Nonlinear Differ. Equ. Appl. (2018) 25:50

1021-9722/18/060001-20
*published online* October 1, 2018
https://doi.org/10.1007/s00030-018-0540-8

**Nonlinear Differential Equations and Applications NoDEA**

# Scalar reduction techniques for weakly coupled Hamilton–Jacobi systems

Antonio Siconolfi and Sahar Zabad

**Abstract.** We study a class of weakly coupled systems of Hamilton–Jacobi equations at the critical level. We associate to it a family of scalar discounted equation. Using control-theoretic techniques we construct an algorithm which allows obtaining a critical solution to the system as limit of a monotonic sequence of subsolutions. We moreover get a characterization of isolated points of the Aubry set and establish semiconcavity properties for critical subsolutions.

**Mathematics Subject Classification.** 35F21, 49L25, 37J50.

**Keywords.** Weakly coupled systems, Hamilton–Jacobi equations, Viscosity solutions, Aubry set, Optimal control, Critical value.

## 1. Introduction

This paper deals with weakly coupled systems of Hamilton–Jacobi equations, on the torus $\mathbb{T}^N$, of the form

$$H_i(x, Du_i) + \sum_{j=1}^{m} a_{ij} u_j(x) = \alpha \qquad \text{for every } i \in \{1, \ldots, m\},$$

where $\alpha$ is a real constant and $H_1, \ldots, \ldots, H_m$ are continuous Hamiltonians, convex and coercive in the momentum variable, and $A := (a_{ij})$ is an $m \times m$ coupling matrix satisfying

$$a_{ij} \leq 0 \text{ for every } i \neq j, \quad \sum_{j=1}^{m} a_{ij} = 0 \text{ for any } i \in \{1, 2, \ldots, m\}.$$

This is the so-called degenerate case, because of the vanishing condition on any row of $A$. It corresponds, for a single equation, to require it being of Eikonal type, namely independent of the unknown function.

We focus on the critical system, obtained taking as $\alpha$ the minimum value for which the systems of the above family admit viscosity subsolutions. As first pointed out in [3] and [10], it is the unique system of the family for which there are viscosity solutions on the whole torus.

The critical setting has been deeply investigated through PDE techniques in [6], with the purpose of building an adapted Weak KAM theory. It has been in particular proved the existence of an Aubry set, denoted in what follows by $\mathcal{A}$, enjoying properties similar to the corresponding object in the scalar case. Recently the results of [6] have been recovered and improved in [8,9] by means of a more geometric approach to the matter in a suitable probabilistic frame. In [5] the same method has been applied to a time-dependent version of the system with the aim of studying an associated Lax–Oleinik semigroup.

There are still several issues to be investigated and understood in the field, especially about the structure and the property of the Aubry set. The major open question being to get regularity results for critical subsolutions on $\mathcal{A}$, as in the case of a single equation.

The paper is centered on a method, which seems new, to tackle this kind of problems: namely the scalar reduction technique mentioned in the title. It simply consists in associating to the system a family of scalar discounted equations. These are roughly speaking obtained by picking one of the equations in the system, and freezing all the components of a given critical subsolution except the one corresponding to the index of the selected equation. It becomes the unknown of the discounted equation. This approach advantageously allows exploiting the wide knowledge of this kind of equations to gather information on the system. We in particular use that a comparison principle holds for discounted equations, the solutions can be represented as infima of integral functionals, and the fact that corresponding optimal trajectories do exist.

Our achievements are as follows: we provide a constructive algorithm for getting a critical solution by suitably modifying an initial critical subsolution outside the Aubry set. The solution is obtained as uniform limit of a monotonic sequence of subsolutions. The procedure can be useful for numerical approximation of a critical solution and of the Aubry set as well, see Remark 3.4 for more details.

We moreover give a characterization of isolated points of the Aubry set, adapting the notion of equilibrium point to systems. This enables us to also show the strict differentiability of any critical subsolution on such points. The final outcome is about a semiconcavity property for critical subsolutions on the Aubry set, and on the whole torus for solutions. We more precisely prove that the superdifferential is nonempty. These results are clearly related to the aforementioned open problem about the differentiability of critical subsolutions on the Aubry set. They can be viewed as a partial positive answer to the regularity issue. We hope they will be useful to fully crack the problem.

The paper is organized as follows: Sect. 2 contains the notations, some preliminary results and the setting. In Sect. 3 we introduce the family of scalar discounted equations associated to the critical system, and describe an algorithm to get a critical solution starting from any subsolution. In Sect. 4 we study, through the scalar reduction technique, the nature of isolated points of $\mathcal{A}$ and prove semiconcavity properties for critical subsolutions. Finally, the appendix is devoted to recall some results for critical equations in the scalar case.

## 2. Preliminaries

### 2.1. Notations and basic results

Throughout the paper, we denote by $\mathbb{R}^N$ (resp. $\mathbb{R}^m$), $\mathbb{T}^N$ the $N$-dimensional Euclidean space and flat torus, respectively, where $N$ (resp. $m$) is a positive integer number. We further indicate with $\mathbb{R}^m_+$ the set made up by the vectors of $\mathbb{R}^m$ with nonnegative components. We denote by bold characters vectors in $\mathbb{R}^m$ and functions taking values in $\mathbb{R}^m$. The vector of $\mathbb{R}^m$ with all the components equal to 1 is accordingly denoted by $\mathbf{1}$. The partial order relations between elements of $\mathbb{R}^m$, denoted by $\leq, <$ must be understood componentwise. Accordingly, we write $\mathbf{u} \leq \mathbf{v}$ (resp. $\mathbf{u} < \mathbf{v}$), for given functions $\mathbf{u}, \mathbf{v} : \mathbb{T}^N \to \mathbb{R}^m$, if $\mathbf{u}(x) \leq \mathbf{v}(x)$ (resp. $\mathbf{u}(x) < \mathbf{v}(x)$) for every $x \in \mathbb{T}^N$.

Given an upper semicontinuous (respectively, lower semicontinuous) function $f$ on $\mathbb{T}^N$ or on $\mathbb{R}^N$, we say that a continuous function $\phi$ is supertangent (res. subtangent) to $f$ at some point $x_0$ if $\phi(x_0) = f(x_0)$ and $\phi \geq f$ (resp. $\phi \leq f$) in some neighborhood of $x_0$. The set made up by the differentials of $C^1$ functions supertangent (resp. subtangent) to $f$ at $x_0$ is called the superdifferential (resp. subdifferential) of $f$ at $x_0$, and is indicated with $D^+f(x_0)$ (resp. $D^-f(x_0)$).

If $f$ is locally Lipschitz continuous function then it is almost everywhere differentiable with locally bounded gradient, in force of Rademacher's Theorem. The (Clarke) generalized gradient of $f$, denoted by $\partial f$, is defined for every $x$ via the formula

$$\partial f(x) = \mathrm{co}\{p = \lim_n Df(x_n), f \text{ is differentiable at } x_n, \lim_n x_n = x\},$$

where $\mathrm{co}(\cdot)$ denotes the convex hull. The function $f$ is called strictly differentiable at $x$ if $f$ is differentiable at $x$ and $Df$ is continuous at x. More precisely, if

$$\lim_i Df(x_n) = Df(x) \qquad \text{for all sequences of differentiability points } x_n \text{ of } f \text{ converging to } x.$$

This is equivalent to require $\partial f(x)$ to be a singleton, namely $\partial f(x) = \{Df(x)\}$. We recall from [5], Lemma 1.4, a result on the a.e. derivative of $f$ along an absolutely continuous curve

**Lemma 2.1.** *Let $f : \mathbb{T}^N \to \mathbb{R}$, $\eta : (-\infty, 0] \to \mathbb{T}^N$ be a locally Lipschitz continuous function and an absolutely continuous curve, respectively. Let $s$ be such that $t \mapsto f\big(\eta(t)\big)$ and $t \mapsto \eta(t)$ are both differentiable at $s$. Then*

$$\frac{d}{dt} f(\eta(t))\Big|_{t=s} = p \cdot \dot\eta(s) \quad \textit{for some } p \in \partial f(\eta(s)).$$

We refer to [1] or [2] for the definition of viscosity solution,and more generally for a comprehensive treatment of viscosity solutions theory.

We record for later use

**Proposition 2.2.** *Let $u$ be an USC subsolution (resp. LSC supersolution) of a Hamilton–Jacobi equation of the form*

$$F(x,u,Du)=0 \qquad \textit{in } \mathbb{T}^N,$$

*where $F$ is continuous in all arguments and nondecreasing in $u$. Let $\psi$ be a Lipschitz-continuous supertangent (resp. subtangent) to $u$ at some point $x_0$. Then*

$$F(x_0,u(x_0),p)\le 0 \quad (\textit{resp.} \ge 0) \quad \textit{for some } p\in\partial\psi(x).$$

*Proof.* Given $\varepsilon>0$, we define the $\varepsilon$-inf-convolution of $\psi$ via

$$\psi_\varepsilon(x)=\min_{y\in\mathbb{T}^N}\left(\psi(y)+\frac{1}{2\varepsilon}|y-x|^2\right).$$

We can assume, without loosing generality, $\psi$ to be strict supertangent, and so $x_0$ to be the unique maximizer of $u-\psi$ in a suitable closed ball $B$ centered at $x_0$. From this uniqueness property we deduce that any sequence $x_\epsilon$ of maximizers of $u-\psi_\epsilon$ in $B$ converges to $x_0$. Hence $x_\epsilon$ is in the interior of $B$ for $\epsilon$ sufficiently small, and then for such $\epsilon$, $\psi_\epsilon$ is supertangent to $u$ at $x_\epsilon$. The inf-convolution being semiconcave, we deduce that

$$H(x_\epsilon,u(x_\epsilon),p_\epsilon)\le 0 \qquad \text{for any } p_\epsilon\in\partial\psi_\epsilon(x_\epsilon). \tag{2.1}$$

Further

$$u(x_\epsilon)-\psi_\epsilon(x_\epsilon)=\max_B u-\psi_\epsilon \to \max_B u-\psi=u(x_0)-\psi(x_0),$$

which, implies, bearing in mind that $\lim\psi_\epsilon(x_\epsilon)=\psi(x_0)$

$$\lim_{\epsilon\to 0}u(x_\epsilon)=u(x_0). \tag{2.2}$$

It is also well known that for any $y_\epsilon$ realizing the minimum in the formula defining $\psi_\epsilon$ at $x_\epsilon$, we have

$$\partial\psi_\epsilon(x_\epsilon)\cap D^+\psi(y_\epsilon)\subset\partial\psi_\epsilon(x_\epsilon)\cap\partial\psi(y_\epsilon)\neq\emptyset.$$

Taking into account that $y_\epsilon\to x_0$, as $\epsilon$ goes to 0, exploiting (2.2), plus the continuity properties of $F$ and generalized gradients, we find for $q_\epsilon\in\partial\psi_\epsilon(x_\epsilon)\cap\partial\psi(y_\epsilon)$

$$q_\epsilon\to p\in\partial\psi(x)$$
$$F(x_\epsilon,u(x_\epsilon),q_\epsilon)\to F(x,u(x),p)$$

then, thanks to (2.1)

$$F(x,u(x),p)\le 0 \quad \text{and} \quad p\in\partial\psi(x)$$

as claimed. □

### 2.2. Setting of the problem

Here we introduce the system under investigation as well as some basic assumptions and preliminary facts. We basically follow the approach of [6].

We deal with a one-parameter family of weakly coupled systems of Hamilton–Jacobi equations of the form

$$H_i(x, Du_i) + \sum_{j=1}^{m} a_{ij} u_j(x) = \alpha \quad \text{in } \mathbb{T}^N \qquad \text{for every } i \in \{1, \dots, m\}, \qquad (\mathrm{HJ}\alpha)$$

where $\alpha$ is a real constant and $H_1, \dots, H_m$ are Hamiltonians satisfying, for any $i \in \{1, \dots, m\}$, the following set of assumptions:

(H1) $H_i : \mathbb{T}^N \times \mathbb{R}^N \to \mathbb{R}$ is continuous;
(H2) $p \mapsto H_i(x, p)$ is convex for every $x \in \mathbb{T}^N$;
(H3) $p \mapsto H_i(x, p)$ is coercive for every $x \in \mathbb{T}^N$.

The above conditions will be assumed throughout the paper without further mentioning. For some specific results we will also need the following additional requirements for $i \in \{1, 2, \dots, m\}$:

(H4) $p \mapsto H_i(x, p)$ is strictly convex for every $x \in \mathbb{T}^N$;
(H5) $(x, p) \mapsto H_i(x, p)$ is locally Lipschitz continuous in $\mathbb{T}^N \times \mathbb{R}^N$ .

The $m \times m$ coupling matrix $A := (a_{ij})$ satisfies:

(A1) $a_{ij} \leq 0$ for every $i \neq j$;
(A2) $\sum\limits_{j=1}^{m} a_{ij} = 0$ for any $i \in \{1, \dots, m\}$;
(A3) $A$ is irreducible, i.e for every $W \subsetneq \{1, 2, \dots, m\}$ there exists $i \in W$ and $j \notin W$ such that $a_{ij} < 0$.

The relevant consequence of the irreducibility condition in our context is that the system cannot split into independent subsystems. Under our assumptions the coupling matrix $A$ is singular with rank $m - 1$ and kernel spanned by $\mathbf{1}$. Moreover $\mathrm{Im}A$, which has dimension $m - 1$, cannot contain vectors with strictly positive or negative components. This in particular implies that $\mathrm{Im}(A) \cap \ker(A) = \{0\}$.

We also derive:

**Proposition 2.3.** *All the diagonal elements of the coupling matrix $A$ are strictly positive.*

*Proof.* It is clear that $a_{ii} \geq 0$ for any $i$. Indeed, if $a_{kk} = 0$ for some $k \in \{1, 2, \dots, m\}$, then condition $(A2)$ would imply $a_{kj} = 0$ for every $j \in \{1, 2, \dots, m\}$, which contradicts the irreducibility character of the matrix. □

The notion of viscosity (sub/super) solution can be easily adapted to systems as (HJ$\alpha$).

**Definition 2.4.** (*Viscosity solution*) We say that a continuous function $\mathbf{u} : \mathbb{T}^N \to \mathbb{R}^m$ is a viscosity subsolution of (HJ$\alpha$) if for every $(x, i) \in \mathbb{T}^N \times \{1, 2, \dots, m\}$, we have

$$H_i(x,p) + \sum_{j=1}^{m} a_{ij} u_j(x) \le \alpha \quad \text{for every } p \in D^+ u_i(x).$$

Symmetrically, $\mathbf{u}$ is a viscosity supersolution of (HJ$\alpha$) if for every $(x,i) \in \mathbb{T}^N \times \{1,2,\ldots,m\}$, we have

$$H_i(x,p) + \sum_{j=1}^{m} a_{ij} u_j(x) \ge \alpha \quad \text{for every } p \in D^- u_i(x).$$

Finally, if $\mathbf{u}$ is both a viscosity sub and supersolution, then it is called a viscosity solution.

From now on, we will drop the term viscosity since no other kind of weak solutions will be considered. As already pointed out in the Introduction, we will be especially interested in the critical weakly coupled system

$$H_i(x, Du_i) + \sum_{j=1}^{m} a_{ij} u_j(x) = \beta \quad \text{in } \mathbb{T}^N \qquad \text{for every } i \in \{1,\ldots,m\}, \tag{HJ$\beta$}$$

where

$$\beta = \min\{\alpha \in \mathbb{R} \mid \text{ (HJ}\alpha\text{) admits subsolutions}\}$$

We straightforwardly derive from the coercivity condition:

**Proposition 2.5.** *Let $\alpha \ge \beta$. The family of all subsolutions to* (HJ$\alpha$) *is equi-Lipschitz continuous with Lipschitz constant denoted by $\ell_\alpha$.*

Moreover, owing to the convexity of the Hamiltonians, the notion of viscosity and a.e. subsolutions are equivalent for (HJ$\alpha$). Furthermore, we can express the same property using generalized gradients of any component. Namely, $u$ is a subsolution to (HJ$\alpha$) if and only if

$$H_i(x,p) + \sum_{j=1}^{m} a_{ij} u_j(x) \le \alpha$$

for any $x \in \mathbb{T}^N$, $p \in \partial u_i(x)$, $i \in \{1,\ldots,m\}$.

Due to the fact that any subsolution of (HJ$\beta$) is $\ell_\beta$-Lipschitz continuous, we may modify the $H_i$'s outside the compact set $\{(x,p) : |p| \le \ell_\beta\}$, to obtain a new Hamiltonian which is still continuous and convex, and in addition satisfies superlinearity condition, for every $i$. Since the sublevels contained in $B(0,\ell_\beta)$ are not affected, the subsolutions of the system obtained by replacing the $H_i$'s in (HJ$\beta$) by the new Hamiltonians are the same as the original one.
In the remainder of the paper, we will therefore assume without any loss of generality

(H'2) $H_i$ is superlinear in $p$ for any $i \in \{1,2,\ldots,m\}$,

We can thus associate to any $H_i$ a Lagrangian function $L_i$ through the Fenchel transform, i.e.

$$L_i(x,q) = \sup_{p \in \mathbb{R}^N} \{pq - H_i(x,p)\}.$$

The function $L_i$ is continuous on $\mathbb{T}^N \times \mathbb{R}^N$, convex and superlinear in $q$.

As pointed out in [6], there is restriction in the value that a subsolution to the system (HJ$\alpha$) can assume at a given point of the torus. An adaptation of the pull-up method used in the scalar version of the theory gives:

**Proposition 2.6.** *Let $\alpha \geq \beta$. The maximal subsolution of (HJ$\alpha$) among those taking the same admissible value at a given point $y$ is solution to (HJ$\alpha$) in $\mathbb{T}^N \setminus \{y\}$.*

As mentioned in the Introduction, the Aubry set associated to the critical system, denoted by $\mathcal{A}$, is a closed nonempty subset of $\mathbb{T}^N$ where the obstruction of getting global strict subsolution concentrates, see [6,8]. More specifically, there cannot be any critical subsolution which is, in addition, locally strict at a point in $\mathcal{A}$, in the sense of the subsequent definition.

**Definition 2.7.** Given a subsolution $\mathbf{u}$ of (HJ$\beta$). We say that $u_i$ is locally strict at $y \in \mathbb{T}^N$ if there exists a neighborhood $U$ of $y$ and $\delta > 0$ such that

$$H_i(x, Du_i(x)) + \sum_{j=1}^{m} a_{ij} u_j(x) \leq \beta - \delta \quad \text{for a.e. } x \in U.$$

We say that $u_i$ is strict in an open subset $U$ of $\mathbb{T}^N$ if it is locally strict at any $y \in U$. We say that $\mathbf{u}$ is locally strict at $y$ or strict in an open subset of the torus, if the propriety holds for any component.

An useful criterion is the following:

**Lemma 2.8.** *Let $y \in \mathbb{T}^N$ and $\mathbf{u}$ be a subsolution of (HJ$\beta$). The ith component of $\mathbf{u}$ is locally strict at $y$ if and only if*

$$H_i(y, p) + \sum_{j=1}^{m} a_{ij} u_j(y) < \beta \quad \textit{for any } p \in \partial u_i(y).$$

The formal PDE definition of Aubry set is:

**Definition 2.9.** (*Aubry set*) The Aubry set $\mathcal{A}$ is made up by points $y$ such that any maximal subsolution of the critical system among those taking a given admissible value at $y$ is a solution on the whole torus.

We refer to [8] for a more geometric characterization of $\mathcal{A}$ through cycles, in a suitable probabilistic framework. The following results on the Aubry set are taken from [6]. They generalize to systems facts already known in the case of a single equation.

**Theorem 2.10.** *A point $y \in \mathcal{A}$ if and only if no critical subsolution is locally strict at $y$.*

**Theorem 2.11.** *There exists a critical subsolution to the system which is strict in $\mathbb{T}^N \setminus \mathcal{A}$.*

We conclude the section recalling a final property which is peculiar to systems and has no equivalent in the scalar case. There is a remarkable rigidity phenomenon taking place, namely the restriction on admissible values that a critical subsolution can attain at a given point. This becomes severe on the Aubry set. We have

**Theorem 2.12.** [6, Proposition 5.1] *Let* $y \in \mathcal{A}$ *and* $\mathbf{u}, \mathbf{v}$ *be two subsolutions of* (HJ$\beta$)*, then*

$$\mathbf{u}(y) - \mathbf{v}(y) = k\mathbf{1}, \quad k \in \mathbb{R} \tag{2.3}$$

## 3. Scalar reduction

In this section we associate to the critical system some discounted scalar equations. Using these equations, we will thereafter write an algorithm to construct a solution of the critical system by suitably modifying outside $\mathcal{A}$ a given critical subsolution.

We denote by $\mathbf{w} = (w_1, \ldots, w_m)$ the initial subsolution of (HJ$\beta$) and freeze all its components except one obtaining for a given $i \in \{1, \ldots, m\}$, the following discounted equation:

$$a_{ii}v(x) + H_i(x, Dv) + \sum_{j \neq i} a_{ij} w_j(x) - \beta = 0, \quad \text{in } \mathbb{T}^N. \tag{3.1}$$

For simplicity we set $f_i(x) = -\sum_{j \neq i} a_{ij} w_j(x) + \beta$, for every $i$.

The discounted equation satisfies a comparison principle. This is well known, however the proof in our setting is simplified to some extent by exploiting the compactness of the ambient space plus the coercivity of the Hamiltonian with respect to the momentum variable. This straightforwardly implies that all subsolutions are Lipschitz-continuous and allows using Proposition 2.2. We provide the argument for reader's convenience.

**Theorem 3.1.** *If* $u$*,* $v$ *an USC continuous subsolution and a LSC supersolution of* (3.1)*, respectively, then* $u \leq v$ *in* $\mathbb{T}^N$*.*

*Proof.* We recall that $u$ is Lipschitz continuous. Let $x_0$ be a point in $\mathbb{T}^N$ where $v - u$ attains its minimum and assume, for purposes of contradiction, that

$$v(x_0) - u(x_0) < 0. \tag{3.2}$$

The function $u(.) + v(x_0) - u(x_0)$ is therefore a Lipschitz continuous subtangent to $v$ at $x_0$ and hence we have

$$a_{ii}v(x_0) + H_i(x_0, p) - f_i(x_0) \geq 0 \quad \text{for some } p \in \partial u(x_0).$$

But $u$ is a viscosity subsolution of (3.1), then

$$a_{ii}u(x_0) + H_i(x_0, p) - f_i(x_0) \leq 0 \quad \text{for all } p \in \partial u(x_0).$$

Subtracting the above two inequalities, we get

$$a_{ii}(v(x_0) - u(x_0)) \geq 0,$$

which contradicts (3.2). We therefore conclude that $\min_{\mathbb{T}^N}(v - u) \geq 0$, which in turn implies $v \geq u$ in $\mathbb{T}^N$, as desired. □

The Eq. (3.1) can be interpreted as the Hamilton–Jacobi–Bellman equation of a control problem with the Lagrangian $L_i$ as cost and $a_{ii}$ as discount factor. The control is given by the velocities which are in principle unbounded, but this is somehow compensated by the coercive character of the Hamiltonian. The corresponding value function $v^i : \mathbb{T}^N \to \mathbb{R}$ is defined by

$$v^i(x) = \inf_{\gamma} \int_{-\infty}^{0} e^{a_{ii}s} \left( L_i(\gamma(s), \dot\gamma(s)) + f_i(\gamma(s)) \right) ds, \tag{3.3}$$

where the infimum is taken over all absolutely continuous curves $\gamma : ]-\infty, 0] \to \mathbb{T}^N$ with $\gamma(0) = x$.

We have:

**Theorem 3.2.** [4, Appendix 2 ] *The discounted value function $v^i$ is the unique continuous viscosity solution of (3.1). Moreover, for every $x \in \mathbb{T}^N$ there exists a curve $\gamma : (-\infty, 0] \to \mathbb{T}^N$ with $\gamma(0) = x$ such that*

$$v^i(x) = \int_{-\infty}^{0} e^{a_{ii}s} \left( L_i(\gamma(s), \dot\gamma(s)) + f_i(\gamma(s)) \right) ds. \tag{3.4}$$

**Corollary 3.3.** *We have*

$$w_i \le v^i \text{ in } \mathbb{T}^N \quad \text{and} \quad w_i = v^i \text{ in } \mathcal{A}. \tag{3.5}$$

*Proof.* Since $w_i$ is a viscosity subsolution of (3.1), we derive from Theorem 3.1 the inequality in (3.5). Taking into account that $a_{ij} \le 0$ for every $i \ne j$, we further derive that the vector valued function $\widetilde{\mathbf{w}}$ obtained from $\mathbf{w}$ by replacing $w_i$ with $v^i$ and keeping all other components unaffected, is still a subsolution of (HJ$\beta$). The equality in (3.5) then comes from the rigidity phenomenon in $\mathcal{A}$. In fact, if $y \in \mathcal{A}$ then $\widetilde{w}_j(y) = w_j(y)$ for $j \ne i$, which implies by Theorem 2.12 $\mathbf{w}(y) - \overline{\mathbf{w}}(y) = k\,\mathbf{1}$ with $k = 0$. □

### 3.1. Algorithm

In this subsection we will construct a monotonic sequence of critical subsolutions $(\mathbf{v}_n)$ which converges, up to a subsequence, to a solution of (HJ$\beta$). The algorithm depends on the comparison principle stated in Theorem 3.1 plus the properties of Aubry set. We have summarized the relevant points we exploit in Corollary 3.3.

**step 1: Construction of the sequence** $(\mathbf{v}_n)$.

Let $\mathbf{w} = \mathbf{v}_0 = (v_0^1, v_0^2, \dots, v_0^m)$ be any subsolution of (HJ$\beta$).
The first element $\mathbf{v}_1 = (v_1^1, v_1^2, \dots, v_1^m)$ of $(\mathbf{v}_n)$ is defined component by component as follows :
For $k = 1 \dots m$, $v_1^k$ is the solution of the discounted equation

$$H_k(x, Du) + a_{kk}u + \sum_{j<k} a_{kj} v_1^j(x) + \sum_{j>k} a_{kj} v_0^j(x) = \beta,$$

where a possible empty sum in the above formula is counted as 0. By construction, see the proof of Corollary 3.3, $\mathbf{v}_1$ is a critical subsolution of the system. In addition, using Corollary 3.3, we get

$$\mathbf{w} = \mathbf{v}_0 \le \mathbf{v}_1 \quad \text{and} \quad \mathbf{w} = \mathbf{v}_0 = \mathbf{v}_1 \text{ on } \mathcal{A}.$$

We iterate the above procedure to construct $\mathbf{v}_n$, for any $n \in \mathbb{N}$, $n > 1$, starting from $\mathbf{v}_{n-1}$.

We get that any element $\mathbf{v}_n$ is a critical subsolution of the system and

$$\mathbf{v}_{n-1} \leq \mathbf{v}_n \qquad \text{for any } n \tag{3.6}$$

$$\text{all the } \mathbf{v}_n \text{ coincide on } \mathcal{A}. \tag{3.7}$$

**step 2: Convergence of the sequence $(\mathbf{v_n})$.**
We exploit Proposition 2.5 to infer that the functions $(\mathbf{v}_n)$ are equi-Lipschitz. Moreover, all the $(\mathbf{v}_n)$'s take a fixed value on the Aubry set, see (3.7), so they are equibounded as well. We derive by Ascoli-Arzelà Theorem that $(\mathbf{v}_n)$ converge uniformly, up to subsequences, and we in turn deduce convergence of the whole sequence because of its monotonicity, see (3.6).

**step 3: proving the limit is a critical solution .**
We denote by $\mathbf{V} = (V_1, V_2, \ldots, V_m)$ the uniform limit of $\mathbf{v}_n$.
Given $k \in \{1, \ldots, m\}$, we have, by construction, that $v_n^k$ is the solution of

$$F_n^k(x, u, Du) := H_k(x, Du) + a_{kk}u + \sum_{j<k} a_{kj} v_n^j(x) + \sum_{j>k} a_{kj} v_{n-1}^j(x) = \beta.$$

The Hamiltonians $F_n^k$ converge uniformly in $\mathbb{T}^N \times \mathbb{R} \times \mathbb{R}^N$, as $n \to +\infty$, to

$$F^k(x, u, p) := H_k(x, p) + a_{kk}u + \sum_{j \neq k} a_{kj} V_j(x).$$

Consequently, by basic stability properties in viscosity solutions theory, $V_k$ is solution to the limit equation

$$F^k(x, u, Du) = H_k(x, Du) + a_{kk}u + \sum_{j \neq k} a_{kj} V_j(x) = \beta.$$

We conclude that the limit $\mathbf{V} = (V_1, V_2, \ldots, V_m)$ is solution of (HJ$\beta$), as it was claimed.

**Remark 3.4.** (i) As already pointed out in the Introduction, the above algorithm can have a numerical interest to compute critical solutions of the system via the analysis of a sequence of scalar discounted equations. The latter problem has been extensively studied and well tested numerical codes are available. It is clearly required the knowledge of a critical subsolution as starting point, but this is easier than the determination of a solution. In addition, since the initial subsolution is not affected on $\mathcal{A}$ at any step of the procedure, the algorithm can be useful to get an approximation of the Aubry set itself. (ii) In principle we could apply the algorithm also starting from any supercritical subsolution. What happens is that the sequence we construct is not any more anchored at the Aubry set, and we get in the end a sequence of functions positively diverging at any point. We believe that the rate of divergence could be exploited to estimate how far the supercritical value we have chosen is from $\beta$, but we do not investigate any further this issue in the present paper.

## 4. Applications of the scalar reduction

In this section we describe two applications of the scalar reduction method. Namely, we provide a characterization of the isolated points of $\mathcal{A}$ and establish some semiconcavity properties of critical subsolutions to the system. To do that, we need a strengthened form of Theorem 3.2 for points belonging to $\mathcal{A}$.

**Proposition 4.1.** *Let $y \in \mathcal{A}$ and $\mathbf{u}$ be a subsolution of (HJ$\beta$). Then for every $i \in \{1, \dots, m\}$ there exists a curve $\gamma : (-\infty, 0] \to \mathbb{T}^N$ with $\gamma(0) = y$ such that*

$$u_i(y) = \int_{-\infty}^{0} e^{a_{ii}s} \left( L_i(\gamma(s), \dot\gamma(s)) - \sum_{j \neq i} a_{ij} u_j(\gamma(s)) + \beta \right) ds,$$

*and $\gamma(t) \in \mathcal{A}$ for every $t$.*

For the proof we need a preliminary lemma

**Lemma 4.2.** *Let $\mathbf{u}$ be a subsolution of (HJ$\beta$) strict outside $\mathcal{A}$. Then for every $y \in \mathcal{A}$, there exists a critical solution $\mathbf{v}$ such that*

$$\mathbf{u}(y) = \mathbf{v}(y) \quad \textit{and} \quad \mathbf{u} < \mathbf{v} \textit{ on } \mathbb{T}^N \setminus \mathcal{A}.$$

*Proof.* Given $y \in \mathcal{A}$, we consider the maximal critical subsolution $\mathbf{v}$ taking the value $\mathbf{u}(y)$ at $y$. Then, by the very definition of Aubry set, $\mathbf{v}$ is a critical solution, and $\mathbf{v} \geq \mathbf{u}$. If this inequality were not strict at some $x_0 \in \mathbb{T}^N \setminus \mathcal{A}$, then $u_i(x_0) = v_i(x_0)$ for some index $i$, and consequently $u_i$ should be subtangent to $v_i$ at $x_0$, and hence by Proposition 2.2

$$\beta \leq H_i(x_0, p) + \sum_{j=1}^{m} a_{ij} v_j(x_0) \leq H_i(x_0, p) + \sum_{j=1}^{m} a_{ij} u_j(x_0)$$

for some $p \in \partial u_i(x_0)$. This contradicts $u_i$ being locally strict at $x_0$, in view of Lemma 2.8. □

**Proof of the Proposition 4.1** We consider a critical subsolution $\mathbf{w}$ to the system strict outside $\mathcal{A}$, see Theorem 2.11. It is not restrictive, by adding a suitable constant, to assume $u_i(y) = w_i(y)$, where $\mathbf{u}$ is the subsolution appearing in the statement. This in turn implies by the rigidity property on the Aubry set, see Theorem 2.12, $\mathbf{u}(y) = \mathbf{w}(y)$. We in addition denote by $\bar{\mathbf{u}}$ the maximal subsolution taking the value $\mathbf{w}(y) = \mathbf{u}(y)$ at $y$. It is a critical solution to the system in view of Definition 2.9 and, according to Lemma 4.2, we also have

$$\mathbf{w} < \bar{\mathbf{u}} \quad \text{on } \mathbb{T}^N \setminus \mathcal{A}. \tag{4.1}$$

Now, let $\underline{v}$ be the solution of the discounted equation

$$H_i(x, Dv) + a_{ii} v(x) + \sum_{j \neq i} a_{ij} w_j(x) = \beta,$$

and $\bar{v} = \bar{u}_i$ the solution of

$$H_i(x, Dv) + a_{ii} v(x) + \sum_{j \neq i} a_{ij} \bar{u}_j(x) = \beta. \tag{4.2}$$

We deduce from Corollary 3.3

$$\bar{v}(y) = \underline{v}(y) = \bar{u}_i(y) = u_i(y) = w_i(y). \tag{4.3}$$

There exists, in force of Theorem 3.2 with $\bar{u}_i$ the unique solution of (3.1) satisfying $w_j = \bar{u}_j$ for $j \neq i$, a curve $\gamma : (-\infty, 0] \to \mathbb{T}^N$ with $\gamma(0) = y$ such that

$$\bar{u}_i(y) = \int_{-\infty}^{0} e^{a_{ii}s} \left( L_i(\gamma(s), \dot{\gamma}(s)) - \sum_{j \neq i} a_{ij} \bar{u}_j(\gamma(s)) + \beta \right) ds.$$

Assume, for purposes of contradiction, that the support of $\gamma$ is not contained in $\mathcal{A}$ then, taking into account (4.1), that $a_{ij} < 0$ for $i \neq j$ by (A1) plus irreducibility of $A$, we get

$$\begin{aligned} w_i(y) &\leq \int_{-\infty}^{0} e^{a_{ii}s} \left( L_i(\gamma(s), \dot{\gamma}(s)) - \sum_{j \neq i} a_{ij} w_j(\gamma(s)) + \beta \right) ds \\ &< \int_{-\infty}^{0} e^{a_{ii}s} \left( L_i(\gamma(s), \dot{\gamma}(s)) - \sum_{j \neq i} a_{ij} \bar{u}_j(\gamma(s)) + \beta \right) ds \\ &= \bar{u}_i(y), \end{aligned}$$

which is impossible in view of (4.3). By the maximality property of $\bar{u}$, we also have

$$\begin{aligned} u_i(y) &\leq \int_{-\infty}^{0} e^{a_{ii}s} \left( L_i(\gamma(s), \dot{\gamma}(s)) - \sum_{j \neq i} a_{ij} u_j(\gamma(s)) + \beta \right) ds \\ &\leq \int_{-\infty}^{0} e^{a_{ii}s} \left( L_i(\gamma(s), \dot{\gamma}(s)) - \sum_{j \neq i} a_{ij} \bar{u}_j(\gamma(s)) + \beta \right) ds \\ &= \bar{u}_i(y), \end{aligned}$$

which proves that $\gamma$ is also optimal for $u_i(y)$ and concludes the proof.

### 4.1. Equilibria

In this section we provide a characterization of the isolated points of $\mathcal{A}$. To this aim, we introduce the notion of equilibrium points of the weakly coupled system, which we compactly write in the form

$$\mathbf{H}(x, Du) + Au = \beta \mathbf{1}, \tag{4.4}$$

where the Hamiltonian $\mathbf{H} : \mathbb{T}^N \times \mathbb{R}^{mN}$ has the separated variable form

$$\mathbf{H}(x, p_1, \ldots, p_m) = (H_1(x, p_1), \ldots, H_m(x, p_m)).$$

We consider the equilibrium distribution $\mathbf{o} \in \mathbb{R}^m$ which is uniquely identified by the following conditions:

1) $\mathbf{o}\, A = 0$
2) $\mathbf{o} \cdot \mathbf{1} = 1$

It is an immediate consequence of the condition $\mathrm{Im}\, A \cap \mathbb{R}^m_+ = \{0\}$ plus dim Im $A = m - 1$, that all the vectors orthogonal to Im $A$ have either strictly positive or strictly negative components. Consequently $\mathbf{o}$ is a probability vector, i.e. all its components are nonnegative and sum up to 1.

Multiplying the system (4.4) by $\mathbf{o}$ we get that all subsolutions $\mathbf{u}$ satisfy

$$\mathbf{o} \cdot \mathbf{H}(x, p_1, \ldots, p_m) \leq \beta \qquad \text{for any } x \in \mathbb{T}^N,\ p_i \in \partial u_i(x). \tag{4.5}$$

For $x \in \mathbb{T}^N$, we set $\min_p \mathbf{H}(x,p) := \left(\min_p H_1(x,p), \ldots, \min_p H_m(x,p)\right)$.

Then we deduce from (4.5) that

$$\mathbf{o} \cdot \min_p \mathbf{H}(x,p) \leq \beta \qquad \text{for any } x.$$

We call a point $x$ equilibrium if

$$\mathbf{o} \cdot \min_p \mathbf{H}(x,p) = \beta.$$

We see from the above definition that if $x$ is an equilibrium and $\mathbf{u}$ a critical subsolution then for any $i$ and $q \in \partial u_i(x)$ we have

$$H_i(x,q) = \min_p H_i(x,p) \tag{4.6}$$

$$\beta = H_i(x,q) + \sum_{j=1}^m a_{ij}\, u_j(x) \tag{4.7}$$

This implies that any subsolution also satisfies the supersolution property at an equilibrium point, and we deduce from Proposition 2.6 and Theorem 2.10 that the set of equilibria is contained in the Aubry set.

The next proposition is a partial converse of this fact, it provides a characterization of the isolated points of Aubry set, which is a generalization of the scalar case.

**Proposition 4.3.** *Any isolated point of the Aubry set is an equilibrium.*

*Proof.* Let $x$ be an isolated point of Aubry set and $\mathbf{u}$ be a critical subsolution of the system. Then, in view of Proposition 4.1, for every $i \in \{1, \ldots, m\}$ there exists a curve $\gamma$ with $\gamma(0) = x$ such that

$$u_i(x) = \int_{-\infty}^0 e^{a_{ii} s} \left( L_i(\gamma(s), \dot\gamma(s)) - \sum_{j \neq i} a_{ij} u_j(\gamma(s)) + \beta \right) ds,$$

and the support of $\gamma$ is contained in $\mathcal{A}$. Exploiting the fact that $x$ is isolated, we get $\gamma(t) \equiv x$ for every $t$ and hence

$$u_i(x) = \frac{1}{a_{ii}} \left( L_i(x,0) - \sum_{j \neq i} a_{ij} u_j(x) + \beta \right), \qquad \text{for every } i \in \{1, \ldots, m\}.$$

Then

$$\begin{aligned} A\mathbf{u}(x) &= (L_1(x,0), \ldots, L_m(x,0)) + \beta \mathbf{1} \\ &= -\min_p \mathbf{H}(x,p) + \beta \mathbf{1}. \end{aligned}$$

Multiplying by $\mathbf{o}$ and taking into account that $\mathbf{o}$ is a probability vector orthogonal to $\mathrm{Im}(A)$, we get

$$\mathbf{o} \cdot \min_p \mathbf{H}(x,p) = \beta$$

as desired. □

Assuming the strict convexity assumption (H4), we get a regularity result.

**Proposition 4.4.** *Under the additional assumption (H4), any critical subsolution is strictly differentiable at every isolated point of* $\mathcal{A}$.

*Proof.* Let $x_0$ be an isolated point of $\mathcal{A}$ and $\mathbf{u}$ be a subsolution of (HJ$\beta$). Then for every $i \in \{1,\dots,m\}$, we have

$$H_i(x_0,p_i) + \sum_{j=1}^{m} a_{ij} u_j(x_0) \leq \beta \quad \text{for every } p_i \in \partial u_i(x_0).$$

This implies

$$\sum_{i=1}^{m} o_i H_i(x_0,p_i) \leq \beta.$$

Taking into account that $x_0$ is equilibrium we deduce from the above inequality that

$$\beta = \sum_{i=1}^{m} o_i \min_p H_i(x_0,p) \leq \sum_{i=1}^{m} o_i H_i(x_0,p_i) \leq \beta,$$

which in turn gives that

$$H_i(x_0,p_i) = \min_p H_i(x_0,p), \qquad \text{for every } p_i \in \partial u_i(x_0),\ i \in \{1,\dots,m\}.$$

Due to $H_i$ being strictly convex, the above minimum is unique and hence $\partial u_i(x_0)$ reduces to a singleton. This implies strict differentiability of $\mathbf{u}$ at $x_0$. □

### 4.2. Semiconcavity-type properties for critical subsolutions

In this section we study a family of scalar Eikonal equations derived from the critical system. The main information we gather through this approach, under the additional assumptions (H4), (H5), is that the superdifferential of any critical solution of (HJ$\beta$) is nonempty at every point of the torus. The same property holds true for any critical subsolution on the Aubry set.
We start by stating and proving a consequence of Theorem 3.2.

**Proposition 4.5.** *Let* $\mathbf{u}$, $x$, $i$ *be a critical solution to the system, a point in* $\mathbb{T}^N$ *and an index in* $\{1,\dots,m\}$, *respectively. There is a curve* $\gamma$ *defined in* $(-\infty,0]$ *such that* $\gamma(0)=x$ *and*

$$\frac{d}{dt}u_i(\gamma(t)) = L_i(\gamma(t),\dot\gamma(t)) - \sum_j a_{ij}\, u_j(\gamma(t)) + \beta \qquad \text{for a.e. } t \in (-\infty,0).$$

*Proof.* Taking into account that $u_i$ is the solution of the discounted equation (3.1) with $u_j$ in place of $w_j$, we know by Theorem 3.2 that there is an optimal curve $\gamma$ defined in $(-\infty,0]$ with $\gamma(0)=x$ such that

$$u_i(x) = \int_{-\infty}^{0} e^{a_{ii}s}\left[L_i(\gamma(s),\dot\gamma(s)) - \sum_{j\neq i} a_{ij}u_j(\gamma(s)) + \beta\right] ds. \tag{4.8}$$

We claim that $\gamma$ also satisfies the statement of the proposition. We define

$$g(t) = e^{a_{ii}t} u_i(\gamma(t)) \qquad \text{for } t \in (-\infty, 0),$$

accordingly

$$\frac{d}{dt}\, g(t) = a_{ii}\, e^{a_{ii}t} u_i(\gamma(t)) + e^{a_{ii}t}\, p(t) \cdot \dot{\gamma}(t) \tag{4.9}$$

for a.e. $t$, where $p(t)$ is a suitable element of $\partial u_i(\gamma(t))$ satisfying $\frac{d}{dt} u_i(\gamma(t)) = p(t) \cdot \dot{\gamma}(t)$ for a.e. $t$, see Lemma 2.1. We further get taking into account that $\mathbf{u}$ is a solution to the critical system

$$p(t)\cdot\dot{\gamma}(t) \le H_i(\gamma(t), p(t)) + L_i(\gamma(t), \dot{\gamma}(t)) \le -\sum_j a_{ij} u_j(\gamma(t)) + \beta + L_i(\gamma(t), \dot{\gamma}(t)). \tag{4.10}$$

We derive from (4.8), (4.9), (4.10)

$$\begin{aligned} u_i(x) &= \lim_{t \to -\infty} g(0) - g(t) = \int_{-\infty}^0 \frac{d}{dt}\, g(t)\, dt \\ &\le \int_{-\infty}^0 a_{ii}\, e^{a_{ii}t} u_i(\gamma(t)) - e^{a_{ii}t} \left( \sum_j a_{ij} u_j(\gamma(t)) - \beta - L_i(\gamma(t), \dot{\gamma}(t)) \right) dt \\ &= \int_{-\infty}^0 e^{a_{ii}t} \left[ L_i(\gamma(t), \dot{\gamma}(t)) - \sum_{j \ne i} a_{ij}\, u_j(\gamma(t)) + \beta \right] dt = u_i(x). \end{aligned}$$

This in turn implies

$$\frac{d}{dt} u_i(\gamma(t)) = p(t) \cdot \dot{\gamma}(t) = -\sum_j a_{ij} u_j(\gamma(t)) + \beta + L_i(\gamma(t), \dot{\gamma}(t)) \qquad \text{for a.e. } t,$$

as it was to be proved. □

In the case where the point $x$ belongs in addition to $\mathcal{A}$, we get, thanks to Proposition 4.1, a strengthened form of the previous assertion.

**Corollary 4.6.** *The statement of Proposition 4.5 holds true for any critical subsolution* $\mathbf{u}$*, provided* $x \in \mathcal{A}$*. The curve* $\gamma$ *is in addition contained in* $\mathcal{A}$*.*

*Proof.* If $\mathbf{u}$ is any critical subsolution, we know from Proposition 4.1 that there is an optimal curve $\gamma$ for $u_i(x)$ which is in addition contained in $\mathcal{A}$. We then prove that $\gamma$ satisfies the assertion arguing as in Proposition 4.5. □

We recognize that the integrand appearing in the statement of Proposition 4.5 is nothing but the Lagrangian associated through Fenchel transform to the Hamiltonian

$$H_i^{\mathbf{u}}(x, p) = H_i(x, p) + \sum_{j=1}^m a_{ij} u_j(x). \tag{4.11}$$

Given a critical subsolution $\mathbf{u}$ to the system, we therefore consider the Eikonal equation

$$H_i^{\mathbf{u}}(x, Dv) = \beta \qquad \text{in } \mathbb{T}^N, \tag{4.12}$$

and denote by $\sigma_i^{\mathbf{u}}$, $S_i^{\mathbf{u}}$ the corresponding support function and intrinsic distance, respectively, given by suitably adapting (A.3) and (A.2). Since $u_i$ is a subsolution to (4.12), it is clear that the critical value of $H_i^{\mathbf{u}}$ is less than or equal to $\beta$. We in addition have:

**Proposition 4.7.** *The critical value of $H_i^{\mathbf{u}}(x,p)$ is equal to $\beta$, for any critical subsolution $\mathbf{u}$ to the system, any index $i \in \{1,\dots,m\}$. In addition the limit points, as $t \to -\infty$,of any curve satisfying the statement of Proposition 4.5/Corollary 4.6 belong to the corresponding Aubry set.*

*Proof.* We fix $\mathbf{u}$ and $i$. Let us consider $x \in \mathcal{A}$ and an optimal curve $\gamma$ as in the statement with $\gamma(0) = x$. We denote by $y$ a limit point of $\gamma$ as $t \to -\infty$. If the set of such limit points reduces to $y$, then there is a sequence $t_n \to -\infty$ with

$$\frac{d}{dt} u_i(\gamma(t_n)) = L_i(\gamma(t_n), \dot\gamma(t_n)) - \sum_j a_{ij}\, u_j(\gamma(t_n)) + \beta \quad \text{and} \quad \dot\gamma(t_n) \to 0,$$

therefore

$$0 = \lim_{t_n \to -\infty} \frac{d}{dt} u_i(\gamma(t_n)) = \lim_{t_n \to -\infty} L_i(\gamma(t_n), \dot\gamma(t_n)) - \sum_j a_{ij}\, u_j(\gamma(t_n)) + \beta.$$

By continuity of $L_i$, $u_j$ we deduce

$$L_i(y, 0) - \sum_j a_{ij}\, u_j(y) = -\beta$$

or equivalently $\min_p H_i^u(y,p) = \beta$. Since we know that $\beta$ is supercritical for $H_i^{\mathbf{u}}$, this implies that $\beta$ is actually the critical value of $H_i^{\mathbf{u}}$ and $y$ belongs to the corresponding Aubry set, by Proposition A.3.

If instead the limit set of $\gamma$, as $t \to -\infty$, is not a singleton, then we find $\gamma(t_n)$ converging to $y$ such that the curves $\gamma_n := \gamma\big|_{[t_n, t_{n+1}]}$ possess Euclidean length bounded from below by a positive constant. We have

$$\int_{t_n}^{t_{n+1}} \sigma_i^{\mathbf{u}}(\gamma(s), \dot\gamma(s))\, ds \le \int_{t_n}^{t_{n+1}} \left[ L_i(\gamma(s), \dot\gamma(s)) - \sum_j a_{ij}\, u_j(\gamma(s)) + \beta \right] ds$$
$$= u_i(\gamma(t_{n+1})) - u_i(\gamma(t_n)).$$

We deduce from Proposition A.1 that the leftmost inequality in the above formula must actually be an equality. This shows that the intrinsic length $\int_{t_n}^{t_{n+1}} \sigma_i^{\mathbf{u}}(\gamma(s), \dot\gamma(s))\, ds$ is infinitesimal as $n \to +\infty$.

We construct a sequence of cycles $\eta_n$ based on $y$ by concatenating the segment linking $y$ to $\gamma(t_n)$, $\gamma_n$ and the segment linking $\gamma(t_{n+1})$ to $y$. We find that the intrinsic lengths of such cycles are infinitesimal, as $n \to +\infty$, while the Euclidean lengths stay bounded from below by a positive constant. Taking into account the very definition of Aubry set for scalar Eikonal equations, we derive also in this case that $\beta$ is the critical value of $H_i^u$, and $y$ belongs to the corresponding Aubry set. This concludes the proof. □

We denote by $\mathcal{A}_i^{\mathbf{u}}$ the Aubry set associated with $H_i^{\mathbf{u}}$ at the critical level $\beta$, for $i \in \{1,\dots,m\}$.

**Proposition 4.8.** *We have that*

$$\mathcal{A}_i^{\mathbf{u}} \cap \mathcal{A} \neq \emptyset \qquad \textit{for any susbsolution } \mathbf{u} \textit{ to } (\mathrm{HJ}\beta), \textit{ any } i. \tag{4.13}$$

*If, in addition,* $\mathbf{u}$ *is strict on* $\mathbb{T}^N \backslash \mathcal{A}$*, then*

$$\mathcal{A}_i^{\mathbf{u}} \subseteq \mathcal{A} \qquad \textit{for every } i \in \{1, \ldots, m\} \tag{4.14}$$

*Proof.* Formula (4.13) is a direct consequence of Corollary 4.6 and Proposition 4.7. To show (4.14), let us consider $y \notin \mathcal{A}$, then $u_i$ is locally strict at $y$ for every $i \in \{1, \ldots, m\}$. Hence, there exists an open neighborhood $W$ of $y$ and $\delta > 0$ such that

$$H_i(x, Du_i(x)) + \sum_{j=1}^{m} a_{ij} u_j(x)$$
$$< -\delta + \beta \quad \text{for a.e. } x \in W, \qquad \text{for every } i \in \{1, \ldots, m\}.$$

Therefore, $u_i$ is a critical subsolution of (4.12) which is locally strict at $y$ and consequently $y \notin \mathcal{A}_i^{\mathbf{u}}$. □

As already announced, we assume (H4), (H5) to establish the final result. Note that, due to the Lipschitz character of any subsolution to the system, the Hamiltonians $H_i^u$ are locally Lipschitz-continuous in $\mathbb{T}^N \times \mathbb{R}^N$, for any subsolution $\mathbf{u}$ of (HJ$\beta$), any index $i$.

In this setting we obtain:

**Theorem 4.9.** *We assume (H4), (H5). If* $\mathbf{u}$ *is a critical subsolution of* (HJ$\beta$)*, then*

$$D^+ u_i(x) \neq \emptyset \qquad \textit{for every } i \in \{1, 2, \ldots, m\},\ x \in \mathcal{A}.$$

*If, in addition,* $\mathbf{u}$ *is a solution to* (HJ$\beta$) *then the above property holds true for any* $x \in \mathbb{T}^N$*.*

*Proof.* First assume $\mathbf{u}$ to be subsolution of (HJ$\beta$). If $x_0 \in \mathcal{A}_i^{\mathbf{u}}$ then $u_i$ is differentiable at $x_0$, according to Proposition A.3. This proves the assertion. If instead $x_0 \in \mathcal{A} \backslash \mathcal{A}_i^{\mathbf{u}}$, then we derive from the proof of Proposition 4.7 that

$$u_i(x_0) \geq \min_{y \in \mathcal{A}_i^{\mathbf{u}}} \{u_i(y) + S_i^{\mathbf{u}}(y, x_0)\}. \tag{4.15}$$

By Proposition A.4, the function on the right hand-side of the above formula is the maximal subsolution to (4.12) with trace $u_i$ on $\mathcal{A}_i^{\mathbf{u}}$, this implies that equality must prevail in (4.15). There is then an element $y_0 \in \mathcal{A}_i^{\mathbf{u}}$ such that

$$u_i(x_0) = u_i(y_0) + S_i^{\mathbf{u}}(y_0, x_0).$$

Hence $u_i(y_0) + S_i^{\mathbf{u}}(y_0, .)$ is supertangent to $u_i$ at $x_0$, and so by Proposition A.2 the superdifferential of $u_i$ is nonempty at $x_0$, as it was claimed. If $\mathbf{u}$ is in addition solution of (HJ$\beta$), the same argument of above gives that $D^+ u_i(x_0) \neq \emptyset$ at any $x_0 \in \mathbb{T}^N$. This concludes the proof. □

## Acknowledgements

Funding was provided by Progetto Ateneo 20154213.

## A. Scalar Eikonal equations

We refer readers to [7] for the material presented in this appendix. We consider a continuous Hamiltonian $H : \mathbb{T}^N \times \mathbb{R}^N \to \mathbb{R}$ satisfying the following assumptions:

(E1) $p \mapsto H(x,p)$ is convex for every $x \in \mathbb{T}^N$;
(E2) $p \mapsto H(x,p)$ is superlinear for every $x \in \mathbb{T}^N$.

For som e results we need the following additional conditions:

(E3) $p \mapsto H(x,p)$ is strictly convex for every $x \in \mathbb{T}^N$;
(E4) $(x,p) \mapsto H(x,p)$ is locally Lipschitz-continuous in $\mathbb{T}^N \times \mathbb{R}^N$.

We consider the family of Hamilton–Jacobi equations

$$H(x, Du) = a \quad \text{in } \mathbb{T}^N, \tag{A.1}$$

where $a$ is a real parameter. We introduce an intrinsic semidistance $S_a$ on the torus, related to (A.1), via

$$S_a(x,y) = \inf_{\xi} \left\{ \int_0^1 \sigma_a(\xi(s), \dot{\xi}(s)) ds \right\}, \tag{A.2}$$

where $\xi$ varies in the family of absolutely continuous curves taking the value $x$, $y$ at $s = 0$ and $s = 1$, respectively, and $\sigma_a(x,q)$ is the support function of the $a$-sublevel of $H$, namely

$$\sigma_a(x,q) := \max\{p \cdot q : H(x,p) \leq a\}. \tag{A.3}$$

The following inequality holds true

$$\sigma_a(x,q) \leq L(x,q) + a \qquad \text{for any } x \in \mathbb{T}^N,\ q \in \mathbb{R}^N, \tag{A.4}$$

where $L$ is the Lagrangian associated to $H$ via the Fenchel transform thanks to (E2).

**Proposition A.1.** *A function $u$ is subsolution to* (A.1) *if and only if*

$$u(x) - u(y) \leq S_a(y,x) \quad \textit{for every } x, y \in \mathbb{T}^N.$$

It is well known that there exists a unique value of $a$, denoted by $c$ and called critical, for which the Eq. (A.1) has a solution on the whole torus. It is defined as

$$c = \inf\{a : \ (A.1) \text{ has a subsolution}\}.$$

We then focus on the critical equation

$$H(x, Du) = c. \tag{A.5}$$

In what follows we will recall some important facts and results about (A.5), taken from [7], that will play an important role for the semi-concavity results of Sect. 4.2.

**Proposition A.2.** *Under the additional assumptions (E3), (E4) the function*

$$x \mapsto S_c(y,x)$$

*possess nonempty superdifferential at $x$ for any $y$ and any $x \neq y$.*

In the analysis of the behavior of critical subsolutions, a special role is played by the so-called Aubry set, denoted by $\mathcal{A}_e$, defined as the collection of points $y \in \mathbb{T}^N$ such that

$$\inf_{\xi} \left\{ \int_0^1 \sigma_c(\xi(s), \dot{\xi}(s)) ds \right\} = 0,$$

where $\xi$ varies in the class of absolutely continuous cycles based on $y$ with Euclidean length uniformly estimated from below by some positive constant. It is not difficult to see that the definition is independent of such constant. In the next statement we recall some relevant properties of $\mathcal{A}_e$.

**Proposition A.3.** *(i) Any point $y$ with $\min_p H(y,p) = c$ belongs to $\mathcal{A}_e$;*

*(ii) A point $y \notin \mathcal{A}_e$ if and only if there exists a critical subsolution which is locally strict at $y$;*

*(iii) Under the additional assumptions (E3), (E4), any subsolution of (A.5) is strictly differentiable at every point of $\mathcal{A}_e$.*

We finally record:

**Proposition A.4.** *Given a continuous function $u_0$ defined on a closed set $C \subset \mathcal{A}_e$ such that*

$$u_0(x) - u_0(y) \le S_c(y,x) \quad \textit{for every } \ x, y \in C,$$

*then the function*

$$u := \min_{y \in C} \{u_0(y) + S_c(y, .)\}$$

*is the maximal subsolution of (A.5) in $\mathbb{T}^N$ agreeing with $u_0$ on $C$, and is a solution as well.*

## References

[1] Bardi, M., CapuzzoDolcetta, I.: Optimal Control Theory and Viscosity Solutions of Hamilton–Jacobi–Bellman Equations. Birkhäuser, Boston (1997). https://doi.org/10.1007/978-0-8176-4755-1

[2] Barles, G.: Solutions de viscosité des équations de Hamilton–Jacobi, Mathématiques & Applications (Berlin) 17. Springer, Paris (1994)

[3] Camilli, F., Ley, O., Loreti, P., Nguyen, V.D.: Large time behavior of weakly coupled systems of first-order Hamilton–Jacobi equations. NoDEA Nonlinear Differ. Equ. Appl. **19**(6), 719–749 (2012). https://doi.org/10.1007/s00030-011-0149-7

[4] Davini, A., Fathi, A., Iturriaga, R., Zavidovique, M.: Convergence of the solutions of the discounted equation. Inventiones Mathematicae **206**(1), 29–55 (2016). https://doi.org/10.1007/s00222-016-0648-6

[5] Davini, A., Siconolfi, A., Zavidovique, M.: Random Lax–Oleinik Semigroups for Hamilton–Jacobi Systems, arXiv:1608.01836

[6] Davini, A., Zavidovique, M.: Aubry sets for weakly coupled systems of Hamilton–Jacobi equations. SIAM J. Math. Anal. **46**(5), 3361–3389 (2014). https://doi.org/10.1137/120899960

[7] Fathi, A., Siconolfi, A.: PDE aspects of Aubry–Mather theory for quasiconvex Hamiltonians. Calc. Var. Partial Differ. Equ. **22**(2), 185–228 (2005). https://doi.org/10.1007/s00526-004-0271-z

[8] Ibrahim, H., Siconolfi, A., Zabad, S.: Cycle Chacterization of the Aubry Set for Weakly Coupled Hamilton–Jacobi Systems, arXiv: 1604.08012

[9] Mitake, H., Siconolfi, A., Tran, H.V., Yamada, N.: A Lagrangian approach to weakly coupled Hamilton–Jacobi systems. SIAM J. Math. Anal. **48**(2), 821–846 (2016). https://doi.org/10.1137/15M1010841

[10] Mitake, H., Tran, H.V.: Remarks on the large time behavior of viscosity solutions of quasi-monotone weakly coupled systems of Hamilton–Jacobi equations. Asymptot. Anal. **77**(1–2), 43–70 (2012). https://doi.org/10.3233/ASY-2011-1071

Antonio Siconolfi and Sahar Zabad
Mathematics Department
University of Rome "La Sapienza"
Rome
Italy
e-mail: siconolf@mat.uniroma1.it

Sahar Zabad
e-mail: zabad@mat.uniroma1.it

Received: 5 November 2017.
Accepted: 14 September 2018.