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Nehari method for locally Lipschitz functionals with examples in problems in the space of bounded variation functions

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Abstract. In this work we prove some abstract results about the existence of a minimizer for locally Lipschitz functionals, over a set which has its definition inspired in the Nehari manifold. As applications we present a result of existence of ground state bounded variation solutions of problems involving the 1-laplacian and the 1-biharmonic operator, where the nonlinearity satisfies mild assumptions.

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Contents

1. Introduction and some abstract results

Since its appearance, the Nehari method has been used in a number of situations in order to get ground state solutions to elliptic problems. In [17] Szulkin

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and Weth have elegantly described in a systematic way the real essence of the Nehari method. Their main theorem, describing in an abstract framework sufficient conditions to get ground state solutions, has been applied to study problems like

$$\begin{cases} -\Delta_p u - \lambda|u|^{p-2}u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda < \lambda_1$, λ_1 the first eigenvalue of $-\Delta_p$ and f a subcritical power-type nonlinearity. Also, they considered elliptic problems in $\mathbb{R}^N$ like

$$\begin{cases} -\Delta u + V(x)u = f(x,u) & \text{in } \mathbb{R}^N, \\ u(x) \to 0, \qquad |x| \to \infty, \end{cases}$$

under some conditions on V and f.

These two problems (and other two also considered in [17]) have a common feature which allows the use of the standard Nehari method described by Szulkin and Weth. In fact, the energy functionals associated to them has the "principal part" p-homogenous. This condition, by its side, has been dropped out in the paper of Figueiredo and Ramos [11], in which they conclude the same as Szulkin and Weth, but with weaker assumptions. The applications of Figueiredo and Ramos results treat with non homogenous elliptic problems. They consider, for instance, quasilinear elliptic problems which includes the (p,q)-Laplacian operator, Kirchhoff-type problems and an anisotropic equation (see [11] for details).

Leaving aside the questions about the homogeneity, a common feature which seems to never be dropped out in results based on a Nehari approach, is the differentiability of Φ. In fact this is a natural assumption that researchers have always considered when dealing with sets which resemble the Nehari manifold, since some directional derivatives of the functional is involved in its definition.

In this work we present some abstract results whose assumptions are enough to give a reasonable sense to a Nehari set, which contains all the critical points of functionals which are neither C^1, nor even differentiable and where all minimizers are critical points. More specifically, we deal with functionals defined in Banach spaces, which are written like a combination of a locally Lipschitz and a C^1 functional and satisfy some natural assumptions. In fact, despite the lack of smoothness in our functional, we ask it to have at least directional derivatives in some directions which are enough to get the result. Although this can sound a little bit artificial, in a lot of examples this assumption is satisfied. For example, as we will show later in Sect. 3, in problems involving the 1-laplacian (the formal limit of Δ_p as $p \to 1$), the 1-biharmonic operator and some other operators derived from these ones, the associated energy functionals are not differentiable in all directions.

Our main results are the following.

Theorem 1. *Let E, F be Banach spaces, E compactly embedded into F and such that for all bounded sequence $(u_n) \subset E$ such that $u_n \to u$ in F, it holds that $u \in E$. Let $\Phi, I_0 : E \to \mathbb{R}$, $I : F \to \mathbb{R}$ be functionals such that $\Phi = I_0 - I|_E$,* *where I_0 is locally Lipschitz continuous, $I \in C^1(E) \cap C^0(F)$, $I(0) = 0$ and for all $u \in E$, there exists the following limit*

$$I_0'(su)u := \lim_{t \to 0} \frac{I_0(su + tu) - I_0(su)}{t}, \quad \forall s \in \mathbb{R}.$$

Suppose also the following conditions are satisfied:

(i) *I_0 is lower semicontinuous in the topology of F, i.e., if (u_n) is a bounded sequence in E, and $u \in E$ are such that, up to a subsequence, $u_n \to u$ in F, then $I_0(u) \leqslant \liminf_{n \to \infty} I_0(u_n)$;*

(ii) *There exist $\rho, \alpha_0 > 0$, such that $\Phi(u) \geqslant \alpha_0 > \Phi(0)$, for every $u \in E$ with $\|u\| = \rho$;*

(iii) *$\forall u \in E$, $\Phi(u) \geqslant \|u\| - I(u)$;*

(iv) *$t \mapsto \Phi'(tu)u$ is strictly decreasing in $(0, +\infty)$;*

(v) *For each bounded sequence $(v_n) \subset E$ such that $v_n \to v \neq 0$ in F, it follows that*

$$\lim_{t \to \infty} \frac{\Phi(tv_n)}{t} = -\infty, \quad \textit{uniformly for } n \in \mathbb{N}.$$

Then, the infimum of Φ on the following set

$$\mathcal{N} := \{u \in E \backslash \{0\};\ I_0'(u)u = I'(u)u\}$$

is achieved.

It is important to say that, since Theorem 1 is going to be applied in non-reflexive Banach spaces, it is not possible to work with weak convergence in E, this way, being necessary to introduce the space F.

Since we are dealing with locally Lipschitz functionals, before saying something about the existence of critical points, we have to give the precise definition of it. We say that $u_0 \in E$ is a critical point of Φ, if $0 \in \partial\Phi(u_0)$, where $\partial\Phi(u_0)$ denotes the generalized gradient of Φ in u_0, as defined in [6] for instance.

Finally, our last theorem ensures that all minimizers of Φ on the Nehari set, $\mathcal{N}$, are in fact critical points.

Theorem 2. *Suppose all conditions of Theorem 1 hold, then if $u_0 \in \mathcal{N}$ is such that $\Phi(u_0) = \min_{\mathcal{N}} \Phi$, then u_0 is a critical point of Φ in E.*

In [18] the author studies all the usual minimax theorems like Mountain Pass Theorem, Saddle Point Theorem, etc., for functionals written as $\Phi = I_0 - I$, where I_0 is locally Lipschitz, lower semicontinuous and I is C^1. However, any mention about arguments which resembles a Nehari set has been made.

As a first application of these results we address the question of finding critical points of a functional involving the absolute value of the total variation of a function in the space of functions of bounded variation, $BV(\Omega)$, in a setting in which coerciveness and smoothness are lost. In fact, a lot of attention has been paid recently to the space $BV(\Omega)$, since it is the natural environment in which minimizers of many problems can be found, especially in problems involving interesting physical situations, in capillarity theory and existence of minimal surfaces. In fact, this space turns to be the natural domain of relaxed versions of some functionals defined in usual Sobolev spaces.

In [10], Degiovanni and Magrone study the version of Brézis–Nirenberg problem to the 1-laplacian operator, corresponding to

$$\begin{cases} -\Delta_1 u = \lambda \dfrac{u}{|u|} + |u|^{1^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $1^* = N/(N-1)$ and $\Delta_1 u = \mathrm{div}\left(\dfrac{\nabla u}{|\nabla u|}\right)$. In this work, the authors extend Φ in a suitable $L^p(\Omega)$ space, which have better compactness properties, even though its continuity is lost.

This kind of argument, which consists in extending the functional defined in $BV(\Omega)$, to some $L^p(\Omega)$ in order to recover the Palais–Smale condition, is generally used in dealing with Δ_1 operator. For example, in [7], Chang uses this approach to study the spectrum of the 1-Laplacian operator, proving the existence of a sequence of eigenvalues.

In the first application we study the energy functional whose Euler–Lagrange equation is a relaxed form of

$$\begin{cases} -\Delta_1 u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N \geqslant 2$ and f satisfies the following set of assumptions.

(f_1) $f \in C^0(\mathbb{R})$;
(f_2) $f(0) = 0$;
(f_3) there exist constants $c_1, c_2 > 0$ and $p \in (1, 1^*)$ such that

$$|f(s)| \leqslant c_1 + c_2|s|^{p-1}, \quad s \in \mathbb{R};$$

(f_4) $\displaystyle\lim_{t\to\pm\infty} \frac{F(t)}{t} = \pm\infty$, where $F(t) = \int_0^t f(s)ds$;
(f_5) f is increasing for $s \in \mathbb{R}$.

Let us consider $I_0, I : BV(\Omega) \to \mathbb{R}$ defined by

$$I_0(u) = \int_\Omega |Du| + \int_{\partial\Omega} |u| d\mathcal{H}_{N-1}$$

and

$$I(u) = \int_\Omega F(u)dx.$$

It follows that $\Phi : BV(\Omega) \to \mathbb{R}$ given by $\Phi(u) = I_0(u) - I(u)$ is the functional whose Euler–Lagrange equation is a relaxed form of (1.2). Since Φ is just a locally Lipschitz functional (since I_0 lacks smoothness), we have first to explain what we mean by saying that $u \in BV(\Omega)$ is a critical point of Φ. Since I_0 is a convex locally Lipchitz functional and $I \in C^1(BV(\Omega))$, then $u \in BV(\Omega)$ is going to be called a *critical point of* Φ, or a *bounded variation solution of*

(1.2), if $I'(u) \in \partial I_0(u)$, where $\partial I_0(u)$ denotes the subdifferential of the convex function I_0. This is equivalent to

$$I_0(v) - I_0(u) \geqslant I'(u)(v-u), \quad \forall v \in BV(\Omega). \tag{1.3}$$

In the second application of our main results, we study a problem involving the 1-biharmonic operator

$$\begin{cases} \Delta_1^2 u = f(u) & \text{in } \Omega, \\ u = \dfrac{\Delta u}{|\Delta u|} = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N \geqslant 2$ and f satisfies the same set of assumptions $(f_1)-(f_5)$, with $1 < p < N/(N-2)$ in (f_3). The 1-biharmonic operator is formally defined as

$$\Delta_1^2 u = \Delta\left(\frac{\Delta u}{|\Delta u|}\right)$$

and can be seen as the limit case as $p \to 1$ of the p-biharmonic operator

$$\Delta_p^2 u := \Delta\left(|\Delta u|^{p-2}\Delta u\right),$$

which in turn, has been used to study systems of second order elliptic problems (see [4] for example). As in the first application, we get ground state solutions of (1.4) by proving that the energy functional Φ satisfies all the assumptions of Theorems 1 and 2. However, this time the energy functional has to be defined in a different space, $BL_0(\Omega) = \left\{u \in W_0^{1,1}(\Omega) : \Delta u \in \mathcal{M}(\Omega)\right\}$, and is given by $\Phi : BL_0(\Omega) \to \mathbb{R}$, such that $\Phi(u) = \tilde{I}_0(u) - I(u)$, where

$$\tilde{I}_0(u) = \int_\Omega |\Delta u|,$$

where $\int_\Omega |\Delta u|$ is the total variation of the Radon measure Δu.

The paper by Parini et al. [14] seems to be the very first work dealing with the 1-biharmonic operator and treating in particular the related eigenvalue problem. Actually their result is even more complete, in the sense that it provides also information about the shape of the domain Ω that maximizes the first eigenvalue of such operator. The natural space in which problems involving this operator takes place is $BL_0(\Omega) := \{u \in W_0^{1,1}(\Omega); \Delta u \in \mathcal{M}(\Omega)\}$, a space which has similar properties of $BV(\Omega)$. In [16], the same authors still deal with the 1-biharmonic operator since they study the following minimization problem

$$\Lambda_{1,1}^c(\Omega) = \inf_{u \in C_c^\infty(\Omega)\setminus\{0\}} \frac{\int_\Omega |\Delta u| dx}{|u|_1}.$$

It turns that since $C_c^\infty(\Omega)$ is not a dense subset of $BL_0(\Omega)$ in the topology of the norm, the minimizing problems above are in fact different. As in their first work [14], in [16] the authors also study the shape of the subset that maximizes the quantity $\Lambda_{1,1}^c(\Omega)$. In [15] these authors investigate some optimal constants of the Sobolev embeddings of some spaces of functions which are associated to the 1-biharmonic operator.

To finish, the wide range of situations in which Nehari method has been applied in problems involving operators like $-\Delta$ and $-\Delta_p$, together with the lack of study of different geometric situations when dealing with the 1-laplacian and 1-biharmonic operators, lead us to think that this paper can give the tools to shed some light in several questions which could be raised to problems in $BV(\Omega)$, in analogy to situations involving $-\Delta$ and $-\Delta_p$ operators.

In Sect. 2 we prove the abstract results stated in this introduction. In Sect. 3, we start with a subsection, in order to provide the basic notation and results about functionals defined in $BV(\Omega)$ and in similar spaces. Later, we present two applications of our abstract results. To finish, in an Appendix we present an alternative proof of Theorem 2 which uses a version of the Lagrange Multiplier rule to locally Lipschitz functionals (see [8]), in the concrete case where the function f is C^2.

2. Proof of the abstract results

Proof of Theorem 1. Let us start proving that $\mathcal{N} \neq \emptyset$. In order to do so let us prove that for all $w \in E\backslash\{0\}$, there exists a unique $t_w > 0$ such that $t_w w \in \mathcal{N}$.

For $w \in E\backslash\{0\}$ consider $\gamma_w : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$\gamma_w(t) = \Phi(tw).$$

Note that γ_w is a smooth function such that, $\gamma_w(0) = \Phi(0)$ and $\gamma_w'(t) = I_0'(tw)w - I'(tw)w$. By *ii*), for $t_0 = \rho/\|w\|$, it follows that $\gamma_w(t_0) \geqslant \alpha_0 > \Phi(0)$. Moreover, by *v*), it follows that

$$\lim_{t\to\infty} \gamma_w(t) = -\infty.$$

Then there exists a $t_w > 0$ such that $\gamma_w'(t_w) = 0$ and consequently, $t_w w \in \mathcal{N}$.

In order to verify the uniqueness of such a t_w, note that, supposing that $\gamma_w'(t) = \gamma_w'(s) = 0$, for $t, s > 0$, then it follows that $\Phi'(tw)w = \Phi'(sw)w$. Then by *iv*) it follows that $t = s$.

Note that for all $u \in \mathcal{N}$, it follows by *ii*)

$$\Phi(u) = \max_{t\geqslant 0} \Phi(tu) \geqslant \Phi\left(\frac{\rho}{\|u\|}u\right) \geqslant \alpha_0 > \Phi(0). \tag{2.1}$$

Then there exists $(u_n) \subset \mathcal{N}$ such that

$$\lim_{n\to\infty} \Phi(u_n) = \inf_{\mathcal{N}} \Phi =: c.$$

Note that there exists $\delta > 0$ such that

$$\|u\| \geqslant \delta \quad \text{for all } u \in \mathcal{N}. \tag{2.2}$$

In fact, otherwise it would exists $(w_n) \subset \mathcal{N}$ such that $w_n \to 0$ in E, which would contradict (2.1).

Let us prove that the minimizing sequence (u_n) is bounded in E. Assume by contradiction that $\|u_n\| \to \infty$, as $n \to \infty$ and let $v_n = \dfrac{u_n}{\|u_n\|}$. Since $(v_n) \subset E$ is bounded, the compactness of the embedding $E \hookrightarrow F$, implies that there exists $v \in F$ such that $v_n \to v$ in F, up to a subsequence. Then it follows that $I(v_n) \to I(v)$.

If $v = 0$, then by (iii), for all $t \geqslant 0$

$$\begin{aligned} c + o_n(1) &= \Phi(u_n) = \Phi(v_n \|u_n\|) \\ &= \max_{s \geqslant 0} \Phi(s v_n) \\ &\geqslant \Phi(t v_n) \\ &\geqslant t - I(t v_n) \\ &= t + o_n(1), \end{aligned}$$

which is a clear contradiction with the fact that $c \in \mathbb{R}$.

If $v \neq 0$, then by (v)

$$o_n(1) = \lim_{n\to\infty} \frac{c}{\|u_n\|} = \lim_{n\to\infty} \frac{\Phi(u_n)}{\|u_n\|} = \lim_{n\to\infty} \frac{\Phi(v_n\|u_n\|)}{\|u_n\|} = -\infty.$$

Then we get a contradiction.

Hence (u_n) is bounded in E and then there exists $u \in F$ such that $u_n \to u$ in F, up to a subsequence. Then, by hypothesis, it follows that $u \in E$.

If $u = 0$, then by (2.2), for all $t \geqslant 0$,

$$\begin{aligned} c + o_n(1) &= \Phi(u_n) \\ &= \max_{s \geqslant 0} \Phi(s u_n) \\ &\geqslant \Phi(t u_n) \\ &\geqslant t\|u_n\| - I(t u_n) \\ &\geqslant t\delta - I(t u_n) \\ &= t\delta + o_n(1), \end{aligned}$$

which give us a contradiction.

Hence $u \neq 0$ and let $t_u > 0$ be such that $t_u u \in \mathcal{N}$. Then, by $i)$

$$\begin{aligned} c &\leqslant \Phi(t_u u) \\ &= I_0(t_u u) - I(t_u u) \\ &\leqslant \liminf_{n\to\infty} I_0(t_u u_n) - \lim_{n\to\infty} I(t_u u_n) \\ &= \liminf_{n\to\infty} \Phi(t_u u_n) \\ &\leqslant \liminf_{n\to\infty} \Phi(u_n) = c \end{aligned}$$

and then $\Phi(t_u u) = c$ and $t_u u \in \mathcal{N}$, which proves the theorem. □

Now let us present the proof of our second result.

Proof of Theorem 2. Suppose by contradiction that $0 \notin \partial\Phi(u_0)$, then $\beta(u_0) > 0$, where $\beta(u_0) = \inf\{\|z\|_*;\ z \in \partial\Phi(u_0)\}$, since by ([6, p. 105]), such infimum is attained, where the dual norm and the subdifferential of Φ_0 are taken with respect to E. Since $u \mapsto \beta(u)$ is lower semicontinuous (again by [6, p. 105], for instance), it follows that there exists $\kappa > 0$ such that

$$\beta(u) > \frac{\beta(u_0)}{2} > 0, \quad \forall u \in B_\kappa(u_0). \tag{2.3}$$

Let us denote $J = \left[1 - \frac{\kappa}{4}, 1 + \frac{\kappa}{4}\right] \subset \mathbb{R}$ and define $g : J \to E$ by

$$g(t) = tu_0.$$

Denoting $c := \Phi(u_0) = \inf_{\mathcal{N}} \Phi$, note that

$$\Phi(g(t)) < c, \quad \forall t \neq 1.$$

Moreover, note that

$$\max\left\{\Phi\left(g\left(1 - \frac{\kappa}{4}\right)\right), \quad \Phi\left(g\left(1 + \frac{\kappa}{4}\right)\right)\right\} = c_0 < c.$$

By using the version of the Deformation Lemma to locally Lipschitz functionals, without the Palais–Smale condition (see [9, Lemma 2.3]), there exists $\epsilon > 0$ such that

$$\epsilon < \epsilon_0 := \min\left\{\frac{c - c_0}{2}, \frac{\kappa\beta(u_0)}{16}\right\},$$

and an homeomorphism $\eta : E \to E$ such that

(i) $\eta(x) = x$ for all $x \notin \Phi^{-1}([c - \epsilon_0, c + \epsilon_0]) \cap B_\kappa(u_0)$;
(ii) $\eta(\Phi_{c+\epsilon} \cap B_{\kappa/2}(u_0)) \subset \Phi_{c-\epsilon}$;
(iii) $\Phi(\eta(x)) \leqslant \Phi(x)$, for all $x \in E$,

where, for $d \in \mathbb{R}$, $\Phi_d = \{u \in E;\ \Phi(u) \leqslant d\}$.

Let us define now $h : J \to E$ by $h(t) = \eta(g(t))$ and two functions, $\Psi_0, \Psi_1 : J \to \mathbb{R}$ by

$$\Psi_0(t) = \Phi'(tu_0)u_0$$

and

$$\Psi_1(t) = \frac{1}{t}\Phi'(h(t))h(t).$$

Since for $t \in \left\{\left(1 - \frac{\kappa}{4}\right), \left(1 + \frac{\kappa}{4}\right)\right\}$, $\Phi(g(t)) \leqslant c_0 < c - \epsilon_0$, then $h(t) = \eta(g(t)) = g(t) = tu_0$ for $t \in \left\{\left(1 - \frac{\kappa}{4}\right), \left(1 + \frac{\kappa}{4}\right)\right\}$. Hence

$$\Psi_0(t) = \Psi_1(t), \quad \forall t \in \left\{\left(1 - \frac{\kappa}{4}\right), \left(1 + \frac{\kappa}{4}\right)\right\}. \tag{2.4}$$

Let us briefly justify that the Brouwer Degree $d(\Psi_0, J, 0)$ is equal to -1. In fact, by the proof of Theorem 1, it follows that $\Psi_0(t) = \gamma'_{u_0}(t)$ is continuous, strictly decreasing, $\Psi_0(1 - \kappa/4) > 0 > \Psi_0(1 + \kappa/4)$, and $t = 1$ is the unique point in J where Ψ_0 vanishes. Since there is no information about the smoothness of Ψ_0, in order to calculate $d(\Psi_0, J, 0)$, we should approximate Ψ_0 by some $C^1(J, \mathbb{R})$ function. By considering a strictly decreasing function $\overline{\Psi} \in C^1(J, \mathbb{R})$, such that $\overline{\Psi}(t - \kappa/4) = \Psi_0(t - \kappa/4)$, $\overline{\Psi}(t + \kappa/4) = \Psi_0(t + \kappa/4)$ and $\overline{\Psi}(1) = 0$, note that by the very definition of Brouwer Degree and by its invariance by homotopy, $d(\Psi_0, J, 0) = d(\overline{\Psi}, J, 0) = \mathrm{sgn}(\overline{\Psi}'(1)) = -1$.

Since $\Psi_0 = \Psi_1$ on ∂J and $d(\Psi_0, J, 0) = -1$, by degree theory, $d(\Psi_1, J, 0) = -1$. Then there exists $t \in J$ such that $h(t) \in \mathcal{N}$. This implies that

$$c \leqslant \Phi(h(t)) = \Phi(\eta(g(t))).$$

But note that $\Phi(g(t)) < c + \epsilon$ and also $g(J) \subset B_{\kappa/2}(u_0)$. Then, by *ii)*

$$\Phi(\eta(g(t))) < c - \epsilon$$

which contradicts the last inequality. Then the result follows. □

3. Applications

3.1. Preliminaries

First of all let us introduce the space of functions of bounded variation, $BV(\Omega)$. We say that $u \in BV(\Omega)$, or is a function of bounded variation, if $u \in L^1(\Omega)$, and its distributional derivative Du is a vectorial Radon measure, i.e.,

$$BV(\Omega) = \left\{u \in L^1(\Omega);\, Du \in \mathcal{M}(\Omega, \mathbb{R}^N)\right\}.$$

It can be proved that u belongs to $BV(\Omega)$ is equivalent to

$$\int_\Omega |Du| := \sup\left\{\int_\Omega u \mathrm{div}\phi dx;\ \phi \in C_c^1(\Omega, \mathbb{R}^N), \text{ s.t. } |\phi|_\infty \leqslant 1\right\} < +\infty.$$

The space $BV(\Omega)$ is a Banach space when endowed with the norm

$$\|u\|_{BV(\Omega)} := \int_\Omega |Du| + |u|_1,$$

which is continuously embedded into $L^r(\Omega)$ for all $r \in [1, N/(N-1)]$ and is compactly embedded for $r \in [1, N/(N-1))$ (see [3, Theorems 10.1.3, 10.1.4]).

Because of Trace Theorem [3, Theorem 10.2.1] and the continuous embedding of $BV(\Omega)$ into $L^1(\Omega)$, it follows that

$$\|u\| := \int_\Omega |Du| + \int_{\partial\Omega} |u| d\mathcal{H}_{N-1},$$

where $\mathcal{H}_{N-1}$ denotes the usual $(N-1)$-dimensional Hausdorff measure, defines in $BV(\Omega)$ an equivalent norm, which we use as the standard norm in $BV(\Omega)$ from now on.

It can be proved that $I_0 : BV(\Omega) \to \mathbb{R}$ given by

$$I_0(u) = \int_\Omega |Du| + \int_{\partial\Omega} |u| d\mathcal{H}_{N-1} \tag{3.1}$$

is a convex functional and Lipschitz continuous in $BV(\Omega)$. In fact, note that by the version of Poincaré inequality to $BV(\Omega)$ (see [13] for example) and the continuous embedding $BV(\Omega) \hookrightarrow L^1(\partial\Omega)$, it follows that I_0 is a norm in $BV(\Omega)$, equivalent to the usual one.

It follows also that $BV(\Omega)$ is a *lattice* (see [1] for instance), i.e., if $u, v \in BV(\Omega)$, then $\max\{u, v\}, \min\{u, v\} \in BV(\Omega)$ and also

$$I_0(\max\{u, v\}) + I_0(\min\{u, v\}) \leqslant I_0(u) + I_0(v), \quad \forall u, v \in BV(\Omega). \tag{3.2}$$

For a vectorial Radon measure $\mu \in \mathcal{M}(\Omega, \mathbb{R}^N)$, we denote by $\mu = \mu^a + \mu^s$ the usual decomposition stated in the Radon Nikodyn Theorem, where μ^a and μ^s are, respectively, the absolute continuous and the singular parts with respect to the N-dimensional Lebesgue measure $\mathcal{L}^N$. We denote by $|\mu|$, the absolute value of μ, the scalar Radon measure defined like in [3, p. 125]. By $\frac{d\mu}{d|\mu|}(x)$ we denote the usual Lebesgue derivative of μ with respect to $|\mu|$, given by

$$\frac{d\mu}{d|\mu|}(x) = \limsup_{r\to 0} \frac{\mu(B_r(x))}{|\mu|(B_r(x))}.$$

In order to remark the properties of differentiability (or lack thereof) of I_0, let us recall the result of Anzellotti in [2, Theorem 2.4]. For $g : \Omega \times \mathbb{R}^N \to \mathbb{R}$, let us define

$$g^0(x,p) = \lim_{t\to 0^+} g\left(x, \frac{p}{t}\right) t.$$

Suppose that g is differentiable in p for all $x \in \Omega$, $p \in \mathbb{R}^N$ and $g^0(x,p)$ is differentiable for all $x \in \Omega$, $p \in \mathbb{R}^N \backslash \{0\}$ and also that there exists $M > 0$ such that

$$|g_p(x,p)| \leqslant M, \quad |g_p^0(x,p)| \leqslant M.$$

Then $\mathcal{J}_g : BV(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{J}_g(u) = \int_\Omega g(x, Du) := \int_\Omega g(x, (Du)^a(x))dx + \int_\Omega g^0\left(x, \frac{Du}{|Du|}(x)\right)|Du|^s$$

is differentiable at the point $u \in BV(\Omega)$ in the direction $v \in BV(\Omega)$ if and only if $|Dv|^s$ is absolutely continuous with respect to $|Du|^s$, and in such a case one has

$$\mathcal{J}_g'(u)v = \int_\Omega g_p(x, (Du)^a(x))(Dv)^a(x)dx + \int_\Omega g_p^0\left(x, \frac{Du}{|Du|}(x)\right)\frac{Dv}{|Dv|}(x)|Dv|^s.$$

Since

$$I_0(u) = \mathcal{J}_g(u) + \int_{\partial\Omega} |u| d\mathcal{H}_{N-1}$$

with $g(x,p) = g^0(x,p) = |p|$, we have that, given $u \in BV(\Omega)$, $I_0'(u)v$ is well defined for every $v \in BV(\Omega)$ such that $|Dv|^s$ is absolutely continuous with respect to $|Du|^s$ and $v(x) = 0$, $\mathcal{H}_{N-1}-$ a.e. on the set $\{x \in \partial\Omega;\, u(x) = 0\}$ and we have that

$$I_0'(u)v = \int_\Omega \frac{(Du)^a(Dv)^a}{|(Du)^a|}dx + \int_\Omega \frac{Du}{|Du|}(x)\frac{Dv}{|Dv|}(x)|(Dv)|^s + \int_{\partial\Omega} \mathrm{sgn}(u) v d\mathcal{H}_{N-1}. \tag{3.3}$$

Now let us just make precise the sense of solutions that we consider in this work. Regarding (1.2), the energy functional associated to it is $\Phi : BV(\Omega) \to \mathbb{R}$ given by

$$\Phi(u) = I_0(u) - I(u),$$

where

$$I(u) = \int_\Omega F(u)dx. \tag{3.4}$$

Since $I \in C^1(BV(\Omega))$ and I_0 is Lipschitz continuous, we say that $u_0 \in BV(\Omega)$ is a solution of (1.2) if $0 \in \partial\Phi(u_0)$, where $\partial\Phi(u_0)$ denotes the generalized gradient of Φ in u_0, as defined in [6]. It follows that this is equivalent to $I'(u_0) \in \partial I_0(u_0)$ and, since I_0 is convex, this is written as

$$I_0(v) - I_0(u_0) \geqslant I'(u_0)(v - u_0), \quad \forall v \in BV(\Omega). \tag{3.5}$$

Hence all $u_0 \in BV(\Omega)$ such that (3.5) holds is going to be called a bounded variation solution of (1.2).

In the second application we deal with the space

$$BL_0(\Omega) = \left\{ u \in W_0^{1,1}(\Omega) : \int_\Omega |\Delta u| < +\infty \right\},$$

when endowed with the norm

$$\|u\|_0 = |u|_1 + |\nabla u|_1 + \int_\Omega |\Delta u|.$$

By the Poincaré inequality in $BV(\Omega)$ (see [13, Proposition 2]) and the results in [5, Theorem 1.2 and Proposition 2.1] by H. Brézis and A. Ponce, we can define a norm in $BL_0(\Omega)$ given by

$$\|u\| = \int_\Omega |\Delta u| \quad \text{for every } u \in BL_0(\Omega), \tag{3.6}$$

which is equivalent to $\|\cdot\|_0$. In fact, if $u \in BL_0(\Omega)$, since $u \in W_0^{1,1}(\Omega)$, then $\int_\Omega |Du| = \int_\Omega |\nabla u| dx$. Then it follows that

$$\begin{aligned} \|u\|_0 &\leqslant (C+1)|\nabla u|_1 + \int_\Omega |\Delta u| \\ &\leqslant C_1 \int_\Omega |\Delta u|. \end{aligned}$$

It follows also that the space $BL_0(\Omega)$ has w.r.t. its norm all the properties that $BV(\Omega)$ has. For example, it can be proved (see [14]) that $BL_0(\Omega)$ is a Banach which is continuously embedded into $L^r(\Omega)$ for all $r \in [1, N/(N-1)]$ (this embedding being compact for $r \in [1, N/(N-1))$). The norm $\|\cdot\|$ given in (3.6) is lower semicontinuous w.r.t. the $L^r(\Omega)$ convergence, for $r \in [1, N/(N-1)]$.

In analogy with the first application, the energy functional associated to (1.4) is given by

$$\Phi(u) = \tilde{I}_0(u) - I(u),$$

where

$$\tilde{I}_0(u) = \int_\Omega |\Delta u| \tag{3.7}$$

and I is given by (3.4). Also as in the first application, Φ is written as the difference between a Lipschitz continuous and a $C^1(BL_0(\Omega))$ functional, in such a way that a solution of (1.4) is going to be considered as a function $u_0 \in BL_0(\Omega)$ such that

$$\tilde{I}_0(v) - \tilde{I}_0(u_0) \geqslant I'(u_0)(v - u_0), \quad \forall v \in BL_0(\Omega). \tag{3.8}$$

3.2. A problem involving the 1-laplacian operator

Let us consider the problem of finding critical points of the functional $\Phi : BV(\Omega) \to \mathbb{R}$ given by

$$\Phi(u) = \int_\Omega |Du| + \int_{\partial\Omega} |u| d\mathcal{H}_{N-1} - \int_\Omega F(u)dx,$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N \geqslant 2$, $F(t) := \displaystyle\int_0^t f(s)ds$, and f satisfy the following assumptions

(f_1) $f \in C^0(\mathbb{R})$;
(f_2) $f(0) = 0$;
(f_3) there exist constants $c_1, c_2 > 0$ and $p \in (1, 1^*)$ such that

$$|f(s)| \leqslant c_1 + c_2|s|^{p-1}, \quad s \in \mathbb{R};$$

(f_4) $\displaystyle\lim_{t \to \pm\infty} \frac{F(t)}{t} = \pm\infty$;
(f_5) f is increasing for $s \in \mathbb{R}$.

In fact the critical points of Φ are weak solutions of a relaxed form of

$$\begin{cases} -\Delta_1 u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.9}$$

where $\Delta_1 u = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)$.

Let us consider the functionals $I_0, I : BV(\Omega) \to \mathbb{R}$ given by (3.1) and (3.4), respectively and define $\Phi : BV(\Omega) \to \mathbb{R}$ by

$$\Phi(u) = I_0(u) - I(u).$$

Clearly the operator Δ_1 is highly singular and some words about this imprecise way to define it have to be stated. The first step is to extend the functionals I_0, I and Φ to $L^{1^*}(\Omega)$, defining $\overline{I_0}, \overline{I}, \overline{\Phi} : L^{1^*}(\Omega) \to \mathbb{R}$, where

$$\overline{I_0}(u) = \begin{cases} I_0(u), \text{ if } u \in BV(\Omega), \\ +\infty, \text{ if } u \in L^{1^*}(\Omega) \backslash BV(\Omega), \end{cases}$$
$$\overline{I}(u) = \int_\Omega F(u)dx$$

and $\overline{\Phi} = \overline{I_0} - \overline{I}$. It is easy to see that $\overline{I}$ belongs to $C^1(L^{1^*}(\Omega))$ and that $\overline{I_0}$ is convex and lower semicontinuous in $L^{1^*}(\Omega)$. Hence the subdifferential of $\overline{I_0}$ is well defined. The following is a crucial result in obtaining an Euler-Lagrange equation satisfied by the critical points of Φ.

Lemma 3. *If $u \in BV(\Omega)$ is such that $0 \in \partial\Phi(u)$, then $0 \in \partial\overline{\Phi}(u)$.*

Proof. Suppose that $u \in BV(\Omega)$ is such that $0 \in \partial\Phi(u)$. Then u satisfies (1.3). Let us verify that

$$\overline{I_0}(v) - \overline{I_0}(u) \geqslant \overline{I}'(u)(v-u), \quad \forall v \in L^{1^*}(\Omega).$$

For $v \in L^{1^*}(\Omega)$, note that:

- if $v \in BV(\Omega) \cap L^{1^*}(\Omega)$, then

$$\begin{aligned}\overline{I_0}(v) - \overline{I_0}(u) &= I_0(v) - I_0(u)\\ &\geqslant I'(u)(v-u)\\ &= \int_\Omega f(u)(v-u)dx\\ &= \overline{I}'(u)(v-u);\end{aligned}$$

- if $u \in L^{1^*}(\Omega)\backslash BV(\Omega)$, since $\overline{I_0}(v) = +\infty$ and $\overline{I_0}(u) < +\infty$, it follows that

$$\begin{aligned}\overline{I_0}(v) - \overline{I_0}(u) &= +\infty\\ &\geqslant \overline{I}'(u)(v-u).\end{aligned}$$

Therefore the result follows. □

Let us assume that $u \in BV(\Omega)$ is a bounded variation solution of (3.9). Since $0 \in \partial\Phi(u)$, by the last result it follows that $0 \in \partial\overline{\Phi}(u)$. Since $\overline{I_0}$ is convex and $\overline{I}$ is smooth, it follows that $\overline{I}'(u) \in \partial\overline{I_0}(u)$. By [12, Proposition 4.23, p. 529], it follows that there exist $z \in L^\infty(\Omega, \mathbb{R}^N)$ such that $|z|_\infty \leqslant 1$,

$$-\operatorname{div} z = f(u) \quad \text{ in } L^N(\Omega) \tag{3.10}$$

and

$$-\int_\Omega u \operatorname{div} z dx = \int_\Omega |Du|, \tag{3.11}$$

where the divergence in (3.10) has to be understood in the distributional sense. Therefore, it follows from (3.10) and (3.11) that u satisfies

$$\begin{cases} \exists z \in L^\infty(\Omega, \mathbb{R}^N),\ |z|_\infty \leqslant 1,\ \operatorname{div} z \in L^N(\Omega),\\ -\displaystyle\int_\Omega u \operatorname{div} z dx = \int_\Omega |Du|,\\ -\operatorname{div} z = f(u), \quad \text{a.e. in } \Omega. \end{cases} \tag{3.12}$$

Hence, (3.12) is the precise version of (3.9).

Note that I_0 is Lipschitz continuous in $BV(\Omega)$ and $I \in C^1(BV(\Omega))$. Moreover, $BV(\Omega)$ is compactly embedded into $L^p(\Omega)$, p as in (f_3), $I(0) = 0$ and, for all $u \in BV(\Omega)$ and $s \in \mathbb{R}$, by (3.3) we have that

$$\begin{aligned}I_0'(su)u &= \lim_{t\to 0}\frac{I_0(su+tu)-I_0(su)}{t} = \int_\Omega \frac{(D(su))^a(Du)^a}{|(D(su))^a|}dx\\ &\quad + \textstyle\int_\Omega \frac{D(su)}{|D(su)|}(x)\frac{Du}{|Du|}(x)|Du|^s + \int_{\partial\Omega}\operatorname{sgn}(su)ud\mathcal{H}_{N-1}\\ &= \int_\Omega |(Du)^a|dx + \int_\Omega |(Du)^s| + \int_{\partial\Omega}|u|d\mathcal{H}_{N-1}\end{aligned}$$

and then $I_0'(u)u = I_0(u)$, for all $u \in BV(\Omega)$.

Let us define the Nehari set by

$$\begin{aligned}\mathcal{N} &= \{u \in BV(\Omega)\backslash\{0\};\, I_0'(u)u = I'(u)u\}\\ &= \left\{u \in BV(\Omega)\backslash\{0\};\, \int_\Omega |Du| + \int_{\partial\Omega}|u|d\mathcal{H}_{N-1} = \int_\Omega f(u)udx\right\}.\end{aligned}$$

In the following result we prove that all nontrivial critical points of Φ belong to $\mathcal{N}$.

Lemma 4. *If $u_0 \in BV(\Omega)$, $u_0 \neq 0$ and $0 \in \partial\Phi(u_0)$, then $u \in \mathcal{N}$.*

Proof. If $0 \in \partial\Phi(u_0)$, then

$$I_0(v) - I_0(u_0) \geqslant \int_\Omega f(u_0)(v-u_0)dx, \quad \forall v \in BV(\Omega).$$

For $t > 0$, by taking $v = u_0 + tu_0$ in the last expression and calculating the limit as $t \to 0^+$ we get

$$I_0(u_0) = \lim_{t\to 0^+}\frac{I_0(u_0+tu_0)-I_0(u_0)}{t} \geqslant \int_\Omega f(u_0)u_0dx.$$

Doing the same for $t < 0$ we get

$$I_0(u_0) = \lim_{t\to 0^-}\frac{I_0(u_0+tu_0)-I_0(u_0)}{t} \leqslant \int_\Omega f(u_0)u_0dx$$

from where it follows the equality in both expressions above. Hence $u_0 \in \mathcal{N}$. □

Note that by the last result, if we manage to prove that the infimum of Φ in $\mathcal{N}$ is achieved and it is a critical point, then we would get a nontrivial critical point of Φ with lowest energy among all nontrivial ones, then, it would be a ground state bounded variation solution of (3.9). In order to do so, let us verify that Φ satisfies all the conditions of Theorem 1.

It is a well know result the fact that I_0 satisfies $i)$ of Theorem 1.

For $ii)$, first of all note that (f_2) and (f_3) imply that for all $\epsilon > 0$, there exists C_ϵ such that

$$|F(s)| \leqslant \epsilon|s| + C_\epsilon|s|^p, \quad \text{for all } s \in \mathbb{R}. \tag{3.13}$$

Then, the embeddings of $BV(\Omega)$ and (3.13) imply that

$$\begin{aligned}\Phi(u) &\geqslant (1-C\epsilon)\|u\| - CC_\epsilon\|u\|^p\\ &= \|u\|(1-\epsilon - CC_\epsilon\|u\|^{p-1})\\ &= \rho(1-\epsilon-CC_\epsilon\rho^{p-1}) =: \alpha_0 > 0 = \Phi(0),\end{aligned}$$

where $\|u\| = \rho$ and ϵ, ρ are positive and small enough, and the constant $C > 0$ is larger than the best constants of the embeddings of $BV(\Omega)$ into $L^1(\Omega)$ and $L^p(\Omega)$.

By definition of Φ it follows that (iii) holds, since in this case we have the equality being satisfied.

In order to verify (iv), just note that

$$t \mapsto I'(tu)u = \int_\Omega f(tu)udx$$

is increasing in $(0, +\infty)$ by (f_5). Taking into account the fact that, by (3.3), $t \mapsto I_0'(tu)u = I_0(u)$ is constant, it follows that $t \mapsto \Phi'(tu)u$ is a decreasing function in $(0, +\infty)$.

Finally, to verify v), let $(v_n) \subset BV(\Omega)$ and $v \in L^p(\Omega)\backslash\{0\}$ such that $v_n \to v$ in $L^p(\Omega)$. Since $\Phi(u) = \|u\| - \int_\Omega F(u)dx$, it is enough to prove that

$$\lim_{t\to\infty} \int_\Omega \frac{F(tv_n)}{t} = +\infty \quad \text{uniformly in } n \in \mathbb{N}.$$

Let $\Gamma = \{x \in \Omega; v(x) \neq 0\}$ and note that $|\Gamma| > 0$. Then by Fatou Lemma, it follows that, for all $t > 0$,

$$\int_\Gamma \frac{F(tv)}{t} dx \leqslant \liminf_{n\to\infty} \int_\Omega \frac{F(tv_n)}{t} dx.$$

Then, by (f_4) we have that

$$\liminf_{t\to\infty} \liminf_{n\to\infty} \int_\Omega \frac{F(tv_n)}{t} dx \geqslant \liminf_{t\to\infty} \int_\Gamma \frac{F(tv)}{t} dx = +\infty.$$

But this means that for every $M > 0$, there exist $t_0 > 0$ and $n_0 \in \mathbb{N}$ such that

$$\int_\Omega \frac{F(tv_n)}{t} dx \geqslant M, \quad \text{for every } t > t_0 \text{ and } n \geqslant n_0,$$

which proves (v).

Now, since all conditions of Theorem 1 are satisfied, it follows the existence of $u_0 \in \mathcal{N}$ such that

$$\Phi(u_0) = \inf_{v\in\mathcal{N}} \Phi(v).$$

Moreover, by Theorem 2, it follows that u_0 is a critical point to Φ and then, a ground state bounded variation solution of (3.9).

3.3. A problem involving the 1-biharmonic operator

In this section we deal with the following problem involving the 1-biharmonic operator

$$\begin{cases} \Delta_1^2 u = f(u) \quad \text{in } \Omega, \\ u = \dfrac{\Delta u}{|\Delta u|} = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.14}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N \geqslant 2$ and $f : \mathbb{R} \to \mathbb{R}$ is assumed to satisfy the same set of assumptions $(f_1) - (f_5)$ as in our first application.

Let us consider $\tilde{I}_0, I : BL_0(\Omega) \to \mathbb{R}$ given by (3.7) and (3.4), respectively and let us define $\Phi : BL_0(\Omega) \to \mathbb{R}$ by

$$\Phi(u) = \tilde{I}_0(u) - I(u).$$

Note that $\tilde{I}_0$ is Lipschitz continuous in $BL_0(\Omega)$ and $I \in C^1(BL_0(\Omega))$. Moreover, $BL_0(\Omega)$ is compactly embedded into $L^r(\Omega)$ for $r \in [1, N/(N-1))$ and $I(0) = 0$.

Note that for all $u \in BL_0(\Omega)$, in a similar way that in (3.3) it is possible to show that

$$\tilde{I}_0'(u)v = \int_\Omega \frac{(\Delta u)^a(\Delta v)^a}{|(\Delta u)^a|}dx + \int_\Omega \frac{\Delta u}{|\Delta u|}(x)\frac{\Delta v}{|\Delta v|}(x)|(\Delta v)|^s. \tag{3.15}$$

Let us define the Nehari set by

$$\begin{aligned} \mathcal{N} &= \left\{u \in BL_0(\Omega)\backslash\{0\};\, \tilde{I}_0'(u)u = I'(u)u\right\} \\ &= \left\{u \in BL_0(\Omega)\backslash\{0\};\, \int_\Omega |\Delta u| = \int_\Omega f(u)udx\right\} \end{aligned}$$

In the following result we state that all nontrivial critical points of Φ belong to $\mathcal{N}$. Its proof is totally analogous of Lemma 4.

Lemma 5. *If $u_0 \in BL_0(\Omega)$, $u_0 \neq 0$ and $0 \in \partial\Phi(u_0)$, then $u \in \mathcal{N}$.*

As in the case of Sect. 3.2, if we manage to prove that the infimum of Φ in $\mathcal{N}$ is achieved and it is a critical point, then we would get a nontrivial critical point of Φ with lowest energy, then, it would be a ground state bounded variation solution of (3.14).

Once more, let us verify that all the conditions of Theorem 1 are satisfied. As in the first application, it is well known (see [14]) that $\tilde{I}_0$ satisfies (i) in Theorem 1.

For (ii), (iii), (iv) and (v), all the calculations are absolutely the same as in Sect. 3.2 and then will be omited.

Now, since all conditions of Theorem 1 are satisfied, it follows that there exists $u_0 \in \mathcal{N}$ such that

$$\Phi(u_0) = \inf_{v \in \mathcal{N}} \Phi(v).$$

Moreover, by Theorem 2, it follows that u_0 is a critical point to Φ and then, a ground state bounded variation solution of (3.14).

Again, as in our first application, by using the same arguments it is possible to prove that the ground state solution $u_0 \in BL_0(\Omega)$ of (3.14) in fact satisfy the following problem, which is the precise version of (3.14),

$$\begin{cases} \exists\, z \in W_0^{1,1}(\Omega) \cap L^\infty(\Omega),\ |z|_\infty \leqslant 1,\ \Delta z \in L^N(\Omega), \\ \displaystyle\int_\Omega u\Delta z dx = \int_\Omega |\Delta u|, \\ \Delta z = f(u), \quad \text{a.e. in } \Omega. \end{cases}$$

Remark 6. *Note that the boundary condition* $\frac{\Delta u}{|\Delta u|} = 0$ *on* $\partial\Omega$ *can be seen as being satisfied by* z*, since* $z \in W_0^{1,1}(\Omega)$ *implies that* $z = 0$ *on* $\partial\Omega$ *in the sense of trace.*

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