Nonlinear Differ. Equ. Appl. (2018) 25:46

1021-9722/18/050001-33
published online August 24, 2018
https://doi.org/10.1007/s00030-018-0537-3

Nonlinear Differential Equations and Applications NoDEA

Robust Stackelberg controllability for the Navier–Stokes equations

Cristhian Montoya and Luz de Teresa

Abstract. In this paper we deal with a robust Stackelberg strategy for the Navier–Stokes system. The scheme is based in considering a robust control problem for the "follower control" and its associated disturbance function. Afterwards, we consider the notion of Stackelberg optimization (which is associated to the "leader control") in order to deduce a local null controllability result for the Navier–Stokes system.

Mathematics Subject Classification. Primary 35Q30, 93B05, 49J20; Secondary 35Q93, 91A65, 93C10.

Keywords. Robust control, Hierarchic control, Navier–Stokes equations, Carleman estimates, Null controllability.

1. Introduction

The theory of robust control began in the late 1970s and early 1980s for finite dimensional systems. Since then, many techniques have been developed to deal with systems with uncertainties. In the late 90s the papers of Bewley et al. [4] presented the first rigorous generalization of the concepts in the case of partial differential equations. What could we understand by robustness in a control system? Well, informally, a controller designed for a particular set of parameters is said to be robust if it also functions correctly under a uncertainty: the controller is designed to work assuming that certain variable will be unknown. In this sense, one could think in the worst-case disturbance of the system, and design a controller which is suited to handle even this extreme situation. Thus, the problem of finding a robust control involves the problem of finding the worst-case disturbance in the spirit of a non-cooperative game (when there is not cooperation between the controller and disturbance function), which is from the mathematical point of view to reach a saddle point for the pair disturbance–controller.

C. Montoya: Supported by Project FORDECYT 265667 of CONACYT. L. de Teresa: The research was partially supported by UNAM-DGAPA-PAPIIT IN102116.

The research on robust control for PDE systems is in an early stage. Much of the literature deals with numerical aspects and much of the theory has been developed for fluid mechanics and for some elliptic problems. See e.g. [1,4,5,20,22]. In this paper we will present a hierarchic strategy to deal with robust control and, simultaneously, with null control for incompressible fluids modelled by the Navier–Stokes equations with Dirichlet boundary conditions.

We will work in the setting of a Stackelberg competition, see [24]. This consists in a non-cooperative decision problem in which one of the participants enforce its strategy on the other participants. We assume that we can act on the dynamics of the system through a hierarchy of controls. In our case the controls are external forces acting on the system, where the leader control has a local null controllability objective while the follower control and perturbation solve a robust control problem.

To be precise: let Ω be a nonempty bounded connected open subset of $\mathbb{R}^N$ ($N = 2$ or $N = 3$) of class C^∞. Let $T > 0$ and let ω and $\mathcal{O}$ be (small) nonempty open subsets of Ω with $\omega \cap \mathcal{O} = \emptyset$. We will use the notation $Q := \Omega \times (0, T)$, $\Sigma := \partial\Omega \times (0, T)$ and $n(x)$ will denote the outward unit normal vector at the point $x \in \partial\Omega$.

Let us consider the Navier–Stokes system with homogeneous Dirichlet boundary conditions

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla)y + \nabla p = h1_\omega + v\chi_{\mathcal{O}} + \psi & \text{in} \quad Q, \\ \nabla \cdot y = 0 & \text{in} \quad Q, \\ y = 0 & \text{on} \quad \Sigma, \\ y(\cdot, 0) = y_0(\cdot) & \text{in} \quad \Omega, \end{cases} \tag{1}$$

where $h = h(x,t) \in L^2(0, T; L^2(\omega)^N)$ is called the "leader control", $v = v(x,t) \in L^2(0, T; L^2(\mathcal{O})^N)$ is the "follower control", $\psi \in L^2(Q)^N$ is an unknown perturbation and y_0 an initial state in a suitable space. Here 1_ω is the characteristic function of the set ω and $\chi_{\mathcal{O}}$ is a smooth non-negative function such that *supp* $\chi_{\mathcal{O}} = \overline{\mathcal{O}}$.

To our knowledge there are not results in the literature concerning a robust Stackelberg strategy for system (1). As far as we know, the first paper on robust Stackelberg controllability is [19], which develops the concept of control for a semi-linear parabolic equation. However, there exist several papers which treat independently robust and hierarchical control for the Navier–Stokes system. In the context of robust control [that is $h \equiv 0$ in (1)], the works [4,6] show the existence and uniqueness of the solution to the robust control problem for the N-dimensional case of system (1), and present an appropriate numerical method to solve it. In their works the authors have used an abstract scheme throughout Leray projection and classical techniques of optimal control theory. In [22], some theoretical and numerical aspects are presented for the optimal and robust control of the Navier–Stokes equations. Additional information on optimal and robust control theory for linear and nonlinear systems can be found in [5,14,22], and references therein.

In the context of hierarchical control [that is $\psi \equiv 0$ in (1)], some recent works such as [2,3,17,19,21] show a strategy with a leader and follower controls for different equations. Some older results on a Stackelberg–Nash control strategy were proved by Diaz and Lions [9] for a linear parabolic problem and by Limaco et al. [21] for a linear parabolic problem with moving boundaries. In both cases, the objective of the leader control is an approximate controllability result. In the case of linear fluid models some approximate controllability of Stackelberg–Nash strategies started with the result of Guillén-González et al. [17] for the Stokes system, and were extended by Araruna et al. [2] for linearized micropolar fluids. In [2] the main arguments are based on a Fenchel–Rockafeller dual variational principle [25]. For semilinear parabolic equations, a Stackelberg–Nash strategy with exact controllability for the leader control is proved in [3] using Carleman inequalities.

In the case of nonlinear fluids there are not, as far as we know, results concerning a Stackelberg control strategy. In this paper we will fill this gap giving an answer for the case of a incompressible fluid flow described by means of Navier–Stokes equations with no-slip boundary conditions, see Theorem 1.3. More precisely, system (1) with $\psi \equiv 0$ and Theorem 1.2 with $\gamma = \gamma_0 = 0$ will allow us to deduce a local null controllability result for the leader control when we apply a Stackelberg minimizing strategy for the Navier–Stokes system. The arguments should be carried out following the schemes of Proposition 1 and Theorem 3.5.

In our work, we follow the ideas introduced in [19] for the Navier–Stokes equations with Dirichlet boundary conditions. However the nonlinearity of (1) will allows only to obtain a local null controllability result for the leader control.

Let us now introduce the usual spaces in the context of incompressible fluids [23]:

$$\begin{aligned} H &:= \{u \in L^2(\Omega)^N : \nabla \cdot u = 0, \text{ in } \Omega, \;\; u \cdot n = 0 \text{ on } \partial\Omega\}, \\ V &:= \{u \in H_0^1(\Omega)^N : \nabla \cdot u = 0 \text{ in } \Omega\}. \end{aligned}$$

Following the scheme for the robust control problem given in [1,6], the general space for the control functions and the disturbance ψ in the right-hand side of (1) is $L^2(0,T;H)$.

Now, we focus our attention on the control problem we are interested in.

1.1. The main problem

Given $h \in L^2(0,T;L^2(\omega)^N)$ a (leader) control, we consider the *secondary* cost functional

$$J_r(\psi, v; h) := \frac{\mu}{2}\iint_{\mathcal{O}_d\times(0,T)} |y - y_d|^2 dxdt + \frac{\ell^2}{2}\iint_Q \chi_{\mathcal{O}}|v|^2 dxdt - \frac{\gamma^2}{2}\iint_Q |\psi|^2 dxdt, \tag{2}$$

where $\ell, \gamma, \mu > 0$ are constants, $\mathcal{O}_d$ is an open subset of Ω, which represents a observability domain, and $y_d \in L^2(0,T;L^2(\mathcal{O}_d)^N)$ is given. The constant μ arises from the physical parameters that govern the motion of fluids such as viscosity, characteristic length and characteristic velocity. The parameters ℓ, γ

are included to make the cost functional consistent and to account the relative weight of each term. Note that the sign of the term associated to the disturbance is opposite to the sign used for the control, this is because we minimize with respect to the control v meanwhile simultaneouly maximize with respect to the disturbance ψ. From another perspective, the term $-\gamma^2\|\psi\|^2_{L^2(Q)^N}$ constrains the magnitude of the disturbance function in the maximization with respect to ψ and, the term associated to $\ell^2\|v\|^2_{L^2(Q)^N}$ constrains the magnitude of the control in the minimization with respect to v.

To explain the robust Stackelberg control problem, we will consider the following two subproblems:

(i) *First problem* For every fixed leader control h, solve the robust control problem for the nonlinear system (1), that is, find the best control v in the presence of the disturbance ψ which maximally spoils the follower control for the Navier–Stokes system (1). The robust control problem to be solved is given in the following definition.

Definition 1.1. Let $h \in L^2(0,T;L^2(\omega)^N)$ be fixed. The disturbance $\overline{\psi} \in L^2(Q)^N$, the control $\overline{v} \in L^2(Q)^N$, and the solution $\overline{y} = y(h,\overline{v}(h),\overline{\psi}(h))$ of (1) associated with $(\overline{\psi}(h),\overline{v}(h))$ are said to solve the robust control problem when a saddle point $(\overline{\psi}(h),\overline{v}(h))$ of the cost functional defined in (2) is reached, that is, if $\forall(\psi,v) \in L^2(Q)^{N\times N}$

$$J_r(\psi,\overline{v}(h);h) \leq J_r(\overline{\psi}(h),\overline{v}(h);h) \leq J_r(\overline{\psi}(h),v(h);h). \tag{3}$$

In this case,

$$\begin{aligned} J_r(\overline{\psi}(h),\overline{v}(h);h) &= \max_{\psi\in L^2(Q)^N}\ \min_{v\in L^2(Q)^N} J_r(\psi,v;h) \\ &= \min_{v\in L^2(Q)^N}\ \max_{\psi\in L^2(Q)^N} J_r(\psi,v;h). \end{aligned}$$

(ii) *Second problem* Once the saddle point has been identified for each leader control h, this is, once the existence of the saddle point $(\overline{\psi}(h),\overline{v}(h))$ for every leader control h is guarantied, we deal with the problem of finding the control h of minimal norm satisfying null controllability constraints. More precisely, we look for an optimal control $\overline{h}$ such that

$$J(\overline{h}) = \min_h \frac{1}{2}\iint_{\omega\times(0,T)} |h|^2 dxdt, \quad \text{subject to} \quad y(\cdot,T) = 0 \text{ in } \Omega. \tag{4}$$

Our main result on the robust hierarchic control is given in the following theorem.

Theorem 1.2. *Assume that $\omega \cap \mathcal{O}_d \neq \emptyset$. Then, for every $T > 0$ and $\mathcal{O},\omega \subset \Omega$ open subsets such that $\mathcal{O}\cap\omega = \emptyset$, there exist γ_0, ℓ_0, δ and a positive function $\rho = \rho(t)$ blowing up $t = T$ such that for any $\gamma \geq \gamma_0$, $\ell \geq \ell_0$, $y_0 \in V$ and $y_d \in L^2(0,T;L^2(\mathcal{O}_d)^N)$ satisfying*

$$\|y_0\|_V \leq \delta \quad \textit{and} \quad \iint_{\mathcal{O}_d\times(0,T)} \rho^2(t)|y_d|^2 dxdt < +\infty, \tag{5}$$

we can find a leader control $h \in L^2(0,T;L^2(\omega)^N)$ *and an unique saddle point* $(\overline{\psi},\overline{v}) \in L^2(Q)^N \times L^2(0,T;L^2(\mathcal{O})^N)$*, for the functional given by* (2)*, and an associated solution* (y,p) *to* (1) *verifying* $y(\cdot,T)=0$ *in* Ω.

As mentioned in the introduction, we have an additional result when $\psi=0$. That is, when we consider a Stackelberg strategy for

$$\begin{cases} y_t - \Delta y + (y\cdot\nabla)y + \nabla p = h1_\omega + v\chi_{\mathcal{O}} & \text{in} \quad Q,\\ \nabla\cdot y = 0 & \text{in} \quad Q,\\ y=0 & \text{on} \quad \Sigma,\\ y(\cdot,0)=y_0(\cdot) & \text{in} \quad \Omega, \end{cases} \tag{6}$$

and we define as the follower functional

$$J_r(v;h) := \frac{\mu}{2}\iint_{\mathcal{O}_d\times(0,T)} |y-y_d|^2 dxdt + \frac{\ell^2}{2}\iint_{\mathcal{O}\times(0,T)} \chi_{\mathcal{O}}|v|^2 dxdt. \tag{7}$$

We have the following result.

Theorem 1.3. *Assume that* $\omega\cap\mathcal{O}_d \neq \emptyset$*. Then, for every* $T>0$ *and* $\mathcal{O},\omega\subset\Omega$ *open subsets such that* $\mathcal{O}\cap\omega=\emptyset$*, there exist* $\ell_0,\delta>0$ *and a positive function* $\rho=\rho(t)$ *blowing up* $t=T$ *such that for any* $\ell\geq\ell_0$, $y_0\in V$ *and* $y_d\in L^2(0,T;L^2(\mathcal{O}_d)^N)$ *satisfying*

$$\|y_0\|_V \leq \delta \quad \textit{and} \quad \iint_{\mathcal{O}_d\times(0,T)} \rho^2(t)|y_d|^2 dxdt < +\infty, \tag{8}$$

we can find a leader control $h\in L^2(0,T;L^2(\omega)^N)$ *and an unique follower control* $\overline{v}$ *on* $L^2(0,T;L^2(\mathcal{O})^N)$ *minimizing* (7) *and an associated solution* (y,p) *to* (6) *verifying* $y(\cdot,T)=0$ *in* Ω.

In order to prove Theorem 1.2, we shall mainly consider two steps: a) the robust control results established in [6] allow us to solve the mentioned-above *first problem*. Here, as consequence of the nonlinearity given by the convection term, constrains either over small data or small time are necessary in order to obtain the robust control; b) The hierarchical control (*second problem*), where the main tools will be news Carleman estimates and fixed point arguments for solving the local null controllability associated to the leader control.

The rest of the paper is organized as follows. In Sect. 2, we present the general scheme of the robust control problem for the system (1). In the first subsection we present the existence and characterization of the robust control for the linearized system (Stokes equation) and in the second subsection the same result for the nonlinear case. In Sect. 3, we solve the robust Stackelberg strategy for the Stokes case. That is, we prove the null controllability for the coupled Stokes system that arises as characterization of the robust control problem. In Sect. 4, we end the proof of Theorem 1.2 throughout an inverse function theorem of the Lyusternik's kind.

2. The robust control problem

As mentioned in the previous section, the main objetive in robust control is to determine the best control function $v\in L^2(0,T;L^2(\mathcal{O})^N)$ in the presence of the disturbance $\psi \in L^2(Q)^N$ which maximally spoils the control. In this section we present some lemmas on the existence, uniqueness and characterisation of a solution to the robust control problem established in Definition 1.1.

The proof of the existence of a solution $(\overline{\psi}, \overline{v})$ to the robust control problem is based on the following result. The interested reader can see [10] for more details.

Lemma 2.1. *Let $\mathcal{J}$ be a functional defined on $X \times Y$, where X and Y are non-empty. closed, unbounded convex sets. If $\mathcal{J}$ satisfies*

(a) $\forall \psi \in X$, $v \longmapsto \mathcal{J}(\psi, v)$ *is convex lower semicontinuous.*
(b) $\forall v \in Y$, $\psi \longmapsto \mathcal{J}(\psi, v)$ *is concave upper semicontinuous.*
(c) $\exists \psi_0 \in X$ *such that* $\lim_{\|v\|_Y \to \infty} \mathcal{J}(\psi_0, v) = +\infty$.
(d) $\exists v_0 \in Y$ *such that* $\lim_{\|\psi\|_X \to \infty} \mathcal{J}(\psi, v_0) = -\infty$.

Then the functional $\mathcal{J}$ has a least one saddle point $(\overline{\psi}, \overline{v})$ and

$$\mathcal{J}(\overline{\psi}, \overline{v}) = \min_{v \in Y} \sup_{\psi \in X} \mathcal{J}(\psi, v) = \max_{\psi \in X} \inf_{v \in Y} \mathcal{J}(\psi, v).$$

2.1. Linear problem

In this section we will treat the corresponding robust Stackelberg strategy for the linearized system. That is we will consider the Stokes system

$$\begin{cases} y_t - \Delta y + \nabla p = h 1_\omega + v \chi_{\mathcal{O}} + \psi & \text{in } \quad Q, \\ \nabla \cdot y = 0 & \text{in } \quad Q, \\ y = 0 & \text{on } \quad \Sigma, \\ y(\cdot, 0) = y_0(\cdot) & \text{in } \quad \Omega. \end{cases} \tag{9}$$

We have the following result:

Lemma 2.2. *Let $h \in L^2(0, T; L^2(\omega)^N)$ be fixed. There exists $\gamma_0 > 0$ such that for every $\gamma > \gamma_0$, there exists a saddle point $(\overline{\psi}, \overline{v})$ and the corresponding solution $y(h, \overline{\psi}, \overline{v})$ of (9) such that*

$$J_r(\psi, \overline{v}; h) \leq J_r(\overline{\psi}, \overline{v}; h) \leq J_r(\overline{\psi}, v; h), \quad \forall (\psi, v) \in L^2(Q)^N \times L^2(0, T; L^2(\mathcal{O})^N).$$

The proof of Lemma 2.2 follows as in [6] where the authors used Lemma 2.1 with $X = Y = L^2(Q)^N$ to prove the existence of a saddle point for a slightly different cost functional $\mathcal{J}$. As consequence of this result, the existence of a solution $(\overline{\psi}, \overline{v})$ to our robust control problem is guaranteed.

Remark 1. In Lemma 2.2, if the condition on γ is not met, we cannot prove the existence of the saddle point. On the other hand, it is known that the existence of a saddle point for the functional J_r implies that for any $\psi \in L^2(Q)^N$, $v \in L^2(0, T; L^2(\mathcal{O})^N)$

$$\frac{\partial J_r}{\partial \psi}(\overline{\psi}, \overline{v}) \cdot \psi = 0, \quad \frac{\partial J_r}{\partial v}(\overline{\psi}, \overline{v}) \cdot v = 0,$$

where

$$\frac{\partial J_r}{\partial \psi}(\overline{\psi}, \overline{v}) \cdot \psi = \iint_{\mathcal{O}_d \times (0,T)} (y - y_d) w_\psi dxdt - \gamma^2 \iint_{\mathcal{O} \times (0,T)} \psi \overline{\psi} dxdt$$

and

$$\frac{\partial J_r}{\partial v}(\overline{\psi},\overline{v})\cdot v = \iint_{\mathcal{O}_d\times(0,T)}(y-y_d)w_v dxdt + \ell^2\iint_{\mathcal{O}\times(0,T)}\chi_{\mathcal{O}} v\overline{v}dxdt,$$

and w_ψ, w_v are the Gâteaux derivatives of y solution to (9) in the directions ψ and v respectively.

Finally, in order to characterize the robust control problem, we introduce the linear adjoint system to (9) with right-hand side related with J_r, that is, we consider

$$\begin{cases} -z_t - \Delta z + \nabla\pi_z = \mu(y-y_d)\chi_{\mathcal{O}_d} & \text{in} \quad Q, \\ \nabla\cdot z = 0 & \text{in} \quad Q, \\ z = 0 & \text{on} \quad \Sigma, \\ z(\cdot,T) = 0 & \text{in} \quad \Omega. \end{cases}$$

In the following result we characterize the saddle point $(\overline{v},\overline{\psi})$ in terms of z. The interested reader can consult [4] for more details.

Lemma 2.3. *Let $h\in L^2(0,T;L^2(\omega)^N)$ and $y_0\in V$ be given. Suppose that $(\overline{\psi},\overline{v})$ is the solution to the robust control problem stated in Definition 1.1. Then*

$$\overline{\psi} = \frac{1}{\gamma^2}z \quad \textit{and} \quad \overline{v} = -\frac{1}{\ell^2}z\chi_{\mathcal{O}},$$

where γ is sufficiently large and the pair (y,z) solves the following coupled system:

$$\begin{cases} y_t - \Delta y + \nabla\pi_y = h1_\omega + (-\ell^{-2}\chi_{\mathcal{O}} + \gamma^{-2})z & \textit{in} \quad Q, \\ -z_t - \Delta z + \nabla\pi_z = \mu(y-y_d)\chi_{\mathcal{O}_d} & \textit{in} \quad Q, \\ \nabla\cdot y = 0, \nabla\cdot z = 0 & \textit{in} \quad Q, \\ y = z = 0 & \textit{on} \quad \Sigma, \\ y(\cdot,0) = y_0(\cdot), \quad z(\cdot,T) = 0 & \textit{in} \quad \Omega. \end{cases} \tag{10}$$

2.2. Nonlinear problem

The analysis is similar to the previous one for the linear case. However, it is well known that the theory of the Navier–Stokes equations is complete in two-dimensional spaces, which do not occur in three-dimensional spaces. Roughly speaking, in three dimensions, the existence of a robust control is restricted to cases of either small data or small T. Additionally, the nonlinearity will require new assumptions on the parameter ℓ. Under the constraint of small data, we need to impose the following condition: there exists $\delta>0$ such that, for every $(v\chi_{\mathcal{O}},\psi)\in L^2(Q)^{N\times N}$ and $y_0\in V$

$$\|v\chi_{\mathcal{O}}\|_{L^2(Q)^N} + \|\psi\|_{L^2(Q)^N} \le \delta \quad \text{and} \quad \|y_0\|_V \le \delta \tag{11}$$

holds.

Lemma 2.4. *Let $h\in L^2(0,T;L^2(\omega)^N)$ be fixed.*

(i) *Case $N=2$. There exist constants $\gamma_0>0$ and $\ell_0>0$ such that for every $\gamma>\gamma_0$ and $\ell>\ell_0$, there exists $(\overline{\psi},\overline{v})$ on $L^2(Q)^N\times L^2(0,T;L^2(\mathcal{O})^N)$ and the associated solution to (1) $y=y(h,\overline{v},\overline{\psi})$ such that*

$$J_r(\psi,\overline{v};h)\le J_r(\overline{\psi},\overline{v};h)\le J_r(\overline{\psi},v;h), \quad \forall\,(\psi,v)\in L^2(Q)^N\times L^2(0,T;L^2(\mathcal{O})^N).$$

That is, $(\overline{\psi}, \overline{v})$ *is a saddle point of* J_r*.*

(ii) *Case* $N = 3$*. Under the hypothesis of the case* $N = 2$*, and that either* $y_0 \in V$ *and* $(v\chi_{\mathcal{O}}, \psi) \in L^2(Q)^{N\times N}$ *satisfies* (11)*, or that* $t = T$ *is small, then there exists* $(\overline{\psi}, \overline{v}) \in L^2(Q)^N \times L^2(0,T; L^2(\mathcal{O})^N)$ *a saddle point of* J_r*.*

Analogously to the linear case, we give the characterization of the robust control problem in the following result.

Lemma 2.5. *Let* $h \in L^2(0,T; L^2(\omega)^N)$ *and* $y_0 \in V$ *be given. Then, there exist positive constants* γ_0, ℓ_0 *such that if* $\gamma > \gamma_0$, $\ell > \ell_0$*, the solution* $(\overline{v}, \overline{\psi})$ *to the robust control problem stated in Definition* 1.1 *exists and is unique. Furthermore,* $(\overline{v}, \overline{\psi})$ *is characterized by*

$$\overline{\psi} = \frac{1}{\gamma^2} z \quad \text{and} \quad \overline{v} = -\frac{1}{\ell^2} z \chi_{\mathcal{O}},$$

where z *is the second component of* (y, z) *solution to the following coupled system:*

$$\begin{cases} y_t - \Delta y + (y\cdot\nabla)y + \nabla\pi_y = h1_\omega + (-\ell^{-2}\chi_{\mathcal{O}} + \gamma^{-2})z & \text{in } Q, \\ -z_t - \Delta z + (z\cdot\nabla^t)y - (y\cdot\nabla)z + \nabla\pi_z = \mu(y - y_d)\chi_{\mathcal{O}_d} & \text{in } Q, \\ \nabla\cdot y = 0, \nabla\cdot z = 0 & \text{in } Q, \\ y = z = 0 & \text{on } \Sigma, \\ y(\cdot,0) = y_0(\cdot), \quad z(\cdot,T) = 0 & \text{in } \Omega. \end{cases} \tag{12}$$

The proof of Lemmas 2.4 and 2.5 can be found in [6].

3. Controllability

In the previous sections we saw that the robust control is characterized in such a way that a coupled system needs to be solved. In order to establish a Stackelberg strategy requiring the leader control to drive the equation to zero we need to find $h \in L^2(0,T; L^2(\omega)^N)$ such that the corresponding y solution to (10) (in the linear case) or to (12) (in the nonlinear case), satisfies $y(T) = 0$. To achieve this objectives, we will obtain first the result in the linear case. To this aim we will prove an observability inequality for the adjoint system to (10) by means of Carleman estimates. The nonlinear case will be obtained by a fixed point argument. The next subsection will be devoted to the obtention of the Carleman inequalities.

3.1. Carleman inequalities

We first define several weight functions which will be useful in the sequel. Let ω_0 be a nonempty open subset of $\mathbb{R}^N$ such that $\omega_0 \subset\subset \omega \cap \mathcal{O}_d$ and $\eta \in C^2(\overline{\Omega})$ such that

$$|\nabla\eta| > 0 \text{ in } \overline{\Omega}\setminus\omega_0, \quad \eta > 0 \text{ in } \Omega \quad \text{and} \quad \eta \equiv 0 \quad \text{on } \partial\Omega.$$

The existence of such a function η is proved in [13]. Then, for some positive real number λ, we consider the following weight functions:

$$\begin{aligned} &\alpha(x,t) = \frac{e^{12\lambda\|\eta\|_\infty} - e^{\lambda(10\|\eta\|_\infty+\eta(x))}}{(t(T-t))^5}, \quad \xi(x,t) = \frac{e^{\lambda(10\|\eta\|_\infty+\eta(x))}}{(t(T-t))^5},\\ &\alpha^*(t) = \max_{x\in\overline{\Omega}} \alpha(x,t), \quad \xi^*(t) = \min_{x\in\overline{\Omega}} \xi(x,t), \\ &\widehat{\alpha}(t) = \min_{x\in\overline{\Omega}} \alpha(x,t), \quad \widehat{\xi}(t) = \max_{x\in\overline{\Omega}} \xi(x,t). \end{aligned} \tag{13}$$

These weight functions have been used by Gueye [16] and Guerrero [15] to obtain Carleman estimates for a Stokes coupled system similar to the presented in our work.

We consider now the non homogeneous adjoint system to (10):

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi_\varphi = g_1 + \mu\theta\chi_{\mathcal{O}_d} & \text{in } \ Q,\\ \theta_t - \Delta\theta + \nabla\pi_\theta = g_2 - \ell^{-2}\varphi\chi_{\mathcal{O}} + \gamma^{-2}\varphi & \text{in } \ Q,\\ \nabla\cdot\varphi = 0,\ \nabla\cdot\theta = 0 & \text{in } \ Q,\\ \varphi = \theta = 0 & \text{on } \ \Sigma,\\ \varphi(\cdot,T) = \varphi_T(\cdot),\ \theta(\cdot,0) = 0 & \text{in } \ \Omega, \end{cases} \tag{14}$$

where $g_1, g_2 \in L^2(Q)^N$ and $\varphi_T \in H$.

Our Carleman estimate is given in the following proposition. In what follows, the constants a_0 and m_0 are fixed, and satisfy

$$\frac{5}{4} \le a_0 < a_0 + 1 < m_0 < 2a_0, \quad m_0 < 2 + a_0. \tag{15}$$

Proposition 1. *Assume that $\omega \cap \mathcal{O}_d \neq \emptyset$ and that ℓ and γ are large enough. Then, there exist a constant $\overline{\lambda} = \overline{\lambda}(\Omega,\omega,\mathcal{O}_d)$ such that for any $\lambda \ge \overline{\lambda}$ there exist two constants $\overline{s}(\lambda) > 0$ and $C = C(\lambda) > 0$ depending only on Ω and ω such that for any $g_1, g_2 \in L^2(Q)^N$ and any $\varphi_T \in H$, the solution of (14) satisfies*

$$\begin{aligned} &\iint_Q e^{-2s\alpha-2a_0s\alpha^*}(s\lambda^2\xi|\nabla(\nabla\times\theta)|^2 + s^3\lambda^4\xi^3|\nabla\times\theta|^2)dxdt \\ &\quad + \iint_Q e^{-2sm_0\alpha}(s\lambda^2\xi|\nabla\varphi|^2 + s^3\lambda^4\xi^3|\varphi|^2 + (s\xi)^{-1}|\Delta\varphi|^2)dxdt \\ &\le C\left(s^{15}\lambda^{24}\iint_{\omega\times(0,T)} e^{-4a_0s\alpha^*+2(m_0-2)s\alpha^*}(\hat{\xi})^{15}|\varphi|^2dxdt \right.\\ &\quad \left. + s^5\lambda^6\iint_Q e^{-2s\hat{\alpha}-2a_0s\alpha^*}(\hat{\xi})^5|g_1|^2dxdt + \iint_Q e^{-2a_0s\alpha^*}|g_2|^2dxdt\right), \end{aligned} \tag{16}$$

for any $s \ge \overline{s}$.

Before giving the proof of Proposition 1, we recall some technical results. We first present a Carleman inequality proved in [12] for a general heat equation with Fourier boundary conditions. Let us introduce the system

$$\begin{cases} -u_t - \Delta u = f_1 + \nabla\cdot f_2 & \text{in } \ Q,\\ (\nabla u + f_2)\cdot n = f_3 & \text{on } \ \Sigma,\\ u(\cdot,T) = u_T(\cdot) & \text{in } \ \Omega, \end{cases} \tag{17}$$

where $f_1 \in L^2(Q)$, $f_2 \in L^2(Q)^N$ and $f_3 \in L^2(\Sigma)$. We have:

Lemma 3.1. *Under the previous assumptions on f_1, f_2 and f_3, there exist positive constants $\overline{\lambda}, \sigma_1, \sigma_2$ and C, only depending on Ω and ω, such that, for any $\lambda \geq \overline{\lambda}$, any $s \geq \overline{s} = \sigma_1(e^{\sigma_2 \lambda} T + T^2)$ and any $u_T \in L^2(\Omega)$, the weak solution to* (17) *satisfies*

$$\begin{aligned}
&\iint_Q e^{-2s\alpha}[s^3\lambda^4\xi^3|u|^2 + s\lambda^2\xi|\nabla u|^2]dxdt \\
&\quad \leq C\left(\iint_Q e^{-2s\alpha}(|f_1|^2 + s^2\lambda^2\xi^2|f_2|^2)dxdt \right. \\
&\qquad \left. + s\lambda \iint_\Sigma e^{-2s\alpha}\xi|f_3|^2 d\sigma dt + s^3\lambda^4 \iint_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^3|u|^2 dxdt\right).
\end{aligned} \tag{18}$$

The second result holds for the solutions of a Stokes system with Dirichlet boundary conditions. The interested reader can see [11] for more details.

Lemma 3.2. *Let $u_0 \in V$ and $f_4 \in L^2(Q)^N$. Then, there exists a constant $C(\Omega, \omega, T) > 0$ such that the solution $u \in L^2(0,T; H^2(\Omega)^N \cap V) \cap L^\infty(0,T;V)$, $p \in L^2(0,T;H^1(\Omega))$, with $\int_{\omega_0} p(x,t)dx = 0$, of*

$$\begin{cases}
u_t - \Delta u + \nabla p = f_4 & in \quad Q, \\
\nabla \cdot u = 0 & in \quad Q, \\
u = 0 & on \quad \Sigma, \\
u(\cdot, 0) = u_0(\cdot) & in \quad \Omega,
\end{cases}$$

satisfies

$$\begin{aligned}
&\iint_Q e^{-2s\alpha}(s\lambda^2\xi|\nabla u|^2 + s^3\lambda^4\xi^3|u|^2)dxdt \\
&\quad \leq C\left(s^{16}\lambda^{40}\iint_{\omega\times(0,T)} e^{-8s\hat{\alpha}+6s\alpha^*}(\hat{\xi})^{16}|u|^2 dxdt \right. \\
&\qquad \left. + s^{15/2}\lambda^{20}\iint_Q e^{-4s\hat{\alpha}+2s\alpha^*}(\hat{\xi})^{15/2}|f_4|^2 dxdt\right),
\end{aligned} \tag{19}$$

for any $\lambda \geq C$ and $s \geq C(T^5 + T^{10})$.

Remark 2. In [11,12] slightly different weight functions are used to prove the above results. However, the inequality remains valid since the key point of the proof is that α goes to 0 when t tends to 0 and T.

The next result concerns the regularity of the solutions to the Stokes system, see [15,23] for more details.

Lemma 3.3. *Let $a \in \mathbb{R}$ and $B \in \mathbb{R}^N$ be constant and let $f_5 \in L^2(0,T;V)$. Then, there exists a unique solution*

$$u \in L^2(0,T;H^3(\Omega)^N \cap V) \cap H^1(0,T;V)$$

for the Stokes system

$$\begin{cases} u_t - \Delta u + au + B\cdot\nabla u + \nabla p = f_5 & in \quad Q, \\ \nabla\cdot u = 0 & in \quad Q, \\ u = 0 & on \quad \Sigma, \\ u(\cdot,0) = 0 & in \quad \Omega, \end{cases} \tag{20}$$

for some $p \in L^2(0,T;H^2(\Omega))$, *and there exists a constant* $C > 0$ *such that*

$$\|u\|_{L^2(0,T;H^3(\Omega)^N)} + \|u\|_{H^1(0,T;L^2(\Omega)^N)} \le C\|f_5\|_{L^2(0,T;H^1(\Omega)^N)}. \tag{21}$$

Moreover, if we assume that $a \equiv B \equiv 0$ *and* $f_5 \in L^2(Q)^N$, u *is actually, together a pressure* p, *the strong solution of* (20), *i.e.*,

$$(u,p) \in L^2(0,T;H^2(\Omega)^N) \cap L^\infty(0,T;V) \cap H^1(0,T;H) \times L^2(0,T;H^1(\Omega)).$$

Furthermore, there exists a constant $C > 0$ *such that*

$$\|u\|_{L^2(0,T;H^2(\Omega)^N)} + \|u\|_{L^\infty(0,T;V)} + \|u\|_{H^1(0,T;L^2(\Omega)^N)} \le C\|f_5\|_{L^2(Q)^N}. \tag{22}$$

Now, we give the proof of Proposition 1.

3.2. Proof of Proposition 1

Proof. **Carleman estimate for** θ

Let define $\theta^* := \rho^*\theta$, $\quad \pi^* := \rho^*\pi$, where $\rho^* = \rho^*(t) = e^{-a_0 s\alpha^*}$ and a_0 fixed satisfying (15). From (14), (θ^*,π^*) is the solution of the following system

$$\begin{cases} \theta^*_t - \Delta\theta^* + \nabla\pi^* = \rho^* g_2 + \rho^*(-\ell^{-2}\varphi\chi_{\mathcal{O}} + \gamma^{-2}\varphi) + \rho^*_t\theta & \text{in} \quad Q, \\ \nabla\cdot\theta^* = 0 & \text{in} \quad Q, \\ \theta^* = 0 & \text{on} \quad \Sigma, \\ \theta^*(\cdot,0) = 0 & \text{in} \quad \Omega. \end{cases}$$

Now, we decompose (θ^*,π^*) as follows:

$$(\theta^*,\pi^*) = (\hat\theta,\hat\pi) + (\tilde\theta,\tilde\pi), \tag{23}$$

where $(\hat\theta,\hat\pi)$ and $(\tilde\theta,\tilde\pi)$ solve respectively

$$\begin{cases} \tilde\theta_t - \Delta\tilde\theta + \nabla\tilde\pi = \rho^* g_2 + \rho^*(-\ell^{-2}\varphi\chi_{\mathcal{O}} + \gamma^{-2}\varphi) & \text{in} \quad Q, \\ \nabla\cdot\tilde\theta = 0 & \text{in} \quad Q, \\ \tilde\theta = 0 & \text{on} \quad \Sigma, \\ \tilde\theta(\cdot,0) = 0 & \text{in} \quad \Omega, \end{cases} \tag{24}$$

and

$$\begin{cases} \hat\theta_t - \Delta\hat\theta + \nabla\hat\pi = \rho^*_t\theta & \text{in} \quad Q, \\ \nabla\cdot\hat\theta = 0 & \text{in} \quad Q, \\ \hat\theta = 0 & \text{on} \quad \Sigma, \\ \hat\theta(\cdot,0) = 0 & \text{in} \quad \Omega. \end{cases} \tag{25}$$

For system (24) we will use Lemma 3.3 and the regularity result estimate (22), meanwhile for the system (25) we will use the ideas of both works [15,16].

We apply the operator $\nabla\times\cdot$ to the Stokes system satisfied by $\hat\theta$. Then, we have

$$(\nabla\times\hat\theta)_t - \Delta(\nabla\times\hat\theta) = \rho^*_t(\nabla\times\theta) \quad \text{in } Q.$$

Using Lemma 3.1 with $f_1 = \rho_t^*(\nabla \times \theta)$, there exists a positive constant $C = C(\Omega, \omega_0)$ such that

$$\begin{aligned} &\iint_Q e^{-2s\alpha}(s\lambda^2\xi|\nabla(\nabla\times\hat{\theta})|^2 + s^3\lambda^4\xi^3|\nabla\times\hat{\theta}|^2)dxdt \\ &\le C\left(\iint_Q e^{-2s\alpha}|\rho_t^*|^2|\nabla\times\theta|^2dxdt\right. \\ &\left.+ s\lambda\iint_\Sigma e^{-2s\alpha}\xi\left|\frac{\partial(\nabla\times\hat{\theta})}{\partial n}\right|^2 d\sigma dt + s^3\lambda^4\iint_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^3|\nabla\times\hat{\theta}|^2dxdt\right), \end{aligned} \tag{26}$$

for any $\lambda_1 := \lambda \ge C$ and $s \ge C(T^{10} + T^9)$.

Now, using the inequality $(a-b)^2 \ge \frac{a^2}{2} - b^2$, for every $a, b \in \mathbb{R}$ with $a = \theta^*$ and $b = \tilde{\theta}$, we get (recall that $\hat{\theta} = \theta^* - \tilde{\theta}$):

$$\begin{aligned} &\frac{1}{2}\iint_Q e^{-2s\alpha-2a_0s\alpha^*}(s\lambda^2\xi|\nabla(\nabla\times\theta)|^2 + s^3\lambda^4\xi^3|\nabla\times\theta|^2)dxdt \\ &\qquad - \iint_Q e^{-2s\alpha}(s\lambda^2\xi|\nabla(\nabla\times\tilde{\theta})|^2 + s^3\lambda^4\xi^3|\nabla\times\tilde{\theta}|^2)dxdt \\ &\le \iint_Q e^{-2s\alpha}(s\lambda^2\xi|\nabla(\nabla\times\hat{\theta})|^2 + s^3\lambda^4\xi^3|\nabla\times\hat{\theta}|^2)dxdt. \end{aligned} \tag{27}$$

The fact that $s^3\lambda^4e^{-2s\alpha}\xi^3$ and $s\lambda^2e^{-2s\alpha}\xi$ are upper bounded allow us to estimate the terms associated to $|\nabla(\nabla\times\tilde{\theta})|^2$ and $|\nabla\times\tilde{\theta}|^2$ through (22). More precisely, we have:

$$\begin{aligned} &s^3\lambda^4\iint_Q e^{-2s\alpha}\xi^3|\nabla\times\tilde{\theta}|^2dxdt + s\lambda^2\iint_Q e^{-2s\alpha}\xi|\nabla(\nabla\times\tilde{\theta})|^2dxdt \\ &\quad\le C_{s,\lambda}\|\tilde{\theta}\|^2_{L^2(0,T;H^1(\Omega)^N)\cap L^2(0,T;H^2(\Omega)^N)} \\ &\quad\le C_{s,\lambda}\|\rho^*g_2\|^2_{L^2(Q)^N} + C_{s,\lambda}\|\rho^*(-\ell^{-2}\varphi\chi_{\mathcal{O}} + \gamma^{-2}\varphi)\|^2_{L^2(Q)^N}, \end{aligned} \tag{28}$$

where $C_{s,\lambda}$ is a positive constant depending on s and λ. i.e., $C_{s,\lambda} = Cs^3\lambda^4$.

On the other hand, taking into account that $|\rho_t^*| \le CsT\rho^*(\xi^*)^{6/5}$ for every $s \ge C$, it follows that

$$\iint_Q e^{-2s\alpha}|\rho_t^*|^2|\nabla\times\theta|^2dxdt \le Cs^2T^2\iint_Q e^{-2s\alpha-2a_0s\alpha^*}(\xi^*)^{12/5}|\nabla\times\theta|^2dxdt,$$

which can be absorbed by the first term in the right-hand side of (27), for every $\lambda \ge 1$, $s \ge C$.

From the identity $\theta^* = \hat{\theta} + \tilde{\theta}$ (recall (23)) and (28), it is easy to estimate the local term that appear in the right-hand side of (26) by:

$$\begin{aligned} & s^3\lambda^4 \iint_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^3|\nabla\times\hat{\theta}|^2 dxdt \\ & \leq Cs^3\lambda^4 \iint_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^3(|\nabla\times\theta^*|^2 + |\nabla\times\tilde{\theta}|^2)dxdt \\ & \leq Cs^3\lambda^4 \iint_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^3|\nabla\times\theta^*|^2 dxdt + C_{s,\lambda}\|\rho^* g_2\|^2_{L^2(Q)^N} \\ & \qquad + C_{s,\lambda}\|\rho^*(-\ell^{-2}\varphi\chi_{\mathcal{O}} + \gamma^{-2}\varphi)\|^2_{L^2(Q)^N}. \end{aligned} \tag{29}$$

Putting together (26), (27) and (29), we have for the moment

$$\begin{aligned} & \iint_Q e^{-2s\alpha-2a_0 s\alpha^*}\left(s\lambda^2\xi|\nabla(\nabla\times\theta)|^2 + s^3\lambda^4\xi^3|\nabla\times\theta|^2\right)dxdt \\ & \leq C\left(s^3\lambda^4 \iint_{\omega_0\times(0,T)} e^{-2s\alpha-2a_0 s\alpha^*}\xi^3|\nabla\times\theta|^2 dxdt \right. \\ & \qquad \left. + s\lambda \iint_\Sigma e^{-2s\alpha}\xi\left|\frac{\partial(\nabla\times\hat{\theta})}{\partial n}\right|^2 d\sigma dt\right) \\ & \qquad + C_{s,\lambda}\|\rho^* g_2\|^2_{L^2(Q)^N} + C_{s,\lambda}\|\rho^*(-\ell^{-2}\varphi\chi_{\mathcal{O}} + \gamma^{-2}\varphi)\|^2_{L^2(Q)^N}, \end{aligned} \tag{30}$$

for every $s \geq C$ and $\lambda_1 := \lambda \geq C$.

The last step will be to estimate the boundary term

$$s\lambda \iint_\Sigma e^{-2s\alpha}\xi\left|\frac{\partial(\nabla\times\hat{\theta})}{\partial n}\right|^2 d\sigma dt.$$

To this end we follow the arguments of [16].

We consider $\zeta \in C^2(\overline{\Omega})$ such that

$$\frac{\partial\zeta}{\partial n} = 1, \quad \zeta = constant \quad \text{on } \partial\Omega.$$

Observe that

$$\iint_\Sigma e^{-2s\alpha}\xi\left|\frac{\partial(\nabla\times\hat{\theta})}{\partial n}\right|^2 d\sigma dt = \iint_\Sigma e^{-2s\alpha}\zeta\xi\left|\frac{\partial(\nabla\times\hat{\theta})}{\partial n}\right|^2 d\sigma dt.$$

Through integrating by parts and using Cauchy–Schwartz's inequality, the previous boundary term can be estimated by

$$\begin{aligned} I_\Sigma := {} & s\lambda \iint_\Sigma e^{-2s\alpha}\xi\left|\frac{\partial(\nabla\times\hat{\theta})}{\partial n}\right|^2 d\sigma dt \\ & \leq Cs\lambda \int_0^T e^{-2s\alpha^*}\xi^*\|\hat{\theta}\|_{H^2(\Omega)^N}\|\hat{\theta}\|_{H^3(\Omega)^N} dt. \end{aligned} \tag{31}$$

Additionally, taking into account that $H^2(\Omega)^N = (H^1(\Omega)^N, H^3(\Omega)^N)_{1/2,2}$, (31) can be replaced by

$$\begin{aligned} I_\Sigma &:= s\lambda \iint_\Sigma e^{-2s\alpha}\xi \left| \frac{\partial(\nabla\times\hat{\theta})}{\partial n}\right|^2 d\sigma dt \\ &\leq Cs\lambda \int_0^T e^{-2s\alpha^*}\xi^* \|\hat{\theta}\|^{1/2}_{H^1(\Omega)^N} \|\hat{\theta}\|^{3/2}_{H^3(\Omega)^N} dt. \end{aligned} \tag{32}$$

From (31), (32) and Young's inequality we have

$$I_\Sigma \leq C\int_0^T e^{-2s\alpha^*}\lambda \left(s^{5/2}(\xi^*)^{5/2}\|\hat{\theta}\|^2_{H^1(\Omega)^N} + s^{1/2}(\xi^*)^{1/2}\|\hat{\theta}\|^2_{H^3(\Omega)^N}\right) dt. \tag{33}$$

Using the divergence free condition $\nabla\cdot\hat{\theta} = 0$, the Dirichlet condition boundary for $\hat{\theta}$ and also the relationship

$$\hat{\theta} = \theta^* - \tilde{\theta},$$

we obtain

$$\begin{aligned} \|\hat{\theta}\|_{H^1(\Omega)^N} &\leq C\|\nabla\times\hat{\theta}\|_{L^2(\Omega)^{2N-3}} \\ &\leq C(\|\nabla\times\tilde{\theta}\|_{L^2(\Omega)^{2N-3}} + \|\nabla\times\theta^*\|_{L^2(\Omega)^{2N-3}}). \end{aligned} \tag{34}$$

Observe that the first term in the right-hand side of (34) can be estimated like in (28). Respect to the second term, it can be absorbed by the left-hand side of (30). Let us estimate the second term in (33). This is done by a bootstrap argument based on a regularity result of the Stokes system. Let $(\Theta, \pi_\Theta) = (\eta(t)\hat{\theta}, \eta(t)\pi)$, where

$$\eta(t) = s^{1/4}\lambda^{1/2}e^{-s\alpha^*}(\xi^*)^{1/4} \quad \text{in } (0,T).$$

Thus, (Θ, π_Θ) satisfies the Stokes system

$$\begin{cases} \Theta_t - \Delta\Theta + \nabla\pi_\Theta = \eta\rho_t^*\theta + \eta_t\hat{\theta} & \text{in} \quad Q, \\ \nabla\cdot\Theta = 0 & \text{in} \quad Q, \\ \Theta = 0 & \text{on} \quad \Sigma, \\ \Theta(\cdot,0) = 0 & \text{in} \quad \Omega. \end{cases} \tag{35}$$

It is easy to prove the right-hand side of this system belongs to $L^2(0,T;V)$. Therefore, Lemma 3.3 allows us to conclude that the solution of (35) satisfies $\Theta \in L^2(0,T;H^3(\Omega)^N \cap V)$. Furthermore,

$$\|\Theta\|_{L^2(0,T;H^3(\Omega)^N)} + \|\Theta\|_{H^1(0,T;L^2(\Omega)^N)} \leq C\|\eta\rho_t^*\theta + \eta_t\hat{\theta}\|_{L^2(0,T;H^1(\Omega)^N)}. \tag{36}$$

Putting together (33)–(36), there exist $C_{s,\lambda} > 0$ such that

$$\begin{aligned} I_\Sigma &\leq C_{s,\lambda}\|\rho^* g_2\|^2_{L^2(Q)^N} + C_{s,\lambda}\|\rho^*(-\ell^{-2}\varphi\chi_{\mathcal{O}} + \gamma^{-2}\varphi)\|^2_{L^2(Q)^N} \\ &\quad + \varepsilon\left(\iint_Q e^{-2s\alpha-2a_0 s\alpha^*}(s\lambda^2\xi|\nabla(\nabla\times\theta)|^2 + s^3\lambda^4\xi^3|\nabla\times\theta|^2)dxdt\right). \end{aligned}$$

for every $\varepsilon > 0$.

From the previous inequality and (30) we conclude the following Carleman estimate for θ:

$$\begin{aligned}&\iint_Q e^{-2s\alpha-2a_0s\alpha^*}(s\lambda^2\xi|\nabla(\nabla\times\theta)|^2+s^3\lambda^4\xi^3|\nabla\times\theta|^2)dxdt\\&\quad\le C_{s,\lambda}\|\rho^*g_2\|^2_{L^2(Q)^N}+Cs^3\lambda^4\iint_{\omega_0\times(0,T)}e^{-2s\alpha-2a_0s\alpha^*}\xi^3|\nabla\times\theta|^2dxdt\\&\qquad+C_{s,\lambda}\|\rho^*(-\ell^{-2}\varphi\chi_{\mathcal{O}}+\gamma^{-2}\varphi)\|^2_{L^2(Q)^N},\end{aligned}\tag{37}$$

for every $s\ge C(T^5+T^{10})$ and $\lambda_1:=\lambda\ge C$.

Carleman estimate for φ

First, assuming that θ is given, we look at φ as the solution of

$$\begin{cases}-\varphi_t-\Delta\varphi+\nabla\pi_\varphi=g_1+\mu\theta\chi_{\mathcal{O}_d}\text{ in }Q,\\\nabla\cdot\varphi=0\text{ in }Q,\\\varphi=0\text{ on }\Sigma,\\\varphi(\cdot,T)=\varphi_T(\cdot)\text{ in }\Omega.\end{cases}$$

Now, we choose π_φ such that $\int_{\omega_0}\pi_\varphi dx=0$ and we apply Lemma 3.2 with $f_4=g_1+\mu\theta\chi_{\mathcal{O}_d}$ and use the weight function $m_0\alpha$ (instead of α), where $a_0+1<m_0\le 2a_0$ and $m_0\le 2+a_0$. We obtain

$$\begin{aligned}&\iint_Q e^{-2m_0s\alpha}[s^{-1}\xi^{-1}|\Delta\varphi|^2+s\lambda^2\xi|\nabla\varphi|^2+s^3\lambda^4\xi^3|\varphi|^2]dxdt\\&\quad\le C\Bigg(s^{16}\lambda^{40}\iint_{\omega_0\times(0,T)}e^{-8m_0s\hat\alpha+6m_0s\alpha^*}(\hat\xi)^{16}|\varphi|^2dxdt\\&\qquad+s^{15/2}\lambda^{20}\iint_{\mathcal{O}_d\times(0,T)}e^{-4m_0s\hat\alpha+2m_0s\alpha^*}(\hat\xi)^{15/2}|\theta|^2dxdt\\&\qquad+s^{15/2}\lambda^{20}\iint_Q e^{-4m_0s\hat\alpha+2m_0s\alpha^*}(\hat\xi)^{15/2}|g_1|^2dxdt\Bigg),\end{aligned}\tag{38}$$

for any $\lambda_2:=\lambda\ge C$ and $s\ge C(T^5+T^{10})$.

Taking into account that $\|\theta\|_{L^2(\Omega)^N}\le C\|\nabla\times\theta\|_{L^2(\Omega)^{2N-3}}$ and the inequality (66) with $\varepsilon=\frac{m_0-a_0-1}{m_0+a_0+1}$, $M_1=-\frac{15}{4(m_0+a_0+1)}$ and $M_2=-\frac{10}{(m_0+a_0+1)}$, the second term in the right-hand side of (38) can be estimated by

$$\iint_Q e^{-2s\alpha^*-2a_0s\alpha^*}|\nabla\times\theta|^2dxdt$$

and therefore it can be absorbed by the left-hand side of (37).

From (37) and (38) we have

$$\begin{aligned}&\iint_Q e^{-2s\alpha-2a_0s\alpha^*}(s\lambda^2\xi|\nabla(\nabla\times\theta)|^2dxdt+s^3\lambda^4\xi^3|\nabla\times\theta|^2)dxdt\\&\qquad+\iint_Q e^{-2m_0s\alpha}[s^{-1}\xi^{-1}|\Delta\varphi|^2+s\lambda^2\xi|\nabla\varphi|^2+s^3\lambda^4\xi^3|\varphi|^2]dxdt\\&\quad\le Cs^{16}\lambda^{40}\iint_{\omega_0\times(0,T)}e^{-8m_0s\hat\alpha+6m_0s\alpha^*}(\hat\xi)^{16}|\varphi|^2dxdt\end{aligned}$$

$$+C_{s,\lambda}\|\rho^*(-\ell^{-2}\varphi\chi_{\mathcal{O}}+\gamma^{-2}\varphi)\|^2_{L^2(Q)^N}$$
$$+Cs^{15/2}\lambda^{20}\iint_Q e^{-4m_0s\hat{\alpha}+2m_0s\alpha^*}(\hat{\xi})^{15/2}|g_1|^2dxdt+C\|\rho^*g_2\|^2_{L^2(Q)^N}$$
$$+Cs^3\lambda^4\iint_{\omega_0\times(0,T)}e^{-2s\alpha-2a_0s\alpha^*}\xi^3|\nabla\times\theta|^2dxdt, \tag{39}$$

for any $\lambda_3:=\max\{\lambda_1,\lambda_2\}\geq C$, $s\geq C(T^5+T^{10})$ and $C_{s,\lambda}$ depending on s,λ.

Choosing ℓ and γ large enough (i.e., $\ell,\gamma >> C_3T^{30/4}e^{C_4/T^{10}}$, where C_3, C_4 are positive constants depending on a_0, m_0, s.), we can absorb the second term in the right-hand side of (39) by the left-hand side.

Let us estimate the local term concerning $\nabla\times\theta$ in terms of φ. To do this, we use the first equation of (14) since $\omega\cap\mathcal{O}_d\neq\emptyset$ and $\omega_0\subset\mathcal{O}_d$. We have

$$-(\nabla\times\varphi)_t-\Delta(\nabla\times\varphi)=\nabla\times g_1+\mu(\nabla\times\theta),\quad \text{in } \omega_0\times(0,T).$$

Then,

$$\begin{aligned}I:=&\,s^3\lambda^4\iint_{\omega_0\times(0,T)}e^{-2s\alpha-2a_0s\alpha^*}\xi^3|\nabla\times\theta|^2dxdt\\=&\,s^3\lambda^4\iint_{\omega_0\times(0,T)}e^{-2s\alpha-2a_0s\alpha^*}\xi^3(\nabla\times\theta)(-(\nabla\times\varphi)_t-\Delta(\nabla\times\varphi)\\&-(\nabla\times g_1))dxdt.\end{aligned}$$

We introduce an open set $\omega_1\subset\subset\omega$ such that $\omega_0\subset\omega_1$ and a positive function $\zeta\in C^2_c(\omega_1)$ such that $\zeta\equiv1$ in ω_0. Then, after several integration by parts in time and space we have:

$$\begin{aligned}I=&\,s^3\lambda^4\iint_{\omega_0\times(0,T)}e^{-2s\alpha-2a_0s\alpha^*}\xi^3(\nabla\times\theta)(-(\nabla\times\varphi)_t-\Delta(\nabla\times\varphi)\\&-(\nabla\times g_1))dxdt\\\leq&\,s^3\lambda^4\iint_{\omega_1\times(0,T)}\zeta\partial_t(e^{-2s\alpha-2a_0s\alpha^*}\xi^3)(\nabla\times\theta)(\nabla\times\varphi)dxdt\\&+s^3\lambda^4\iint_{\omega_1\times(0,T)}e^{-2s\alpha-2a_0s\alpha^*}\xi^3((\nabla\times\theta)_t-\Delta(\nabla\times\theta))(\nabla\times\varphi)dxdt\\&-s^3\lambda^4\iint_{\omega_1\times(0,T)}\Delta(\zeta e^{-2s\alpha-2a_0s\alpha^*}\xi^3)(\nabla\times\theta)(\nabla\times\varphi)dxdt\\&-2s^3\lambda^4\iint_{\omega_1\times(0,T)}\nabla(\zeta e^{-2s\alpha-2a_0s\alpha^*}\xi^3)(\nabla(\nabla\times\theta))(\nabla\times\varphi)dxdt\\&-s^3\lambda^4\iint_{\omega_1\times(0,T)}\zeta e^{-2s\alpha-2a_0s\alpha^*}\xi^3(\nabla\times\theta)(\nabla\times g_1)dxdt.\end{aligned}$$

From the second equation in (14), we have that

$$
\begin{aligned}
I \leq s^3\lambda^4 &\iint_{\omega_1\times(0,T)} \zeta\partial_t(e^{-2s\alpha-2a_0s\alpha^*}\xi^3)(\nabla\times\theta)(\nabla\times\varphi)dxdt \\
&+ s^3\lambda^4\gamma^{-2}\iint_{\omega_1\times(0,T)} e^{-2s\alpha-2a_0s\alpha^*}\xi^3|\nabla\times\varphi|^2dxdt \\
&- s^3\lambda^4 \iint_{\omega_1\times(0,T)} \Delta(\zeta e^{-2s\alpha-2a_0s\alpha^*}\xi^3)(\nabla\times\theta)(\nabla\times\varphi)dxdt \\
&- 2s^3\lambda^4 \iint_{\omega_1\times(0,T)} \nabla(\zeta e^{-2s\alpha-2a_0s\alpha^*}\xi^3)(\nabla(\nabla\times\theta))(\nabla\times\varphi)dxdt \\
&- s^3\lambda^4 \iint_{\omega_1\times(0,T)} \zeta e^{-2s\alpha-2a_0s\alpha^*}\xi^3(\nabla\times\theta)(\nabla\times g_1)dxdt \\
&- s^3\lambda^4 \iint_{\omega_1\times(0,T)} \zeta e^{-2s\alpha-2a_0s\alpha^*}\xi^3(\nabla\times\varphi)(\nabla\times g_2)dxdt.
\end{aligned}
\tag{40}
$$

Using the estimate

$$
|\partial_t(e^{-2s\alpha-2a_0s\alpha^*}\xi^3)| \leq CTse^{-2s\alpha-2a_0s\alpha^*}(\xi)^{4+1/5}, \quad \text{for every } s\geq C
$$

and Young's inequality, we can deduce the following inequalities:

$$
\begin{aligned}
I_1 &:= s^3\lambda^4\iint_{\omega_1\times(0,T)} \zeta\partial_t(e^{-2s\alpha-2a_0s\alpha^*}\xi^3)(\nabla\times\theta)(\nabla\times\varphi)dxdt \\
&\leq CTs^4\lambda^4\iint_{\omega_1\times(0,T)} \zeta e^{-2s\alpha-2a_0s\alpha^*}\xi^{4+1/5}|\nabla\times\theta||\nabla\times\varphi|dxdt \\
&\leq \varepsilon s^3\iint_{\omega_1\times(0,T)} e^{-2s\alpha-2a_0s\alpha^*}\xi^3|\nabla\times\theta|^2dxdt \\
&\qquad + C(\varepsilon)s^5\lambda^8\iint_{\omega_1\times(0,T)} e^{-2s\alpha-2a_0s\alpha^*}(\xi)^{5+2/5}|\nabla\times\varphi|^2dxdt,
\end{aligned}
$$

for every $s\geq C$ and every $\varepsilon>0$.

Now, using the estimate

$$
|\Delta(\zeta e^{-2s\alpha-2a_0s\alpha^*}\xi^3)| \leq Cs^2\lambda^2e^{-2s\alpha-2a_0s\alpha^*}\xi^5, \quad \text{for every } s\geq C
$$

and again the Young's inequality for the third term in the right-hand side of (40), we obtain

$$
\begin{aligned}
I_3 &:= -s^3\lambda^4\iint_{\omega_1\times(0,T)} \Delta(\zeta e^{-2s\alpha-2a_0s\alpha^*}\xi^3)(\nabla\times\theta)(\nabla\times\varphi)dxdt \\
&\leq Cs^5\lambda^6\iint_{\omega_1\times(0,T)} e^{-2s\alpha-2a_0s\alpha^*}\xi^5|\nabla\times\theta||\nabla\times\varphi|dxdt \\
&\leq \varepsilon s^3\iint_{\omega_1\times(0,T)} e^{-2s\alpha-2a_0s\alpha^*}\xi^3|\nabla\times\theta|^2dxdt \\
&\qquad + C(\varepsilon)s^7\lambda^{12}\iint_{\omega_1\times(0,T)} e^{-2s\alpha-2a_0s\alpha^*}\xi^7|\nabla\times\varphi|^2dxdt,
\end{aligned}
$$

for every $s \geq C$ and every $\varepsilon > 0$.

Analogously, we can estimate the fourth term in the right-hand side of (40) by

$$\begin{aligned} I_4 &:= -2s^3\lambda^4 \iint_{\omega_1\times(0,T)} \nabla(\zeta e^{-2s\alpha-2a_0s\alpha^*}\xi^3)(\nabla(\nabla\times\theta))(\nabla\times\varphi)dxdt \\ &\leq \varepsilon s\lambda^2 \iint_{\omega_1\times(0,T)} e^{-2s\alpha-2a_0s\alpha^*}\xi|\nabla(\nabla\times\theta)|^2dxdt \\ &\qquad + C(\varepsilon)s^7\lambda^8 \iint_{\omega_1\times(0,T)} e^{-2s\alpha-2a_0s\alpha^*}\xi^7|\nabla\times\varphi|^2dxdt, \end{aligned}$$

for every $s \geq C$ and every $\varepsilon > 0$.

Additionally, through another integration by part and Young's inequality we can obtain

$$\begin{aligned} I_5 &:= -s^3\lambda^4 \iint_{\omega_1\times(0,T)} \zeta e^{-2s\alpha-2a_0s\alpha^*}\xi^3(\nabla\times\theta)(\nabla\times g_1)dxdt \\ &\leq \varepsilon\left(s\lambda^2 \iint_{\omega_1\times(0,T)} e^{-2s\alpha-2a_0s\alpha^*}\xi|\nabla(\nabla\times\theta)|^2dxdt\right. \\ &\quad \left. + s^3\lambda^4 \iint_{\omega_1\times(0,T)} e^{-2s\alpha-2a_0s\alpha^*}\xi^3|\nabla\times\theta|^2dxdt\right) \\ &\quad + C(\varepsilon)s^5\lambda^6 \iint_Q e^{-2s\alpha-2a_0s\alpha^*}\xi^5|g_1|^2dxdt, \end{aligned}$$

for every $s \geq C$ and every $\varepsilon > 0$.

$$\begin{aligned} I_6 &:= -s^3\lambda^4 \iint_{\omega_1\times(0,T)} \zeta e^{-2s\alpha-2a_0s\alpha^*}\xi^3(\nabla\times\varphi)(\nabla\times g_2)dxdt \\ &\leq C\left(\|\rho^* g_2\|_{L^2(Q)^N} + s^7\lambda^{12} \iint_{\omega_1\times(0,T)} e^{-2s\alpha-2a_0s\alpha^*}\xi^7|\nabla\varphi|^2dxdt\right. \\ &\qquad \left. + s^6\lambda^8 \iint_{\omega_1\times(0,T)} e^{-4s\alpha-2a_0s\alpha^*}\xi^6|\nabla\times(\nabla\times\varphi)|^2dxdt\right). \end{aligned}$$

We use Lemma 6.2 in the Appendix in order to obtain an appropriate upper bound for the last term in the right-hand side on the previous inequality through the following terms:

$$s^7\lambda^{12} \iint_{\omega_1\times(0,T)} e^{-2s\alpha-2a_0s\alpha^*}\xi^7|\nabla\varphi|^2dxdt \quad \text{and} \quad \varepsilon s^{-1} \iint_Q e^{-2m_0s\alpha}\xi^{-1}|\Delta\varphi|^2dxdt,$$

for every $\varepsilon > 0$.

Putting together (39) and the previous estimates, we have

$$\begin{aligned}
&\iint_Q e^{-2s\alpha-2a_0s\alpha^*}(s\lambda^2\xi|\nabla(\nabla\times\theta)|^2dxdt+s^3\lambda^4\xi^3|\nabla\times\theta|^2)dxdt\\
&\qquad+\iint_Q e^{-2m_0s\alpha}[s^{-1}\xi^{-1}|\Delta\varphi|^2+s\lambda^2\xi|\nabla\varphi|^2+s^3\lambda^4\xi^3|\varphi|^2]dxdt\\
&\le Cs^{16}\lambda^{40}\iint_{\omega_0\times(0,T)} e^{-8m_0s\hat{\alpha}+6m_0s\alpha^*}(\hat{\xi})^{16}|\varphi|^2dxdt+C\iint_Q e^{-2a_0s\alpha^*}|g_2|^2dxdt\\
&\qquad+Cs^{15/2}\lambda^{20}\iint_Q e^{-4m_0s\hat{\alpha}+2m_0s\alpha^*}(\hat{\xi})^{15/2}|g_1|^2dxdt\\
&\qquad+Cs^7\lambda^{12}\iint_{\omega_1\times(0,T)} e^{-2s\alpha-2a_0s\alpha^*}\xi^7|\nabla\times\varphi|^2dxdt,
\end{aligned}\tag{41}$$

for any $\lambda_3\ge C$, $s\ge CT^{10}$.

On the other hand, considering open sets $\omega_2,\omega_3\subset\subset\omega$ such that $\omega_1\subset\subset\omega_2\subset\subset\omega_3\subset\omega$, we can deduce that

$$\begin{aligned}
&s^7\lambda^{12}\iint_{\omega_1\times(0,T)} e^{-2s\alpha-2a_0s\alpha^*}\xi^7|\nabla\times\varphi|^2dxdt\\
&\quad\le C(\varepsilon)s^{15}\lambda^{24}\iint_{\omega_3\times(0,T)} e^{2(m_0-2)s\alpha^*-4a_0s\alpha^*}(\hat{\xi})^{15}|\varphi|^2dxdt\\
&\qquad+\varepsilon\left(\iint_Q e^{-2m_0s\alpha}[s^{-1}\xi^{-1}|\Delta\varphi|^2+s\lambda^2\xi|\nabla\varphi|^2+s^3\lambda^4\xi^3|\varphi|^2]dxdt\right),
\end{aligned}$$

for any $\lambda_3\ge C$, $s\ge CT^{10}$ and any $\varepsilon>0$. By defining

$$\tilde{\rho}_1(t):=s^{16}\lambda^{40}e^{-8sm_0\hat{\alpha}+6sm_0\alpha^*}(\hat{\xi})^{16},\quad \tilde{\rho}_2(t)=s^{15}\lambda^{24}e^{2(m_0-2)s\alpha^*-4a_0s\alpha^*}(\hat{\xi})^{15}$$

and taking into account that $m_0>a_0+1$, there exists a constant $C>0$ such that

$$\iint_{\omega_0\times(0,T)}\tilde{\rho}_1(t)|\varphi|^2dxdt\le C\iint_{\omega_3\times(0,T)}\tilde{\rho}_2(t)|\varphi|^2dxdt.\tag{42}$$

From (41)–(42), we conclude the proof of Proposition 1. □

3.3. Null controllability of the linear system

In this section we are concerned in the null controllability of the linear coupled Stokes system

$$\begin{cases}
y_t-\Delta y+\nabla p=f_1+h1_\omega+(-\ell^{-2}\chi_{\mathcal{O}}+\gamma^{-2})z & \text{in}\quad Q,\\
-z_t-\Delta z+\nabla\pi=f_2+\mu(y-y_d)\chi_{\mathcal{O}_d} & \text{in}\quad Q,\\
\nabla\cdot y=0,\nabla\cdot z=0 & \text{in}\quad Q,\\
y=z=0 & \text{on}\quad \Sigma,\\
y(\cdot,0)=y_0(\cdot),\quad z(\cdot,T)=0 & \text{in}\quad \Omega,
\end{cases}\tag{43}$$

where the functions f_1 and f_2 are in appropriate weighted spaces. We look for a control $h\in L^2(0,T;L^2(\omega)^N)$ such that, under suitable properties on f_1,f_2, the solution to (43) satisfies $y(\cdot,T)=0$ in Ω. To do this, let us first state a Carleman inequality with weight functions not vanishing in $t = 0$. Let $\tilde{\ell} \in C^1([0,T])$ be a positive function in $[0,T)$ such that:

$$\tilde{\ell}(t) = T^2/4 \ \forall t \in [0, T/2] \ \text{ and } \ \tilde{\ell}(t) = t(T-t) \ \forall t \in [T/2, T].$$

Now, we introduce the following weight functions

$$\begin{aligned} \beta(x,t) &= \frac{e^{12\lambda\|\eta\|_\infty} - e^{\lambda(10\|\eta\|_\infty + \eta(x))}}{\tilde{\ell}^5(t)}, \quad \tau(x,t) = \frac{e^{\lambda(10\|\eta\|_\infty + \eta(x))}}{\tilde{\ell}^5(t)}, \\ \beta^*(t) &= \max_{x\in\overline{\Omega}} \beta(x,t), \qquad \tau^*(t) = \min_{x\in\overline{\Omega}} \tau(x,t), \\ \widehat{\beta}(t) &= \min_{x\in\overline{\Omega}} \beta(x,t), \qquad \widehat{\tau}(t) = \max_{x\in\overline{\Omega}} \tau(x,t). \end{aligned} \tag{44}$$

Lemma 3.4. *Let s and λ like in Proposition 1. Then, there exists a constant $C > 0$ (depending on $s, \lambda, \omega, \Omega, T$ and μ) such that every solution (φ, θ) of (14) satisfies*

$$\begin{aligned} &\|\varphi(\cdot,0)\|^2_{L^2(Q)^N} + \iint_Q e^{-2m_0 s\beta^*}(\tau^*)^3|\varphi|^2 dxdt \\ &\quad + \iint_Q e^{-2(a_0+1)s\beta^*}(\tau^*)^3|\theta|^2 dxdt \\ &\qquad \le C\left(\iint_Q e^{-2a_0 s\beta^*}(\hat{\tau})^{15}|g_1|^2 dxdt + \iint_Q e^{-2a_0 s\beta^*}|g_2|^2 dxdt \right. \\ &\qquad\qquad \left. + \iint_{\omega\times(0,T)} e^{-4a_0 s\beta^* + 2(m_0-2)s\beta}(\hat{\tau})^{15}|\varphi|^2 dxdt \right). \end{aligned} \tag{45}$$

Proof. By construction $\alpha = \beta$ and $\xi = \tau$ in $\Omega \times (T/2, T)$, so that

$$\begin{aligned} &\int_{T/2}^T \int_\Omega e^{-2(a_0+1)s\alpha^*}(\xi^*)^3|\theta|^2 dxdt + \int_{T/2}^T \int_\Omega e^{-2sm_0\alpha^*}(\xi^*)^3|\varphi|^2)dxdt \\ &\quad = \int_{T/2}^T \int_\Omega (e^{-2(a_0+1)s\beta^*}(\tau^*)^3|\theta|^2 + e^{-2sm_0\beta^*}(\tau^*)^3|\varphi|^2)dxdt. \end{aligned}$$

Therefore, it follows from Proposition 1 the estimate

$$\begin{aligned} &\int_{T/2}^T \int_\Omega (e^{-2(a_0+1)s\beta^*}(\tau^*)^3|\theta|^2 + e^{-2sm_0\beta^*}(\tau^*)^3|\varphi|^2)dxdt \\ &\quad \le C\left(\iint_Q e^{-2a_0 s\alpha^*}(\hat{\xi})^5|g_1|^2 dxdt + \iint_Q e^{-2a_0 s\alpha^*}|g_2|^2 dxdt \right. \\ &\qquad \left. + \iint_{\omega\times(0,T)} e^{-4a_0 s\alpha^* + 2(m_0-2)s\alpha}(\hat{\xi})^{15}|\varphi|^2 dxdt \right). \end{aligned}$$

Since $\tilde{\ell}(t) = t(T-t)$ for any $t \in [T/2, T]$ and

$$e^{-2a_0 s\beta^*} \ge C, \ e^{-2a_0 s\beta^*}(\tau^*)^5 \ge C \text{ and } e^{-4a_0 s\beta^* + 2(m_0-2)s\beta}(\hat{\tau})^{15} \ge C \text{ in } [0, T/2],$$

we readily get

$$\int_{T/2}^{T}\int_{\Omega}(e^{-2(a_0+1)s\beta^*}(\tau^*)^3|\theta|^2+e^{-2sm_0\beta^*}(\tau^*)^3|\varphi|^2)dxdt$$
$$\leq C\left(\iint_Q e^{-2a_0s\beta^*}(\hat{\tau})^5|g_1|^2dxdt+\iint_Q e^{-2a_0s\beta^*}|g_2|^2dxdt\right.$$
$$\left.+\iint_{\omega\times(0,T)} e^{-4a_0s\beta^*+2(m_0-2)s\beta}(\hat{\tau})^{15}|\varphi|^2dxdt\right). \tag{46}$$

Now, we introduce a function $\nu\in C^1([0,T])$ such that $\nu\equiv 1$ in $[0,T/2]$, $\nu\equiv 0$ in $[3T/4,T]$. It is easy to see that $(\nu\varphi,\nu\pi_\varphi)$ and $(\nu\theta,\nu\pi_\varphi)$ satisfies the system

$$\begin{cases} -(\nu\varphi)_t-\Delta(\nu\varphi)+\nabla(\nu\pi_\varphi)=\nu(g_1+\mu\theta\chi_{\mathcal{O}_d})-\nu'\varphi \text{ in } Q,\\ (\nu\theta)_t-\Delta(\nu\theta)+\nabla(\nu\pi_\theta)=\nu(g_2-\ell^{-2}\varphi\chi_{\mathcal{O}}+\gamma^{-2}\varphi)+\nu'\theta \text{ in } Q,\\ \nabla\cdot(\nu\varphi)=\nabla\cdot(\nu\theta)=0, \nabla\cdot q=0 \text{ in } Q,\\ \nu\varphi=\nu\theta=0 \text{ on } \Sigma,\\ (\nu\varphi)(T)=0, \quad (\nu\theta)(0)=0 \text{ in } \Omega. \end{cases} \tag{47}$$

Using classical energy estimate for both $\nu\varphi$ and $\nu\theta$, which solve the Stokes system (47) we get

$$\|\varphi(0)\|^2_{L^2(Q)^N}+\|\varphi\|^2_{L^2(0,T/2;H^1_0(\Omega)^N)}$$
$$\leq C\left(\frac{1}{T^2}\|\varphi\|^2_{L^2(T/2,T/4;L^2(\Omega)^N)}\right.$$
$$\left.+\|\theta\|^2_{L^2(0,3T/4;L^2(\mathcal{O}_d)^N)}+\|g_1\|^2_{L^2(0,3T/2;L^2(\Omega)^N)}\right)$$

and

$$\|\theta\|^2_{L^2(0,T/2;H^1_0(\Omega)^N)}\leq C\left(\frac{1}{T^2}\|\theta\|^2_{L^2(T/2,3T/4;L^2(\Omega)^N)}\right.$$
$$\left.+\|\nu(-\ell^{-2}\varphi\chi_{\mathcal{O}}+\gamma^{-2}\varphi)\|^2_{L^2(0,3T/4;L^2(\Omega)^N)}+\|g_2\|^2_{L^2(0,3T/2;L^2(\Omega)^N)}\right).$$

Taking into account that

$$e^{-2sm_0\beta^*}(\tau^*)^3\geq C>0 \quad e^{-2(a_0+1)s\beta^*}(\tau^*)^3\geq C>0, \quad \forall t\in[T/2,3T/4]$$

and

$$e^{-2a_0s\beta^*}(\hat{\tau})^5\geq C>0 \quad e^{-2a_0s\beta^*}>e^{-4a_0s\beta^*}\geq C>0, \quad \forall t\in[0,3T/4],$$

we have

$$
\begin{aligned}
&\|\varphi(0)\|^2_{L^2(\Omega)^N} + \int_0^{T/2}\int_\Omega e^{-2m_0 s\beta^*}(\tau^*)^3|\varphi|^2 + e^{-2(a_0+1)s\beta^*}(\tau^*)^3|\theta|^2\,dxdt \\
&\le C\left(\int_{T/2}^{3T/2}\int_\Omega [e^{-2m_0 s\beta^*}(\tau^*)^3|\varphi|^2 + e^{-2(a_0+1)s\beta^*}(\tau^*)^3|\theta|^2]dxdt\right. \\
&\qquad + \|\nu\mu e^{-2a_0 s\beta^*}\theta\|^2_{L^2(0,3T/4;L^2(\mathcal{O}_d)^N)} \\
&\qquad + \|\nu(-\ell^{-2}\varphi\chi_{\mathcal{O}} + \gamma^{-2}\varphi)\|^2_{L^2(0,3T/4;L^2(\Omega)^N)} \\
&\qquad \left.+ \int_0^{3T/4}\int_\Omega \left[e^{-2a_0 s\beta^*}(\tau^*)^5|g_1|^2 + e^{-2a_0 s\beta^*}|g_2|^2\right]dxdt\right).
\end{aligned} \tag{48}
$$

Thus, from (46) and (48) we have at this moment

$$
\begin{aligned}
&\|\varphi(0)\|^2_{L^2(\Omega)^N} + \iint_Q (e^{-2(a_0+1)s\beta^*}(\tau^*)^3|\theta|^2 + e^{-2sm_0\beta^*}(\tau^*)^3|\varphi|^2)dxdt \\
&\le C\left(\iint_Q e^{-2a_0 s\beta^*}(\hat{\tau})^5|g_1|^2dxdt + \iint_Q e^{-2a_0 s\beta^*}|g_2|^2dxdt\right. \\
&\quad + \iint_{\omega\times(0,T)} e^{-4a_0 s\beta^* + 2(m_0-2)s\beta}(\hat{\tau})^{15}|\varphi|^2dxdt \\
&\quad + \|\nu\mu e^{-2a_0 s\beta^*}\theta\|^2_{L^2(0,3T/4;L^2(\mathcal{O}_d)^N)} \\
&\quad \left.+ \|\nu(-\ell^{-2}\varphi\chi_{\mathcal{O}} + \gamma^{-2}\varphi)\|^2_{L^2(0,3T/4;L^2(\Omega)^N)}\right).
\end{aligned} \tag{49}
$$

Observe that if ℓ and γ are large enough (again, $\ell,\gamma >> C_3T^{30/4}e^{C_4/T^{10}}$, where C_3, C_4 are positive constants depending on a_0, m_0, s), the last term in the right-hand side of (49) can be absorbed by the left-hand side. In addition, considering $\theta^*(x,t) = e^{-2s\beta^*}\theta(x,t)$ instead $\nu\theta$ in (47) and using standard energy estimate for the system associated to θ, we obtain

$$
\begin{aligned}
&\int_0^{3T/4}\int_\Omega \nu^2\mu^2 e^{-4a_0 s\beta^*}|\theta|^2dxdt \\
&\le C\left(\iint_Q e^{-4a_0 s\beta^*}|g_2|^2dxdt + \frac{1}{\ell^4}\int_0^T\int_{\mathcal{O}_d} e^{-4a_0 s\beta^*}|\varphi|^2dxdt\right. \\
&\qquad \left.+ \frac{1}{\gamma^4}\iint_Q e^{-4a_0 s\beta^*}|\varphi|^2dxdt + \iint_Q e^{-4a_0 s\beta^*}(\tau^*)^{6/5}|\theta|^2dxdt\right).
\end{aligned} \tag{50}
$$

Putting together (49), (50) and taking again ℓ and γ large enough (as above), we obtain the desired inequality (45). □

Remark 3. In order to establish a null controllability result for the system (43) we need adequate weight functions, see Theorem 3.5. As consequence of the inequalities $a_0 + 1 < m_0 \le 2a_0,\ a_0 \ge \frac{5}{4}$ observe that on the left-hand side of

(16) it is possible to add the term

$$\iint_Q e^{-4a_0 s\beta^*}(\tau^*)^3|\theta|^2 dxdt.$$

Now, we are ready to prove the null controllability of system (43). The idea is to look a solution in an appropriate weighted functional space. Let us introduce the following space

$$\begin{aligned} E := \Big\{ &(y,z,\pi_y,\pi_z,h) : e^{a_0 s\beta^*}(\hat\tau)^{-5/2} y \in L^2(Q)^N, e^{a_0 s\beta^*} z \in L^2(Q)^N,\\ &e^{2a_0 s\beta^* - (m_0-2)s\hat\beta}(\hat\tau)^{-15/2} h1_\omega \in L^2(Q)^N,\\ &e^{a_0 s\beta^*}(\hat\tau)^{-15/2} y \in L^2(0,T;H^2(\Omega)^N)\cap L^\infty(0,T;V),\\ &e^{a_0 s\beta^*}(\tau^*)^{-c_0} z \in L^2(0,T;H^2(\Omega)^N)\cap L^\infty(0,T;V),\ c_0 \geq \frac{5}{2},\\ &e^{m_0 s\beta^*}(\hat\tau^*)^{-3/2}(y_t - \Delta y + \nabla\pi_y - (-\ell^{-2}\chi_{\mathcal{O}} + \gamma^{-2})z - h1_\omega) \in L^2(Q)^N,\\ &e^{2a_0 s\beta^*}(\hat\tau^*)^{-3/2}(-z_t - \Delta z + \nabla\pi_z - \mu(y-y_d)\chi_{\mathcal{O}_d}) \in L^2(Q)^N \Big\}. \end{aligned}$$

It is clear that E is a Banach space for the following norm:

$$\begin{aligned} &\|e^{a_0 s\beta^*}(\hat\tau)^{-5/2} y\|_{L^2(Q)^N} + \|e^{a_0 s\beta^*} z\|_{L^2(Q)^N}\\ &\quad + \|e^{2a_0 s\beta^* - (m_0-2)s\hat\beta}(\hat\tau)^{-15/2} h1_\omega\|_{L^2(Q)^N}\\ &\quad + \|e^{a_0 s\beta^*}(\hat\tau)^{-15/2} y\|_{L^2(0,T;H^2(\Omega)^N)} + \|e^{a_0 s\beta^*}(\hat\tau)^{-15/2} y\|_{L^\infty(0,T;V)}\\ &\quad + \|e^{a_0 s\beta^*}(\tau^*)^{-c_0} z\|_{L^2(0,T;H^2(\Omega)^N)} + \|e^{a_0 s\beta^*}(\tau^*)^{-c_0} z\|_{L^\infty(0,T;V)}\\ &\quad + \|e^{m_0 s\beta^*}(\tau^*)^{-3/2}(y_t - \Delta y + \nabla\pi_y - (-\ell^{-2}\chi_{\mathcal{O}} + \gamma^{-2})z - h1_\omega)\|_{L^2(Q)^N}\\ &\quad + \|e^{2a_0 s\beta^*}(\tau^*)^{-3/2}(-z_t - \Delta z + \nabla\pi_z - \mu(y-y_d)\chi_{\mathcal{O}_d})\|_{L^2(Q)^N}. \end{aligned}$$

Remark 4. Observe in particular that $(y,z,\pi_y,\pi_z,h)\in E$ implies $y(\cdot,T)=0$ in Ω.

Theorem 3.5. *Assume the hypothesis of Lemma 3.4 and*

$$y_0 \in V,\quad e^{m_0 s\beta^*}(\tau^*)^{-3/2} f_1 \in L^2(Q)^N,\quad e^{2a_0 s\beta^*}(\tau^*)^{-3/2} f_2 \in L^2(Q)^N. \tag{51}$$

Then, we can find a control $h\in L^2(0,T;L^2(\omega)^N)$ such that the associated solution (y,z,π_y,π_z,h) to (43) satisfies $(y,z,\pi_y,\pi_z,h)\in E$.

Proof. Let us introduce the following constrained extremal problem:

$$\begin{cases} \inf \left\{ \dfrac{1}{2}\left(\displaystyle\iint_Q e^{2a_0 s\beta^*}(\hat\tau)^{-5}|y|^2 dxdt + \iint_Q e^{2a_0 s\beta^*}|z|^2 dxdt \right.\right.\\ \qquad\qquad \left.\left. + \displaystyle\iint_{\omega\times(0,T)} e^{4a_0 s\beta^* - 2(m_0-2)s\hat\beta}(\hat\tau)^{-15}|h|^2 dxdt \right)\right.\\ \text{subject to } h\in L^2(Q),\ supp\, h \subset \omega\times(0,T),\ \text{and}\\ \begin{cases} y_t - \Delta y + \nabla\pi_y = f_1 + h\chi_\omega + (-\ell^{-2}\chi_{\mathcal{O}} + \gamma^{-2})z & \text{in } Q,\\ -z_t - \Delta z + \nabla\pi_z = f_2 + \mu(y-y_d)\chi_{\mathcal{O}_d} & \text{in } Q,\\ \nabla\cdot y = 0, \nabla\cdot z = 0 & \text{in } Q,\\ y = z = 0 & \text{on } \Sigma,\\ y(\cdot,0) = y_0(\cdot),\quad y(\cdot,T)=0,\quad z(\cdot,T)=0 & \text{in } \Omega. \end{cases} \end{cases} \tag{52}$$

Assume that this problem admits a unique solution $(\hat{y}, \hat{z}, \hat{\pi_y}, \hat{\pi_z}, \hat{h})$. Then, from the Lagrange's principle there exists dual variables $(\hat{\varphi}, \hat{\theta}, \hat{\pi}_\varphi, \hat{\pi}_\theta)$ such that

$$\begin{array}{ll} \hat{y} = e^{-2a_0 s\beta^*}(\hat{\tau})^5(-\hat{\varphi}_t - \Delta\hat{\varphi} + \nabla\hat{\pi}_\varphi - \mu\hat{\theta}_{\mathcal{O}_d}) & \text{in } Q, \\ \hat{z} = e^{-2a_0 s\beta^*}(\hat{\theta}_t - \Delta\hat{\theta} + \nabla\hat{\pi}_\theta - (-\ell^{-2}\chi_{\mathcal{O}} + \gamma^{-2})\hat{\varphi}) & \text{in } Q, \\ \hat{h} = e^{-4a_0 s\beta^* + 2(m_0-2)s\hat{\beta}}(\hat{\tau})^{15}\hat{\varphi} & \text{in } Q, \\ \hat{y} = \hat{z} = 0 & \text{on } \Sigma. \end{array} \tag{53}$$

Now, following the arguments established in [11], we introduce the space P_0 of functions $(y, z, \pi_y, \pi_z) \in C^2(\overline{Q})^{2N+2}$ such that

(i) $\nabla \cdot y = \nabla \cdot z = 0$ in Q.
(ii) $y = z = 0$ on Σ.
(iii) $\displaystyle\int_{\omega_0} \pi_\varphi dx = 0$.

We also consider the bilinear form $a(\cdot,\cdot)$ over $P_0 \times P_0$ defined by:

$$\begin{aligned} & a((\hat{\varphi}, \hat{\theta}, \hat{\pi}_\varphi, \hat{\pi}_\theta), (w, z, \pi_w, \pi_z)) \\ & =: \iint_Q e^{-2a_0 s\beta^*}(\hat{\tau})^5(-\hat{\varphi}_t - \Delta\hat{\varphi} + \nabla\hat{\pi}_\varphi - \mu\hat{\theta}_{\mathcal{O}_d})(-y_t - \Delta y + \nabla\pi_y - \mu z_{\mathcal{O}_d})\, dxdt \\ & \quad + \iint_Q e^{-2a_0 s\beta^*}(\hat{\theta}_t - \Delta\hat{\theta} + \nabla\hat{\pi}_\theta - (-\ell^{-2}\chi_{\mathcal{O}} + \gamma^{-2})\hat{\varphi})(z_t - \Delta z + \nabla\pi_z) \\ & \quad - \iint_Q e^{-2a_0 s\beta^*}(\hat{\theta}_t - \Delta\hat{\theta} + \nabla\hat{\pi}_\theta - (-\ell^{-2}\chi_{\mathcal{O}} + \gamma^{-2})\hat{\varphi})(\ell^{-2}\chi_{\mathcal{O}} + \gamma^{-2})w)\, dxdt \\ & \quad + \iint_{\omega\times(0,T)} e^{-4a_0 s\beta^* + 2(m_0-2)s\hat{\beta}}(\hat{\tau})^{15}\hat{\varphi} w\, dxdt, \end{aligned}$$

for every $(w, z, \pi_w, \pi_z) \in P_0$, and a linear form

$$\langle G, (w, z, \pi_w, \pi_z)\rangle := \iint_Q f_1 \cdot w\, dxdt + \iint_Q f_2 \cdot z\, dxdt + \int_\Omega y_0(\cdot)\cdot w(\cdot, 0)\, dx. \tag{54}$$

Taking into account this definitions, one can see that, if the functions $\hat{y}, \hat{z}$ and $\hat{h}$ solve (52), we must have $\forall (w, z, \pi_w, \pi_z) \in P_0$

$$a((\hat{\varphi}, \hat{\theta}, \hat{\pi}_\varphi, \hat{\pi}_\theta), (w, z, \pi_w, \pi_z)) = \langle G, (w, z, \pi_w, \pi_z)\rangle. \tag{55}$$

Observe that Carleman inequality (45) holds for all $(w, z, \pi_w, \pi_z) \in P_0$. Consequently,

$$\iint_Q e^{-2m_0 s\beta^*}(\tau^*)^3|z|^2 dxdt + \iint_Q e^{-2(a_0+1)s\beta^*}(\tau^*)^3|w|^2 dxdt \tag{56}$$

$$+ \iint_Q e^{-2a_0 s\beta^*}(\tau^*)^3|w|^2 dxdt + \|w(0)\|^2_{L^2(\Omega)^N} \tag{57}$$

$$\le Ca((w, z, \pi_w, \pi_z), (w, z, \pi_w, \pi_z)), \tag{58}$$

for every $(w, z, \pi_w, \pi_z) \in P_0$.

Therefore, $a(\cdot,\cdot) : P_0 \times P_0 \longmapsto \mathbb{R}$ is symmetric, definite positive bilinear form on P_0. We denote by P the completion of P_0 for the norm induced by $a(\cdot,\cdot)$. Then, $a(\cdot,\cdot)$ is well-defined, continuous and again definite positive on P. Furthermore, in view of the Carleman inequality (45), the assumption (51) and (56), the linear form $(w,z,\pi_w,\pi_z) \longmapsto \langle G,(w,z,\pi_w,\pi_z)\rangle$ is well-defined and continuous on P. Indeed, for every $(w,z,\pi_w,\pi_z) \in P$,

$$\begin{aligned}
&\langle G,(w,z,\pi_w,\pi_z)\rangle \\
&\quad \le \|e^{(a_0+1)s\beta^*}(\tau^*)^{-3/2}f_1\|_{L^2(Q)^N}\|e^{-(a_0+1)s\beta^*}(\tau^*)^{3/2}w\|_{L^2(Q)^N} \\
&\qquad + \|e^{m_0 s\beta^*}(\tau^*)^{-3/2}f_2\|_{L^2(Q)^N}\|e^{-m_0 s\beta^*}(\tau^*)^{3/2}z\|_{L^2(Q)^N} + \|y_0\|_H\|w(0)\|_H \\
&\quad \le \|e^{m_0 s\beta^*}(\tau^*)^{-3/2}f_1\|_{L^2(Q)^N}\|e^{-(a_0+1)s\beta^*}(\tau^*)^{3/2}w\|_{L^2(Q)^N} \\
&\qquad + \|e^{2a_0 s\beta^*}(\tau^*)^{-3/2}f_2\|_{L^2(Q)^N}\|e^{-m_0 s\beta^*}(\tau^*)^{3/2}z\|_{L^2(Q)^N} + \|y_0\|_H\|w(0)\|_H.
\end{aligned}$$

Using (56) and the density of P_0 in P, we find

$$\begin{aligned}
\langle G,(w,z,\pi_w,\pi_z)\rangle \le C\Big(&\|e^{m_0 s\beta^*}(\tau^*)^{-3/2}f_1\|_{L^2(Q)^N} \\
&+ \|e^{2a_0 s\beta^*}(\tau^*)^{-3/2}f_2\|_{L^2(Q)^N} + \|y_0\|_H\Big)\|(w,z,\pi_w,\pi_z)\|_P.
\end{aligned}$$

Hence, from Lax–Milgram's Lemma, there exists a unique $(\hat\varphi,\hat\theta,\hat\pi_\varphi,\hat\pi_\theta) \in P$ satisfying $\forall (w,z,\pi_w,\pi_z) \in P$:

$$a((\hat\varphi,\hat\theta,\hat\pi_\varphi,\hat\pi_\theta),(w,z,\pi_w,\pi_z)) = \langle G,(w,z,\pi_w,\pi_z)\rangle. \tag{59}$$

Let us set $(\hat y,\hat z,\hat h)$ like in (53) and remark that $(\hat y,\hat z,\hat\pi_y,\hat\pi_z,\hat h)$ verifies

$$\begin{aligned}
a((\hat\varphi,\hat\theta,\hat\pi_\varphi,\hat\pi_\theta),(\hat\varphi,\hat\theta,\hat\pi_\varphi,\hat\pi_\theta)) &= \iint_Q e^{2a_0 s\beta^*}(\hat\tau)^{-5}|\hat y|^2 dxdt \\
+ \iint_Q e^{2a_0 s\beta^*}|\hat z|^2 dxdt &+ \iint_{\omega\times(0,T)} e^{4a_0 s\beta^* - 2(m_0-2)s\hat\beta}(\hat\tau)^{-15}|\hat h| dxdt < +\infty.
\end{aligned}$$

Let us prove that $(\hat y,\hat z)$ is, together with some $(\hat\pi_y,\hat\pi_z)$, the weak solution of the Stokes system in (52) for $h=\hat h$. In fact, we introduce the (weak) solution $(\tilde y,\tilde z,\tilde\pi_y,\tilde\pi_z)$ to the Stokes system

$$\begin{cases}
\tilde y_t - \Delta\tilde y + \nabla\tilde\pi_y = f_1 + \hat h 1_\omega + (-\ell^{-2}\chi_{\mathcal O} + \gamma^{-2})\tilde z & \text{in } Q,\\
-\tilde z_t - \Delta\tilde z + \nabla\tilde\pi_z = f_2 + \mu(\tilde y - \tilde y_d)\chi_{\mathcal O_d} & \text{in } Q,\\
\nabla\cdot\tilde y = 0, \nabla\cdot\tilde z = 0 & \text{in } Q,\\
\tilde y = \tilde z = 0 & \text{on } \Sigma,\\
\tilde y(\cdot,0) = y_0(\cdot), \quad \tilde z(\cdot,T) = 0 & \text{in } \Omega.
\end{cases} \tag{60}$$

Clearly, $(\tilde y,\tilde z)$ is the unique solution of (60) defined by transposition. This means that, for every $(a,b) \in L^2(Q)^{2N}$,

$$\langle(\tilde y,\tilde z),(a,b)\rangle_{L^2(Q)^N} = \langle y_0,\varphi(0)\rangle_{L^2(\Omega)} + \langle(f_1 + \hat h 1_\omega, f_2),(\varphi,\theta)\rangle_{L^2(Q)^N}, \tag{61}$$

where (φ,θ) is, together with some (π_φ,π_θ), the solution to

$$\begin{cases}
L^*(\varphi,\theta) = (a,b) & \text{in } Q,\\
\nabla\cdot\varphi = 0, \quad \nabla\cdot\theta = 0 & \text{in } Q,\\
\varphi = \theta = 0 & \text{on } \Sigma,\\
\varphi(\cdot,T) = 0, \quad \theta(\cdot,0) = 0 & \text{in } \Omega,
\end{cases} \tag{62}$$

and L^* is the adjoint operator of L given by:

$$L(\tilde{y},\tilde{z}) := (\tilde{y}_t - \Delta\tilde{y} + \nabla\tilde{\pi}_y - (-\ell^{-2}\chi_{\mathcal{O}} + \gamma^{-2})\tilde{z}, -\tilde{z}_t - \Delta\tilde{z} + \nabla\tilde{\pi}_z \\ -\mu(\tilde{y} - \tilde{y}_d)\chi_{\mathcal{O}_d}).$$

From (53) and (55), we see that $(\hat{y},\hat{z})$ also satisfies (61). Consequently, $(\hat{y},\hat{z}) = (\tilde{y},\tilde{z})$ and $(\hat{y},\hat{z})$ is, together with some $(\hat{\pi}_y,\hat{\pi}_z) = (\tilde{\pi}_y,\tilde{\pi}_z)$, the weak solution to the system (60).

Finally, we must see that $(\hat{y},\hat{z},\hat{\pi}_y,\hat{\pi}_z,\hat{h}) \in E$. We already know that

$$e^{a_0 s\beta^*}(\hat{\tau})^{-5/2}\hat{y}, \quad e^{a_0 s\beta^*}\hat{z}, \quad e^{2a_0 s\beta^* - (m_0-2)s\hat{\beta}}(\hat{\tau})^{-15/2}\hat{h}1_\omega \in L^2(Q)^N$$

and [see hypothesis (51)]

$$e^{m_0 s\beta^*}(\tau^*)^{-3/2}f_1 \in L^2(Q)^N \quad \text{and} \quad e^{2a_0 s\beta^*}(\tau^*)^{-3/2}f_2 \in L^2(Q)^N.$$

Thus, it only remains to check that

$$e^{a_0 s\beta^*}(\hat{\tau})^{-15/2}\hat{y},\ e^{a_0 s\beta^*}(\tau^*)^{-c_0}z \in L^2(0,T;H^2(\Omega)^N) \cap L^\infty(0,T;V),$$

where $c_0 \geq \frac{5}{2}$.

1. We define the functions

$$\begin{aligned} y^* &:= e^{a_0 s\beta^*}(\hat{\tau})^{-15/2}\hat{y}, & z^* &:= e^{a_0 s\beta^*}(\tau^*)^{-c_0}\hat{z} \\ \pi_y^* &:= e^{a_0 s\beta^*}(\hat{\tau})^{-15/2}\hat{\pi}_y, & \pi_z^* &:= e^{a_0 s\beta^*}(\tau^*)^{-c_0}\hat{\pi}_z \end{aligned}$$

and

$$\begin{aligned} f_1^* &:= e^{a_0 s\beta^*}(\hat{\tau})^{-15/2}(f_1 + h1_\omega), & z^{**} &:= e^{a_0 s\beta^*}(\hat{\tau})^{-15/2}(-\ell^{-2}\chi_{\mathcal{O}} + \gamma^{-2})z \\ f_2^* &:= e^{a_0 s\beta^*}(\tau^*)^{-c_0}f_2, & y^{**} &:= e^{a_0 s\beta^*}(\tau^*)^{-c_0}(y - y_d)\chi_{\mathcal{O}_d}. \end{aligned}$$

Then $(y^*,\pi_y^*,z^*,\pi_z^*)$ satisfies:

$$\begin{cases} y_t^* - \Delta y^* + \nabla\pi_y^* = f_1^* + z^{**} + \left(e^{3/2s\beta^*}(\hat{\tau})^{-15/2}\right)'\hat{y} & \text{in} \quad Q, \\ -z_t^* - \Delta z^* + \nabla\pi_z^* = f_2^* + y^{**} + \left(e^{1/2s\beta^*}(\hat{\tau})^7\right)'\hat{z} & \text{in} \quad Q, \\ \nabla\cdot y^* = 0, \nabla\cdot z^* = 0 & \text{in} \quad Q, \\ y^* = z^* = 0 & \text{on} \quad \Sigma, \\ y^*(\cdot,0) = e^{3/2s\beta^*(0)}(\hat{\tau}(0))^{-15/2}y_0(\cdot), \quad z^*(\cdot,T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{63}$$

2. Now, we prove that the right-hand side of the main equations in (63) is in $L^2(Q)^N$.
   - $|e^{a_0 s\beta^*}(\hat{\tau})^{-15/2}f_1| \leq Ce^{a_0 s\beta^*}|\hat{\tau}|^{-15/2}|f_1| \leq Ce^{m_0 s\beta}|\tau^*|^{-3/2}|f_1|$.
   - $|e^{a_0 s\beta^*}(\hat{\tau})^{-15/2}h1_\omega| \leq Ce^{2a_0 s\beta^* - (m_0-2)s\beta^*}(\hat{\tau})^{-15/2}|h|1_\omega$.
   - $|z^{**}| = |e^{a_0 s\beta^*}(\hat{\tau})^{-15/2}(-\ell^{-2}\chi_{\mathcal{O}} + \gamma^{-2})z| \leq Ce^{a_0 s\beta^*}|\hat{z}|$.
   - $|\left(e^{3/2s\beta^*}(\hat{\tau})^{-15/2}\right)'\hat{y}| \leq Cse^{a_0 s\beta^*}|\tau^*|^{6/5}|\hat{y}| \leq Ce^{a_0 s\beta^*}|\hat{\tau}|^{-5/2}|\hat{y}|$.
   - $|f_2^{**}| = |e^{a_0 s\beta^*}(\tau^*)^{-c_0}f_2| \leq Ce^{(a_0+1)s\beta^*}|\tau^*|^{-c_0}|f_2|$.
   - $|\left(e^{1/2s\beta^*}(\hat{\tau})^7\right)'\hat{z}| \leq Ce^{a_0 s\beta^*}|\hat{z}|$.
   - $$\begin{aligned} |y^{**}| &= |e^{a_0 s\beta^*}(\tau^*)^{-c_0}(y - y_d)\chi_{\mathcal{O}_d}| \\ &\leq Ce^{a_0 s\beta^*}|\hat{\tau}|^{-5/2}|\hat{y}| + Ce^{a_0 s\beta^*}|\tau^*|^{-c_0}|y_d|. \end{aligned}$$

   Observe that $y^{**} \in L^2(Q)^N$ thanks to the hypothesis (5).

Taking into account $a) - b)$ and $y_0 \in V$, we have $y^*, z^* \in L^2(0,T;H^2(\Omega)^N) \cap L^\infty(0,T;V)$ (see Lemma 3.3 in Sect. 3.1).

This concludes the proof of Theorem 3.5. □

Remark 5. Before starting the last section, it is important to consider small data in order to prove our main result, Theorem 1.2. Thus, we impose that

$$\|e^{m_0 s\beta^*}(\tau^*)^{-3/2} f_1\|_{L^2(Q)^N} + \|e^{2a_0 s\beta^*}(\tau^*)^{-3/2} f_2\|_{L^2(Q)^N} + \|y_0\|_V \leq \delta, \quad (64)$$

where δ is a small positive number.

4. Proof of the main result

In this section we give the proof of Theorem 1.2 throughout classical arguments such like in [11]. The results obtained in the previous section allow us to locally invert a nonlinear operator associated to the nonlinear system

$$\begin{cases} y_t - \Delta y + (y\cdot\nabla)y + \nabla\pi_y = h1_\omega + (\ell^{-2}\tilde{\chi}_{\mathcal{O}} + \gamma^{-2})z & \text{in} \quad Q, \\ -z_t - \Delta z + (z,\nabla^t)y - (y,\nabla)z + \nabla\pi_z = \mu(y-y_d)\chi_{\mathcal{O}_d} & \text{in} \quad Q, \\ \nabla\cdot y = 0, \nabla\cdot z = 0 & \text{in} \quad Q, \\ y = z = 0 & \text{on} \quad \Sigma, \\ y(\cdot,0) = y_0(\cdot), \quad z(\cdot,T) = 0 & \text{in} \quad \Omega. \end{cases}$$

To do this, we will apply an inverse function theorem of the Lyusternik's kind [18], which will allow us to complete the proof of Theorem 1.2. More precisely, we will use the following theorem.

Theorem 4.1. *Suppose that $\mathcal{B}_1, \mathcal{B}_2$ are Banach spaces and*

$$\mathcal{A} : \mathcal{B}_1 \to \mathcal{B}_2$$

is a continuously differentiable map. We assume that for $b_1^0 \in \mathcal{B}_1, b_2^0 \in \mathcal{B}_2$ the equality

$$\mathcal{A}(b_1^0) = b_2^0 \quad (65)$$

holds and $\mathcal{A}'(b_1^0) : \mathcal{B}_1 \to \mathcal{B}_2$ is an epimorphism.Then there exists $\delta > 0$ such that for any $b_2 \in \mathcal{B}_2$ which satisfies the condition

$$\|b_2^0 - b_2\|_{\mathcal{B}_2} < \delta$$

there exists a solution $b_1 \in \mathcal{B}_1$ of the equation

$$\mathcal{A}(b_1) = b_2.$$

Proof. We apply Theorem 4.1 for the spaces $\mathcal{B}_1 := E$ and

$$\mathcal{B}_2 := \{(f_1, f_2, y_0) \in X_1^* \times X_2^* \times V : f_1, f_2, y_0 \text{ satisfies } (64)\},$$

where $X_1^* := L^2(e^{m_0 s\beta^*}(\tau^*)^{-3/2}(0,T); L^2(\Omega)^N)$ and

$$X_2^* := L^2(e^{2a_0 s\beta^*}(\tau^*)^{-3/2}(0,T); L^2(\Omega)^N).$$

We define the operator $\mathcal{A}$ by the formula

$$\mathcal{A}(y,z,\pi_y,\pi_z,h) := \big(y_t - \Delta y + (y\cdot\nabla)y + \nabla\pi_y - (\ell^{-2}\tilde{\chi}_{\mathcal{O}} + \gamma^{-2})z - h1_\omega,$$
$$- z_t - \Delta z + (z,\nabla^t)y - (y,\nabla)z + \nabla\pi_z - \mu(y-y_d)\chi_{\mathcal{O}_d}, y(\cdot,0)\big)\,,$$

for every $(y,z,\pi_y,\pi_z,h)\in\mathcal{B}_1$.

Let us see that $\mathcal{A}$ is of class $C^1(\mathcal{B}_1,\mathcal{B}_2)$. Indeed, notice that all the terms in $\mathcal{A}$ are linear, except for $(y\cdot\nabla)y$ and $z,\nabla^t)y-(y,\nabla)z$, then, we only have to check that these nonlinear terms are well-defined and depend continuously on the data. Thus, we will prove that the bilinear operator

$$((y^1,z^1,\pi_y^1,\pi_z^1,h^1),(y^2,z^2,\pi_y^2,\pi_z^2,h^2)) \longmapsto (y^1\cdot\nabla)y^2$$

is continuous from $\mathcal{B}_1\times\mathcal{B}_1$ to X_1^*, and the bilinear forms

$$((y^1,z^1,\pi_y^1,\pi_z^1,h^1),(y^2,z^2,\pi_y^2,\pi_z^2,h^2)) \longmapsto (y^1\cdot\nabla)z^2,$$
$$((y^1,z^1,\pi_y^1,\pi_z^1,h^1),(y^2,z^2,\pi_y^2,\pi_z^2,h^2)) \longmapsto (z^1\cdot\nabla^t)y^2$$

are continuous from $\mathcal{B}_1\times\mathcal{B}_1$ to X_2^*.

In fact, notice that (see the definition of the space E):

$$e^{a_0 s\beta^*}(\hat{\tau})^{-15/2}y\in L^2(0,T;L^\infty(\Omega)^N)$$

and

$$\nabla(e^{a_0 s\beta^*}(\hat{\tau})^{-15/2}y)\in L^\infty(0,T;L^2(\Omega)^{N\times N}).$$

Consequently, we obtain

$$\begin{aligned}&\|e^{m_0 s\beta^*}(\tau^*)^{-3/2}(y^1\cdot\nabla)y^2\|_{L^2(Q)^N}\\ &\quad\le C\|(e^{a_0 s\beta^*}(\hat{\tau})^{-15/2}y^1\cdot\nabla)e^{a_0 s\beta^*}(\hat{\tau})^{-15/2}y^2\|_{L^2(Q)^N}\\ &\quad\le C\|e^{2s\beta^*}(\hat{\tau})^{-15/2}y^1\|_{L^2(0,T;L^\infty(\Omega)^N)}\|e^{a_0 s\beta^*}(\hat{\tau})^{-15/2}y^2\|_{L^\infty(0,T;V)}.\end{aligned}$$

On the other hand, for $c_0\ge 5/2$,

$$e^{a_0 s\beta^*}(\tau^*)^{-c_0}z\in L^2(0,T;L^\infty(\Omega)^N)$$

and

$$\nabla(e^{a_0 s\beta^*}(\tau^*)^{-c_0}z)\in L^\infty(0,T;L^2(\Omega)^{N\times N}).$$

Then,

$$\begin{aligned}&\|e^{2a_0 s\beta^*}(\tau^*)^{-3/2}(y^1\cdot\nabla)z^2\|_{L^2(Q)^N}\\ &\quad\le C\|e^{a_0 s\beta^*}(\hat{\tau})^{-15/2}y^1\|_{L^2(0,T;L^\infty(\Omega)^N)}\|e^{a_0 s\beta^*}(\tau^*)^{-c_0}z^2\|_{L^\infty(0,T;V)},\end{aligned}$$

and analogously,

$$\begin{aligned}&\|e^{2a_0 s\beta^*}(\tau^*)^{-3/2}(z^1\cdot\nabla)y^2\|_{L^2(Q)^N}\\ &\quad\le C\|e^{a_0 s\beta^*}(\tau^*)^{-c_0}z^1\|_{L^2(0,T;L^\infty(\Omega)^N)}\|e^{a_0 s\beta^*}(\hat{\tau})^{-15/2}y^2\|_{L^\infty(0,T;V)}.\end{aligned}$$

Notice that $\mathcal{A}'(0,0,0):\mathcal{B}_1\to\mathcal{B}_2$ is given by

$$(y_t-\Delta y+\nabla\pi_y-(\ell^{-2}\tilde{\chi}_{\mathcal{O}}+\gamma^{-2})z-h1_\omega, -z_t-\Delta z$$
$$+\nabla\pi_z-\mu(y-y_d)\chi_{\mathcal{O}_d}, y(\cdot,0)),$$

for all $(y, z, \pi_y, \pi_z, h) \in \mathcal{B}_1$. In virtue of Theorem 3.5, this functional satisfies $Im(\mathcal{A}'(0,0,0)) = \mathcal{B}_2$.

Let $b_1^0 = (0,0,0)$ and $b_2^0 = (0,0)$. Then Eq. (65) obviously holds. So all necessary conditions to apply Theorem 4.1 are fulfilled. Therefore there exists a positive number δ such that, if $\|y(\cdot,0)\|_V \leq \delta$, we can find a control $h \in L^2(0,T; L^2(\omega)^N)$ and an associated solution (y, z, π_y, π_z) to (1) satisfying $y(\cdot,T) = 0$ in Ω. This finishes the proof of Theorem 1.2. □

5. Conclusion and open problems

We omitted the proof of Theorem 1.3 since it can be done following the proof of Theorem 1.2 an adapting the proof for $\gamma = 0$.

In this article, we mentioned the main results on robust control for the N-dimensional Navier–Stokes system with Dirichlet boundary conditions. These results has also allowed us to characterise the follower control v and its disturbance function ψ through a nonlinear coupled system. Once this step has finished, we used the robust pair (v, ψ) to prove the null controllability of the leader control h. The main novelties are the Carleman inequalities for coupled Stokes system, which involves new relationships between the weight functions and the robustness parameters ℓ, γ, see Proposition 1 and Lemma 6.1. To conclude, we present now some open problems arising from our study:

- If instead of considering in the hierarchical strategy a zero objective for the leader control h in (1), the objective may be a trajectory $(\overline{y}, \overline{\pi})$ of the uncontrolled system:

$$\begin{cases} \overline{y}_t - \Delta\overline{y} + (\overline{y}\cdot\nabla)\overline{y} + \nabla\overline{\pi} = 0 & \text{in} \quad Q, \\ \nabla\cdot\overline{y} = 0, & \text{in} \quad Q, \\ \overline{y} = 0 & \text{on} \quad \Sigma, \\ \overline{y}(\cdot,0) = \overline{y}_0(\cdot) & \text{in} \quad \Omega, \end{cases}$$

  So we may ask if is it possible to prove the local exact controllability to trajectories of system (1). That is, does there exist a control h such that for the corresponding solution to (10) satisfies $y(T) = \overline{y}(T)$?
- Some null controllability results for the N-dimensional Navier–Stokes system [7,8] allow to act on the system by means of few controls. Is it possible to extend these results to a robust Stackelberg strategy? Is is possible to ask the leader control h to have one vanishing component?
- Is it possible to extend the results in this paper to Navier-slip boundary conditions? In other words, can we say something about the existence and uniqueness of saddle points for the Navier–Stokes system with Navier-slip conditions? Does we have the null controllability for the leader control h?
- From a numerical point of view, the implementation of the Stackelberg strategy even for the linear coupled system (10) shows a challenge to overcome.
- Finally, it would be interesting to study the problems proposed in this paper to other models such as water waves (Korteweg–de Vries equation),

interaction fluid–heat (Boussinesq system), micropolar fluids, models of turbulence, among others.

6. Appendix: Some technical results

In the following results, it will be assumed that $N = 2$ or $N = 3$.
From the relation between α^* and $\hat{\alpha}$, it is possible to prove the following inequality.

Lemma 6.1. *For any $\varepsilon > 0$, any $M_1, M_2 \in \mathbb{R}$, there exists $\lambda_0 > 0$ and $C = C(\varepsilon, M_1, M_2) > 0$ such that*

$$e^{s\alpha^*} \leq Cs^{M_1}\lambda^{M_2}(\hat{\xi})^{M_1}e^{s(1+\varepsilon)\hat{\alpha}} \tag{66}$$

for every $\lambda > \lambda_0$.

Proof. Recall that

$$\alpha^*(t) := \max_{x\in\overline{\Omega}} \alpha(x,t), \qquad \widehat{\alpha}(t) := \min_{x\in\overline{\Omega}} \alpha(x,t) \quad \text{and} \quad \widehat{\xi}(t) := \max_{x\in\overline{\Omega}} \xi(x,t).$$

From the definition of α^* and $\widehat{\alpha}$, $\widehat{\alpha}(t) = F(\lambda)\alpha^*$, where

$$F(\lambda) := \frac{e^{2\lambda\|\eta_0\|_\infty} - e^{\lambda\|\eta_0\|_\infty}}{e^{2\lambda\|\eta_0\|_\infty} - 1}.$$

It is easy to check that $F(\lambda) \to 1$ to $\lambda \to +\infty$ and $F(\lambda) \to 1/2$ to $\lambda \to 0^+$. Additionally, by construction of $F(\lambda)$, for any $\varepsilon > 0$, there exists $\lambda_0 > 0$ such that, for every $\lambda \geq \lambda_0$

$$F(\lambda) + \varepsilon F(\lambda) > 1.$$

In consequence, exists a positive constant $C = C(\varepsilon, M_1, \tilde{M}_2)$ such that the inequality

$$\lambda^{\tilde{M}_2}e^{(1-(1+\varepsilon)F(\lambda))s\alpha^*} \leq Cs^{M_1}(\hat{\xi})^{M_1}$$

holds for any $M_1, \tilde{M}_2 \in \mathbb{R}$.

This completes the proof of Lemma 6.1. □

As a consequence of Lemma 6.1, for a_0, m_0 satisfying $2 \leq a_0 < m_0 \leq a_0 + 2$, we can deduce the next result:

Lemma 6.2. *Under the hypothesis of Lemma 6.1, for any $\omega \subset\subset \Omega$ and any $u \in V$, there exists $\lambda_0 > 0$ and $C = C(\varepsilon, \tilde{M}_1, \tilde{M}_2) > 0$ such that*

$$s^{\tilde{M}_1}\lambda^{\tilde{M}_2}\iint_{\omega\times(0,T)} e^{-4s\hat{\alpha}-2a_0s\alpha^*}(\hat{\xi})^{\tilde{M}_1}|\Delta u|^2dxdt \leq Cs^{-1}\iint_Q e^{-2m_0s\alpha^*}(\hat{\xi})^{-1}|\Delta u|^2dxdt.$$

Proof. Sketch Taking $\varepsilon = \dfrac{2}{m_0 - a_0} - 1$, $M_1 = -\dfrac{\tilde{M}_1 + 1}{2(m_0 - a_0)}$ and $M_2 = -\dfrac{\tilde{M}_2 + 1}{2(m_0 - a_0)}$ in (66), the proof is direct. □

References

[1] Abergel, F., Temam, R.: On some control problems in fluid mechanics. Theor. Comput. Fluid Dyn. **1**, 303–325 (1990)

[2] Araruna, F.D., de Menezes, S.D.B., Rojas-Medar, M.A.: On the approximate controllability of Stackelberg–Nash strategies for linearized micropolar fluids. Appl. Math. Optim. **70**, 373–393 (2014). https://doi.org/10.1007/s00245-014-9240-x

[3] Araruna, F.D., Fernández-Cara, E., Santos, M.C.: Stackelberg–Nash exact controllability for linear and semilinear parabolic equations. ESAIM Control Optim. Calc. Var. **21**, 835–856 (2015). https://doi.org/10.1051/cocv/2014052

[4] Bewley, T., Temam, R., Ziane, M.: Existence and uniqueness of optimal control to the Navier–Stokes equations. C. R. Acad. Sci. Paris Sér. I Math. **330**, 1007–1011 (2000). https://doi.org/10.1016/S0764-4442(00)00299-8

[5] Bewley, T.R., Moin, P., Temam, R.: Optimal and robust approaches for linear and nonlinear regulation problems in fluid mechanics. AIAA Paper

[6] Bewley, T.R., Temam, R., Ziane, M.: A general framework for robust control in fluid mechanics. Physica D **138**, 360–392 (2000). https://doi.org/10.1016/S0167-2789(99)00206-7

[7] Carreño, N., Guerrero, S.: Local null controllability of the N-dimensional Navier–Stokes system with $N-1$ scalar controls in an arbitrary control domain. J. Math. Fluid Mech. **15**, 139–153 (2013). https://doi.org/10.1007/s00021-012-0093-2

[8] Coron, J.-M., Lissy, P.: Local null controllability of the three-dimensional Navier–Stokes system with a distributed control having two vanishing components. Invent. Math. **198**, 833–880 (2014). https://doi.org/10.1007/s00222-014-0512-5

[9] Díaz, J.I., Lions, J.L.: On the approximate controllability of Stackelberg–Nash strategies. In: Díaz, J.I. (ed.) Ocean Circulation and Pollution Control—A Mathematical and Numerical Investigation. Springer, Berlin (2004)

[10] Ekeland, I., Témam, R.: Convex analysis and variational problems, vol. 28 of Classics in Applied Mathematics, English edn. Society for Industrial and Applied Mathematics (SIAM), Philadelphia (1999). https://doi.org/10.1137/1.9781611971088. Translated from the French

[11] Fernández-Cara, E., Guerrero, S., Imanuvilov, O.Y., Puel, J.-P.: Local exact controllability of the Navier–Stokes system. J. Math. Pures Appl. (9) **83**, 1501–1542 (2004). https://doi.org/10.1016/j.matpur.2004.02.010

[12] Fernández-Cara, E., González-Burgos, M., Guerrero, S., Puel, J.-P.: Null controllability of the heat equation with boundary Fourier conditions: the linear case. ESAIM Control Optim. Calc. Var. **12**, 442–465 (2006). https://doi.org/10.1051/cocv:2006010

[13] Fursikov, A.V., Imanuvilov, O.Y.: Controllability of Evolution Equations. Lecture Notes Series, vol. 34. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul (1996)

[14] Green, M., Limebeer, D.J.: Linear Robust Control. Courier Corporation, North Chelmsford (2012)

[15] Guerrero, S.: Controllability of systems of Stokes equations with one control force: existence of insensitizing controls. Ann. Inst. H. Poincaré Anal. Non Linéaire **24**, 1029–1054 (2007). https://doi.org/10.1016/j.anihpc.2006.11.001

[16] Gueye, M.: Insensitizing controls for the Navier–Stokes equations. Ann. Inst. H. Poincaré Anal. Non Linéaire **30**, 825–844 (2013). https://doi.org/10.1016/j.anihpc.2012.09.005

[17] Guillén-González, F., Marques-Lopes, F., Rojas-Medar, M.: On the approximate controllability of Stackelberg–Nash strategies for Stokes equations. Proc. Am. Math. Soc. **141**, 1759–1773 (2013). https://doi.org/10.1090/S0002-9939-2012-11459-5

[18] Hamilton, R.S.: The inverse function theorem of Nash and Moser. Bull. Am. Math. Soc. (N.S.) **7**, 65–222 (1982). https://doi.org/10.1090/S0273-0979-1982-15004-2

[19] Hernández-Santamaría, V., de Teresa, L.: Robust stackelberg controllability for a parabolic equation. Preprint arXiv:1610.06149v1

[20] Herzog, R., Schmidt, F.: Weak lower semi-continuity of the optimal value function and applications to worst-case robust optimal control problems. Optimization **61**, 685–697 (2012). https://doi.org/10.1080/02331934.2011.603322

[21] Límaco, J., Clark, H.R., Medeiros, L.A.: Remarks on hierarchic control. J. Math. Anal. Appl. **359**, 368–383 (2009). https://doi.org/10.1016/j.jmaa.2009.05.040

[22] Medjo, T.T., Temam, R., Ziane, M.: Optimal and robust control of fluid flows: some theoretical and computational aspects. Appl. Mech. Rev. **61**, 010802 (2008)

[23] Temam, R.: Navier–Stokes Equations. AMS Chelsea Publishing, Providence (2001). https://doi.org/10.1090/chel/343. (**Theory and numerical analysis, Reprint of the 1984 edition**)

[24] Von Stackelberg, H.: The Theory of the Market Economy. Oxford University Press, Oxford (1952)

[25] Zălinescu, C.: Convex Analysis in General Vector Spaces. World Scientific Publishing Co., Inc., River Edge (2002). https://doi.org/10.1142/9789812777096

Cristhian Montoya and Luz de Teresa
Instituto de Matemáticas
Universidad Nacional Autónoma de México
Circuito Exterior, C.U.
C.P. 04510 Mexico CDMX
Mexico
e-mail: ldeteresa@im.unam.mx

Cristhian Montoya
e-mail: cmontoya@matem.unam.mx

Received: 28 November 2017.
Accepted: 9 August 2018.