(-\Delta)^{\frac{s}{2}}\left(K_1(t-\tau,\cdot)*(f(u(\tau,\cdot))-f(v(\tau,\cdot)))\right)\right\|_2 d\tau$$
$$\lesssim\langle t\rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{s}{2}+\sigma\right)}(\|u\|_X+\|v\|_X)^{p-1}\|u-v\|_X. \qquad(7.15)$$

for every $u,v\in X$,

Proof. We first prove (7.12). We divide the left-hand side of (7.12) into three parts:

$$\int_0^t\left\|(-\Delta)^{\frac{s}{2}}\left(K_1^+(t-\tau,\cdot)*f(u(\tau,\cdot))\right)\right\|_2 d\tau$$

$$=\int_0^{t/2}\left\|(-\Delta)^{\frac{s}{2}}\left(\mathcal{F}^{-1}[\hat{K}_1^+(t-\tau,\cdot)\chi_{low}](t-\tau,\cdot)*f(u(\tau,\cdot))\right)\right\|_2 d\tau$$

$$+\int_{t/2}^t\left\|(-\Delta)^{\frac{s}{2}}\left(\mathcal{F}^{-1}[\hat{K}_1^+(t-\tau,\cdot)\chi_{low}](t-\tau,\cdot)*f(u(\tau,\cdot))\right)\right\|_2 d\tau$$

$$+\int_0^t\left\|(-\Delta)^{\frac{s}{2}}\left(\mathcal{F}^{-1}[\hat{K}_1^+(t-\tau,\cdot)\chi_{mh}](t-\tau,\cdot)*f(u(\tau,\cdot))\right)\right\|_2 d\tau$$

$$:=J_1^+ + J_2^+ + J_3^+\text{ (we put).} \qquad(7.16)$$

Substituting (4.44) with $\vartheta = 0$, $q_1 = q_2 = r$, $s_1 = s$ and $s_2 = 0$ and (7.5) into J_1^+, and using (7.9), we obtain

$$J_1^+ \lesssim \langle t \rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{s}{2}+\sigma\right)} \int_0^{t/2} \langle \tau \rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}\left(\frac{p-1}{r}\right)+p\sigma\right)} d\tau \|u\|_X^p$$

$$\lesssim \langle t \rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{s}{2}+\sigma\right)} \|u\|_X^p. \tag{7.17}$$

By (4.44) with $\vartheta = 0$, $s_1 = s$, $s_2 = [s]$, $q_1 = q_2 = \tilde{q}_s$ (defined by (6.4)), and (6.9), we have

$$J_2^+ \lesssim \int_{t/2}^t \langle t - \tau \rangle^{\frac{1}{2(1-\sigma)}\left(-n\left(\frac{1}{\tilde{q}_s}-\frac{1}{2}\right)-s+[s]+2\sigma\right)} \left\| (-\Delta)^{\frac{[s]}{2}} f(u(\tau, \cdot)) \right\|_{\tilde{q}_s} d\tau$$

$$\lesssim \langle t \rangle^{\frac{1}{1-\sigma}\left(\left(-\frac{n}{2r}+\sigma\right)p+\frac{n}{4}+\frac{1}{2}-\frac{s}{2}\right)} \|u\|_X^p \int_0^{t/2} \langle \tau \rangle^{\frac{1}{2(1-\sigma)}(-1+2\sigma)} d\tau$$

$$\sim \langle t \rangle^{\frac{1}{1-\sigma}\left(\left(-\frac{n}{2r}+\sigma\right)p+\frac{n}{4}-\frac{s}{2}+1\right)} \|u\|_X^p$$

$$= \langle t \rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{s}{2}+\sigma-(p-1)\left(\frac{n}{2r}-\sigma\right)+1\right)} \|u\|_X^p. \tag{7.18}$$

Last we estimate J_3^+. Combining (4.63), (4.69) and (6.10), we have

$$J_3^+ \lesssim \int_0^t e^{-\varepsilon_\sigma(t-\tau)} \langle \tau \rangle^{\frac{1}{1-\sigma}\left(\left(-\frac{n}{2r}+\sigma\right)p+\frac{n}{4}+\frac{1}{2}-\frac{s}{2}\right)} d\tau \|u\|_X^p$$

$$\lesssim \langle t \rangle^{\frac{1}{1-\sigma}\left(\left(-\frac{n}{2r}+\sigma\right)p+\frac{n}{4}+\frac{1}{2}-\frac{s}{2}\right)} \|u\|_X^p$$

$$= \langle t \rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\sigma-\frac{s}{2}-(p-1)\left(\frac{n}{2r}-\sigma\right)+\frac{1}{2}\right)} \|u\|_X^p. \tag{7.19}$$

The assumption (2.13) implies $-(p-1)\left(\frac{n}{2r}-\sigma\right)+1 < 0$. Thus, (7.12) follows from (7.16)–(7.19).

We divide the left-hand side of (7.13) into three parts:

$$\int_0^t \left\| (-\Delta)^{\frac{s}{2}} \left(K_1^-(t-\tau,\cdot) * f(u(\tau,\cdot)) \right) \right\|_2 d\tau$$

$$= \int_0^{t/2} \left\| (-\Delta)^{\frac{s}{2}} \left(\mathcal{F}^{-1}[\hat{K}_1^-(t-\tau,\cdot)\chi_{low}](t-\tau,\cdot) * f(u(\tau,\cdot)) \right) \right\|_2 d\tau$$

$$+ \int_{t/2}^t \left\| (-\Delta)^{\frac{s}{2}} \left(\mathcal{F}^{-1}[\hat{K}_1^-(t-\tau,\cdot)\chi_{low}](t-\tau,\cdot) * f(u(\tau,\cdot)) \right) \right\|_2 d\tau$$

$$+ \int_0^t \left\| (-\Delta)^{\frac{s}{2}} \left(\mathcal{F}^{-1}[\hat{K}_1^-(t-\tau,\cdot)\chi_{mh}](t-\tau,\cdot) * f(u(\tau,\cdot)) \right) \right\|_2 d\tau$$

$$=: J_1^- + J_2^- + J_3^- \text{ (we put).} \tag{7.20}$$

Substituting (4.45) with $\vartheta = 0$, $q_1 = q_2 = r$, $s_1 = s$ and $s_2 = 0$ and (7.5) into J_1^-, and using (7.9), we obtain

$$J_1^- \lesssim \langle t \rangle^{\frac{1}{\sigma}\left(-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{s}{2}+\sigma\right)} \int_0^{t/2} \langle \tau \rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}\left(\frac{p-1}{r}\right)+p\sigma\right)} d\tau \|u\|_X^p$$

$$\lesssim \langle t \rangle^{\frac{1}{\sigma}\left(-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{s}{2}+\sigma\right)} \|u\|_X^p. \tag{7.21}$$

Since $\sigma < 1 - \sigma$, the right-hand side of (4.45) is dominated by that of (4.44). Hence, J_2^- and J_3^- are estimated by the right-hand sides of (7.18) and (7.19), respectively, and thus,

$$J_2^- \lesssim \langle t \rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}(\frac{1}{r}-\frac{1}{2})-\frac{s}{2}+\sigma-(p-1)(\frac{n}{2r}-\sigma)+1\right)} \|u\|_X^p, \tag{7.22}$$

$$J_3^- \lesssim \langle t \rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}(\frac{1}{r}-\frac{1}{2})-\frac{s}{2}+\sigma-(p-1)(\frac{n}{2r}-\sigma)+\frac{1}{2}\right)} \|u\|_X^p. \tag{7.23}$$

Substituting (7.21), (7.22) and (7.23) into (7.20), we obtain (7.13).

Inequality (7.14) follows from (7.12) and (7.13), since $\sigma < 1 - \sigma$ and $-(p-1)(\frac{n}{2r} - \sigma) + 1 < 0$.

By using assumtion (1.2), we can prove (7.15) in the same way. $\qquad \square$

7.2. Diffusion estimate

Lemma 11. *Let δ and ν be an arbitrary number satisfying the assumption of Theorem 3. Let $X = X_{1,\delta,\bar{s}}$, where $X_{1,\delta,\bar{s}}$ is defined by (6.1). Then we have*

$$\left\| \int_0^t K_1(t-\tau,\cdot) * f(u(\tau,\cdot))d\tau - G_\sigma(t,\cdot) \int_0^\infty \int_{\mathbb{R}^n} f(u(\tau,y))dy d\tau \right\|_2$$
$$\lesssim t^{\max\left\{ \frac{1}{1-\sigma}\left(-\frac{n}{4}+\sigma-\min\{(p-1)\left(\frac{n}{2}\right)-1,1-2\sigma,\nu\}\right), \frac{1}{\sigma}\left(-\frac{n}{4}+\sigma\right) \right\}} \|u\|_X^p. \tag{7.24}$$

Proof. We have

$$\left\| \int_0^t K_1(t-\tau,\cdot) * f(u(\tau,\cdot))d\tau - G_\sigma(t,\cdot) \int_0^\infty \int_{\mathbb{R}^n} f(u(\tau,y))dy d\tau \right\|_2$$
$$\leq L_1 + L_2 + L_3 + L_4, \tag{7.25}$$

where

$$L_1 := \int_{\frac{t}{2}}^t \|K_1(t-\tau,\cdot) * f(u(\tau,\cdot))\|_2 d\tau,$$

$$L_2 := \int_0^{t/2} \left\| K_1(t-\tau,\cdot) * f(u(\tau,\cdot)) - G_\sigma(t-\tau,\cdot) \int_{\mathbb{R}^n} f(u(\tau,y))dy \right\|_2 d\tau,$$

$$L_3 := \int_0^{t/2} \left\| G_\sigma(t-\tau,\cdot) - G_\sigma(t,\cdot) \right\|_2 \left| \int_{\mathbb{R}^n} f(u(\tau,y))dy \right| d\tau,$$

$$L_4 := \|G_\sigma(t,\cdot)\|_2 \left| \int_{t/2}^\infty \int_{\mathbb{R}^n} f(u(\tau,y))dy d\tau \right|.$$

First we estimate L_1 by dividing the integrand as

$$\|K_1(t-\tau,\cdot) * f(u(\tau,\cdot))\|_2 = \|K_{1,low}(t-\tau,\cdot) * f(u(\tau,\cdot))\|_2$$
$$+ \|K_{1,mh}(t-\tau,\cdot) * f(u(\tau,\cdot))\|_2.$$

Taking $q_1 = q_2 = \frac{2n}{n+4\sigma}$ and $s_1 = s_2 = \vartheta = 0$ in (4.46), we obtain

$$\|K_{1,low}(t-\tau,\cdot) * f(u(\tau,\cdot))\|_2 \lesssim \|f(u(\tau,\cdot))\|_{\frac{2n}{n+4\sigma}} \lesssim \|u(\tau,\cdot)\|_{\frac{2np}{n+4\sigma}}^p. \tag{7.26}$$

Since $\bar{s} \geq 1$, assumption (2.13) implies

$$\frac{2np}{n+4\sigma} > \frac{2n}{n+2\delta},$$

and (2.7) implies

$$\frac{2np}{n+4\sigma} \leq \frac{2n}{n-2\bar{s}}$$

if $2\bar{s} < n$. Thus, we can apply (6.8) with $r = 1$ and $q = \frac{2np}{n+4\sigma}$ to obtain

$$\|u(\tau,\cdot)\|^p_{\frac{2np}{n+4\sigma}} \lesssim \langle\tau\rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}(p-\frac{1}{2}-\frac{2\sigma}{n})+p\sigma\right)}\|u\|^p_X$$

$$= \langle\tau\rangle^{\frac{1}{1-\sigma}\left(-(p-1)\left(\frac{n}{2}-\sigma\right)-\frac{n}{4}+2\sigma\right)}\|u\|^p_X.$$

From the inequality above and (7.26), it follows that

$$\int_{\frac{t}{2}}^t \|K_{1,low}(t-\tau,\cdot) * f(u(\tau,\cdot))\|_2 d\tau \lesssim \langle t\rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{4}+\sigma-(p-1)\left(\frac{n}{2}-\sigma\right)+1\right)}\|u\|^p_X. \tag{7.27}$$

By (7.19) and (7.23) with $r = 1$ and $s = 0$, we have

$$\int_{\frac{t}{2}}^t \|K_{1,mh}(t-\tau,\cdot) * f(u(\tau,\cdot))\|_2 d\tau \lesssim \langle t\rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{4}+\sigma+\frac{1}{2}-(p-1)\left(\frac{n}{2}-\sigma\right)\right)}\|u\|^p_X,$$

which together with (7.27) yields

$$L_1 \lesssim \langle t\rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{4}+\sigma-(p-1)\left(\frac{n}{2}-\sigma\right)+1\right)}\|u\|^p_X. \tag{7.28}$$

By (5.1) with $u_1 = f(u(\tau,\cdot))$ and $\theta = 2\tilde{\nu}$, we have

$$L_2 \lesssim \langle t\rangle^{\max\{\frac{1}{1-\sigma}\left(-\frac{n}{4}+3\sigma-1\right),\frac{1}{\sigma}\left(-\frac{n}{4}+\sigma\right)\}} \int_0^{t/2} \|f(u(\tau,\cdot))\|_1 d\tau$$

$$+ e^{-\frac{\varepsilon_\sigma t}{2}} \int_0^{t/2} \|f(u(\tau,\cdot))\|_{H^{-2\sigma}} d\tau \tag{7.29}$$

$$+ t^{\frac{1}{1-\sigma}\left(-\frac{n}{4}+\sigma-\tilde{\nu}\right)} \int_0^{t/2} \||\cdot|^{2\tilde{\nu}} f(u(\tau,\cdot))\|_1 d\tau$$

$$=: L_{2,1} + L_{2,2} + L_{2,3} \text{ (we put)}.$$

Inequality (7.5) and (7.9) with $r = 1$ yield

$$\int_0^{t/2} \|f(u(\tau,\cdot))\|_1 d\tau \lesssim \int_0^{t/2} \langle\tau\rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}(p-1)+p\sigma\right)} d\tau \|u\|^p_X \lesssim \|u\|^p_X.$$

Thus

$$L_{2,1} \lesssim \langle t\rangle^{\max\{\frac{1}{1-\sigma}\left(-\frac{n}{4}+3\sigma-1\right),\frac{1}{\sigma}\left(-\frac{n}{4}+\sigma\right)\}}\|u\|^p_X. \tag{7.30}$$

Since $\bar{s} \geq 1$, (2.7) and (2.15) with $r = 1$ imply $2p > \frac{2n}{n+2\delta}$, and moreover $2p \leq \frac{2n}{n-2\bar{s}}$ if $2\bar{s} < n$. Hence, we can use (6.8) with $r = 1$ and $q = 2p$ to obtain

$$\|f(u(\tau,\cdot))\|_{H^{-2\sigma}} \lesssim \|u(t,\cdot)\|^p_{2p} \lesssim \langle t\rangle^{\frac{p}{1-\sigma}\left(-\frac{n}{2}(1-\frac{1}{2p})+\sigma\right)}\|u\|^p_X \lesssim \|u\|^p_X.$$

Thus

$$L_{2,2} \leq e^{-\varepsilon_\sigma t/2} \|u\|_X^p. \tag{7.31}$$

We estimate $L_{2,3}$. Let $\tilde{\nu}$ be an arbitrary number satisfying

$$0 < \nu < \tilde{\nu} < \min \left\{ \frac{n}{4}(p-2) + \frac{1}{2} p\delta, \delta \right\}. \tag{7.32}$$

Assume moreover that

$$\tilde{\nu} \leq \frac{\delta}{2\bar{s}} \left(n - \frac{p}{2}(n - 2\bar{s}) \right), \tag{7.33}$$

if $\bar{s} < \frac{n}{2}$. By Hölder's inequality, we have

$$\left\| |\cdot|^{2\tilde{\nu}} f(u(\tau, \cdot)) \right\|_1 \lesssim \left\| |\cdot|^{\frac{2\tilde{\nu}}{p}} u(\tau, \cdot) \right\|_p^p \leq \left\| (|\cdot|^\delta |u(\tau, \cdot)|) \right\|_{\frac{p\delta}{\tilde{\nu}}}^{\frac{2\tilde{\nu}}{p\delta}} \left\| |u(\tau, \cdot)|^{1 - \frac{2\tilde{\nu}}{p\delta}} \right\|_q^p$$

$$= \left\| |\cdot|^\delta u(\tau, \cdot) \right\|_2^{\frac{2\tilde{\nu}}{\delta}} \left\| u(\tau, \cdot) \right\|_{\tilde{q}}^{p - \frac{2\tilde{\nu}}{\delta}}, \tag{7.34}$$

where $q = \frac{p\delta}{\delta - \tilde{\nu}}$ and $\tilde{q} = q(1 - \frac{2\tilde{\nu}}{p\delta})$. The assumption (7.32) implies

$$\tilde{q} = q \left(1 - \frac{2\tilde{\nu}}{p\delta} \right) = \frac{p\delta - 2\tilde{\nu}}{\delta - \tilde{\nu}} > \frac{2n}{n + 2\delta}.$$

In fact, the condition $\tilde{\nu} < \frac{n}{4}(p-2) + \frac{p\delta}{2}$ is equivalent to $\tilde{q} = \frac{p\delta - 2\tilde{\nu}}{\delta - \tilde{\nu}} > \frac{2n}{n + 2\delta}$. The condition (7.33) is equivalent to $\tilde{q} = \frac{p\delta - 2\tilde{\nu}}{\delta - \tilde{\nu}} \leq \frac{2n}{n - 2\bar{s}}$ in the case $n > 2\bar{s}$. Hence, using (6.8) with taking q as $\tilde{q}$, and definition of $\|\cdot\|_X$ with $r = 1$ (see (6.2)) in the right-hand side of (7.34), we obtain

$$\left\| |\cdot|^{2\tilde{\nu}} f(u(\tau, \cdot)) \right\|_1 \lesssim \langle\tau\rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}(1-\frac{1}{2})+\sigma+\frac{\delta}{2}\right)\frac{2\tilde{\nu}}{\delta}} \langle\tau\rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}(1-\frac{1}{\tilde{q}})+\sigma\right)(p-\frac{2\tilde{\nu}}{\delta})} \|u\|_X^p$$

$$= \langle\tau\rangle^{\frac{1}{1-\sigma}\left((-\frac{n}{2}+\sigma)p+\frac{n}{2}+\tilde{\nu}\right)} \|u\|_X^p. \tag{7.35}$$

Thus,

$$L_{2,3} \lesssim \langle t\rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{4}+\sigma-\tilde{\nu}\right)} \int_0^{t/2} \langle\tau\rangle^{\frac{1}{1-\sigma}\left((-\frac{n}{2}+\sigma)p+\frac{n}{2}+\tilde{\nu}\right)} d\tau \|u\|_X^p,$$

which yields

$$L_{2,3} \lesssim \begin{cases} \langle t\rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{4}+\sigma-(p-1)\left(\frac{n}{2}-\sigma\right)+1\right)} & \text{if} \quad \frac{1}{1-\sigma}\left(-\frac{n}{2}(p-1)+p\sigma+\tilde{\nu}\right) > -1, \\ \langle t\rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{4}+\sigma-\tilde{\nu}\right)} \log(\langle t\rangle + 1) & \text{if} \quad \frac{1}{1-\sigma}\left(-\frac{n}{2}(p-1)+p\sigma+\tilde{\nu}\right) = -1, \\ \langle t\rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{4}+\sigma-\tilde{\nu}\right)} & \text{if} \quad \frac{1}{1-\sigma}\left(-\frac{n}{2}(p-1)+p\sigma+\tilde{\nu}\right) < -1. \end{cases}$$

$$\lesssim \langle t\rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{4}+\sigma-\min\{(p-1)\left(\frac{n}{2}-\sigma\right)-1,\nu\}\right)}. \tag{7.36}$$

Inequalities (7.30), (7.31) and (7.36) yield

$$L_2 \lesssim \langle t\rangle^{\max\left\{\frac{1}{1-\sigma}\left(-\frac{n}{4}+\sigma-\min\{1-2\sigma,(p-1)\left(\frac{n}{2}-\sigma\right)-1,\nu\}\right), \frac{1}{\sigma}\left(-\frac{n}{4}+\sigma\right)\right\}} \|u\|_X^p. \tag{7.37}$$

We estimate L_3. By the definition of G_σ,

$$\mathcal{F}(G_\sigma(t - \tau, \cdot) - G_\sigma(t, \cdot)) = \tau \int_0^1 \frac{\partial \widehat{G_\sigma}}{\partial t}(t - \theta\tau, \xi)d\theta$$

$$= \tau \int_0^1 |\xi|^{2(1-2\sigma)} e^{-|\xi|^{2(1-\sigma)}(t-\theta\tau)} d\theta.$$

Inequality (5.8) with $\theta = 2 - 2\sigma$ implies

$$\left\| |\xi|^{2(1-2\sigma)} e^{-|\xi|^{2(1-\sigma)}(t-\theta\tau)} \right\|_2 \sim t^{\frac{1}{1-\sigma}\left(-\frac{n}{4}-1+2\sigma\right)} \tag{7.38}$$

uniformly to $\theta \in [0, 1]$ and $\tau \in [0, t/2]$. Then by using (7.5) with $r = 1$ together, we have

$$\begin{aligned} L_3 &\lesssim t^{\frac{1}{1-\sigma}\left(-\frac{n}{4}-1+2\sigma\right)} \int_0^{t/2} \tau \|f(u(\tau, \cdot))\|_1 d\tau \\ &\lesssim t^{\frac{1}{1-\sigma}\left(-\frac{n}{4}-1+2\sigma\right)} \int_0^{t/2} \tau \langle\tau\rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}(p-1)+p\sigma\right)} d\tau, \end{aligned} \tag{7.39}$$

which yields

$$L_3 \lesssim \begin{cases} t^{\frac{1}{1-\sigma}\left(-\frac{n}{4}+\sigma-(p-1)\left(\frac{n}{2}-\sigma\right)+1\right)} & \text{if} \quad \frac{1}{1-\sigma}\left(-\frac{n}{2}(p-1)+p\sigma\right) > -2 \\ t^{\frac{1}{1-\sigma}\left(-\frac{n}{4}-1+2\sigma\right)} \log(\langle t\rangle + 1) & \text{if} \quad \frac{1}{1-\sigma}\left(-\frac{n}{2}(p-1)+p\sigma\right) = -2 \\ t^{\frac{1}{1-\sigma}\left(-\frac{n}{4}-1+2\sigma\right)} & \text{if} \quad \frac{1}{1-\sigma}\left(-\frac{n}{2}(p-1)+p\sigma\right) < -2. \end{cases}$$

$$\leq t^{\frac{1}{1-\sigma}\left(-\frac{n}{4}+\sigma-\min\{(p-1)\left(\frac{n}{2}-\sigma\right)-1,1-2\sigma\}\right)}. \tag{7.40}$$

Last we estimate L_4. Since $n \geq 2$, the assumptions (2.14) and (2.13) imply

$$\frac{2n}{n+2\delta} \leq \frac{2n}{2n-2} = 1 + \frac{1}{n-1} \leq 1 + \frac{2}{n-2\sigma} < p.$$

By this inequality and (2.7), we can apply (6.8) with $r = 1$ and $q = p$ to obtain

$$\|u(\tau, \cdot)\|_p \lesssim \langle\tau\rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}(1-\frac{1}{p})+\sigma\right)} \|u\|_X.$$

This together with (7.9) yields

$$\int_{t/2}^\infty \left| \int_{\mathbb{R}^n} f(u(\tau, y)) dy \right| d\tau \lesssim \int_{t/2}^\infty \langle\tau\rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}(p-1)+p\sigma\right)} d\tau \|u\|_X^p$$

$$\sim \langle t\rangle^{\frac{1}{1-\sigma}\left(-(\frac{n}{2}-\sigma)(p-1)+1\right)} \|u\|_X^p. \tag{7.41}$$

Taking the product of (2.4) and (7.41), we obtain

$$L_4 \lesssim t^{\frac{1}{1-\sigma}\left(-\frac{n}{4}+\sigma-(p-1)\left(\frac{n}{2}-\sigma\right)+1\right)}. \tag{7.42}$$

Substituting (7.28), (7.37), (7.40) and (7.42) into (7.25), we obtain (7.24). $\square$

8. Proof of Proposition and Theorems

8.1. Proof of Proposition 1

Let $\varepsilon > 0$, and

$$X(\varepsilon) = X_{r,\delta,\bar{s}}(\varepsilon) := \{\varphi \in X_{r,\delta,\bar{s}}; \|\varphi\|_{X_{r,\delta,\bar{s}}} < \varepsilon\},$$

where $X_{r,\delta,\bar{s}}$ and $\|\varphi\|_{X_{r,\delta,\bar{s}}}$ are defined by (6.1) and (6.2), respectively. We put $X = X_{r,\delta,\bar{s}}$ and $\|\cdot\|_X = \|\varphi\|_{X_{r,\delta,\bar{s}}}$, throughout this subsection.

If u is a solution of (1.1), then Duhamel's principles implies

$$u(t,x) = K_0(t,\cdot) * u_0 + K_1(t,\cdot) * u_1 + \int_0^t K_1(t-\tau,\cdot) * f(u(\tau,\cdot))d\tau,$$

where K_0 and K_1 are defined by (4.2) and (4.3). Taking account of the formula above, we define the mapping Φ on $X(\varepsilon)$ by

$$(\Phi u)(t) := K_0(t,\cdot) * u_0 + K_1(t,\cdot) * u_1 + \int_0^t K_1(t-\tau,\cdot) * f(u(\tau,\cdot))d\tau. \tag{8.1}$$

We prove that Φ is a contraction mapping on $X(\varepsilon)$ provided ε and initial data are sufficiently small.

First we estimate $K_1(t,\dot{)} * u_1$. By (2.14), we see that assumtion (4.43) of Lemma 2 is satisfied for $\vartheta = 0$ and δ, $s_1 = s_2 = 0$, $q_1 = r$ and $q_2 = \frac{nr}{n-r\vartheta}(\in [r,2))$ (that is, $\frac{1}{q_2} = \frac{1}{r} - \frac{\vartheta}{n}$). Then, (4.46) gives estimate of low frequency part. The high and middle frequency parts are given by (4.64) and (4.70). Then we have

$$\left\|\langle\cdot\rangle^\delta (K_1(t,\cdot) * u_1)\right\|_2 \lesssim \langle t\rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}(\frac{1}{r}-\frac{1}{2})+\frac{\delta}{2}+\sigma\right)}(\|u_1\|_r' + \|\langle\cdot\rangle^\delta u_1\|_{\frac{nr}{n-r\delta},2})$$
$$+ e^{-\varepsilon_\sigma t}\|(1-\Delta)^{-\frac{1}{2}}\langle\cdot\rangle^\delta u_1\|_2. \tag{8.2}$$

Assumption (2.14) implies $\frac{n-r\delta}{r} - 1 \leq \frac{n}{2}$, and therefore, sharp Sobolev's embedding theorem (Lemma E) yields

$$\|(1-\Delta)^{-\frac{1}{2}}\langle\cdot\rangle^\delta u_1\|_2 \leq \|\langle\cdot\rangle^\delta u_1\|_{\frac{nr}{n-r\delta},2}.$$

Substituting this inequality into (8.2), we obtain

$$\left\|\langle\cdot\rangle^\delta (K_1(t,\cdot) * u_1)\right\|_2 \lesssim \langle t\rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}(\frac{1}{r}-\frac{1}{2})+\frac{\delta}{2}+\sigma\right)}(\|u_1\|_r' + \|\langle\cdot\rangle^\delta u_1\|_{\frac{nr}{n-r\delta},2}). \tag{8.3}$$

Inequality (4.46) with $\vartheta = 0$, $s_1 = \bar{s}$, $s_2 = 0$, $q_1 = q_2 = r$, and inequalities (4.63) and (4.69) with $s = \bar{s}$ yield

$$\left\|(-\Delta)^{\frac{\bar{s}}{2}} (K_1(t,\cdot) * u_1)\right\|_2$$
$$\lesssim \langle t\rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}(\frac{1}{r}-\frac{1}{2})-\frac{\bar{s}}{2}+\sigma\right)}\|u_1\|_r' + e^{-\varepsilon_\sigma t}\|(-\Delta)^{\frac{\bar{s}}{2}}(1-\Delta)^{-\frac{1}{2}}u_1\|_2. \tag{8.4}$$

Next we estimate $K_0(t,\dot{)} * u_0$. By (2.14), we see that the assumption of Lemma 2 is satisfied for

$$\vartheta = 0 \text{ and } \delta, \quad s_1 = s_2 = 0, \quad q_3 = \frac{nr}{n-2r\sigma}, \quad q_4 = \frac{nr}{n-r(\vartheta+2\sigma)}.$$

Then, (4.49) gives estimate of low frequency part. Inequalities (4.66) and (4.72) give estimate of high and middle frequency parts. Then we obtain

$$\left\|\langle\cdot\rangle^\delta \left(K_0(t,\cdot) * u_0\right)\right\|_2$$
$$\lesssim \langle t\rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{\delta}{2}+\sigma\right)}\left(\|u_0\|_{\frac{nr}{n-2r\sigma},2} + \|\langle\cdot\rangle^\delta u_0\|_{\frac{nr}{n-r(\delta+2\sigma)},2}\right) + e^{-\varepsilon_\sigma t}\|\langle\cdot\rangle^\delta u_0\|_2.$$

By Corollary A, we have

$$\|u_0\|_{\frac{nr}{n-2r\sigma},2} \lesssim \||x|^{-\delta}\|_{\frac{n}{\delta},\infty}\|\langle\cdot\rangle^\delta u_0\|_{\frac{nr}{n-r(\delta+2\sigma)},2} \lesssim \|\langle\cdot\rangle^\delta u_0\|_{\frac{nr}{n-r(\delta+2\sigma)},2}. \quad (8.5)$$

Hence, we have

$$\left\|\langle\cdot\rangle^\delta \left(K_0(t,\cdot) * u_0\right)\right\|_2$$
$$\lesssim \langle t\rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)+\frac{\delta}{2}+\sigma\right)}\|\langle\cdot\rangle^\delta u_0\|_{\frac{nr}{n-r(\delta+2\sigma)},2} + e^{-\varepsilon_\sigma t}\|\langle\cdot\rangle^\delta u_0\|_2. \quad (8.6)$$

Inequality (4.49) with $\vartheta = 0$, $s_1 = \bar{s}$, $s_2 = 0$ and $q_3 = q_4 = \frac{nr}{n-2r\sigma}$ and inequalities (4.65) and (4.71) with $s = \bar{s}$ imply

$$\left\|(-\Delta)^{\frac{\bar{s}}{2}} \left(K_0(t,\cdot) * u_0\right)\right\|_2$$
$$\lesssim \langle t\rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{\bar{s}}{2}+\sigma\right)}\|u_0\|_{\frac{nr}{n-2r\sigma},2} + e^{-\varepsilon_\sigma t}\|u_0\|_{H^{\bar{s}}}$$
$$\lesssim \langle t\rangle^{\frac{1}{1-\sigma}\left(-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{\bar{s}}{2}+\sigma\right)}\|\langle\cdot\rangle^\delta u_0\|_{\frac{nr}{n-r(\delta+2\sigma)},2} + e^{-\varepsilon_\sigma t}\|u_0\|_{H^{\bar{s}}}. \quad (8.7)$$

For the last inequality, we used (8.5).

By (8.1), (8.3), (8.4), (8.6), (8.7), (7.2) and (7.14) with $s = 0, \bar{s}$, we have

$$\|\Phi u\|_X \lesssim \|\langle\cdot\rangle^\delta u_0\|_{\frac{nr}{n-r(\delta+2\sigma)},2} + \|\langle\cdot\rangle^\delta u_0\|_2 + \|u_0\|_{H^{\bar{s}}}$$
$$+ \|u_1\|'_r + \|\langle\cdot\rangle^\delta u_1\|_{\frac{nr}{n-r\delta},2} + \|(-\Delta)^{\frac{\bar{s}}{2}}(1-\Delta)^{-\frac{1}{2}}u_1\|_2 + \|u\|_X^p. \quad (8.8)$$

(i) First we consider the case $r = 1$. By (8.8), there is a positive constant C_1 independent of initial data such that

$$\|\Phi u\|_X \leq C_1\left(\|\langle\cdot\rangle^\delta u_0\|_{\frac{n}{n-(\delta+2\sigma)},2} + \|\langle\cdot\rangle^\delta u_0\|_2 + \|u_0\|_{H^{\bar{s}}}\right.$$
$$\left. + \|u_1\|_1 + \|\langle\cdot\rangle^\delta u_1\|_{\frac{n}{n-\delta},2} + \|(-\Delta)^{\frac{\bar{s}}{2}}(1-\Delta)^{-\frac{1}{2}}u_1\|_2 + \|u\|_X^p\right).$$

Hence, taking $\varepsilon_1 > 0$ such that $C_1\varepsilon_1^{p-1} \leq \frac{1}{2}$, and assuming u_0 and u_1 satisfy

$$C_1\left(\|\langle\cdot\rangle^\delta u_0\|_{\frac{n}{n-(\delta+2\sigma)},2} + \|\langle\cdot\rangle^\delta u_0\|_2 + \|(-\Delta)^{\frac{\bar{s}}{2}}u_0\|_2\right.$$
$$\left. + \|u_1\|_1 + \|\langle\cdot\rangle^\delta u_1\|_{\frac{n}{n-\delta},2} + \|(-\Delta)^{\frac{\bar{s}}{2}}(1-\Delta)^{-\frac{1}{2}}u_1\|_2\right) \leq \frac{\varepsilon_1}{2},$$

we see that Φ is a mapping from $X(\varepsilon_1)$ to $X(\varepsilon_1)$.

By (7.3) and (7.15), there is a positive constant C_2 independent of initial data such that

$$\|\Phi u - \Phi v\|_X \leq C_2(\|u\|_X + \|v\|_X)^{p-1}\|u - v\|_X \quad (8.9)$$

for every $u, v \in X$. We take $\varepsilon \in (0, \varepsilon_1)$ such that

$$C_2(2\varepsilon)^{p-1} < 1.$$

Then we see that Φ is a contraction mapping from $X(\varepsilon)$ to $X(\varepsilon)$, and therefore Φ has the only one fixed point u, which is the unique solution.

(ii) Next we consider the case $r \in (1, \frac{2n}{n+2\sigma}]$. By Corollary A, we have

$$\|u_1\|_r' = \|u_1\|_{r,2} \lesssim \||x|^{-\delta}\|_{\frac{n}{\delta},\infty}\|\langle\cdot\rangle^\delta u_1\|_{\frac{nr}{n-r\delta},2} \lesssim \|\langle\cdot\rangle^\delta u_1\|_{\frac{nr}{n-r\delta},2}.$$

Substituting this inequality into (8.8), we obtain

$$\|\Phi u\|_X \leq C_3\Big(\|\langle\cdot\rangle^\delta u_0\|_{\frac{nr}{n-r(\delta+2\sigma)},2} + \|\langle\cdot\rangle^\delta u_0\|_2 + \|u_0\|_{H^{\bar{s}}}$$
$$+\|\langle\cdot\rangle^\delta u_1\|_{\frac{nr}{n-r\delta},2} + \|(-\Delta)^{\frac{\bar{s}}{2}}(1-\Delta)^{-\frac{1}{2}}u_1\|_2 + \|u\|_X^p\Big), \quad (8.10)$$

for a positive constant C_3 independent of initial data. Hence, taking $\varepsilon_2 > 0$ such that $C_3\varepsilon_2^{p-1} \leq \frac{1}{2}$, and assuming u_0 and u_1 satisfy

$$C_3\Big(\|\langle\cdot\rangle^\delta u_0\|_{\frac{nr}{n-r(\delta+2\sigma)},2} + \|\langle\cdot\rangle^\delta u_0\|_2 + \|u_0\|_{H^{\bar{s}}}$$
$$+ \|\langle\cdot\rangle^\delta u_1\|_{\frac{nr}{n-r\delta},2} + \|(-\Delta)^{\frac{\bar{s}}{2}}(1-\Delta)^{-\frac{1}{2}}u_1\|_2\Big) \leq \frac{\varepsilon_2}{2},$$

we see that Φ is a mapping from $X(\varepsilon_2)$ to $X(\varepsilon_2)$. In the same way as (8.9), we see that there is a positive constant C_4 independent of initial data satisfying

$$\|\Phi u - \Phi v\|_X \leq C_4(\|u\| + \|v\|)^{p-1}\|u - v\|_X$$

for every $u, v \in X$. We take $\varepsilon \in (0, \varepsilon_2)$ such that $C_4(2\varepsilon)^{p-1} < 1$. Then Φ becomes a contraction mapping $X(\varepsilon)$ to $X(\varepsilon)$, and therefore Φ has the only one fixed point u, which is the unique solution.

8.2. Proof of Theorems 2 and 3

Proof of Theorems 2. We prove Theorem 2 by using Proposition 1.

(Case 1) In the case $p_\sigma < p \leq 1 + \frac{4}{n}$, we can $\eta > 0$ sufficiently small such that r defined by

$$\frac{1}{r} = \frac{2}{n}\left(\frac{1}{p-1} + \sigma\right) + \eta \quad (8.11)$$

satisfies

$$\frac{1}{2} + \frac{2\sigma}{n} < \frac{1}{r} < 1, \quad \text{that is,} \quad r \in \left(1, \frac{2n}{n+4\sigma}\right). \quad (8.12)$$

Then $\frac{1}{r} > \frac{2}{n}\left(\frac{1}{p-1} + \sigma\right)$ implies assumtion (2.13).

(Case 1-1) First we consider the case $p_\sigma < p \leq 1 + \frac{4}{n+2-4\sigma}$. We put

$$\delta' = 2\left(\frac{1}{p-1} + \sigma\right) - \frac{n}{2} - 1 + n\eta. \quad (8.13)$$

By definition (8.11), we have

$$\delta' = n\left(\frac{1}{r} - \frac{1}{2}\right) - 1. \quad (8.14)$$

Comparing (2.9) and (8.13), we see that

$$\delta' \leq \delta \quad (8.15)$$

if $\eta > 0$ is sufficiently small. Hence, taking $\eta > 0$ sufficiently small, we can assume that r and δ' defined above satisfy (8.12) and (8.15). We check that the assumptions (2.14) and (2.15) of Proposition 1 are satisfied with δ replaced δ'. Since $2\sigma < 1$, (2.14) is trivial by (8.14). Since $n \geq 2$ and $2\sigma < 1$, we have

$$p \leq 1 + \frac{4}{n + 2 - 4\sigma} < 1 + \frac{1}{1 - 2\sigma} = \frac{2 - 2\sigma}{1 - 2\sigma},$$

from which it follows that

$$2\left(\frac{1}{p-1} + \sigma\right) - 1 > \frac{2}{p}\left(\frac{1}{p-1} + \sigma\right).$$

From this and the definition of δ' and r, it follows that

$$\delta' = 2\left(\frac{1}{p-1} + \sigma\right) - 1 - \frac{n}{2} + n\eta > \frac{2}{p}\left(\frac{1}{p-1} + \sigma\right) + n\eta - \frac{n}{2} = n\left(\frac{1}{pr} - \frac{1}{2}\right), \tag{8.16}$$

that is, (2.15) is satisfied with δ replaced by δ'. Hence the assumption of Proposition 1 is satisfied. Let $\hat{q}_j$ $(j = 0, 1)$ be the constants defined by (2.16) with $\delta = \delta'$ and r defined above. Then

$$\frac{1}{\hat{q}_0} = \frac{n - r(\delta' + 2\sigma)}{nr} = \frac{n + 2 - 4\sigma}{2n}, \quad \frac{1}{\hat{q}_1} = \frac{n - r\delta'}{nr} = \frac{n + 2}{2n}, \quad (8.17)$$

that is, $\hat{q}_j = q_j$ $(j = 0, 1)$. Since $\delta' \leq \delta$, the assumptions (2.10) and (2.11) imply (2.17) and (2.19) with δ replaced by δ'. Thus, Proposition 1 guarantees the existence of the solution $u \in C^1([0,\infty), H^{\bar{s}}) \cap C([0,\infty), H^{\bar{s}-1})$ if ε is sufficiently small. By the standard argument, the uniqueness holds in the class $C^1([0,\infty), H^{\bar{s}}) \cap C([0,\infty), H^{\bar{s}-1})$.

(Case 1-2) Next we consider the case

$$1 + \frac{4}{n + 2 - 4\sigma} < p \leq 1 + \frac{4}{n}. \tag{8.18}$$

We show that $\delta = 0$ satisfies the assumptions (2.14) and (2.15), that is,

$$2\left(\frac{1}{p-1} + \sigma\right) + n\eta - \frac{n}{2} - 1 = n\left(\frac{1}{r} - \frac{1}{2}\right) - 1 \leq 0 < n\left(\frac{1}{r} - \frac{1}{2}\right) - 2\sigma, \tag{8.19}$$

$$\frac{2}{p}\left(\frac{1}{p-1} + \sigma\right) + \frac{n\eta}{p} - \frac{n}{2} = n\left(\frac{1}{pr} - \frac{1}{2}\right) \leq 0, \tag{8.20}$$

if $\eta > 0$ is sufficiently small.

The assumption (8.12) implies $n(\frac{1}{r} - \frac{1}{2}) - 2\sigma > 0$. The assumption $1 + \frac{4}{n+2-4\sigma} < p$ is equivalent to

$$2\left(\frac{1}{p-1} + \sigma\right) - \frac{n}{2} - 1 < 0. \tag{8.21}$$

Hence (8.19) holds if η is sufficiently small.

From the assumption that $p \geq 1 + \frac{4}{n+2-4\sigma}$, it follows that

$$\frac{2}{p}\left(\frac{1}{p-1}+\sigma\right) - \frac{n}{2} \leq \frac{2(n+2-4\sigma)}{n+6-4\sigma}\left(\frac{n+2-4\sigma}{4}+\sigma\right) - \frac{n}{2}$$
$$= -\frac{n+4\sigma-2}{n+6-4\sigma} < 0,$$

and thus assumtion (8.20) holds if $\eta > 0$ is sufficiently small. Hence the assumption of Proposition 1 is satisfied with $\delta = 0$. Let $\hat{q}_j$ $(j = 0,1)$ be the constants defined by (2.16) with $\delta = 0$, and r be defined by (8.11) for sufficiently small η satisfying conditions described above. Then

$$\frac{1}{\hat{q}_0} = \frac{1}{r} - \frac{2\sigma}{n} = \frac{2}{n(p-1)} + \eta, \quad \frac{1}{\hat{q}_1} = \frac{1}{r} == \frac{2}{n(p-1)} + \frac{2\sigma}{n} + \eta. \tag{8.22}$$

Then by the assumption of q_j $(j = 0,1)$, we have

$$q_j < \hat{q}_j \quad (j = 0,1), \tag{8.23}$$

if $\eta > 0$ is sufficiently small. Since $p \leq 1 + \frac{4}{n}$,

$$\frac{1}{\hat{q}_j} > \frac{2}{n(p-1)} \geq \frac{1}{2}, \quad \text{that is,} \quad \hat{q}_j < 2 \quad (j = 0,1). \tag{8.24}$$

By (8.23) and (8.24),

$$L_{\hat{q}_j,2} \subset L_{q_j,2} \cap L_2.$$

Hence (2.11) implies (2.19). Thus the conclusion holds by Proposition 1 in the same way as above.

(Case 2) Last we consider the case $p \geq 1 + \frac{4}{n}$. We define r by

$$\frac{1}{r} = \frac{1}{2} + \frac{2\sigma}{n} + \eta \quad (\eta > 0). \tag{8.25}$$

Since $2\sigma < 1$, (2.14) holds for $\delta = 0$ if $\eta > 0$ is sufficiently small. The assumption

$$n\left(\frac{1}{pr} - \frac{1}{2}\right) = \frac{n}{p}\left(\frac{1}{2} + \frac{2\sigma}{n} + \eta\right) - \frac{n}{2} < 0$$

is equivalent to

$$1 + \frac{4\sigma}{n} + 2\eta < p,$$

which holds if $\eta > 0$ is sufficiently small, since $\sigma < 1$ and $p \geq 1 + \frac{4}{n}$. Hence, defining r by (8.25) with sufficiently small $\eta > 0$, we can take $\delta = 0$ in Proposition 1. Let $\hat{q}_j$ $(j = 0,1)$ be the constants defined by (2.16) with $\delta = 0$ and r defined above. Then, considering the asumption of q_j $(j = 0,1)$, we see that

$$\frac{1}{\hat{q}_0} = \frac{1}{r} - \frac{2\sigma}{n} = \frac{1}{2} + \eta < \frac{1}{q_0}, \quad \frac{1}{\hat{q}_1} = \frac{1}{r} = \frac{1}{2} + \frac{2\sigma}{n} + \eta < \frac{1}{q_1}, \tag{8.26}$$

if $\eta > 0$ is sufficiently small. This imply that $q_j < \hat{q}_j < 2$ $(j = 0,1)$. Hence (2.12) implies (2.19) with $\delta = 0$, and the conclusion holds by Proposition 1.

$\square$

Proof of Theorem 3. Since $u = \Phi u$ (Φ is defined by (8.1)), we can write

$$u(t, \cdot) - \Theta G_\sigma(t, x) = (K_0(t, \cdot) * u_0 + K_1(t, \cdot) * u_1 - \vartheta_1 G_\sigma(t, x))$$

$$+ \left(\int_0^t K_1(t - \tau, \cdot) * f(u(\tau, \cdot))d\tau - G_\sigma(t, x) \int_0^\infty \int_{\mathbb{R}^n} f(u(\tau, y))dyd\tau \right),$$
$$(8.27)$$

where ϑ_1 is defined by (2.3). Since $K_0(t, \cdot) * u_0 + K_1(t, \cdot) * u_1$ is a solution of the linear equation (1.3), the first term of the right-hand side of (8.27) is estimated by Theorem 1. The second term is estimated in (7.24). Combining these estimates, we obtain the assertion. □

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References

[1] Bergh, J., Löfström, J.: Interpolation Spaces, An Introduction. Springer, New York (1976)

[2] D'Abbicco, M., Ebert, M.R.: Diffusion phenomena for the wave equation with structural damping in the $L^p - L^q$ framework. J. Differ. Equ. **256**, 2307–2336 (2014)

[3] D'Abbicco, M., Ebert, M.R.: An application of $L^p - L^q$ decay estimates to the semi-linear wave equation with parabolic-llike structural damping. Nonlinear Anal. **99**, 16–34 (2014)

[4] D'Abbicco, M., Ebert, M.R.: A new phenomenon in the critical exponent for structurally damped semi-linear evolution equations. Nonlinear Anal. **149**, 1–40 (2017)

[5] D'Abbicco, M., Reissig, M.: Semilinear structural damped waves. Math. Methods Appl. Sci. **37**, 1570–1592 (2014)

[6] Galley, T.H., Raugel, G.: Scaling variables and asymptotic expansions in damped wave equations. J. Differ. Equ. **150**, 42–97 (1998)

[7] Hayashi, N., Kaikina, R.I., Naumkin, P.I.: Damped wave equation with super critical nonlinearities. Differ. Integr. Equ. **17**, 637–652 (2004)

[8] Hosono, T., Ogawa, T.: Large time behavior and $L^q - L^p$ estimate of solutions of 2-dimensional nonlinear damped wave equations. J. Differ. Equ. **203**, 82–118 (2004)

[9] Ikeda, M., Inui, T., Wakasugi, Y.: The Cauchy problem for the nonlinear damped wave equation with slowly decaying data. Nonlinear Differ. Equ. Appl. **24**(10), 1–53 (2017)

[10] Ikehata, R., Ohta, M.: Critical exponents for semilinear dissipative wave equations in $\mathbb{R}^N$. J. Math. Anal. Appl. **269**, 8797 (2002)

[11] Ikehata, R., Takeda, H.: Asymptotic profiles of solutions for structural damped wave equations. J. Dyn. Differ. Equ. **31**, 537–571 (2019)

[12] Ikehata, R., Tanizawa, K.: Global existence of solutions for semilinear damped wave equationsn $\mathbb{R}^n$ with noncompactly supported initial data. Nonlinear Anal. **61**, 1189–1208 (2005)

[13] Karch, G.: Selfsimilar profiles in large time aysmptotics of solutions to damped wave equation. Stud. Math. **143**(2), 175–197 (2000)

[14] Kawakami, T., Takeda, H.: Higher order asymptotic expansions to the solutions for a nonlinear damped wave equation. Nonlinear Differ. Equ. Appl. **23**(54), 1–30 (2016)

[15] Li, T.T., Zhou, Y.: Breakdown of solutions to $\Box u + u_t = |u|^{1+\alpha}$. Discrete Contin. Dyn. Syst. **1**, 503–520 (1995)

[16] Matsumura, A.: On the asymptotic behavior of solution of semi-linear wave equations. Publ. Res. Inst. Math. Sci. **12**, 169–189 (1976)

[17] Nakao, M., Ono, K.: Existence of global solutions to the Cauchy problem for the semilinear dissipative wave equations. Math. Z. **214**, 325–342 (1993)

[18] Narazaki, T.: $L^p - L^q$ estimates for the damped wave equation and their applications to semi-linear problem. J. Math. Soc. Jpn. **56**, 585–626 (2004)

[19] Narazaki, T., Nishihara, K.: Asymptotic behavior of solutions for damped wave equation with slowly decaying data. J. Math. Anal. Appl. **338**, 803–819 (2008)

[20] Narazaki, T., Reissig, M.: L^1 estimates for oscillating integrals related to structural damped wave models. In: Cicognani, M., Colombini, F., Del Santo, D. (eds.) Studies in Phase Space Analysis with Applications to PDEs. Nonlinear Differential Equations Applications, pp. 215–218. Birkhäuser, New York (2013)

[21] Nishihara, K.: $L^p - L^q$ estimates of solutions to the damped wave equation in 3-dimensional space and their application. Math. Z. **244**, 631–649 (2003)

[22] O'Neil, R.: Convolution operators and $L(p, q)$ spaces. Duke Math. J. **30**, 129–142 (1963)

[23] Todorova, G., Yordanov, B.: Critical exponent for a nonnlinear wave equation with damping. J. Differ. Equ. **174**, 464–489 (2001)

[24] Zhang, Q.S.: A blow-up result for a nonlinear wave equation with damping: the critical case. C. R. Acad. Sci. Paris Série **I**(333), 109–114 (2001)

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