Nonlinear Differ. Equ. Appl. (2018) 25:40

1021-9722/18/050001-23
published online August 9, 2018
https://doi.org/10.1007/s00030-018-0531-9

Nonlinear Differential Equations and Applications NoDEA

Existence and asymptotic behavior of sign-changing solutions for fractional Kirchhoff-type problems in low dimensions

Sitong Chen, Xianhua Tang and Fangfang Liao

Abstract. This paper is dedicated to studying the following fractional Kirchhoff-type equation

$$\left(a+b\int_{\mathbb{R}^N}|(-\triangle)^{\alpha/2}u|^2\mathrm{d}x\right)(-\triangle)^{\alpha}u+V(|x|)u=f(|x|,u),\qquad x\in\mathbb{R}^N,$$

where $a,b>0$, either $N=2$ and $\alpha\in(1/2,1)$ or $N=3$ and $\alpha\in(3/4,1)$ holds, $V\in\mathcal{C}(\mathbb{R}^N,[0,\infty))$ and $f\in\mathcal{C}(\mathbb{R}^N\times\mathbb{R},\mathbb{R})$. By combining the constraint variational method with some new inequalities, we prove that the above problem possesses a radial sign-changing solution u_b for $b\geq 0$ without the usual Nehari-type monotonicity condition on f, and its energy is strictly larger than twice that of the ground state radial solutions of Nehari-type. Moreover, we establish the convergence property of u_b as $b\searrow 0$. In particular, our results unify both asymptotically cubic and super-cubic cases, which improve and complement the existing ones in the literature.

Mathematics Subject Classification. 35J20, 35Q55, 35Q51, 53C35.

Keywords. Fractional Kirchhoff type problems, Sign-changing solution, Asymptotic behavior.

1. Introduction

In this paper, we study the existence of sign-changing solutions for the following fractional Kirchhoff-type equation

$$\left(a+b\int_{\mathbb{R}^N}|(-\triangle)^{\alpha/2}u|^2\mathrm{d}x\right)(-\triangle)^{\alpha}u+V(x)u=f(x,u),\qquad x\in\mathbb{R}^N,\quad (1.1)$$

where a,b are positive constants, either $N=2$ and $\alpha\in(1/2,1)$ or $N=3$ and $\alpha\in(3/4,1)$ holds, and $V:\mathbb{R}^N\to\mathbb{R}$ and $f:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ satisfy the following basic assumptions:

(V1) $V \in \mathcal{C}(\mathbb{R}^N, [0, \infty))$, $V(x) = V(|x|)$, and the operator $(-\triangle)^\alpha + V(x) : H^\alpha(\mathbb{R}^N) \to H^{-\alpha}(\mathbb{R}^N)$ satisfies

$$\inf_{\substack{u \in H^\alpha(\mathbb{R}^N) \\ \|u\|_2 = 1}} \int_{\mathbb{R}^N} \left(a|(-\triangle)^{\alpha/2} u|^2 + V(x)u^2\right) \mathrm{d}x > 0;$$

(F1) $f \in \mathcal{C}(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, and there exist $C_0 > 0$ and $2 < p < 2^*_\alpha := 2N/(N-2\alpha)$ such that

$$|f(x,t)| \le C_0 \left(1 + |t|^{p-1}\right), \quad \forall\ (x,t) \in \mathbb{R}^N \times \mathbb{R};$$

(F2) $f(x,t) = o(|t|)$ as $t \to 0$ uniformly in $x \in \mathbb{R}^N$.

The fractional Laplacian $(-\triangle)^\alpha$ in $\mathbb{R}^N$ is a nonlocal pseudo-differential operator taking the form

$$(-\triangle)^\alpha u(x) = C_{N,s} \mathrm{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2\alpha}} \mathrm{d}y, \quad x \in \mathbb{R}^N,\ u \in \mathcal{S}(\mathbb{R}^N), \tag{1.2}$$

where P.V. is the principal value, $C_{N,\alpha}$ is a normalization constant and $\mathcal{S}(\mathbb{R}^N)$ is the Schwartz space of rapidly decaying $\mathcal{C}^1$ functions in $\mathbb{R}^N$, see [3,24] for more details. In this paper, we consider the fractional Laplacian in the weak sense. As usual, for any $\alpha \in (0,1)$, we let

$$\int_{\mathbb{R}^N} (-\triangle)^{\alpha/2} u (-\triangle)^{\alpha/2} v = C_{N,\alpha} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{[u(x) - u(y)][v(x) - v(y)]}{|x-y|^{N+2\alpha}} \mathrm{d}x\mathrm{d}y,$$

$$\|(-\triangle)^{\alpha/2} u\|_2^2 = C_{N,\alpha} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{[u(x) - u(y)]^2}{|x-y|^{N+2\alpha}} \mathrm{d}x\mathrm{d}y,$$

and define the fractional Sobolev space

$$H^\alpha(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : (-\triangle)^{\alpha/2} u \in L^2(\mathbb{R}^N) \right\}$$

endowed with scalar product and norm

$$(u,v)_{H^\alpha} = \int_{\mathbb{R}^N} \left[(-\triangle)^{\alpha/2} u (-\triangle)^{\alpha/2} v + uv\right] \mathrm{d}x,$$

$$\|u\|_{H^\alpha} = \left(\int_{\mathbb{R}^N} \left[|(-\triangle)^{\alpha/2} u|^2 + u^2\right] \mathrm{d}x \right)^{1/2}.$$

Problem (1.1) has a strong physical meaning because it models, as a special significant case, the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string, see Fiscella and Valdinoci [21] for more details. In particular, when $a = 1$ and $b = 0$, (1.1) reduces to the following fractional Schrödinger equation

$$(-\triangle)^\alpha u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N, \tag{1.3}$$

which has been proposed by Laskin [24] in fractional quantum mechanics as a result of extending the Feynman integrals from the Brownian like to the Lèvy like quantum mechanical paths. Pioneered from [19,20] via variational methods, the existence and multiplicity of solutions for problems like (1.3) have been intensively studied under various hypotheses on V and f. Concerning the existence of sign-changing solutions, we refer to [10,18,28,40,41] and so on. In

[10], by using the s-harmonic extension introduced by Caffarelli and Silvestre [9] (see also [32]), via the method of invariant sets of descent flow, sign-changing solutions of the following equation

$$\begin{cases}(-\triangle)^s u = f(x,u), & \text{in } \Omega,\\ u=0, & \text{in } \mathbb{R}^N\backslash\Omega,\end{cases} \tag{1.4}$$

was obtained, where $s\in(0,1)$ is fixed, $N>2s$, Ω denotes an open bounded set in $\mathbb{R}^N$ with smooth boundary. We must note that the s-harmonic extension transform the nonlocal problem in to a local problem in the half cylinder $\Omega\times[0,+\infty)$. In this sense the authors in [9] still dealt with a local problem. While in [18], the authors investigated the existence of sign-changing solutions of (1.4) directly by the definition (1.2).

If $b=0$ and $\alpha=1$, then (1.1) formally becomes the semilinear Schrödinger equation. As far as we know, there are different ways to get the sign-changing solutions for such a local equation, such as, constructing invariant sets and descending flow (Bartsch et al. [5]), the Ekeland's variational principle and the implicit function theorem (Noussair and Wei [33]), variational method together with the Brouwer degree theory (Bartsh and Weth [6]), we refer the reader to the book [46] of Zou for more discussions. These methods rely on fact that if u belongs to the corresponding nodal Nehari manifold, then $u^{\pm}$ belongs to the Nehari manifold related to (1.3) with $\alpha=1$ due to the following decomposition:

$$\|\nabla(u^+ + u^-)\|_2^2 = \|\nabla u^+\|_2^2 + \|\nabla u^-\|_2^2, \qquad \forall\ u\in H^1(\mathbb{R}^N),$$

where

$$u^+(x):=\max\{u(x),0\} \quad \text{ and } \quad u^-(x):=\min\{u(x),0\}.$$

If $b>0$ and $\alpha=1$, then (1.1) formally reduces to the following well-known Kirchhoff equation

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\right)\triangle u + V(x)u = f(x,u), \qquad x\in\mathbb{R}^N, \tag{1.5}$$

which is related to the stationary analogue of the equations

$$u_{tt}-\left(a+b\int_{\Omega}|\nabla u|^2\mathrm{d}x\right)\triangle u = f(x,u),$$

when $V=0$ and $\mathbb{R}^N$ is replaced by a bounded domain $\Omega\subset\mathbb{R}^N$. Such a equation is first proposed by Kirchhoff [23] as an extension of the classical d'Alembert wave equations for free vibrations of elastic strings. For more mathematical and physical background on Kirchhoff type problems, we refer the readers to [4,7,8,16] and the references therein. After Lions [26] proposed an abstract functional analysis framework to Kirchhoff equation (1.5), problems like (1.5) received more and more attention on mathematical studies. As far as we know, there are many papers on sign-changing solutions, see e.g. [15,17,29,30,35,36,44] when $f(x,t)$ is super-cubic or asymptotically cubic at $t=\infty$.

More recently, many researchers began to focus on problems like (1.1), especially on the existence of positive solutions, multiple solutions and ground state solutions, see for example, [2,22,27,34,43] and the references therein. However, there exist very few results on sign-changing solutions of (1.1). In fact, in the case $\alpha \in (0,1)$, we have the following decomposition:

$$\begin{aligned}\|(-\triangle)^{\alpha/2}u^+ + (-\triangle)^{\alpha/2}u^-\|_2^2 &= \|(-\triangle)^{\alpha/2}u^+\|_2^2 + \|(-\triangle)^{\alpha/2}u^-\|_2^2\\ &-4C_{N,\alpha}\int_{\mathbb{R}^{2N}}\frac{u^+(x)u^-(y)}{|x-y|^{N+2\alpha}}\mathrm{d}x\mathrm{d}y, \quad \forall\ u \in H^\alpha(\mathbb{R}^N).\end{aligned} \tag{1.6}$$

The difficulty in finding sign-changing solutions of (1.1) results from two nonlocal terms: $(-\triangle)^\alpha u$ and $\|(-\triangle)^{\alpha/2}u\|_2^2(-\triangle)^\alpha u$. In this sense, (1.1) is different from the classical case $\alpha = 1$ and the methods of finding sign-changing solutions for (1.3) with $\alpha \in (0,1]$ and (1.5) can not be directly applied to (1.1). This causes some mathematical difficulties which make the study of sign-changing solutions for (1.1) particularly interesting. To the best of our knowledge, there seems to be only one paper [11] dealing with this problem. More precisely, when $V \in \mathcal{C}(\mathbb{R}^N,\mathbb{R})$ is coercive and $\inf_{\mathbb{R}^N} V > 0$, inspired by Alves and Nòbrega [1], by the minimization argument on the nodal Nehari manifold and quantitative deformation lemma [42], Cheng and Gao [11] proved the existence and asymptotic behavior of sign-changing solutions for (1.1), where f satisfies (F1), (F2) and the following assumptions:

(F3′) $\lim_{|t|\to\infty}\frac{f(x,t)}{t^3} = +\infty$ uniformly in $x \in \mathbb{R}^N$;
(F4′) $\frac{f(x,t)}{|t|^3}$ is nondecreasing in t on $(-\infty,0)\cup(0,\infty)$ for every $x \in \mathbb{R}^N$.

It is easy to see that (F1) and (F3′) imply that $2_\alpha^* = 2N/(N-2\alpha) > 4$, i.e., $N < 4\alpha$, which is the focus of the present paper, and (F4′) implies the weak Ambrosetti-Rabinowitz type condition:

(AR) $f(x,t)t \geq 4F(x,t) > 0$ for $x \in \mathbb{R}^N,\ t \in \mathbb{R}\backslash\{0\}$,

which would readily imply the boundedness of Palais-Smale sequences. We point out that the method used in [11] relies heavily on (F3′) and (F4′). Indeed, because $\|(-\triangle)^{\alpha/2}u\|_2^4$ is homogeneous of degree 4, in [11], the super-cubic condition (F3′) plays a crucial role in using Miranda's theorem [31] to show that the nodal Nehari manifold related to (1.1) is not empty, and the monotonicity condition (F4′) is essential to show that the minimizer is a critical point. Obviously, the method is no longer applicable for (1.1) with more general nonlinearities, even for the special form $f(x,u) = u^3$.

Now, a natural question is whether (F3′) and (F4′) can be relaxed to obtain the existence and asymptotic behavior of sign-changing solutions for (1.1).

Motivated by the above works, in the present paper, we shall solve the above problem by using the following conditions:

(F3) $\lim_{|t|\to\infty}\frac{|t|^{4\alpha-N}f(x,t)}{t^3} = +\infty$ uniformly in $x \in \mathbb{R}^N$ $(N < 4\alpha)$;
(F4) $\frac{f(x,t)-V(x)t}{|t|^3}$ is nondecreasing in t on both $(-\infty,0)$ and $(0,\infty)$ for every $x \in \mathbb{R}^N$,

instead of (F3′) and (F4′). More precisely, by combining the constraint variational method with some new inequalities, we will prove that (1.1) with $b \geq 0$ has a radial sign-changing solution u_b, and its energy is strictly larger than twice that of the ground state radial solutions of Nehari-type. Furthermore, we establish the convergence property of u_b as the parameter $b \searrow 0$. Note that (F3′) implies (F3) due to $2 \leq N < 4\alpha$, and (F4′) implies (F4) due to $\inf_{x\in\mathbb{R}^N} V(x) \geq 0$. We would like to mention that the nodal sets and nodal domains of sign-changing solutions for (1.1) is very difficult, even if for Eq. (1.4), see [18, Remark 1.1].

Throughout this paper, we define

$$H_r^\alpha(\mathbb{R}^N) = \left\{u \in H^\alpha(\mathbb{R}^N) : u(x) = u(|x|)\right\},$$

and denote the fractional Sobolev space for (1.1) by

$$E = \left\{u \in H_r^\alpha(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 \mathrm{d}x < \infty\right\}$$

equipped with scalar product and norm

$$(u,v) = \int_{\mathbb{R}^N} \left[a(-\triangle)^{\alpha/2}u(-\triangle)^{\alpha/2}v + V(x)uv\right] \mathrm{d}x,$$
$$\|u\| = \left(\int_{\mathbb{R}^N} \left[a|(-\triangle)^{\alpha/2}u|^2 + V(x)u^2\right] \mathrm{d}x\right)^{1/2}.$$

The embedding $E \hookrightarrow H_r^\alpha(\mathbb{R}^N)$ is continuous due to (V1) and $a > 0$. In view of P.L. Lions [25], the embedding $E \hookrightarrow L^q(\mathbb{R}^N)$ is compact for $2 < q < 2_\alpha^*$ when $\alpha \in (0,1)$ and $N \geq 2$.

Define the energy functional $\Phi_b : E \to \mathbb{R}$ by

$$\Phi_b(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left[a|(-\triangle)^{2/\alpha}u|^2 + V(x)u^2\right] \mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^N} |(-\triangle)^{2/\alpha}u|^2 \mathrm{d}x\right)^2 - \int_{\mathbb{R}^N} F(x,u)\mathrm{d}x. \tag{1.7}$$

Using (F1) and (F2), it is easy to see that $\Phi_b \in \mathcal{C}^1(E,\mathbb{R})$. Moreover, for any $u, \varphi \in E$, we have

$$\langle \Phi_b'(u), \varphi\rangle = \int_{\mathbb{R}^N} \left[a(-\triangle)^{2/\alpha}u(-\triangle)^{2/\alpha}\varphi + V(x)u\varphi\right] \mathrm{d}x + b\int_{\mathbb{R}^N} |(-\triangle)^{2/\alpha}u|^2 \mathrm{d}x \int_{\mathbb{R}^N} (-\triangle)^{2/\alpha}u(-\triangle)^{2/\alpha}\varphi \mathrm{d}x - \int_{\mathbb{R}^N} f(x,u)\varphi \mathrm{d}x. \tag{1.8}$$

Clearly, critical points of Φ_b are the weak solutions for (1.1) in E. If $u \in E$ is a solution of (1.1) and $u^\pm \neq 0$, we say that u is a radial sign-changing solution of (1.1). Let

$$\mathcal{M}_b := \left\{u \in E : u^\pm \neq 0, \langle \Phi_b'(u), u^+\rangle = \langle \Phi_b'(u), u^-\rangle = 0\right\},$$
$$m_b := \inf_{u\in\mathcal{M}_b} \Phi_b(u), \quad \forall\, b \geq 0,$$
$$\mathcal{N}_b := \{u \in E : u \neq 0, \langle \Phi_b'(u), u\rangle = 0\}, \quad c_b := \inf_{u\in\mathcal{N}_b} \Phi_b(u), \quad \forall\, b \geq 0.$$

By (1.6), (1.7) and a simple calculation, one has

$$\begin{aligned}\Phi_b(u) &= \Phi_b(u^+) + \Phi_b(u^-) + 2aP(u^+,u^-) + \frac{b}{2}\|(-\triangle)^{\alpha/2}u^+\|_2^2\|(-\triangle)^{\alpha/2}u^-\|_2^2 \\ &\quad + 2bP(u^+,u^-)\left[\|(-\triangle)^{\alpha/2}u^+\|_2^2 + \|(-\triangle)^{\alpha/2}u^-\|_2^2 + 2P(u^+,u^-)\right] \\ &> \Phi_b(u^+) + \Phi_b(u^-), \quad \forall\, u \in E,\ u^+, u^- \neq 0, \end{aligned} \tag{1.9}$$

where

$$P(u^+,u^-) := -C_{N,\alpha}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^+(x)u^-(y)}{|x-y|^{N+2\alpha}}\mathrm{d}x\mathrm{d}y > 0, \quad \forall\, u \in E,\ u^+, u^- \neq 0. \tag{1.10}$$

Similarly, one has

$$\begin{aligned}&\langle \Phi_b'(u), u^+\rangle > \langle \Phi_b'(u^+), u^+\rangle, \quad \langle \Phi_b'(u), u^-\rangle > \langle \Phi_b'(u^-), u^-\rangle,, \\ &\forall\, u \in E,\ u^+, u^- \neq 0, \end{aligned} \tag{1.11}$$

which implies that $u^\pm \notin \mathcal{N}_b$ for $u \in \mathcal{M}_b$.

To state our results, we introduce the following assumption:

(V2) there exists a sequence $\{t_n\} \subset (0,\infty)$ such that $t_n \to \infty$ and $\sup_{x\in\mathbb{R}^N, n\in\mathbb{N}} \frac{V(t_n x)}{t_n^{N+2-4\alpha}V(x)} < \infty$.

Now, we state the main results of this paper.

Theorem 1.1. *Assume that (V1), (V2) and (F1)–(F4) hold. Then* (1.1) *with* $b \geq 0$ *has a radial sign-changing solution* $u_b \in \mathcal{M}_b$ *such that* $\Phi_b(u_b) = \inf_{\mathcal{M}_b}\Phi_b > 0$.

Theorem 1.2. *Assume that (V1), (V2) and (F1)–(F4) hold. Then* (1.1) *with* $b \geq 0$ *has a radial solution* $\bar{u}_b \in \mathcal{N}_b$ *such that* $\Phi_b(\bar{u}_b) = \inf_{\mathcal{N}_b}\Phi_b > 0$. *Moreover,* $m_b > 2c_b$ *for all* $b \geq 0$.

Theorem 1.3. *Assume that (V1), (V2) and (F1)–(F4) hold. For any sequence* $\{b_n\}$ *with* $b_n \searrow 0$ *as* $n \to \infty$, *there exists a subsequence which we label in the same way such that* $u_{b_n} \to v_0$ *in* E, *where* $v_0 \in \mathcal{M}_0$ *is a radial sign-changing solution of* (1.3) *with* $\Phi_0(v_0) = \inf_{\mathcal{M}_0}\Phi_0 > 0$.

To obtain Theorem 1.1, we prove that the minimizers of m_b is radial sign-changing solutions of (1.1). Here, we must overcome three main difficulties: (I) showing $\mathcal{M}_b \neq \emptyset$ under (F3) instead of (F3′) (because the term $\|(-\triangle)^{\alpha/2}u\|_2^4$ is homogeneous of degree 4); (II) verifying the boundedness of the minimizing sequence of Φ_b on $\mathcal{M}_b$ (due to the lack of the condition (AR)); (III) proving the minimizer of Φ_b on $\mathcal{M}_b$ is a critical point (because the Nehari-type monotonicity condition (F4′) is not assumed, and $f \notin \mathcal{C}^1$).

To overcome difficulty I), we first show that for $b \geq 0$ the following set

$$\begin{aligned}\mathcal{E}_b := \Big\{ u \in E : b\|(-\triangle)^{\alpha/2}u\|_2^2\int_{\mathbb{R}^N}(-\triangle)^{\alpha/2}u(-\triangle)^{\alpha/2}u^\pm \mathrm{d}x \Big\} \\ + \int_{\mathbb{R}^N}\left[V(x)(u^\pm)^2 - f(x,u^\pm)u^\pm\right]\mathrm{d}x < 0 \Big\} \end{aligned} \tag{1.12}$$

is not empty by scaling technique (see Lemma 2.5), then prove that for each $u \in \mathcal{E}_b$, there is a unique pair $(s,t) \in (\mathbb{R}^+ \times \mathbb{R}^+)$ such that $su^+ + tu^- \in \mathcal{M}_b$ by combining some new inequalities with Miranda's Theorem [31] (see Lemma 2.6). On the basis of these, we establish the following new minimax characterization:

$$m_b = \inf_{u\in\mathcal{E}_b} \max_{s,t\geq 0} \Phi_b(su^+ + tu^-), \quad \forall\, b \geq 0,$$

(see Lemma 2.7). Under (F4) instead of (F4′), we use some new tricks to overcome difficulties (II) and (III) (see Lemma 2.8).

To prove Theorem 1.2, we consider the minimization problem c_b and prove that the minimizers are ground state solutions for (1.1). To this end, we construct the following minimax characterization:

$$c_b = \inf_{u\in\bar{\mathcal{E}}_\lambda} \max_{t\geq 0} \Phi_b(tu), \quad \forall\, b \geq 0,$$

where

$$\bar{\mathcal{E}}_b := \left\{ u \in E : \ b\|(-\triangle)^{\alpha/2}u\|_2^4 + \int_{\mathbb{R}^N} \left[V(x)u^2 - f(x,u)u \right] \mathrm{d}x < 0 \right\}. \tag{1.13}$$

To prove Theorem 1.3, different from the super-cubic case, that is (F3′) holds, we use some new inequalities to prove that there exists a constant $\Lambda_0 > 0$ independent of b such that $\Phi_b(u_b) \leq \Lambda_0$ for $b \in [0,1]$, and $\lim_{n\to\infty} m_{b_n} = m_0$ with $b_n \searrow 0$ (see (3.5) and (3.10)).

Example 1.4. *If $V \in L^\infty(\mathbb{R}^N)$ or $V(x) = 1 + |x|^{N+2-4\alpha}$ for $x \in \mathbb{R}^N$, then it follows from (V1) that (V2) holds naturally. There are also many unbounded and non-monotonous functions satisfying (V1) and (V2), for example, $V(x) = 1 + |x|^{1-s}\left[1 + \sin^2(\pi|x|)\right]$ with $t_n = n$.*

Example 1.5. *When $\inf_{x\in\mathbb{R}^N} V(x) \geq 0$, there are many functions satisfying (F1)-(F4), but do not satisfy (F3′) or (F4′). Let $K \in \mathcal{C}(\mathbb{R}^N, [a_1,a_2])$ with $a_1, a_2 > 0$,*

$$f_1(x,t) = K(x)t^3 - K_1(x)|t|^{q-1}t \quad \textit{with} \quad K_1 \in \mathcal{C}(\mathbb{R}^N, [0,\infty)) \ \textit{and}\ 1 < q < 3;$$
$$f_2(x,t) = K(x)|t|^3 t + K_2(x)|t|t \quad \textit{with} \quad K_2 \in \mathcal{C}(\mathbb{R}^N, (0,\infty));$$
$$f_3(x,t) = K(x)t^3 - |t|^{3/2}t + |t|t.$$

Obviously, f_1 satisfies (F1)-(F4) when $\inf_{x\in\mathbb{R}^N} V(x) \geq 0$, but does not satisfy (F3′), It is easy to see that f_2, f_3 satisfy (F1)-(F4) when $\inf_{x\in\mathbb{R}^N} V(x) \geq 1$. Moreover, f_2 does not satisfy (F4′), and f_3 does not satisfy either (F3′) or (F4′).

Remark 1.6. Due to (F3), we treat asymptotically cubic and super-cubic nonlinearities in a unified way. Our results improve and complement the corresponding ones in the literatures.

Remark 1.7. Our results are available for Kirchhoff-type Eq. (1.5) with slight modification. In fact, if $\alpha = 1$, then (F3) becomes the following limits hold uniformly in $x \in \mathbb{R}^N$

$$\lim_{|t|\to\infty} \frac{f(x,t)}{t} = +\infty \text{ if } N=2; \qquad \lim_{|t|\to\infty} \frac{tf(x,t)}{|t|^3} = +\infty \text{ if } N=3.$$

From this point of view, we give an extension of the corresponding results in [15,17,29,30,35,36,44], which considered (1.5) with super-cubic and asymptotically cubic nonlinearities respectively.

Throughout this paper, we denote the usual norm of $L^s(\cdot)$ by $\|u\|_s$ for $s \geq 2$, $B_r(x) = \{y \in \mathbb{R}^3 : |y-x| < r\}$, and positive constants possibly different in different places, by $C_1, C_2, \ldots$.

2. Preliminary lemmas

In this section, we give some preliminary lemmas for proving our results.

Lemma 2.1. [40] *For all $u \in H^\alpha(\mathbb{R}^N)$, the following facts hold.*

i. $\|(-\triangle)^{\alpha/2}u^{\pm}\|_2 \leq \|(-\triangle)^{\alpha/2}u\|_2$;

ii.

$$\int_{\mathbb{R}^N} (-\triangle)^{\alpha/2}u(-\triangle)^{\alpha/2}u^{\pm}\mathrm{d}x = \|(-\triangle)^{\alpha/2}u^{\pm}\|_2^2 - 2C_{N,s}\int_{\mathbb{R}^{2N}} \frac{u^+(x)u^-(y)}{|x-y|^{N+2\alpha}}\mathrm{d}x\mathrm{d}y. \tag{2.1}$$

Inspired by [36, Lemma 2.1] (or [13,14,37–39,45]), we establish some new inequalities, which are key points in the present paper.

Lemma 2.2. *Assume that (V1), (F1), (F2) and (F4) hold. Then*

$$\begin{aligned}\Phi_b(u) \geq \Phi_b(su^+ + tu^-) &+ \frac{1-s^4}{4}\langle \Phi_b'(u), u^+\rangle + \frac{1-t^4}{4}\langle \Phi_b'(u), u^-\rangle \\ &+ \frac{a(1-s^2)^2}{4}\|(-\triangle)^{\alpha/2}u^+\|_2^2 + \frac{a(1-t^2)^2}{4}\|(-\triangle)^{\alpha/2}u^-\|_2^2, \\ &\forall\ u \in E,\ \ s,t \geq 0.\end{aligned} \tag{2.2}$$

Proof. By (F4), one has

$$\begin{aligned}&\frac{1-t^4}{4}\tau f(x,\tau) + F(x,t\tau) - F(x,\tau) + \frac{V(x)}{4}(1-t^2)^2\tau^2 \\ &= \int_t^1 \left[\frac{f(x,\tau)-V(x)\tau}{\tau^3} - \frac{f(x,s\tau)-V(x)s\tau}{(s\tau)^3}\right] s^3\tau^4 \mathrm{d}s \geq 0, \\ &\forall\ t \geq 0, \quad \tau \in \mathbb{R}\backslash\{0\}.\end{aligned} \tag{2.3}$$

Thus, it follows from (1.6), (1.7), (1.8), (1.10) and (2.3) that

$$\begin{aligned}&\Phi_b(u) - \Phi_b(su^+ + tu^-) \\ &= \frac{a}{2}\left(\|(-\triangle)^{\alpha/2}u\|_2^2 - \|s(-\triangle)^{\alpha/2}u^+ + t(-\triangle)^{\alpha/2}u^-\|_2^2\right) \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^N} V(x)\left[u^2 - (su^+ + tu^-)^2\right]\mathrm{d}x \\ &\quad + \frac{b}{4}\left(\|(-\triangle)^{\alpha/2}u\|_2^4 - \|s(-\triangle)^{\alpha/2}u^+ + t(-\triangle)^{\alpha/2}u^-\|_2^4\right)\end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^N}[F(x,su^++tu^-)-F(x,u)]\mathrm{d}x\\
=&\frac{a}{2}\left[(1-s^2)\|(-\triangle)^{\alpha/2}u^+\|_2^2+(1-t^2)\|(-\triangle)^{\alpha/2}u^-\|_2^2+4(1-st)P(u^+,u^-)\right]\\
&+\frac{1-s^2}{2}\int_{\mathbb{R}^N}V(x)(u^+)^2\mathrm{d}x+\frac{1-t^2}{2}\int_{\mathbb{R}^N}V(x)(u^-)^2\mathrm{d}x\\
&+\frac{b}{4}\left[\|(-\triangle)^{\alpha/2}u^+\|_2^2+\|(-\triangle)^{\alpha/2}u^+\|_2^2+4P(u^+,u^-)\right]^2\\
&-\frac{b}{4}\left[s^2\|(-\triangle)^{\alpha/2}u^+\|_2^2+t^2\|(-\triangle)^{\alpha/2}u^+\|_2^2+4stP(u^+,u^-)\right]^2\\
&+\int_{\mathbb{R}^N}[F(x,su^+)+F(x,tu^-)-F(x,u^+)-F(x,u^-)]\mathrm{d}x\\
=&\frac{1-s^4}{4}\left\{a\|(-\triangle)^{\alpha/2}u^+\|_2^2+2P(u^+,u^-)+\int_{\mathbb{R}^N}V(x)(u^+)^2\mathrm{d}x\right.\\
&+b\|(-\triangle)^{\alpha/2}u^++(-\triangle)^{\alpha/2}u^-\|_2^2\left[\|(-\triangle)^{\alpha/2}u^+\|_2^2+2P(u^+,u^-)\right]\\
&\left.-\int_{\mathbb{R}^N}f(x,u^+)u^+\mathrm{d}x\right\}\\
&+\frac{1-t^4}{4}\left\{a\|(-\triangle)^{\alpha/2}u^-\|_2^2+2P(u^+,u^-)+\int_{\mathbb{R}^N}V(x)(u^-)^2\mathrm{d}x\right.\\
&+b\|(-\triangle)^{\alpha/2}u^++(-\triangle)^{\alpha/2}u^-\|_2^2\left[\|(-\triangle)^{\alpha/2}u^-\|_2^2+2P(u^+,u^-)\right]\\
&\left.-\int_{\mathbb{R}^N}f(x,u^-)u^-\mathrm{d}x\right\}\\
&+\frac{a(1-s^2)^2}{4}\|(-\triangle)^{\alpha/2}u^+\|_2^2+\frac{a(1-t^2)^2}{4}\|(-\triangle)^{\alpha/2}u^-\|_2^2\\
&+\frac{b(s^2-t^2)^2}{4}[P(u^+,u^-)]^2\\
&+\frac{a}{2}\left[(s^2-t^2)^2+2(st-1)^2\right]P(u^+,u^-)+\frac{b(s^2-t^2)^2}{4}\\
&\|(-\triangle)^{\alpha/2}u^+\|_2^2\|(-\triangle)^{\alpha/2}u^-\|_2^2\\
&+\frac{b(s-t)^2}{2}\left\{\left[(s+t)^2+2s^2\right]\|(-\triangle)^{\alpha/2}u^+\|_2^2\right.\\
&\left.+\left[(s+t)^2+2t^2\right]\|(-\triangle)^{\alpha/2}u^-\|_2^2\right\}P(u^+,u^-)\\
&+\int_{\mathbb{R}^N}\left[\frac{1-s^4}{4}f(x,u^+)u^++F(x,su^+)-F(x,u^+)+\frac{(1-s^2)^2}{4}V(x)(u^+)^2\right]\mathrm{d}x\\
&+\int_{\mathbb{R}^N}\left[\frac{1-t^4}{4}f(x,u^-)u^-+F(x,tu^-)-F(x,u^-)+\frac{(1-t^2)^2}{4}V(x)(u^-)^2\right]\mathrm{d}x\\
\geq&\frac{1-s^4}{4}\langle\Phi_b'(u),u^+\rangle+\frac{1-t^4}{4}\langle\Phi_b'(u),u^-\rangle\\
&+\frac{a(1-s^2)^2}{4}\|(-\triangle)^{\alpha/2}u^+\|_2^2+\frac{a(1-t^2)^2}{4}\|(-\triangle)^{\alpha/2}u^-\|_2^2,\\
&\quad\forall\, u\in E,\ s,t\geq 0.
\end{aligned}
$$

This shows that (2.2) holds. □

Lemma 2.3. *Assume that (V1), (F1), (F2) and (F4) hold. Then*

$$\Phi_b(u) \geq \Phi_b(tu) + \frac{1-t^4}{4}\langle \Phi_b'(u), u\rangle + \frac{a(1-t^2)^2}{4}\|(-\triangle)^{\alpha/2}u\|_2^2, \quad \forall\, u \in E,\ t \geq 0. \tag{2.4}$$

Proof. From (1.7), (1.8) and (2.3), we can deduce that (2.4) holds. □

Corollary 2.4. *Assume that (V1), (F1), (F2) and (F4) hold. Then*

$$\Phi_b(u^+ + u^-) = \max_{s,t\geq 0}\Phi_b(su^+ + tu^-), \qquad \forall\, u = u^+ + u^- \in \mathcal{M}_b, \tag{2.5}$$

$$\Phi_b(u) = \max_{t\geq 0}\Phi_b(tu), \qquad \forall\, u \in \mathcal{N}_b. \tag{2.6}$$

Unlike the super-cubic case, to show $\mathcal{M}_b$ and $\mathcal{N}_b$ are not empty in our situation, we have to overcome the competing effect of $\|(-\triangle)^{\alpha/2}u\|_2^4$. To this end, we establish the following lemma.

Lemma 2.5. *Assume that (V1), (V2) and (F1)–(F4) hold. Then $\mathcal{E}_b \neq \emptyset$ and $\bar{\mathcal{E}}_b \neq \emptyset$. Moreover, $\mathcal{M}_b \subset \mathcal{E}_b$ and $\mathcal{N}_b \subset \bar{\mathcal{E}}_b$, where $\mathcal{E}_b$ and $\bar{\mathcal{E}}_b$ are defined by* (1.12) *and* (1.13).

Proof. Let

$$\phi(u, u^{\pm}) := \|(-\triangle)^{\alpha/2}u\|_2^2 \int_{\mathbb{R}^N} (-\triangle)^{\alpha/2}u(-\triangle)^{\alpha/2}u^{\pm}\mathrm{d}x, \qquad \forall\, u \in E. \tag{2.7}$$

Set $u_t(x) = u(t^{-1}x)$ for $t > 0$. For any given $u \in E$ with $u^{\pm} \neq 0$, by (2.7) and an elementary computation, one has

$$\begin{aligned}\phi\left(tu_t, t(u^{\pm})_t\right) &= \|(-\triangle)^{\alpha/2}(tu_t)\|_2^2 \int_{\mathbb{R}^N} (-\triangle)^{\alpha/2}tu_t(-\triangle)^{\alpha/2}t(u^{\pm})_t\mathrm{d}x \\ &= t^{2N+4-4\alpha}\|(-\triangle)^{\alpha/2}u\|_2^2 \int_{\mathbb{R}^N} (-\triangle)^{\alpha/2}u(-\triangle)^{\alpha/2}u^{\pm}\mathrm{d}x \\ &= t^{2N+4-4\alpha}\phi(u, u^{\pm}), \qquad \forall\, t > 0. \end{aligned}\tag{2.8}$$

By (V2), there exists $\beta > 0$ such that

$$V(t_n x) \leq \beta V(x) t_n^{N+2-4\alpha}, \qquad \forall\, x \in \mathbb{R}^N,\ n \in \mathbb{N}. \tag{2.9}$$

Then, it follows from (2.8) and (2.9) that

$$\begin{aligned} &b\phi\left(t_n u_{t_n}, t_n(u^{\pm})_{t_n}\right) + \int_{\mathbb{R}^N}\left[V(x)|t_n(u^{\pm})_{t_n}|^2 - f(x, t_n(u^{\pm})_{t_n})t_n(u^{\pm})_{t_n}\right]\mathrm{d}x \\ &= t_n^{2N+4-4\alpha}\left[b\phi(u,u^{\pm}) + \int_{\mathbb{R}^N}\frac{V(t_n x)}{t_n^{N+2-4\alpha}}(u^{\pm})^2\mathrm{d}x - \int_{\mathbb{R}^N}\frac{f(t_n x, t_n u^{\pm})t_n u^{\pm}}{t_n^{N+4-4\alpha}}\mathrm{d}x\right] \\ &\leq t_n^{2N+4-4\alpha}\left[b\phi(u,u^{\pm}) + \beta\int_{\mathbb{R}^N}V(x)(u^{\pm})^2\mathrm{d}x - \int_{\mathbb{R}^N}\frac{f(t_n x, t_n u^{\pm})t_n u^{\pm}}{t_n^{N+4-4\alpha}}\mathrm{d}x\right], \end{aligned}$$

which, together with (F3) and $2 \leq N < 4\alpha$, yields

$$b\phi\left(t_n u_{t_n}, t_n(u^{\pm})_{t_n}\right) + \int_{\mathbb{R}^N}\left[V(x)|t_n(u^{\pm})_{t_n}|^2 - f(x, t_n(u^{\pm})_{t_n})t_n(u^{\pm})_{t_n}\right] \mathrm{d}x \to -\infty, \quad \text{as} \quad n \to \infty.$$

Taking $v_n = t_n u_{t_n}$, we have $v_{n_0} \in \mathcal{E}_b$ for n_0 large enough. Since

$$b\|(-\triangle)^{\alpha/2} v_{n_0}^{\pm}\|_2^4 \le b\|(-\triangle)^{\alpha/2} v_{n_0}\|_2^2 \int_{\mathbb{R}^N} (-\triangle)^{\alpha/2} v_{n_0} (-\triangle)^{\alpha/2} v_{n_0}^{\pm} \mathrm{d}x$$
$$= b\phi(v_{n_0}, v_{n_0}^{\pm}),$$

we have $v_{n_0}^{\pm} \in \bar{\mathcal{E}}_b$. Hence, $\mathcal{E}_b \neq \emptyset$ and $\bar{\mathcal{E}}_b \neq \emptyset$. By (1.8), we get $\mathcal{M}_b \subset \mathcal{E}_b$ and $\mathcal{N}_b \subset \bar{\mathcal{E}}_b$. □

With the help of Lemma 2.5, we prove that $\mathcal{M}_b$ and $\mathcal{N}_b$ are not empty in the following lemma.

Lemma 2.6. *Assume that (V1), (V2) and (F1)-(F4) hold. Then*

i. *for $u \in \mathcal{E}_b$, there exists a unique pair (s_u, t_u) of positive numbers such that $s_u u^+ + t_u u^- \in \mathcal{M}_b$;*
ii. *for $u \in \bar{\mathcal{E}}_b$, there exists a unique $\bar{t}_u > 0$ such that $\bar{t}_u u \in \mathcal{N}_b$.*

Proof. Let

$$\begin{aligned} g_1(s,t) &= \langle \Phi_b'(su^+ + tu^-), su^+\rangle \\ &= a\int_{\mathbb{R}^N} (-\triangle)^{\alpha/2}(su^+ + tu^-)(-\triangle)^{\alpha/2} su^+ \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} \left[V(x)(su^+)^2 - f(x, su^+)su^+\right] \mathrm{d}x \\ &\quad + b\|(-\triangle)^{\alpha/2}(su^+ + tu^-)\|_2^2 \int_{\mathbb{R}^N} (-\triangle)^{\alpha/2}(su^+ + tu^-)(-\triangle)^{\alpha/2}(su^+)\mathrm{d}x, \end{aligned} \tag{2.10}$$

$$\begin{aligned} g_2(s,t) &= \langle \Phi_b'(su^+ + tu^-), tu^-\rangle \\ &= a\int_{\mathbb{R}^N} (-\triangle)^{\alpha/2}(su^+ + tu^-)(-\triangle)^{\alpha/2} tu^- \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} \left[V(x)(tu^-)^2 - f(x, tu^-)tu^-\right] \mathrm{d}x \\ &\quad + b\|(-\triangle)^{\alpha/2}(su^+ + tu^-)\|_2^2 \int_{\mathbb{R}^N} (-\triangle)^{\alpha/2}(su^+ + tu^-)(-\triangle)^{\alpha/2}(tu^-)\mathrm{d}x. \end{aligned} \tag{2.11}$$

Using (F4), one has

$$f(x, s\tau)s\tau \ge f(x,\tau)\tau s^4 - V(x)(s^2-1)(s\tau)^2, \quad \forall\, x \in \mathbb{R}^N,\ s \ge 1,\ \tau \in \mathbb{R}, \tag{2.12}$$

which implies

$$\int_{\mathbb{R}^N} \left[V(x)(su^+)^2 - f(x, su^+)su^+\right] \mathrm{d}x \le s^4 \int_{\mathbb{R}^N} \left[V(x)(u^+)^2 - f(x, u^+)u^+\right] \mathrm{d}x, \quad \forall\ s \ge 1. \tag{2.13}$$

From (2.10) and (2.13), we derive that

$$g_1(s,s) = as^2\left[\|(-\triangle)^{\alpha/2}u^+\|_2^2 + 2P(u^+, u^-)\right] + bs^4\phi(u, u^+)$$
$$+ \int_{\mathbb{R}^N} \left[V(x)(su^+)^2 - f(x, su^+)su^+\right] \mathrm{d}x$$

$$\leq as^2\left[\|(-\triangle)^{\alpha/2}u^+\|_2^2+2P(u^+,u^-)\right]+s^4\left\{b\phi(u,u^+)\right.$$
$$\left.+\int_{\mathbb{R}^N}\left[V(x)(u^+)^2-f(x,u^+)u^+\right]\mathrm{d}x\right\},$$
$$\forall\ s\geq 1. \tag{2.14}$$

Using (2.14), it is easy to verify that $g_1(s,s)<0$ for s large due to $u\in\mathcal{E}_b$. Similarly, we have $g_2(t,t)<0$ for t large. Jointly with (2.10) and (2.11), there exist $0<r<R$ such that

$$g_1(r,r)>0,\quad g_2(r,r)>0;\qquad g_1(R,R)<0,\quad g_2(R,R)<0. \tag{2.15}$$

By (2.10) and (2.11), we have $g_1(s,\cdot)$ is increasing for any fixed $s>0$, and $g_2(\cdot,t)$ is increasing for any fixed $t>0$. Thus, it follows from (2.10), (2.11) and (2.15) that

$$g_1(r,t)>0,\quad g_1(R,t)<0,\qquad \forall\ t\in[r,R], \tag{2.16}$$
$$g_2(s,r)>0,\quad g_2(s,R)<0,\qquad \forall\ s\in[r,R]. \tag{2.17}$$

Applying Miranda's Theorem [31], there exists some point (s_u,t_u) with $s_u,t_u\in[r,R]$ such that $g_1(s_u,t_u)=g_2(s_u,t_u)=0$. Therefore, $s_uu^++t_uu^-\in\mathcal{M}_b$.

Next, we prove the uniqueness. Let (s_1,t_1) and (s_2,t_2) such that $s_iu^++t_iu^-\in\mathcal{M}_b, i=1,2$. In view of Lemma 2.2, one has

$$\Phi_b(s_1u^++t_1u^-)\geq\Phi_b(s_2u^++t_2u^-)+\frac{a(s_1^2-s_2^2)^2}{s_1^2}\|(-\triangle)^{\alpha/2}u^+\|_2^2$$
$$+\frac{a(t_1^2-t_2^2)^2}{t_1^2}\|(-\triangle)^{\alpha/2}u^-\|_2^2,$$
$$\Phi_b(s_2u^++t_2u^-)\geq\Phi_b(s_1u^++t_1u^-)+\frac{a(s_1^2-s_2^2)^2}{s_2^2}\|(-\triangle)^{\alpha/2}u^+\|_2^2$$
$$+\frac{a(t_1^2-t_2^2)^2}{t_2^2}\|(-\triangle)^{\alpha/2}u^-\|_2^2.$$

The above inequalities imply $(s_1,t_1)=(s_2,t_2)$. This shows that i) holds.

To obtain ii), we let $g(t):=\langle\Phi_b'(tu),tu\rangle$ for $u\in\bar{\mathcal{E}}_b$, it follows from (1.8) and (2.12) that

$$g(t)\leq at^2\|(-\triangle)^{\alpha/2}u\|_2^2+t^4\left\{b\|(-\triangle)^{\alpha/2}u\|_2^4+\int_{\mathbb{R}^N}\left[V(x)u^2-f(x,u)u\right]\mathrm{d}x\right\},$$
$$\forall\ t\geq 1, \tag{2.18}$$

which implies that there exists $R_0>0$ large enough such that $g(R_0)<0$. It is easy to see that $g(r_0)>0$ for $r_0>0$ small enough. Thus, there exists $\bar{t}_u>0$ such that $g(\bar{t}_u)=0$ for $u\in\bar{\mathcal{E}}_b$. Similar to the proof of i), we can deduce from Lemma 2.3 that $\bar{t}_u$ is unique, and so ii) holds. □

Lemma 2.7. *Assume that (V1), (V2) and (F1)–(F4) hold. Then*

$$\inf_{u\in\mathcal{M}_b}\Phi_b(u)=m_b=\inf_{u\in\mathcal{E}_b}\max_{s,t\geq 0}\Phi_b(su^++tu^-),$$
$$\inf_{u\in\mathcal{N}_b}\Phi_b(u)=c_b=\inf_{u\in\bar{\mathcal{E}}_b}\max_{t\geq 0}\Phi_b(tu).$$

Proof. Corollary 2.4 and Lemmas 2.5 and 2.6 imply the above lemma. □

Lemma 2.8. *Assume that (V1), (V2) and (F1)–(F4) hold. Then* $m_b > 0$ *and* $c_b > 0$ *are achieved.*

Proof. By (F1) and (F2), for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(x,t)t| \le \varepsilon t^2 + C_\varepsilon |t|^p, \quad |F(x,t)| \le \varepsilon t^2 + C_\varepsilon |t|^p, \quad \forall\ (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{2.19}$$

By (V1), there exists $\gamma_0 > 0$ such that

$$\gamma_0 \|u\|_{H^\alpha}^2 \le \|u\|^2, \quad \forall\ u \in E. \tag{2.20}$$

First, we prove that $m_b > 0$ and $c_b > 0$. For $u \in \mathcal{M}_b$, by (1.8), (1.10), (2.1), (2.19), (2.20) and Sobolev embedding theorem, one has

$$\begin{aligned} \gamma_0 \|u^\pm\|_{H^\alpha}^2 \le \|u^\pm\|^2 &\le a\|(-\triangle)^{\alpha/2}(u^\pm)\|_2 + 2aP(u^+,u^-) + \int_{\mathbb{R}^N} V(x)(u^\pm)^2 \mathrm{d}x \\ &\quad + b\|(-\triangle)^{\alpha/2}u\|_2^2 \int_{\mathbb{R}^N} (-\triangle)^{\alpha/2}u(-\triangle)^{\alpha/2}u^\pm \mathrm{d}x \\ &= \int_{\mathbb{R}^N} f(x,u^\pm)u^\pm \mathrm{d}x \qquad (2.21) \\ &\le \frac{\gamma_0}{2}\|u^\pm\|_2^2 + C_1\|u^\pm\|_p^p \le \frac{\gamma_0}{2}\|u^\pm\|_{H^\alpha}^2 + C_2\|u^\pm\|_{H^\alpha}^p, \qquad (2.22) \end{aligned}$$

which implies that there exists a constant $\varrho > 0$ independent of b such that

$$\|u^\pm\| \ge \sqrt{\gamma_0}\|u^\pm\|_{H^\alpha} \ge \varrho, \quad \forall\ u \in \mathcal{M}_b. \tag{2.23}$$

Similarly, there exists a constant $\varrho_0 > 0$ independent of b such that

$$\|u\| \ge \sqrt{\gamma_0}\|u\|_{H^\alpha} \ge \varrho_0, \quad \forall\ u \in \mathcal{N}_b. \tag{2.24}$$

Since $\mathcal{M}_b \subset \mathcal{N}_b$, we have $m_b \ge c_b$. By (2.4) with $t = 0$, one has

$$\Phi_b(u) = \Phi_b(u) - \frac{1}{4}\langle \Phi_b'(u), u\rangle \ge \frac{a}{4}\|(-\triangle)^{\alpha/2}u\|_2^2, \quad \forall\ u \in \mathcal{N}_b, \tag{2.25}$$

which implies $c_b = \inf_{\mathcal{N}_b} \Phi_b \ge 0$.

Now, we show that $c_b > 0$. To this end, we choose $\{u_n\} \subset \mathcal{N}_b$ be such that $\Phi_b(u_n) \to c_b$. There are two possible cases: (1) $\inf_{n\in\mathbb{N}} \|(-\triangle)^{\alpha/2}u_n\|_2 > 0$; (2) $\inf_{n\in\mathbb{N}} \|(-\triangle)^{\alpha/2}u_n\|_2 = 0$.

Case 1. $\inf_{n\in\mathbb{N}} \|(-\triangle)^{\alpha/2}u_n\|_2 := \varrho_1 > 0$. In this case, from (2.25), one has

$$c_b + o(1) = \Phi_b(u_n) \ge \frac{a}{4}\|(-\triangle)^{\alpha/2}u_n\|_2^2 \ge \frac{a}{4}\varrho_1^2. \tag{2.26}$$

Case 2. $\inf_{n\in\mathbb{N}} \|(-\triangle)^{\alpha/2}u_n\|_2 = 0$. Since $\|u_n\|^2 \ge \varrho_0^2 > 0$, passing to a subsequence, we have

$$\|(-\triangle)^{\alpha/2}u_n\|_2 \to 0, \quad \int_{\mathbb{R}^N} V(x)u_n^2 \mathrm{d}x \ge \varrho_2 > 0 \tag{2.27}$$

for some constant $\varrho_2 > 0$. Let $t_n = [\int_{\mathbb{R}^N} V(x)u_n^2 \mathrm{d}x]^{-1/2}$, then (2.27) implies that $t_n \le \varrho_2^{-1/2}$. From (2.19), (2.20) and Sobolev inequality, we derive that

$$\left|\int_{\mathbb{R}^N} F(x,t_nu_n)\mathrm{d}x\right| \le \int_{\mathbb{R}^N}\left[\frac{\gamma_0}{4}t_n^2u_n^2+C_3|t_nu_n|^{2_\alpha^*}\right]\mathrm{d}x$$
$$\le \frac{t_n^2}{4}\|u_n\|^2+C_3|t_n|^{2_\alpha^*}S_\alpha^{-2_\alpha^*/2}\|(-\triangle)^{\alpha/2}u_n\|_2^{2_\alpha^*}, \tag{2.28}$$

where

$$\mathcal{S}_\alpha=\inf_{u\in\mathcal{D}^{\alpha,2}(\mathbb{R}^N)}\frac{\int_{\mathbb{R}^N}|(-\triangle)^{\alpha/2}u|^2\mathrm{d}x}{\left[\int_{\mathbb{R}^N}|u|^{2_\alpha^*}\mathrm{d}x\right]^{2/2_\alpha^*}}.$$

Since $u_n\in\mathcal{N}_b$, it follows from (1.7), (2.6), (2.27) and (2.28) that

$$\begin{aligned} c_b+o(1) &= \Phi_b(u_n)\ge\Phi_b(t_nu_n)\\ &=\frac{at_n^2}{2}\|(-\triangle)^{\alpha/2}u_n\|_2^2+\frac{t_n^2}{2}\int_{\mathbb{R}^N}V(x)u_n^2\mathrm{d}x+\frac{bt_n^4}{4}\|(-\triangle)^{\alpha/2}u_n\|_2^4\\ &\quad-\int_{\mathbb{R}^N}F(x,t_nu_n)\mathrm{d}x\\ &\ge\frac{t_n^2}{4}\int_{\mathbb{R}^N}V(x)u_n^2\mathrm{d}x-C_3|t_n|^{2_\alpha^*}S_\alpha^{-2_\alpha^*/2}\|(-\triangle)^{\alpha/2}u_n\|_2^{2_\alpha^*}=\frac{1}{4}+o(1).\end{aligned}$$

Cases 1) and 2) show that $c_b=\inf_{u\in\mathcal{N}_b}\Phi_b(u)>0$. Hence, $m_b\ge c_b>0$.

Next, we prove that m_b can be achieved. Let $\{u_n\}\subset\mathcal{M}_b$ be such that $\Phi_b(u_n)\to m_b$. Then, (2.25) implies that

$$m_b+o(1)\ge\Phi_b(u_n)-\frac{1}{4}\langle\Phi_b'(u_n),u_n\rangle\ge\frac{a}{4}\|(-\triangle)^{\alpha/2}u_n\|_2^2. \tag{2.29}$$

This shows that $\{\|(-\triangle)^{\alpha/2}u_n\|_2\}$ is bounded. To obtain the boundedness of $\{\|u_n\|\}$, it suffices to show that $\int_{\mathbb{R}^N}V(x)u_n^2\mathrm{d}x$ is bounded. Arguing by contradiction, suppose that $\int_{\mathbb{R}^N}V(x)u_n^2\mathrm{d}x\to\infty$. Let $t_n=2(m_b+1)^{1/2}[\int_{\mathbb{R}^N}V(x)u_n^2\mathrm{d}x]^{-1/2}$, then $t_n\to0$, and (2.28) still holds. Thus, it follows from (1.7), (2.5) and (2.28) that

$$\begin{aligned} m_b+o(1) &= \Phi_b(u_n)\ge\Phi_b(t_nu_n)\\ &=\frac{at_n^2}{2}\|(-\triangle)^{\alpha/2}u_n\|_2^2+\frac{t_n^2}{2}\int_{\mathbb{R}^N}V(x)u_n^2\mathrm{d}x+\frac{bt_n^4}{4}\|(-\triangle)^{\alpha/2}u_n\|_2^4\\ &\quad-\int_{\mathbb{R}^N}F(x,t_nu_n)\mathrm{d}x\\ &\ge\frac{t_n^2}{4}\int_{\mathbb{R}^N}V(x)u_n^2\mathrm{d}x-C_3|t_n|^{2_\alpha^*}S_\alpha^{2_\alpha^*/2}\|(-\triangle)^{\alpha/2}u_n\|_2^{2_\alpha^*}=m_b+1+o(1).\end{aligned} \tag{2.30}$$

This contradiction shows that $\{u_n\}$ is bounded in E. Passing to a subsequence, we may assume that $u_n^\pm\rightharpoonup u_b^\pm$ in E and $u_n^\pm\to u_b^\pm$ in $L^q(\mathbb{R}^N)$ for $q\in(2,2_\alpha^*)$. Then, by a standard argument, one has

$$\int_{\mathbb{R}^N}f(x,u_n^\pm)u_n^\pm\mathrm{d}x=\int_{\mathbb{R}^N}f(x,u_b^\pm)u_b^\pm\mathrm{d}x+o(1). \tag{2.31}$$

From (2.21), (2.23) and (2.31), we deduce that

$$0<\varrho^2\leq\|u_n^{\pm}\|^2\leq\int_{\mathbb{R}^N}f(x,u_n^{\pm})u_n^{\pm}\mathrm{d}x=\int_{\mathbb{R}^N}f(x,u_b^{\pm})u_b^{\pm}\mathrm{d}x+o(1),\quad(2.32)$$

which yields $u_b^{\pm}\neq 0$. By (2.1), (2.31), the weak semicontinuity of norm and Fatou's Lemma, one has

$$\begin{aligned}&a\|(-\triangle)^{\alpha/2}(u_b^{\pm})\|_2+2aP(u_b^+,u_b^-)+\int_{\mathbb{R}^N}V(x)(u_b^{\pm})^2\mathrm{d}x\\&\quad+b\|(-\triangle)^{\alpha/2}u_b\|_2^2\int_{\mathbb{R}^N}(-\triangle)^{\alpha/2}u_b(-\triangle)^{\alpha/2}u_b^{\pm}\mathrm{d}x\\&\leq\liminf_{n\to\infty}\Big[a\|(-\triangle)^{\alpha/2}(u_n^{\pm})\|_2+2aP(u_n^+,u_n^-)+\int_{\mathbb{R}^N}V(x)(u_n^{\pm})^2\mathrm{d}x\\&\quad+\ b\|(-\triangle)^{\alpha/2}u_n\|_2^2\int_{\mathbb{R}^N}(-\triangle)^{\alpha/2}u_n(-\triangle)^{\alpha/2}u_n^{\pm}\mathrm{d}x\Big]\\&=\liminf_{n\to\infty}\int_{\mathbb{R}^N}f(x,u_n^{\pm})u_n^{\pm}\mathrm{d}x=\int_{\mathbb{R}^N}f(x,u_b^{\pm})u_b^{\pm}\mathrm{d}x,\end{aligned}\quad(2.33)$$

which implies

$$\langle\Phi_b'(u_b),u_b^{\pm}\rangle\leq 0.\quad(2.34)$$

Using (1.8), it is easy to verify that $u_b\in\mathcal{E}_b$. In view of Lemma 2.6, there exist $\hat{s},\hat{t}>0$ such that $\hat{s}u_b^++\hat{t}u_b^-\in\mathcal{M}_b$. By (2.3) with $t=0$, one has

$$\frac{1}{4}f(x,\tau)\tau-F(x,\tau)+\frac{1}{4}V(x)\tau^2\geq 0,\qquad x\in\mathbb{R}^N,\ \tau\in\mathbb{R}.\quad(2.35)$$

Thus, it follows from (1.7), (1.8), (2.2), (2.34), (2.35), the weak semicontinuity of norm, Fatou's Lemma and Lemma 2.7 that

$$\begin{aligned}m_b&=\lim_{n\to\infty}\Big[\Phi_b(u_n)-\frac{1}{4}\langle\Phi_b'(u_n),u_n\rangle\Big]\\&=\lim_{n\to\infty}\Big\{\frac{a}{4}\|(-\triangle)^{\alpha/2}u_n\|_2^2+\int_{\mathbb{R}^N}\Big[\frac{1}{4}f(x,u_n)u_n-F(x,u_n)+\frac{1}{4}V(x)u_n^2\Big]\mathrm{d}x\Big\}\\&\geq\frac{a}{4}\|(-\triangle)^{\alpha/2}u_b\|_2^2+\int_{\mathbb{R}^N}\Big[\frac{1}{4}f(x,u_b)u_b-F(x,u_b)+\frac{1}{4}V(x)u_b^2\Big]\mathrm{d}x\\&=\Phi_b(u_b)-\frac{1}{4}\langle\Phi_b'(u_b),u_b\rangle\\&\geq\sup_{s,t\geq0}\Big[\Phi_b(su_b^++tu_b^-)+\frac{1-s^4}{4}\langle\Phi_b'(u_b),u_b^+\rangle+\frac{1-t^4}{4}\langle\Phi_b'(u_b),u_b^-\rangle\Big]\\&\quad-\frac{1}{4}\langle\Phi_b'(u_b),u_b\rangle\\&\geq\sup_{s,t\geq0}\Phi_b(su_b^++tu_b^-)\geq\Phi_b(\hat{s}u_b^++\hat{t}u_b^-)\geq m_b,\end{aligned}$$

which implies that $\Phi_b(u_b)=m_b$ and $u_b\in\mathcal{M}_b$. Similar to the above argument, we can prove that there exists $\bar{u}_b\in\mathcal{N}_b$ such that $\Phi_b(\bar{u}_b)=c_b$. □

In the same way as [15, Lemma 2.7] and [12, Lemma 2.9], we can prove that the minimizers of $\inf_{\mathcal{M}_b}\Phi_b$ and $\inf_{\mathcal{N}_b}\Phi_b$ are critical points, respectively.

Lemma 2.9. *Assume that (V1), (V2) and (F1)–(F4) hold. Let* $u_b \in \mathcal{M}_b$ *and* $\bar{u}_b \in \mathcal{N}_b$ *be such that* $\Phi_b(u_b) = m_b$ *and* $\Phi_b(\bar{u}_b) = c_b$*, then* u_b *and* $\bar{u}_b$ *are critical points of* Φ_b*.*

3. Proofs of results

In this section, we give the proofs of Theorems 1.1–1.3.

Proof of Theorem 1.1. In view of Lemmas 2.6, 2.8 and 2.9, there exists $u_b \in \mathcal{M}_b \subset \mathcal{E}_b$ such that $\Phi_b(u_b) = m_b$ and $\Phi_b'(u_b) = 0$. Thus, $u_b \in \mathcal{E}_b$ is a sign-changing solution of (1.1). □

Proof of Theorem 1.2. In view of Lemmas 2.6, 2.8 and 2.9, there exists $\bar{u}_b \in \mathcal{N}_b \subset \bar{\mathcal{E}}_b$ such that $\Phi_b(\bar{u}_b) = c_b > 0$ and $\Phi_b'(\bar{u}_b) = 0$.

In what follows, we let $u_b \in \mathcal{E}_b$ is a sign-changing solution of (1.1) obtained in Theorem 1.1. Noting that $u_b^{\pm} \in \bar{\mathcal{E}}_b$, it follows from (1.6), (1.7), (1.10), Lemma 2.2, Corollary 2.4 and Lemma 2.7 that

$$\begin{aligned} m_b &= \Phi_b(u_b) = \sup_{s,t\geq 0} \Phi_b(su_b^+ + tu_b^-) \\ &= \sup_{s,t\geq 0} \Big\{ \Phi_b(su_b^+) + \Phi_b(tu_b^-) + 2astP(u_b^+, u_b^-) \\ &\quad + \frac{b}{2}s^2t^2\|(-\triangle)^{\alpha/2}u_b^+\|_2^2\|(-\triangle)^{\alpha/2}u_b^-\|_2^2 \\ &\quad + 2bst\left[s^2\|(-\triangle)^{\alpha/2}u_b^+\|_2^2 + t^2\|(-\triangle)^{\alpha/2}u_b^-\|_2^2 + 2stP(u_b^+, u_b^-)\right] P(u_b^+, u_b^-)\Big\} \\ &> \sup_{s\geq 0} \Phi_b(su_b^+) + \sup_{t\geq 0} \Phi_b(tu_b^-) \geq 2c_b, \qquad \forall\, b \geq 0. \end{aligned} \tag{3.1}$$

□

Proof of Theorem 1.3. To obtain the convergence property of u_{b_n}, we now prove that $\{u_{b_n}\}$ is bounded in E. In view of Lemma 2.5, we can choose $w_0 \in \mathcal{E}_1$, then

$$\phi(w_0, w_0^{\pm}) + \int_{\mathbb{R}^N} \left[V(x)(w_0^{\pm})^2 - f(x, w_0^{\pm})w_0^{\pm}\right] \mathrm{d}x < 0. \tag{3.2}$$

By (2.3), one has

$$F(x, t\tau) \geq \frac{t^4-1}{4} f(x,\tau)\tau + F(x,\tau) - \frac{1-2t^2+t^4}{4} V(x)\tau^2, \quad \forall\, x \in \mathbb{R}^N,\ t \geq 0,\ \tau \in \mathbb{R},$$

which implies

$$\begin{aligned} \frac{1}{2}V(x)(t\tau)^2 - F(x,t\tau) \leq \frac{t^4}{4}\left[V(x)\tau^2 - F(x,\tau)\right] \\ + \frac{1}{4}\left[V(x)\tau^2 + f(x,\tau)\tau - 4F(x,\tau)\right], \\ \forall\, x \in \mathbb{R}^N,\ t \geq 0,\ \tau \in \mathbb{R}. \end{aligned} \tag{3.3}$$

By Young's inequality, one has

$$y_1^4 + 3y_2^4 - 4y_1y_2^3 \geq 0, \quad \forall\, y_1, y_2 \geq 0. \tag{3.4}$$

Then it follows from (1.7), (3.3), (3.2), (3.4) and Lemma 2.7 that for all $b \in [0,1]$,

$$\begin{aligned}
\Phi_b(u_b) &= m_b \leq \max_{s,t\geq 0} \Phi_b(sw_0^+ + tw_0^-)\\
&= \max_{s,t\geq 0}\left\{\frac{as^2}{2}\|(-\triangle)^{\alpha/2}w_0^+\|_2^2 + \frac{bs^4}{4}\phi(w_0,w_0^+)\right.\\
&\quad + \int_{\mathbb{R}^N}\left[\frac{1}{2}V(x)(sw_0^+)^2 - F(x,sw_0^+)\right]\mathrm{d}x\\
&\quad + \frac{at^2}{2}\|(-\triangle)^{\alpha/2}w_0^-\|_2^2 + \frac{bt^4}{4}\phi(w_0,w_0^-)\\
&\quad + \int_{\mathbb{R}^N}\left[\frac{1}{2}V(x)(tw_0^-)^2 - F(x,tw_0^-)\right]\mathrm{d}x\\
&\quad + 2astP(w_0^+,w_0^-) + \frac{b}{4}\left[\|(-\triangle)^{\alpha/2}(sw_0^+ + tw_0^-)\|_2^4\right.\\
&\quad \left.\left. - s^4\phi(w_0,w_0^+) - t^4\phi(w_0,w_0^-)\right]\right\}\\
&\leq \max_{s,t\geq 0}\left\{\frac{as^2}{2}\|(-\triangle)^{\alpha/2}w_0^+\|_2^2 + \frac{s^4}{4}\left(\phi(w_0,w_0^+)\right.\right.\\
&\quad \left. + \int_{\mathbb{R}^N}\left[V(x)(w_0^+)^2 - f(x,w_0^+)w_0^+\right]\mathrm{d}x\right)\\
&\quad + \frac{at^2}{2}\|(-\triangle)^{\alpha/2}w_0^-\|_2^2 + \frac{t^4}{4}\left(\phi(w_0,w_0^-)\right.\\
&\quad \left. + \int_{\mathbb{R}^N}\left[V(x)(w_0^-)^2 - f(x,w_0^-)w_0^-\right]\mathrm{d}x\right)\\
&\quad + \frac{1}{4}\int_{\mathbb{R}^N}\left[V(x)(w_0^+)^2 + f(x,w_0^+)w_0^+ - 4F(x,w_0^+)\right]\mathrm{d}x\\
&\quad + \frac{1}{4}\int_{\mathbb{R}^N}\left[V(x)(w_0^-)^2 + f(x,w_0^-)w_0^- - 4F(x,w_0^-)\right]\mathrm{d}x\\
&\quad + a(s^2+t^2)P(w_0^+,w_0^-)\\
&\quad - \frac{P(w_0^+,w_0^-)}{2}\left[\left(3s^4 + t^4 - 4s^3t\right)\|(-\triangle)^{\alpha/2}w_0^+\|_2^2\right.\\
&\quad \left. + \left(s^4 + 3t^4 - 4st^3\right)\|(-\triangle)^{\alpha/2}w_0^-\|_2^2\right]\\
&\quad \left. - \frac{(s^2+t^2)^2}{4}\left\{\|(-\triangle)^{\alpha/2}w_0^+\|_2^2\|(-\triangle)^{\alpha/2}w_0^-\|_2^2 + 8[P(w_0^+,w_0^-)]^2\right\}\right\}\\
&:= \Lambda_0,
\end{aligned} \tag{3.5}$$

where $\Lambda_0 > 0$ is a constant independent of b. For any sequence b_n with $b_n \searrow 0$ as $n \to \infty$, we deduce from (2.29) and (3.5) that $\{\|(-\triangle)^{\alpha/2}u_{b_n}\|_2\}$ is bounded. Arguing as in (2.30), we can prove that $\{u_{b_n}\}$ is bounded in E. Hence, there exists a subsequence of $\{b_n\}$, still denoted by $\{b_n\}$, and $v_0 \in E$ such that

$u_{b_n} \rightharpoonup v_0$ in E, $u_{b_n} \to v_0$ in $L^s(\mathbb{R}^N)$ for $s \in (2, 2_\alpha^*)$ and $u_{b_n} \to v_0$ a.e. in $x \in \mathbb{R}^N$. Similar to the proof of (2.31) and (2.32), we conclude that $v_0^\pm \neq 0$. Note that

$$\begin{aligned}\langle \Phi_0'(v_0), \varphi\rangle &= \int_{\mathbb{R}^N} \left[a(-\triangle)^{2/\alpha} v_0 (-\triangle)^{2/\alpha}\varphi + V(x)v_0\varphi\right) \mathrm{d}x - \int_{\mathbb{R}^N} f(x, v_0)\varphi \mathrm{d}x \\ &= \lim_{n\to\infty} \left[\left(a + b_n\|(-\triangle)^{2/\alpha} u_{b_n}\|_2^2\right)(u_{b_n}, \varphi) - \int_{\mathbb{R}^N} f(x, u_{b_n})\varphi \mathrm{d}x\right], \\ &\quad \forall\ \varphi \in E.\end{aligned}$$

This shows that $\Phi_0'(v_0) = 0$, and so $v_0 \in \mathcal{M}_0$ and $\Phi_0(v_0) \geq m_0$. A standard argument shows that $u_{b_n} \to v_0$ in E.

Next, we prove that $\Phi_0(v_0) = m_0$. Since $u_0 \in \mathcal{M}_0$ is a sign-changing solution of (1.1) with $b = 0$ satisfying $\Phi_0(u_0) = m_0$, then we have

$$\begin{aligned} 0 &= \langle \Phi_0'(u_0), u_0^\pm\rangle \\ &= \int_{\mathbb{R}^N} \left[a(-\triangle)^{2/\alpha} u_0 (-\triangle)^{2/\alpha} u_0^\pm + V(x)(u_0^\pm)^2\right] \mathrm{d}x - \int_{\mathbb{R}^N} f(x, u_0)u_0^\pm \mathrm{d}x,\end{aligned}$$

which implies

$$\int_{\mathbb{R}^N} \left[V(x)(u_0^\pm)^2 - f(x, u_0^\pm)u_0^\pm\right] \mathrm{d}x < -\frac{a}{2}\int_{\mathbb{R}^N} (-\triangle)^{\alpha/2} u_0 (-\triangle)^{\alpha/2} u_0^\pm \mathrm{d}x. \tag{3.6}$$

Let $\Lambda_1 = \min\left\{a/2\|(-\triangle)^{2/\alpha} u_0\|_2^2, 1\right\}$ and $b_n \in [0, \Lambda_1]$ for all $n \in \mathbb{N}$. By (3.6), one has

$$\begin{aligned} &b_n\phi(u_0, u_0^\pm) + \int_{\mathbb{R}^N} \left[V(x)(u_0^\pm)^2 - f(x, u_0^\pm)u_0^\pm\right] \mathrm{d}x \\ &< \left(\Lambda_1\|(-\triangle)^{2/\alpha} u_0\|_2^2 - \frac{a}{2}\right)\int_{\mathbb{R}^N} (-\triangle)^{\alpha/2} u_0 (-\triangle)^{\alpha/2} u_0^\pm \mathrm{d}x \leq 0. \end{aligned} \tag{3.7}$$

Then, $u_0 \in \mathcal{E}_{b_n}$ for all $n \in \mathbb{N}$. Thus, it follows from (1.7), (1.10), (2.1), (3.3), (3.4) and (3.7) that there exists $K_0 > 0$ such that for all $s \geq K_0$ or $t \geq K_0$,

$$\begin{aligned} &\Phi_{b_n}(su_0^+ + tu_0^-) \\ &= \frac{as^2}{2}\|(-\triangle)^{\alpha/2} u_0^+\|_2^2 + 2astP(u_0^+, u_0^-) + \frac{at^2}{2}\|(-\triangle)^{\alpha/2} u_0^-\|_2^2 + \frac{b_n s^4}{4}\phi(u_0, u_0^+) \\ &\quad + \int_{\mathbb{R}^N} \left[\frac{1}{2}V(x)(su_0^+)^2 - F(x, su_0^+)\right] \mathrm{d}x + \frac{b_n t^4}{4}\phi(u_0, u_0^-) \\ &\quad + \int_{\mathbb{R}^N} \left[\frac{1}{2}V(x)(tu_0^-)^2 - F(x, tu_0^-)\right] \mathrm{d}x \\ &\quad + \frac{b_n}{4}\left[\|(-\triangle)^{\alpha/2}(su_0^+ + tu_0^-)\|_2^4 - s^4\phi(u_0, u_0^+) - t^4\phi(u_0, u_0^-)\right] \\ &\leq \frac{as^2}{2}\|(-\triangle)^{\alpha/2} u_0^+\|_2^2 + a(s^2 + t^2)P(u_0^+, u_0^-) + \frac{at^2}{2}\|(-\triangle)^{\alpha/2} u_0^-\|_2^2 \\ &\quad + \frac{s^4}{4}\left\{b_n\phi(u_0, u_0^+) + \int_{\mathbb{R}^N} \left[V(x)(u_0^+)^2 - f(x, u_0^+)u_0^+\right] \mathrm{d}x\right\} \\ &\quad + \frac{t^4}{4}\left\{b_n\phi(u_0, u_0^-) + \int_{\mathbb{R}^N} \left[V(x)(u_0^-)^2 - f(x, u_0^-)u_0^-\right] \mathrm{d}x\right\} \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4}\int_{\mathbb{R}^N}\Big[V(x)(u_0^+)^2+f(x,u_0^+)u_0^+-4F(x,u_0^+)+V(x)(u_0^-)^2\\
&+f(x,u_0^-)u_0^--4F(x,u_0^-)\Big]\,\mathrm{d}x\\
&-\frac{b_n}{2}P(u_0^+,u_0^-)\Big[\Big(3s^4+t^4-4s^3t\Big)\|(-\triangle)^{\alpha/2}u_0^+\|_2^2\\
&+\Big(s^4+3t^4-4st^3\Big)\|(-\triangle)^{\alpha/2}u_0^-\|_2^2\Big]\\
&-\frac{b_n(s^2+t^2)^2}{4}\Big\{\|(-\triangle)^{\alpha/2}u_0^+\|_2^2\|(-\triangle)^{\alpha/2}u_0^-\|_2^2+8[P(u_0^+,u_0^-)]^2\Big\}<0. \quad (3.8)
\end{aligned}
$$

In view of Lemma 2.6, there exists (s_n, t_n) such that $s_n u_0^+ + t_n u_0^- \in \mathcal{M}_{b_n}$, which, together with (3.8), implies $0 < s_n, t_n < K_0$. Hence, from (1.7), (1.8) and (2.2), we have

$$
\begin{aligned}
m_0 &= \Phi_0(u_0)\\
&= \Phi_{b_n}(u_0)-\frac{b_n}{4}\|(-\triangle)^{\alpha/2}u_0\|_2^4\\
&\geq \Phi_{b_n}(s_nu_0^++t_nu_0^-)+\frac{1-s_n^4}{4}\langle\Phi'_{b_n}(u_0),u_0^+\rangle+\frac{1-t_n^4}{4}\langle\Phi'_{b_n}(u_0),u_0^-\rangle\\
&\quad -\frac{b_n}{4}\|(-\triangle)^{\alpha/2}u_0\|_2^4\\
&\geq m_{b_n}+\frac{1-s_n^4}{4}b_n\|(-\triangle)^{\alpha/2}u_0\|_2^2\int_{\mathbb{R}^N}(-\triangle)^{\alpha/2}u_0(-\triangle)^{\alpha/2}u_0^+\mathrm{d}x\\
&\quad +\frac{1-t_n^4}{4}b_n\|(-\triangle)^{\alpha/2}u_0\|_2^2\int_{\mathbb{R}^N}(-\triangle)^{\alpha/2}u_0(-\triangle)^{\alpha/2}u_0^-\mathrm{d}x\\
&\quad -\frac{b_n}{4}\|(-\triangle)^{\alpha/2}u_0\|_2^4\\
&\geq m_{b_n}-\frac{K_0^4b_n}{4}\|(-\triangle)^{\alpha/2}u_0\|_2^4,
\end{aligned}
$$

which yields

$$
\limsup_{n\to\infty} m_{b_n} \leq m_0. \tag{3.9}
$$

Since $u_{b_n} \to v_0$ in E, it follows from (1.7) and (3.9) that

$$
m_0 \leq \Phi_0(v_0) = \limsup_{n\to\infty}\Phi_{b_n}(u_{b_n}) = \limsup_{n\to\infty} m_{b_n} \leq m_0. \tag{3.10}
$$

This shows that $\Phi_0(v_0) = m_0$. □

Acknowledgements

This work was supported by the NNSF (11701487, 11626202), Hunan Provincial Natural Science Foundation of China (2016JJ6137), Scientific Research Fund of Hunan Provincial Education Department (15B223). The authors thank the anonymous referees for their valuable suggestions and comments.

References

[1] Alves, C.O., Nòbrega, A.B.: Nodal ground state solution to a biharmonic equation via dual method. J. Differ. Equ. **260**, 5174–5201 (2016)

[2] Ambrosio, V., Isernia, T.: A multiplicity result for a fractional Kirchhoff equation in $\mathbb{R}^N$ with a general nonlinearity. Commun. Contemp. Math. **1**, 1750054 (2017). https://doi.org/10.1142/S0219199717500547

[3] Applebaum, D.: Lévy processes, from probability fo finance and quantum groups. Notices Am. Math. Soc. **51**, 1336–1347 (2004)

[4] Arosio, A., Panizzi, S.: On the well-posedness of the Kirchhoff string. Trans. Am. Math. Soc. **348**, 305–330 (1996)

[5] Bartsch, T., Liu, Z., Weth, T.: Sign changing solutions of superlinear Schrödinger equations. Commun. Partial Differ. Equ. **29**, 25–42 (2004)

[6] Bartsch, T., Weth, T.: Three nodal solutions of singularly perturbed elliptic equations on domains without topology. Ann. Inst. H. Poincaré Anal. Non Linéaire **22**, 259–281 (2005)

[7] Bernstein, S.: Sur une class d'équationsfonctionnelles aux dérivéespartielles. Bull. Acad. Sci. URSS Sér. Math. (Izv. Akad. Nauk SSSR) **4**, 17–26 (1940)

[8] Cavalcanti, M.M., Domingos Cavalcanti, V.N., Soriano, J.A.: Global existence and uniform decay rates for the Kirchhoff–Carrier equation with nonlinear dissipation. Adv. Differ. Equ. **6**, 701–730 (2001)

[9] Caffarelli, L., Silvestre, L.: An extension problem related to the fractional Laplacian. Commun. Partial Differ. Equ. **32**, 1245–1260 (2007)

[10] Chang, X., Wang, Z.Q.: Nodal and multiple solutions of nonlinear problems involving the fractional Laplacian. J. Differ. Equ. **256**, 2965–2992 (2014)

[11] Chang, K., Gao, Q.: Sign-changing solutions for the stationary Kirchhoff problems involving the fractional Laplacian in $\mathbb{R}^N$, arXiv:1701.03862 (2017)

[12] Chen, S.T., Tang, X.H.: Ground state solutions for asymptotically periodic Kirchhoff-type equations with asymptotically cubic or super-cubic nonlinearities. Mediterr. J. Math. **14**, 209 (2017)

[13] Chen, S.T., Tang, X.H.: Improved results for Klein–Gordon–Maxwell systems with general nonlinearity. Discrete Contin. Dyn. Syst. A **38**, 2333–2348 (2018)

[14] Chen, S.T., Tang, X.H.: Infinitely many solutions and least energy solutions for Klein–Gordon–Maxwell systems with general superlinear nonlinearity. Comput. Math. Appl. **75**, 3358–3366 (2018)

[15] Cheng, B.T., Tang, X.H.: Ground state sign-changing solutions for asymptotically 3-linear Kirchhoff-type problems. Complex Var. Elliptic **62**, 1093–1116 (2017)

[16] Chipot, M., Lovat, B.: Some remarks on nonlocal elliptic and parabolic problems. Nonlinear Anal. **30**, 4619–4627 (1997)

[17] Deng, Y., Peng, S., Shuai, W.: Existence and asymptotic behavior of nodal solutions for the Kirchhoff-type problems in $\mathbb{R}^3$. J. Funct. Anal. **269**, 3500–3527 (2015)

[18] Deng, Y., Shuai, W.: Sign-changing solutions for non-local elliptic equations involving the fractional Laplacian. Adv. Differ. Equ. **23**, 109–134 (2018)

[19] Dipierro, S., Palatucci, G., Valdinoci, E.: Existence and symmetry results for a Schrödinger type problem involving the fractional Laplacian. Le Matematiche **68**, 201–216 (2013)

[20] Felmer, P., Quaas, A., Tan, J.G.: Positive solutions of nonlinear Schrödinger equation with the fractional Laplacian. Proc. R. Soc. Edinb. Sect. A **142**, 1237–1262 (2012)

[21] Fiscella, A., Valdinoci, E.: A critical Kirchhoff type problem involving a nonlocal operator. Nonlinear Anal. **94**, 156–170 (2014)

[22] Jin, H., Liu, W.: Fractional Kirchhoff equation with a general critical nonlinearity. Appl. Math. Lett. **74**, 140–146 (2017)

[23] Kirchhoff, G.: Mechanik. Teubner, Leipzig (1883)

[24] Laskin, N.: Fractional schrödinger equation. Phys. Rev. E **66**, 056108 (2002)

[25] Lions, P.L.: Symètrie et compacitè dans les espaces de Sobolev. J. Funct. Anal. **49**, 315–334 (1982)

[26] Lions, J.L.: On some questions in boundary value problems of mathematical physics. In: Contemporary Developments in Continuum Mechanics and Partial Differential Equations, Proc. Internat. Sympos. Inst. Mat., Univ. Fed. Rio de Janeiro, 1997, in: North-Holland Math. Stud., vol. 30. North-Holland, Amsterdam, pp. 284–346 (1978)

[27] Liu, Z.S., Squassina, M., Zhang, J.J.: Ground states for fractional Kirchhoff equations with critical nonlinearity in low dimension. Nonlinear Differ. Equ. Appl. **24**, 50 (2017)

[28] Luo, H.X.: Sign-changing solutions for non-local elliptic equations. Electron. J. Differ. Equ. **2017**, 1–15 (2017)

[29] Mao, A.M., Zhang, Z.T.: Sign-changing and multiple solutions of Kirchhoff type problems without the P.S. condition. Nonlinear Anal. **70**, 1275–1287 (2009)

[30] Mao, A.M., Luan, S.X.: Sign-changing solutions of a class of nonlocal quasilinear elliptic boundary value problems. J. Math. Anal. Appl. **383**, 239–243 (2011)

[31] Miranda, C.: Un'osservazione su un teorema di Brouwer. Boll. Unione Mat. Ital. **3**, 5–7 (1940)

[32] Molčanov, S.A., Ostrovskii, E.: Symmetric stable processes as traces of degenerate diffusion processes. Teor. Verojatnost. i Primenen. **14**, 127–130 (1969)

[33] Noussair, E.S., Wei, J.: On the effect of the domain geometry on the existence and profile of nodal solution of some singularly perturbed semilinear Dirichlet problem. Indiana Univ. Math. J. **46**, 1321–1332 (1997)

[34] Nyamoradi, N.: Existence of three solutions for Kirchhoff nonlocal operators of elliptic type. Math. Commun. **18**, 489–502 (2013)

[35] Shuai, W.: Sign-changing solutions for a class of Kirchhoff-type problem in bounded domains. J. Differ. Equ. **259**, 1256–1274 (2015)

[36] Tang, X.H., Cheng, B.T.: Ground state sign-changing solutions for Kirchhoff type problems in bounded domains. J. Differ. Equ. **261**(4), 2384–2402 (2016)

[37] Tang, X.H., Chen, S.T.: Ground state solutions of Nehari–Pohozaev type for Kirchhoff-type problems with general potentials. Calc. Var. Partial Differ. Equ. **56**, 110 (2017)

[38] Tang, X.H., Chen, S.T.: Ground state solutions of Nehari–Pohozaev type for Schrödinger–Poisson problems with general potentials. Discrete Contin. Dyn. Syst. A **37**, 4973–5002 (2017)

[39] Tang, X.H., Lin, X.Y., Yu, J.S.: Nontrivial solutions for Schrödinger equation with local super-quadratic conditions. J. Dyn. Differ. Equ. **1**, 1–15 (2018). https://doi.org/10.1007/s10884-018-9662-2

[40] Teng, K., Wang, K., Wang, R.: A sign-changing solution for nonlinear problems involving the fractional Laplacian. Electron. J. Differ. Equ. **109**, 1–12 (2015)

[41] Wang, Z., Zhou, H.S.: Radial sign-changing solution for fractional Schrödinger equation. Discrete Contin. Dyn. Syst. **36**, 499–508 (2016)

[42] Willem, M.: Minimax Theorems. Birkhäuser, Boston (1996)

[43] Xiang, M., Zhang, B., Guo, X.: Infinitely many solutions for a fractional Kirchhoff type problem via Fountain Theorem. Nonlinear Anal. **120**, 299–313 (2015)

[44] Zhang, Z.T., Perera, K.: Sign changing solutions of Kirchhoff type problems via invariant sets of descent flow. J. Math. Anal. Appl. **317**(2), 456–463 (2006)

[45] Zhang, W., Tang, X.H., Mi, H.: On fractional Schrödinger equation with periodic and asymptotically periodic conditions. Comput. Math. Appl. **74**, 1321–1332 (2017)

[46] Zou, W.M.: Sign-Changing Critical Point Theory. Springer, New York (2008)

Sitong Chen and Xianhua Tang
School of Mathematics and Statistics
Central South University
Changsha 410083 Hunan
People's Republic of China
e-mail: tangxh@mail.csu.edu.cn

Sitong Chen
e-mail: mathsitongchen@163.com

Fangfang Liao
Department of Mathematics
Xiangnan University
Chenzhou 423000 Hunan
People's Republic of China
e-mail: liaofangfang1981@126.com

Received: 4 January 2018.
Accepted: 25 July 2018.