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Semi-linear fractional σ-evolution equations with mass or power non-linearity

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Abstract. In this paper we study the global (in time) existence of small data solutions to semi-linear fractional σ-evolution equations with mass or power non-linearity. Our main goal is to explain on the one hand the influence of the mass term and on the other hand the influence of higher regularity of the data on qualitative properties of solutions. Using modified Bessel functions we prove some polynomial decay in $L^p - L^q$ estimates for solutions to the corresponding linear fractional σ-evolution equations with vanishing right-hand sides. By a fixed point argument the existence of small data solutions is proved for some admissible range of powers p.

Mathematics Subject Classification. Primary 35R11; Secondary 35A01.

Keywords. Fractional equations, σ-Evolution equations, Global in time existence, Small data solutions.

1. Introduction

Recently, in [3] the authors studied the following Cauchy problem for semi-linear fractional wave equations

$$\begin{aligned} &\partial_t^{1+\alpha} u - \Delta u = |u|^p, \\ &u(0,x) = u_0(x), \quad u_t(0,x) = 0, \end{aligned} \tag{1.1}$$

where $\alpha \in (0,1)$, $\partial_t^{1+\alpha} u = D_t^\alpha(u_t)$ with

$$D_t^\alpha(f) = \partial_t(I_t^{1-\alpha} f) \quad \text{and} \quad I_t^\beta f = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s)\, ds \;\; \text{for} \;\; \beta > 0.$$

Here $D_t^\alpha(f)$ and $I_t^\beta f$ denote the fractional Riemann–Liouville derivative and the fractional Riemann–Liouville integral of f on $[0,t]$, respectively. Moreover, Γ is the Euler Gamma function. The authors proved the following results.

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Proposition 1.1. *Let*

$$p > \overline{p} := \max\left\{p_\alpha(n); \frac{1}{1-\alpha}\right\}, \quad \text{where } p_\alpha(n) := 1 + \frac{2(1+\alpha)}{(n-2)(1+\alpha)+2}.$$

Then there exist positive constants ε and $\overline{\delta}$ such that for any $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with $\|u_0\|_{L^1 \cap L^\infty} \leq \varepsilon$ and for any $\delta \in (0, \overline{\delta})$ there exists a uniquely determined global (in time) Sobolev solution

$$u \in C\big([0,\infty), L^{1+\delta}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big)$$

to (1.1). The solution satisfies the following estimate for any $t \geq 0$:

$$\|u(t,\cdot)\|_{L^q} \leq C(1+t)^{-\beta_q+\alpha}\|u_0\|_{L^{1+\delta}\cap L^\infty}, \quad q \in [1+\delta, \infty], \tag{1.2}$$

where

$$\beta_q = \beta_q(\delta) := \min\left\{\frac{n(1+\alpha)}{2}\left(\frac{1}{1+\delta} - \frac{1}{q}\right); 1\right\}.$$

Proposition 1.2. *Let $p \in (1, p_\alpha(n)]$ and $u_0 \in L^1(\mathbb{R}^n)$ be such that*

$$\int_{\mathbb{R}^n} u_0(x)\, dx > 0.$$

Then there does not exist any global (in time) Sobolev solution

$$u \in L^p_{loc}\big([0,\infty) \times \mathbb{R}^n\big).$$

This paper is devoted to the Cauchy problem for the semi-linear fractional σ-evolution equations with mass or power non-linearity

$$\begin{aligned} &\partial_t^{1+\alpha} u + (-\Delta)^\sigma u + m^2 u = |u|^p, \\ &u(0,x) = u_0(x), \quad u_t(0,x) = 0, \end{aligned} \tag{1.3}$$

where $\alpha \in (0,1)$, $m \geq 0$, $\sigma \geq 1$, $(t,x) \in [0,\infty) \times \mathbb{R}^n$. Our main goal is to understand on the one hand the improving influence of the mass term and on the other hand the influence of higher regularity of the data u_0 on the solvability behavior.

First of all we explain Cauchy conditions. We will construct solutions $u = u(t,x)$ in evolution spaces $C([0,\infty), B)$, where B is a suitable Banach space. Different results base on different Banach spaces B. The continuity in t allows to understand the Cauchy condition $u(0,x) = u_0(x)$ as the restriction of the solution u as an element of the evolution space to $t = 0$.

Now, let us come to the second Cauchy condition $u_t(0,x) = 0$. This condition is understood in a very weak sense. To explain it we use the integro-differential equation

$$\begin{aligned} \partial_t u &= I_t^\alpha\big(-(-\Delta)^\sigma u - m^2 u + |u|^p\big) \\ &= I_t^\alpha\big(-(-\Delta)^\sigma u - m^2 u\big) + I_t^\alpha\big(|u|^p\big). \end{aligned}$$

We have solutions u in $C([0,\infty), B)$, where B stands for a space $L^q(\mathbb{R}^n)$ or a space $H^\gamma_r(\mathbb{R}^n)$. Then, depending on the case with or without mass, we may conclude

$$-(-\Delta)^\sigma u - m^2 u \in C([0,\infty), B_1), \quad |u|^p \in C([0,\infty), B_2),$$

where in the massless case B_1 stands for $\dot{H}_r^{\gamma-2\sigma}(\mathbb{R}^n)$ or for $\dot{H}_r^{-2\sigma}(\mathbb{R}^n)$ (depends on the regularity of the data) and B_2 stands for a suitable space $L^q(\mathbb{R}^n)$. In the case of models with mass B_1 stands for $H_r^{\gamma-2\sigma}(\mathbb{R}^n)$ or for $H_r^{-2\sigma}(\mathbb{R}^n)$ (depends on the regularity of the data) while B_2 stands for a suitable space $L^q(\mathbb{R}^n)$. Using the weak singular structure of I_t^α and the continuity up to $t=0$ of the integrand we verify $\lim_{t\to+0}\|\partial_t u(t,\cdot)\|_{B_k}=0$ for $k=1,2$.
Consequently, if we interpret $u_t(0,x)=0$ in this weak sense, then the Cauchy problem (1.3) may be written in the form of the following Cauchy problem for an integro-differential equation:

$$\partial_t u = I_t^\alpha\big(-(-\Delta)^\sigma u - m^2 u + |u|^p\big), \tag{1.4}$$

$$u(0,x)=u_0(x). \tag{1.5}$$

A solution to (1.3) is defined as a solution of (1.4). For this reason we may restrict ourselves in the further considerations to the study of (1.4) to obtain results for (1.3). Our results of global (in time) existence of small data Sobolev solutions are given in the next section.

2. Main results

2.1. Fractional $\boldsymbol{\sigma}$-evolution models

In the first two results we assume low regularity for the data u_0. We distinguish between conditions for the spatial dimension n.

Theorem 2.1. *Let us assume $0<\alpha<1$, $\alpha\le\lambda<\frac{1+\alpha}{2}$, $\sigma\ge1$ and $r\ge1$. We assume that $n\ge\frac{2\sigma r}{1+\alpha}$. Moreover, the exponent p satisfies the condition*

$$p>p_{\alpha,\lambda,\sigma,r}(n):=\max\left\{p^r_{\alpha,\lambda,\sigma}(n);\frac{1}{1-\lambda}\right\},$$

$$\textit{where } p^r_{\alpha,\lambda,\sigma}(n):=1+\frac{n(r-1)(1+\alpha)+2\sigma r(1+\lambda)}{(n-2\sigma r)(1+\alpha)+2\sigma r(1+\alpha-\lambda)}.$$

Then there exists a positive constant ε such that for any data

$$u_0\in L^r(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)\quad \textit{with } \|u_0\|_{L^r\cap L^\infty}\le\varepsilon$$

we have a uniquely determined global (in time) Sobolev solution

$$u\in C\big([0,\infty),L^r(\mathbb{R}^n)\cap L^q(\mathbb{R}^n)\big)\ \textit{for all } q\in[r,\infty)$$

to the Cauchy problem

$$\partial_t^{1+\alpha}u+(-\Delta)^\sigma u=|u|^p,\ \ u(0,x)=u_0(x),\quad u_t(0,x)=0.$$

Moreover, the solution satisfies the following estimate for any $t\ge0$ and for all sufficiently small $\delta>0$:

$$\|u(t,\cdot)\|_{L^q}\le C(1+t)^{-\beta^{r,\delta}_{\alpha,q,\sigma}+\lambda}\|u_0\|_{L^r\cap L^\infty}\ \textit{for all } q\in[r,\infty],$$

where

$$\beta^{r,\delta}_{\alpha,q,\sigma}:=\min\left\{\frac{n(1+\alpha)}{2\sigma}\left(\frac{1}{r}-\frac{1}{q}\right);1-\delta\right\}.$$

The constant C is independent of u_0.

Theorem 2.2. *Let us assume* $0 < \alpha < 1$, $\alpha \leq \lambda < \frac{1+\alpha}{2}$, $1 \leq \sigma < \frac{\alpha+1}{2\lambda}$ *and* $1 \leq r < \frac{\alpha+1}{2\sigma\lambda}$. *We assume that* $1 \leq n < \frac{2\sigma r}{1+\alpha}$. *Moreover, the exponent* p *satisfies the condition*

$$p > p^r_{\alpha,\lambda,\sigma}(n), \ \ \text{where } p^r_{\alpha,\lambda,\sigma}(n) := 1 + \frac{n(r-1)(1+\alpha)+2\sigma r(1+\lambda)}{(n-2\sigma r)(1+\alpha)+2\sigma r(1+\alpha-\lambda)}.$$

Then there exists a positive constant ε *such that for any data*

$$u_0 \in L^r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \quad \text{with } \|u_0\|_{L^r\cap L^\infty} \leq \varepsilon$$

we have a uniquely determined global (in time) Sobolev solution

$$u \in C\big([0,\infty), L^r(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)\big) \ \text{for all } q \in [r,\infty)$$

to the Cauchy problem

$$\partial_t^{1+\alpha} u + (-\Delta)^\sigma u = |u|^p, \ \ u(0,x) = u_0(x), \quad u_t(0,x) = 0.$$

Moreover, the solution satisfies the following estimate for any $t \geq 0$*:*

$$\|u(t,\cdot)\|_{L^q} \leq C(1+t)^{-\beta^r_{\alpha,q,\sigma}+\lambda}\|u_0\|_{L^r\cap L^\infty} \ \text{for all } q \in [r,\infty],$$

where

$$\beta^r_{\alpha,q,\sigma} := \frac{n(1+\alpha)}{2\sigma}\left(\frac{1}{r} - \frac{1}{q}\right).$$

The constant C *is independent of* u_0.

In the next two results we assume higher regularity for the data u_0 but with additional regularity L^∞. We distinguish between conditions for the spatial dimension n.

Theorem 2.3. *Let us assume* $0 < \alpha < 1$, $\alpha \leq \lambda < \frac{1+\alpha}{2}$, $\sigma \geq 1$, $1 < r < \infty$ *and* $\gamma \geq 0$. *We assume that* $n \geq \frac{2\sigma r}{1+\alpha}$. *The exponent* p *satisfies the condition*

$$p > p_{\alpha,\lambda,\sigma,r,\gamma} := \max\left\{p^r_{\alpha,\lambda,\sigma}(n); \frac{2}{1-\lambda}; \gamma\right\},$$

where

$$p^r_{\alpha,\lambda,\sigma}(n) := 1 + \frac{n(r-1)(1+\alpha)+2\sigma r(1+\lambda)}{(n-2\sigma r)(1+\alpha)+2\sigma r(1+\alpha-\lambda)}.$$

Then there exists a positive constant ε *such that for any data*

$$u_0 \in H^\gamma_r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \quad \text{with } \|u_0\|_{H^\gamma_r\cap L^\infty} \leq \varepsilon$$

we have a uniquely determined global (in time) Sobolev solution

$$u \in C\big([0,\infty), H^\gamma_r(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)\big) \ \text{for all } q \in [r,\infty)$$

to the Cauchy problem

$$\partial_t^{1+\alpha} u + (-\Delta)^\sigma u = |u|^p, \ \ u(0,x) = u_0(x), \quad u_t(0,x) = 0.$$

The solution satisfies the following estimate for any $t \geq 0$ *and for all sufficiently small* $\delta > 0$*:*

$$\|u(t,\cdot)\|_{L^q} \leq C(1+t)^{-\beta^{r,\delta}_{\alpha,q,\sigma}+\lambda}\|u_0\|_{H^\gamma_r\cap L^\infty}, \ \ q \in [r,\infty],$$

where

$$\beta^{r,\delta}_{\alpha,q,\sigma} := \min\left\{\frac{n(1+\alpha)}{2\sigma}\left(\frac{1}{r}-\frac{1}{q}\right); 1-\delta\right\}.$$

Moreover, the solution satisfies the estimate

$$\|u(t,\cdot)\|_{H^\gamma_r} \leq C(1+t)^\lambda \|u_0\|_{H^\gamma_r \cap L^\infty}.$$

The constants C are independent of u_0.

Theorem 2.4. *Let us assume $0<\alpha<1$, $\alpha \leq \lambda < \frac{1+\alpha}{2}$, $1 \leq \sigma < \frac{\alpha+1}{2\lambda}$, $1 < r < \frac{\alpha+1}{2\sigma\lambda}$ and $\gamma \geq 0$. We assume that $1 \leq n < \frac{2\sigma r}{1+\alpha}$. Moreover, the exponent p satisfies the condition*

$$p > \max\{p^r_{\alpha,\lambda,\sigma}(n);\gamma\},$$

$$\textit{where } p^r_{\alpha,\lambda,\sigma}(n) := 1 + \frac{n(r-1)(1+\alpha)+2\sigma r(1+\lambda)}{(n-2\sigma r)(1+\alpha)+2\sigma r(1+\alpha-\lambda)}.$$

Then there exists a positive constant ε such that for any data

$$u_0 \in H^\gamma_r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \quad \textit{with } \|u_0\|_{H^\gamma_r \cap L^\infty} \leq \varepsilon$$

we have a uniquely determined global (in time) Sobolev solution

$$u \in C\big([0,\infty), H^\gamma_r(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)\big) \textit{ for all } q \in [r,\infty)$$

to the Cauchy problem

$$\partial_t^{1+\alpha} u + (-\Delta)^\sigma u = |u|^p, \ \ u(0,x) = u_0(x), \quad u_t(0,x) = 0.$$

The solution satisfies the following estimate for any $t \geq 0$:

$$\|u(t,\cdot)\|_{L^q} \leq C(1+t)^{-\beta^r_{\alpha,q,\sigma}+\lambda} \|u_0\|_{H^\gamma_r \cap L^\infty}, \ \ q \in [r,\infty],$$

where

$$\beta^r_{\alpha,q,\sigma} := \frac{n(1+\alpha)}{2\sigma}\Big(\frac{1}{r}-\frac{1}{q}\Big).$$

Moreover, the solution satisfies the estimate

$$\|u(t,\cdot)\|_{H^\gamma_r} \leq C(1+t)^\lambda \|u_0\|_{H^\gamma_r \cap L^\infty}.$$

The constants C are independent of u_0.

2.2. Fractional σ-evolution models with mass term

Theorem 2.5. *Let us assume $0<\alpha<1$, $\sigma \geq 1$, $r \geq 1$ and $p > \frac{1}{1-\alpha}$. Then there exists a positive constant ε such that for any data*

$$u_0 \in L^r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \quad \textit{with} \quad \|u_0\|_{L^r \cap L^\infty} \leq \varepsilon$$

we have a uniquely determined global (in time) Sobolev solution

$$u \in C\big([0,\infty), L^r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big)$$

to the Cauchy problem

$$\partial_t^{1+\alpha} u + (-\Delta)^\sigma u + m^2 u = |u|^p, \ \ u(0,x) = u_0(x), \quad u_t(0,x) = 0.$$

Moreover, the solution satisfies the decay estimate

$$\|u(t,\cdot)\|_{L^q} \leq C(1+t)^{\alpha-1} \|u_0\|_{L^r \cap L^\infty} \quad \textit{for all} \quad t \geq 0, \quad q \in [r,\infty].$$

The constant C is independent of u_0.

Theorem 2.6. *Let us assume $0 < \alpha < 1$, $\sigma \geq 1$, $\gamma \geq 0$, $1 < r < \infty$ and $p > \max\{2; \frac{1}{1-\alpha}; \gamma\}$. Then there exists a positive constant ε such that for any data*

$$u_0 \in H^\gamma_r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \quad \text{with} \quad \|u_0\|_{H^\gamma_r \cap L^\infty} \leq \varepsilon,$$

we have a uniquely determined global (in time) Sobolev solution

$$u \in C\big([0,\infty), H^\gamma_r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big)$$

to the Cauchy problem

$$\partial_t^{1+\alpha} u + (-\Delta)^\sigma u + m^2 u = |u|^p, \; u(0,x) = u_0(x), \quad u_t(0,x) = 0.$$

Moreover, the solution satisfies the decay estimate

$$\|u(t,\cdot)\|_{H^\gamma_r \cap L^\infty} \leq C(1+t)^{\alpha-1} \|u_0\|_{H^\gamma_r \cap L^\infty}.$$

The constant C is independent of u_0.

Remark 2.7. If we compare Theorem 2.1 with the corresponding result for (1.1) from [3], then we feel the improving influence of the power σ and the order of regularity r in two facts. On the one hand $p_{\alpha,\alpha,1,1}(n) = \overline{p}$ and on the other hand $u \in C\big([0,\infty), L^r(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)\big)$ for all $q < \infty$. In Theorem 2.3 we explain the influence of the regularity of the data on the critical exponent and we have $p_{\alpha,\alpha,1,1}(n) \geq \overline{p}$. If we compare Theorem 2.5 with the corresponding result for (1.1) from [3], then we feel the improving influence of the mass term in three facts. On the one hand $\overline{p} = \frac{1}{1-\alpha}$, $u \in C\big([0,\infty), L^r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big)$ and on the other hand $\beta_q = 1$ in (1.2). In the case of Theorem 2.6 we also feel the influence of the regularity of the data on the exponent and we obtain an exponent larger then $\overline{p}$. Besides some stronger restrictions to the critical exponent the statements of Theorems 2.3, 2.4 and 2.6 are regularity results. If the data u_0 is more regular, then we expect more regularity with respect to the spatial variables for the solution.

3. Some preliminaries

The Cauchy problem (1.4) with $\sigma \geq 1$ and $m \geq 0$ can be formally converted to an integral equation and its solution is given by

$$u(t,x) = (G^m_{\alpha,\sigma}(t) * u_0)(t,x) + N^m_{\alpha,\sigma}(u)(t,x) \tag{3.1}$$

with

$$G^m_{\alpha,\sigma}(t,x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} E_{\alpha+1}\big(-t^{\alpha+1}\langle\xi\rangle^2_{m,\sigma}\big)\, d\xi, \tag{3.2}$$

$$N^m_{\alpha,\sigma}(u)(t,x) = \int_0^t \big(G^m_{\alpha,\sigma}(t-s) * I^\alpha_s(|u|^p)\big)(t,s,x)\, ds, \tag{3.3}$$

where $\{G^m_{\alpha,\sigma}(t)\}_{t\geq 0}$ denotes the semigroup of operators which is defined via Fourier transform by

$$(\widehat{G^m_{\alpha,\sigma}(t) * f})(t,\xi) = E_{\alpha+1}\big(-t^{\alpha+1}\langle\xi\rangle^2_{m,\sigma}\big)\widehat{f}(\xi) \quad \text{with} \quad \langle\xi\rangle^2_{m,\sigma} = |\xi|^{2\sigma} + m^2.$$

Here $E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k+1)}$, $\beta \in \mathbb{C}$ with $\Re\beta > 0$, denotes the Mittag-Leffler function (see Sect. 7.2).

A representation of solutions of the linear integro-differential equation associated to (1.4) or (1.3) with $\sigma \geq 1$ and $m \geq 0$ (and without the term $|u|^p$) is given by

$$u(t,x) = (G^m_{\alpha,\sigma}(t) * u_0)(t,x).$$

Indeed, we put

$$\begin{aligned} v(t,\xi) &= F_{x\to\xi}(u(t,x))(t,\xi) = F_{x\to\xi}\big((G^m_{\alpha,\sigma}(t) * u_0)(t,x)\big)(t,\xi) \\ &= E_{\alpha+1}(-t^{\alpha+1}\langle\xi\rangle^2_{m,\sigma})\widehat{u_0}(\xi). \end{aligned}$$

By using (1.4) and (7.1) we have

$$\begin{aligned} &F^{-1}_{\xi\to x}\left(F_{x\to\xi}\left(\int_0^t I^\alpha_s(-(-\Delta)^\sigma u - m^2 u)(s,x)\,ds\right)(t,\xi)\right)(t,x) \\ &= F^{-1}_{\xi\to x}\left(\langle\xi\rangle^2_{m,\sigma}\int_0^t I^\alpha_s((\widehat{G^m_{\alpha,\sigma}(\tau) * u_0})(\tau,\xi))\,ds\right)(t,x) \\ &= F^{-1}_{\xi\to x}\left(\frac{\langle\xi\rangle^2_{m,\sigma}}{\Gamma(\alpha)}\int_0^t\int_0^s (s-\tau)^{\alpha-1}(\widehat{G^m_{\alpha,\sigma}(\tau) * u_0})(\tau,\xi)\,d\tau\,ds\right)(t,x) \\ &= F^{-1}_{\xi\to x}\left(\frac{\langle\xi\rangle^2_{m,\sigma}}{\Gamma(\alpha)}\int_0^t\int_0^s (s-\tau)^{\alpha-1}E_{\alpha+1}(-\tau^{\alpha+1}\langle\xi\rangle^2_{m,\sigma})\widehat{u_0}(\xi)\,d\tau\,ds\right)(t,x) \\ &= F^{-1}_{\xi\to x}\left(\frac{\langle\xi\rangle^2_{m,\sigma}}{\Gamma(\alpha)}\int_0^t\int_\tau^t (s-\tau)^{\alpha-1}\,ds\,E_{\alpha+1}(-\tau^{\alpha+1}\langle\xi\rangle^2_{m,\sigma})\widehat{u_0}(\xi)\,d\tau\right)(t,x) \\ &= F^{-1}_{\xi\to x}\left(\frac{\langle\xi\rangle^2_{m,\sigma}}{\Gamma(\alpha+1)}\int_0^t (t-\tau)^{\alpha}E_{\alpha+1}(-\tau^{\alpha+1}\langle\xi\rangle^2_{m,\sigma})\widehat{u_0}(\xi)\,d\tau\right)(t,x) \\ &= F^{-1}_{\xi\to x}\left(\left(E_{\alpha+1}(-t^{\alpha+1}\langle\xi\rangle^2_{m,\sigma}) - 1\right)\widehat{u_0}(\xi)\right)(t,x) \\ &= F^{-1}_{\xi\to x}\left(E_{\alpha+1}(-t^{\alpha+1}\langle\xi\rangle^2_{m,\sigma})\widehat{u_0}(\xi) - \widehat{u_0}(\xi)\right)(t,x) \\ &= F^{-1}_{\xi\to x}\left(v(t,\xi) - \widehat{u_0}(\xi)\right)(t,x) = u(t,x) - u_0(x). \end{aligned}$$

Consequently, we have shown (after application of the Fourier inversion formula in S') that

$$u = G^m_{\alpha,\sigma}(t) * u_0$$

is a formal solution to

$$u = u_0(x) + \int_0^t I^\alpha_s(-(-\Delta)^\sigma u - m^2 u)\,ds.$$

In the moment we will not provide any function spaces to which the formal solution will belong.
But, as pointed out by the referee the continuity of solutions with respect to the time variable requires a special treatment. Later we will come back to this issue. But, from the above considerations we can formally conclude the following relation (if the convolution really exist)

$$\begin{aligned} &u(t,\cdot) - u_0 \\ &\quad = \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-\tau)^\alpha \big(F^{-1}_{\xi\to x}\big(\langle\xi\rangle^2_{m,\sigma} E_{\alpha+1}(-\tau^{\alpha+1}\langle\xi\rangle^2_{m,\sigma})\big) * u_0\big)\, d\tau. \end{aligned} \tag{3.4}$$

Later we will use this relation for the discussion of continuity in time of solutions for models with mass.

4. L^p estimates for model oscillating integrals

At first we derive L^p estimates for the model oscillating integral

$$F^{-1}_{\xi\to x}\big(E_{\alpha+1}(-t^{1+\alpha}|\xi|^{2\sigma})\big).$$

Proposition 4.1. *The following estimate holds in $\mathbb{R}^n$ for $\sigma > 0$, $\alpha \geq 0$:*

$$\|F^{-1}_{\xi\to x}\big(E_{\alpha+1}(-t^{1+\alpha}|\xi|^{2\sigma})\big)(t,\cdot)\|_{L^p} \lesssim t^{-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p})} \tag{4.1}$$

for $p \in [1,\infty]$, $t > 0$ and for all $n \geq 1$ satisfying $n(1-\frac{1}{p}) < 2\sigma$.

Here and in the following we use for non-negative functions f and g the notation $f \lesssim g$ if there exists a constant C which is independent of $y \in D$ such that $f(y) \leq Cg(y)$ for all $y \in D$.

Proof. The proof of (4.1) uses the Propositions 5 and 12 of [12]. In a first step we estimate the following oscillating integrals:

$$F^{-1}_{\xi\to x}\big(e^{-c_1 t|\xi|^{2\rho}}\cos(c_2 t|\xi|^{2\rho})\big) \quad \text{and} \quad F^{-1}_{\xi\to x}\big(e^{-\tau t|\xi|^{2\rho}}\big),$$

where $c_1 = -\cos(\frac{\pi}{1+\alpha}), c_2 = \sqrt{1-c_1^2}, \rho = \frac{\sigma}{1+\alpha}$ and $\tau > 0$. We prove instead of (4.1) the polynomial type decay estimates

$$\|F^{-1}_{\xi\to x}\big(e^{-c_1 t|\xi|^{2\rho}}\cos(c_2 t|\xi|^{2\rho})\big)(t,\cdot)\|_{L^p} \lesssim t^{-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p})}, \tag{4.2}$$

$$\|F^{-1}_{\xi\to x}\big(e^{-\tau t|\xi|^{2\rho}}\big)(t,\cdot)\|_{L^p} \lesssim (\tau t)^{-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p})} \tag{4.3}$$

for all $p \in [1,+\infty]$ and $t > 0$. Then, we deduce (see Sect. 7.2)

$$\big\|F^{-1}_{\xi\to x}\big(\exp\big(a_{1+\alpha}(t^{\frac{1+\alpha}{2}}|\xi|^\sigma)\big) + \exp\big(b_{1+\alpha}(t^{\frac{1+\alpha}{2}}|\xi|^\sigma)\big)\big)(t,\cdot)\big\|_{L^p} \lesssim t^{-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p})} \tag{4.4}$$

for all $p \in [1,+\infty]$ and $t > 0$. It remains to prove that (see Sect. 7.2)

$$\big\|F^{-1}_{\xi\to x}\big(l_{1+\alpha}(t^{\frac{1+\alpha}{2}}|\xi|^\sigma)\big)(t,\cdot)\big\|_{L^p} \lesssim t^{-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p})} \tag{4.5}$$

for all $p \in [1,+\infty]$ and $t>0$. Therefore we use the formula (see Sect. 7.2)

$$l_{1+\alpha}(t^{\frac{1+\alpha}{2}}|\xi|^\sigma) \sim \int_0^\infty \frac{\exp\big(-t|\xi|^{\frac{2\sigma}{1+\alpha}}s^{\frac{1}{1+\alpha}}\big)}{s^2+2s\cos((1+\alpha)\pi)+1}\,ds.$$

Taking account of the definition of modified Bessel functions (see Sect. 7.1) we get

$$\begin{aligned}
&F^{-1}_{\xi\to x}\big(l_{1+\alpha}(t^{\frac{1+\alpha}{2}}|\xi|^\sigma)\big)(t,x)\\
&\quad=\int_0^\infty\left(\int_0^\infty \frac{\exp\Big(-tr^{\frac{2\sigma}{1+\alpha}}s^{\frac{1}{1+\alpha}}\Big)}{s^2+2s\cos((1+\alpha)\pi)+1}r^{n-1}\tilde{J}_{\frac{n}{2}-1}(r|x|)dr\right)ds\\
&\quad=\int_0^\infty \frac{1}{s^2+2s\cos((1+\alpha)\pi)+1}\\
&\qquad\times\left(\int_0^\infty \exp\Big(-tr^{\frac{2\sigma}{1+\alpha}}s^{\frac{1}{1+\alpha}}\Big)r^{n-1}\tilde{J}_{\frac{n}{2}-1}(r|x|)dr\right)ds\\
&\quad=\int_0^\infty \frac{1}{s^2+2s\cos((1+\alpha)\pi)+1}\left(F^{-1}_{\xi\to x}\Big(e^{-s^{\frac{1}{1+\alpha}}t|\xi|^{\frac{2\sigma}{1+\alpha}}}\Big)(x)\right)ds.
\end{aligned}$$

The estimate

$$\big\|F^{-1}_{\xi\to x}\big(e^{-s^{\frac{1}{1+\alpha}}t|\xi|^{\frac{2\sigma}{1+\alpha}}}\big)(t,\cdot)\big\|_{L^p} \lesssim s^{-\frac{n}{2\sigma}(1-\frac{1}{p})}t^{-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p})}$$

implies

$$\begin{aligned}
&\big\|F^{-1}_{\xi\to x}\big(l_{1+\alpha}(t^{\frac{1+\alpha}{2}}|\xi|^\sigma)\big)(t,\cdot)\big\|_{L^p}\\
&\quad\lesssim t^{-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p})}\int_0^\infty \frac{s^{-\frac{n}{2\sigma}(1-\frac{1}{p})}}{s^2+2s\cos((1+\alpha)\pi)+1}\,ds \lesssim t^{-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p})}
\end{aligned}$$

if $n(1-\frac{1}{p})<2\sigma$. □

Now let us turn to L^p estimates for the model oscillating integral (see Sect. 7.2)

$$F^{-1}_{\xi\to x}\big(E_{\alpha+1}(-t^{1+\alpha}\langle\xi\rangle^2_{m,\sigma})\big) \text{ with } m>0.$$

At the first glance one might expect an exponential type decay estimate. We are able to prove a potential type decay estimate only.

Proposition 4.2. *The following estimate holds in $\mathbb{R}^n$ for $\sigma>0$, $m>0$, $\alpha\in[0,1)$ and for all $n\geq 1$:*

$$\big\|F^{-1}_{\xi\to x}\big(E_{\alpha+1}(-t^{1+\alpha}\langle\xi\rangle^2_{m,\sigma})\big)(t,\cdot)\big\|_{L^p} \lesssim (1+t)^{-(1+\alpha)} \tag{4.6}$$

for $p\in[1,\infty]$ and $t>0$.

Proof. The proof of (4.6) uses ideas of [10]. In a first step we estimate the following oscillating integrals:

$$F^{-1}_{\xi\to x}\big(e^{-ct\langle\xi\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle\xi\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\big) \quad\text{and}\quad F^{-1}_{\xi\to x}\big(e^{-\tau t\langle\xi\rangle^{2\kappa}_{m,\sigma}}\big),$$

where $c = -\cos(\frac{\pi}{1+\alpha}), \kappa = \frac{1}{1+\alpha} \in (\frac{1}{2}, 1)$ and $\tau > 0$. We shall derive the exponential type decay estimate

$$\begin{aligned} &\big\|F^{-1}_{\xi\to x}\big(e^{-ct\langle\xi\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle\xi\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\big)(t,\cdot)\big\|_{L^p} \\ &\qquad + \big\|F^{-1}_{\xi\to x}\big(e^{-\tau t\langle\xi\rangle^{2\kappa}_{m,\sigma}}\big)(t,\cdot)\big\|_{L^p} \lesssim e^{-Ct} \end{aligned} \tag{4.7}$$

with a suitable positive $C = C(m,\alpha)$, for $p \in [1,\infty]$ and $t \geq 0$. By using modified Bessel functions (see Sect. 7.1) we have for $n = 3$

$$\begin{aligned} &F^{-1}_{\xi\to x}\Big(e^{-ct\langle\xi\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle\xi\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\Big)(t,x) \\ &\quad = \int_0^\infty e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r^2\tilde{J}_{\frac{1}{2}}(r|x|)\,dr \\ &\quad = -\frac{1}{|x|^2}\int_0^\infty e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r\partial_r\tilde{J}_{-\frac{1}{2}}(r|x|)\,dr \\ &\quad = -\frac{\sqrt{2}}{\sqrt{\pi}|x|^2}\int_0^\infty e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r\partial_r(\cos(r|x|))\,dr. \end{aligned}$$

Using twice integration by parts we obtain

$$\begin{aligned} &-\frac{\sqrt{\pi}|x|^4}{\sqrt{2}}F^{-1}_{\xi\to x}\big(e^{-ct\langle\xi\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle\xi\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\big)(t,x) \\ &\quad = \int_0^\infty\Big(t\big(h_1(r)r^{2\sigma-2}\langle r\rangle^4_{m,\sigma} + h_2(r)r^{4\sigma-2}\langle r\rangle^2_{m,\sigma} + h_3(r)r^{6\sigma-2}\big)\langle r\rangle^{2\kappa-6}_{m,\sigma} \\ &\qquad + t^2\big(h_4(r)r^{4\sigma-2}\langle r\rangle^2_{m,\sigma} + h_5(r)r^{6\sigma-2}\big)\langle r\rangle^{4\kappa-6}_{m,\sigma} + t^3h_6(r)r^{6\sigma-2}\langle r\rangle^{6\kappa-6}_{m,\sigma}\Big) \\ &\qquad \times e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(r|x|)\,dr, \end{aligned} \tag{4.8}$$

where $h_i(r) = a_i\cos(g(r)) + b_i\sin(g(r))$, $i = 1,\cdots,6$, $g(r) = t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2}$ and a_i, b_i, $i = 1,\cdots,6$, are constants which depend on α and σ only.
To estimate (4.8) we use the inequality

$$\langle r\rangle^{2\kappa}_{m,\sigma} \geq 2^{\kappa-1}\langle r\rangle^{2\kappa}_{2^{-1/2}m,\sigma} + 2^{-1}m^{2\kappa}. \tag{4.9}$$

Then, we get

$$\big|F^{-1}_{\xi\to x}\big(e^{-ct\langle\xi\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle\xi\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\big)(t,x)\big| \lesssim \frac{e^{-\frac{c}{2}tm^{2\kappa}}}{\langle x\rangle^4_m}.$$

For the oscillating integral $F^{-1}_{\xi\to x}\big(e^{-\tau t\langle\xi\rangle^{2\kappa}_{m,\sigma}}\big)$ we have

$$\begin{aligned} F^{-1}_{\xi\to x}\big(e^{-\tau t\langle\xi\rangle^{2\kappa}_{m,\sigma}}\big)(t,x) &= \int_0^\infty e^{-\tau t\langle r\rangle^{2\kappa}_{m,\sigma}}r^2\tilde{J}_{\frac{1}{2}}(r|x|)\,dr \\ &= -\frac{1}{|x|^2}\int_0^\infty e^{-\tau t\langle r\rangle^{2\kappa}_{m,\sigma}}r\partial_r\tilde{J}_{-\frac{1}{2}}(r|x|)\,dr \\ &= -\frac{\sqrt{2}}{\sqrt{\pi}|x|^2}\int_0^\infty e^{-\tau t\langle r\rangle^{2\kappa}_{m,\sigma}}r\partial_r(\cos(r|x|))\,dr. \end{aligned}$$

Using twice integration by parts we obtain

$$
\begin{aligned}
&-\frac{\sqrt{\pi}|x|^4}{\sqrt{2}} F^{-1}_{\xi\to x}\big(e^{-\tau t\langle\xi\rangle^{2\kappa}_{m,\sigma}}\big)(t,x)\\
&\quad=\int_0^\infty\Big(-2\sigma(4\sigma^2-1)\kappa\tau t r^{2\sigma-2}\langle r\rangle^{2\kappa-2}_{m,\sigma}-24\sigma^3\kappa(\kappa-1)\tau t r^{4\sigma-2}\langle r\rangle^{2\kappa-4}_{m,\sigma}\\
&\qquad-8\sigma^3\kappa(\kappa-1)(\kappa-2)\tau t r^{6\sigma-2}\langle r\rangle^{2\kappa-6}_{m,\sigma}+24\sigma^3\kappa^2\tau^2t^2r^{4\sigma-2}\langle r\rangle^{4\kappa-4}_{m,\sigma}\\
&\qquad+8\sigma^3\kappa^2(\kappa-1)\tau^2t^2r^{6\sigma-2}\langle r\rangle^{4\kappa-6}_{m,\sigma}-8\sigma^3\kappa^3\tau^3t^3r^{6\sigma-2}\langle r\rangle^{6\kappa-6}_{m,\sigma}\Big)\\
&\qquad\quad\times e^{-\tau t\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(r|x|)\,dr.
\end{aligned}
$$

This leads to the estimate

$$
\big|F^{-1}_{\xi\to x}\big(e^{-\tau t\langle\xi\rangle^{2\kappa}_{m,\sigma}}\big)(t,x)\big|\lesssim\frac{e^{-\frac{\tau}{2}tm^{2\kappa}}}{\langle x\rangle^4_m}.
$$

Summarizing all estimates we proved the statement (4.7) in the case $n=3$. Now, let us study the case n odd and $n\geq 4$. Then we carry out $\frac{n+1}{2}$ steps of partial integration. We obtain after $\frac{n-1}{2}$ steps and by applying the rules (see Sect. 7.1)

$$
\tilde{J}_{\mu+1}(r|x|)=-\frac{1}{r|x|^2}\partial_r\tilde{J}_\mu(r|x|)
$$

for real non-negative μ the relation

$$
\begin{aligned}
&F^{-1}_{\xi\to x}\big(e^{-ct\langle\xi\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle\xi\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\big)(t,x)\\
&\quad=\int_0^\infty e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r^{n-1}\tilde{J}_{\frac{n}{2}-1}(r|x|)\,dr\\
&\quad=(-1)^{\frac{n-1}{2}}\frac{1}{|x|^{n-1}}\int_0^\infty\Big(\frac{\partial}{\partial r}\frac{1}{r}\Big)^{\frac{n-1}{2}}\Big(e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r^{n-1}\Big)\\
&\qquad\times\tilde{J}_{-\frac{1}{2}}(r|x|)\,dr\\
&\quad=(-1)^{\frac{n-1}{2}}\frac{1}{|x|^{n-1}}\int_0^\infty\Big(\frac{\partial}{\partial r}\frac{1}{r}\Big)^{\frac{n-1}{2}}\Big(e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r^{n-1}\Big)\\
&\qquad\times\cos(r|x|)\,dr\\
&\quad=(-1)^{\frac{n+1}{2}}\frac{1}{|x|^{n+1}}\sqrt{\frac{2}{\pi}}\\
&\qquad\times\int_0^\infty\Big(\frac{\partial^2}{\partial r^2}\Big)\Big(\frac{\partial}{\partial r}\frac{1}{r}\Big)^{\frac{n-1}{2}}\Big(e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r^{n-1}\Big)\\
&\qquad\quad\times\cos(r|x|)\,dr.
\end{aligned}
$$

All integrals have the form

$$
\begin{aligned}
&\int_0^\infty\langle r\rangle^\rho_{m,\sigma}r^\delta e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\cos(r|x|)\,dr\\
&\quad\text{or}\ \int_0^\infty\langle r\rangle^\rho_{m,\sigma}r^\delta e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\sin(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\cos(r|x|)\,dr,
\end{aligned}
$$

where ρ is a negative integer depending on κ and n and δ is non-negative real depending on σ and n. For this reason we conclude the estimate

$$\left|F^{-1}_{\xi\to x}\big(e^{-ct\langle\xi\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle\xi\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\big)(t,x)\right| \lesssim \frac{e^{-\frac{c}{2}tm^{2\kappa}}}{\langle x\rangle^{n+1}_m}.$$

Analogously, we obtain the same estimate for

$$F^{-1}_{\xi\to x}\big(e^{-\tau t\langle\xi\rangle^{2\kappa}_{m,\sigma}}\big)(t,x).$$

All together implies the statement (4.7) for odd $n\geq 4$.
For $n=2$ we have

$$\begin{aligned}&F^{-1}_{\xi\to x}\big(e^{-ct\langle\xi\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle\xi\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\big)(t,x)\\ &\qquad=\int_0^\infty e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r\tilde{J}_0(r|x|)\,dr.\end{aligned}$$

From the relation (see Sect. 7.1)

$$J_0(s)=\frac{1}{s}J_1(s)+\frac{d}{ds}J_1(s)$$

it follows that

$$\tilde{J}_0(r|x|)=2\tilde{J}_1(r|x|)+r\partial_r\tilde{J}_1(r|x|)=\frac{1}{r}\partial_r(r^2\tilde{J}_1(r|x|)).$$

Then, we get

$$\begin{aligned}&F^{-1}_{\xi\to x}\big(e^{-ct\langle\xi\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle\xi\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\big)(t,x)\\
&\quad=-\int_0^\infty 2\kappa\sigma tr^{4\sigma-1}\langle r\rangle^{2\kappa-2}_{m,\sigma}e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos\Big(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)\\
&\qquad\times\tilde{J}_1(r|x|)\,dr\\
&\quad=-\int_0^{\frac{1}{|x|}} 2\kappa\sigma tr^{4\sigma-1}\langle r\rangle^{2\kappa-2}_{m,\sigma}e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos\Big(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)\\
&\qquad\times\tilde{J}_1(r|x|)\,dr\\
&\qquad-\int_{\frac{1}{|x|}}^\infty 2\kappa\sigma tr^{4\sigma-1}\langle r\rangle^{2\kappa-2}_{m,\sigma}e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos\Big(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)\\
&\qquad\times\tilde{J}_1(r|x|)\,dr.\end{aligned}$$

Using the boundedness of $\tilde{J}_1(s)$ for $s\in[0,1]$ (see Sect. 7.1) the first integral can be estimated by

$$e^{-\frac{c}{2}tm^{2\kappa}}\langle x\rangle_m^{-(4\sigma+2\kappa-2)}.$$

Remark that $4\sigma+2\kappa-2>2$. To estimate the second integral we apply the following asymptotic formula (see Sect. 7.1) for $\tilde{J}_1(s)$ for $s\geq 1$:

$$\tilde{J}_1(s)=cs^{-\frac{3}{2}}\cos\Big(s-\frac{3}{4}\pi\Big)+O(|s|^{-\frac{5}{2}}).$$

Consequently, the integral can be estimated as follows:

$$\int_{\frac{1}{|x|}}^{\infty} r^{4\sigma-1}\langle r\rangle_{m,\sigma}^{2\kappa-2} e^{-ct\langle r\rangle_{m,\sigma}^{2\kappa}} O((r|x|)^{-\frac{5}{2}})\,dr \lesssim |x|^{-\frac{5}{2}} e^{-\frac{c}{2}tm^{2\kappa}}.$$

It remains to estimate

$$\frac{1}{|x|^{\frac{3}{2}}}\int_{\frac{1}{|x|}}^{\infty} r^{\frac{8\sigma-5}{2}}\langle r\rangle_{m,\sigma}^{2\kappa-2} e^{-ct\langle r\rangle_{m,\sigma}^{2\kappa}} \cos\Big(t\langle r\rangle_{m,\sigma}^{2\kappa}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)\cos(r|x|)\,dr,$$
$$\frac{1}{|x|^{\frac{3}{2}}}\int_{\frac{1}{|x|}}^{\infty} r^{\frac{8\sigma-5}{2}}\langle r\rangle_{m,\sigma}^{2\kappa-2} e^{-ct\langle r\rangle_{m,\sigma}^{2\kappa}} \cos\Big(t\langle r\rangle_{m,\sigma}^{2\kappa}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)\sin(r|x|)\,dr.$$

We explain only the first integral because the second one can be treated in the same way. We write the first integral as follows:

$$\frac{1}{|x|^{\frac{3}{2}}}\int_{\frac{1}{|x|}}^{\infty} r^{\frac{8\sigma-5}{2}}\langle r\rangle_{m,\sigma}^{2\kappa-2} e^{-ct\langle r\rangle_{m,\sigma}^{2\kappa}} \cos\Big(t\langle r\rangle_{m,\sigma}^{2\kappa}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)\cos(r|x|)\,dr$$
$$= \frac{1}{|x|^{\frac{3}{2}}}\int_{\frac{1}{|x|}}^{1} r^{\frac{8\sigma-5}{2}}\langle r\rangle_{m,\sigma}^{2\kappa-2} e^{-ct\langle r\rangle_{m,\sigma}^{2\kappa}} \cos\Big(t\langle r\rangle_{m,\sigma}^{2\kappa}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)\cos(r|x|)\,dr \tag{4.10}$$
$$+ \frac{1}{|x|^{\frac{3}{2}}}\int_{1}^{\infty} r^{\frac{8\sigma-5}{2}}\langle r\rangle_{m,\sigma}^{2\kappa-2} e^{-ct\langle r\rangle_{m,\sigma}^{2\kappa}} \cos\Big(t\langle r\rangle_{m,\sigma}^{2\kappa}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)\cos(r|x|)\,dr. \tag{4.11}$$

The integral in (4.10) is equal to

$$\frac{1}{|x|^{\frac{5}{2}}}\int_{\frac{1}{|x|}}^{1} r^{\frac{8\sigma-5}{2}}\langle r\rangle_{m,\sigma}^{2\kappa-2} e^{-ct\langle r\rangle_{m,\sigma}^{2\kappa}} \cos\Big(t\langle r\rangle_{m,\sigma}^{2\kappa}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)\partial_r(\sin(r|x|))\,dr. \tag{4.12}$$

After partial integration and by using (4.9) the limit terms can be estimated by $|x|^{-\frac{5}{2}} e^{-\frac{c}{2}tm^{2\kappa}}$. The new integral is equal to

$$\frac{1}{|x|^{\frac{5}{2}}}\int_{\frac{1}{|x|}}^{1}\Big(c_1 r^{\frac{8\sigma-7}{2}}\langle r\rangle_{m,\sigma}^{2\kappa-2}\cos\Big(t\langle r\rangle_{m,\sigma}^{2\kappa}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)$$
$$+ c_2 r^{\frac{12\sigma-7}{2}}\langle r\rangle_{m,\sigma}^{2\kappa-4}\cos\Big(t\langle r\rangle_{m,\sigma}^{2\kappa}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)$$
$$+ c_3 r^{\frac{12\sigma-7}{2}}\langle r\rangle_{m,\sigma}^{4\kappa-4}\sin\Big(t\langle r\rangle_{m,\sigma}^{2\kappa}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)$$
$$+ c_4 r^{\frac{12\sigma-7}{2}}\langle r\rangle_{m,\sigma}^{4\kappa-4}\cos\Big(t\langle r\rangle_{m,\sigma}^{2\kappa}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)\Big) e^{-ct\langle r\rangle_{m,\sigma}^{2\kappa}}\sin(r|x|)\,dr.$$

It can be estimated by $|x|^{-\frac{5}{2}} e^{-\frac{c}{2}tm^{2\kappa}}$, too. After integration by parts the integral in (4.11) can be estimated by

$$\frac{1}{|x|^{\frac{5}{2}}}\int_{1}^{\infty} r^{\frac{12\sigma-7}{2}}\langle r\rangle_{m,\sigma}^{2\kappa-2} e^{-ct\langle r\rangle_{m,\sigma}^{2\kappa}}\,dr.$$

The latter integral can be estimated by $|x|^{-\frac{5}{2}}e^{-\frac{c}{2}tm^{2\kappa}}$. Finally, we have for the oscillating integral $F^{-1}_{\xi\to x}\big(e^{-\tau t\langle\xi\rangle^{2\kappa}_{m,\sigma}}\big)$ the relation

$$\begin{aligned}F^{-1}_{\xi\to x}(e^{-\tau t\langle\xi\rangle^{2\kappa}_{m,\sigma}})(t,x) &= \int_0^\infty e^{-\tau t\langle r\rangle^{2\kappa}_{m,\sigma}} r\tilde{J}_0(r|x|)\,dr\\ &= \int_0^\infty e^{-\tau t\langle r\rangle^{2\kappa}_{m,\sigma}}\partial_r(r^2\tilde{J}_1(r|x|))\,dr\\ &= \int_0^\infty 2\sigma\kappa\tau t r^{2\sigma+1}\langle r\rangle^{2\kappa-2}_{m,\sigma}e^{-\tau t\langle r\rangle^{2\kappa}_{m,\sigma}}\tilde{J}_1(r|x|)\,dr.\end{aligned}$$

Then, we derive the same estimates as we did before for estimating the oscillating integral

$$F^{-1}_{\xi\to x}\big(e^{-ct\langle\xi\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle\xi\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\big).$$

Summarizing all estimates yields the statement (4.7) for $n=2$.
Now for the case of even $n\geq 4$ we carry out $\frac{n}{2}-1$ steps of partial integration. In this way we obtain

$$\begin{aligned}&F^{-1}_{\xi\to x}\big(e^{-ct\langle\xi\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle\xi\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\big)(t,x)\\ &\quad= \int_0^\infty e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r^{n-1}\tilde{J}_{\frac{n}{2}-1}(r|x|)\,dr\\ &\quad= \frac{1}{|x|^{n-2}}\int_0^\infty\Big(\frac{\partial}{\partial r}\frac{1}{r}\Big)^{\frac{n-2}{2}}\Big(e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r^{n-1}\Big)\tilde{J}_0(r|x|)\,dr\\ &\quad= \frac{1}{|x|^{n-2}}\int_0^\infty\Big(\frac{\partial}{\partial r}\frac{1}{r}\Big)^{\frac{n-2}{2}}\Big(e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r^{n-1}\Big)\\ &\qquad\qquad\times\frac{1}{r}\partial_r(r^2\tilde{J}_1(r|x|))\,dr\\ &\quad= \frac{1}{|x|^{n-2}}\int_0^\infty\Big(\frac{\partial}{\partial r}\frac{1}{r}\Big)^{\frac{n}{2}}\Big(e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r^{n-1}\Big)r^2\tilde{J}_1(r|x|)\,dr\\ &\quad= \frac{1}{|x|^{n-2}}\int_0^{\frac{1}{|x|}}\Big(\frac{\partial}{\partial r}\frac{1}{r}\Big)^{\frac{n}{2}}\Big(e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r^{n-1}\Big)r^2\tilde{J}_1(r|x|)\,dr \qquad (4.13)\\ &\qquad+\frac{1}{|x|^{n-2}}\int_{\frac{1}{|x|}}^\infty\Big(\frac{\partial}{\partial r}\frac{1}{r}\Big)^{\frac{n}{2}}\Big(e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r^{n-1}\Big)r^2\tilde{J}_1(r|x|)\,dr. \qquad (4.14)\end{aligned}$$

For the integral in (4.13) we are able to derive the following estimate:

$$\begin{aligned}&\Big|\frac{1}{|x|^{n-2}}\int_0^{\frac{1}{|x|}}\Big(\frac{\partial}{\partial r}\frac{1}{r}\Big)^{\frac{n}{2}}\Big(e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r^{n-1}\Big)r^2\tilde{J}_1(r|x|)\,dr\Big|\\ &\qquad\lesssim \frac{1}{|x|^{n-2}}e^{-\frac{c}{2}tm^{2\kappa}}\int_0^{\frac{1}{|x|}}r^{2\sigma+1}\langle r\rangle^{2\kappa-2}_{m,\sigma}\,dr \lesssim e^{-\frac{c}{2}tm^{2\kappa}}\langle x\rangle_m^{-(n+2\kappa+2\sigma-2)}.\end{aligned}$$

For the integral in (4.14) we follow the same arguments to obtain the estimate

$$\Big|\frac{1}{|x|^{n-2}}\int_{\frac{1}{|x|}}^{\infty}\Big(\frac{\partial}{\partial r}\frac{1}{r}\Big)^{\frac{n}{2}}\Big(e^{-ct\langle r\rangle_{m,\sigma}^{2\kappa}}\cos(t\langle r\rangle_{m,\sigma}^{2\kappa}\sqrt{1-c^2})r^{n-1}\Big)r^2\tilde{J}_1(r|x|)\,dr\Big|$$
$$\lesssim e^{-\frac{c}{2}tm^{2\kappa}}\langle x\rangle_m^{-(n+\frac{1}{2})}.$$

In the same way we can estimate the oscillating integral $F^{-1}_{\xi\to x}\big(e^{-\tau t\langle\xi\rangle_{m,\sigma}^{2\kappa}}\big)$. All together implies the statement (4.7) for even $n\ge 4$. To complete the proof it remains to show

$$\big\|F^{-1}_{\xi\to x}\big(l_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\big)(t,\cdot)\big\|_{L^p}\lesssim(1+t)^{-(1+\alpha)}$$

for $p\in[1,\infty]$ and $t\ge 0$. Therefore we use the formula (see Sect. 7.2)

$$l_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\sim\int_0^\infty\frac{\exp\big(-t\langle\xi\rangle_{m,\sigma}^{\frac{2}{1+\alpha}}s^{\frac{1}{1+\alpha}}\big)}{s^2+2s\cos((1+\alpha)\pi)+1}\,ds.$$

Taking account of the definition of modified Bessel functions (see Sect. 7.1) we get

$$\begin{aligned}
&F^{-1}_{\xi\to x}\big(l_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\big)(t,x)\\
&\quad=\int_0^\infty\Big(\int_0^\infty\frac{\exp\big(-t\langle r\rangle_m^{\frac{2}{1+\alpha}}s^{\frac{1}{1+\alpha}}\big)}{s^2+2s\cos((1+\alpha)\pi)+1}r^{n-1}\tilde{J}_{\frac{n}{2}-1}(r|x|)\,dr\Big)\,ds\\
&\quad=\int_0^\infty\frac{1}{s^2+2s\cos((1+\alpha)\pi)+1}\\
&\qquad\times\Big(\int_0^\infty\exp\big(-t\langle r\rangle_m^{\frac{2}{1+\alpha}}s^{\frac{1}{1+\alpha}}\big)r^{n-1}\tilde{J}_{\frac{n}{2}-1}(r|x|)\,dr\Big)\,ds\\
&\quad=\int_0^\infty\frac{1}{s^2+2s\cos((1+\alpha)\pi)+1}\Big(F^{-1}_{\xi\to x}\big(e^{-s^{\frac{1}{1+\alpha}}t\langle\xi\rangle_{m,\sigma}^{\frac{2}{1+\alpha}}}\big)(t,x)\Big)\,ds.
\end{aligned}$$

The estimate

$$\big\|F^{-1}_{\xi\to x}\big(e^{-s^{\frac{1}{1+\alpha}}t\langle\xi\rangle_{m,\sigma}^{\frac{2}{1+\alpha}}}\big)(t,\cdot)\big\|_{L^p}\lesssim e^{-\frac{1}{2}s^{\frac{1}{1+\alpha}}tm^{\frac{2}{1+\alpha}}}$$

implies

$$\begin{aligned}
&\big\|F^{-1}_{\xi\to x}\big(l_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\big)(t,\cdot)\big\|_{L^p}\\
&\quad\lesssim\int_0^\infty\frac{1}{s^2+2s\cos((1+\alpha)\pi)+1}\big\|F^{-1}_{\xi\to x}\big(e^{-s^{\frac{1}{1+\alpha}}t\langle\xi\rangle_{m,\sigma}^{\frac{2}{1+\alpha}}}\big)(t,\cdot)\big\|_{L^p}\,ds\\
&\quad\lesssim\int_0^\infty\frac{e^{-\frac{1}{2}s^{\frac{1}{1+\alpha}}tm^{\frac{2}{1+\alpha}}}}{s^2+2s\cos((1+\alpha)\pi)+1}\,ds.
\end{aligned}$$

For $t\in(0,1]$ we may conclude

$$\big\|F^{-1}_{\xi\to x}\big(l_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\big)(t,\cdot)\big\|_{L^p}\lesssim\int_0^\infty\frac{1}{s^2+2s\cos((1+\alpha)\pi)+1}\,ds\lesssim 1.$$

For $t \geq 1$ we have

$$\big\|F^{-1}_{\xi\to x}\big(l_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\big)(t,\cdot)\big\|_{L^p} \lesssim \int_0^\infty \frac{e^{-\frac{1}{2}s^{\frac{1}{1+\alpha}}tm^{\frac{2}{1+\alpha}}}}{s^2+2s\cos((1+\alpha)\pi)+1}\,ds$$
$$\lesssim \int_0^\infty \exp\big(-\tilde{C}_1 ts^{\frac{1}{1+\alpha}}\big)\,ds.$$

After the change of variables $\tau := ts^{\frac{1}{1+\alpha}}$ it follows

$$\big\|F^{-1}_{\xi\to x}\big(l_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\big)(t,\cdot)\big\|_{L^p} \lesssim \int_0^\infty \exp\big(-\tilde{C}_1 ts^{\frac{1}{1+\alpha}}\big)ds$$
$$\lesssim t^{-(1+\alpha)}\int_0^\infty \tau^\alpha \exp\big(-\tilde{C}_1\tau\big)d\tau \lesssim t^{-(1+\alpha)}.$$

We deduce for all $p \in [1,\infty]$ the estimate

$$\big\|F^{-1}_{\xi\to x}\big(l_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\big)(t,\cdot)\big\|_{L^p} \lesssim (1+t)^{-(1+\alpha)} \quad \text{for all} \quad t \geq 0.$$

Summarizing all the estimates we may conclude

$$\begin{aligned}
&\big\|F^{-1}_{\xi\to x}\big(E_{1+\alpha}(-t^{1+\alpha}\langle\xi\rangle^2_{m,\sigma})\big)(t,\cdot)\big\|_{L^p}\\
&\quad\lesssim \big\|F^{-1}_{\xi\to x}\big(\exp\big(a_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\big)(t,\cdot)\big\|_{L^p}\\
&\qquad+\big\|F^{-1}_{\xi\to x}\big(\exp\big(b_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\big)(t,\cdot)\big\|_{L^p}\\
&\qquad+\big\|F^{-1}_{\xi\to x}\big(l_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\big)(t,\cdot)\big\|_{L^p}\\
&\quad\lesssim e^{-Ct}+(1+t)^{-(1+\alpha)} \lesssim (1+t)^{-(1+\alpha)}.
\end{aligned}$$

This completes the proof. □

The following proposition is helpful for the treatment of σ-evolution models with a mass term.

Proposition 4.3. *The following estimate holds in* $\mathbb{R}^n$ *for* $\sigma > 0$, $m > 0$, $\alpha \in [0,1)$ *and for all* $n \geq 1$*:*

$$\big\|F^{-1}_{\xi\to x}\big(\langle\xi\rangle^2_{m,\sigma}E_{1+\alpha}(-t^{1+\alpha}\langle\xi\rangle^2_{m,\sigma})\big)(t,\cdot)\big\|_{L^p} \lesssim (1+t)^{-(1+\alpha)} \tag{4.15}$$

for $p \in [1,\infty]$ *and* $t > 0$.

Proof. The proof is similar to the proof of the previous proposition. In a first step we estimate the oscillating integrals

$$F^{-1}_{\xi\to x}\big(\langle\xi\rangle^2_{m,\sigma}e^{-ct\langle\xi\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle\xi\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\big) \quad \text{and} \quad F^{-1}_{\xi\to x}\big(\langle\xi\rangle^2_{m,\sigma}e^{-\tau t\langle\xi\rangle^{2\kappa}_{m,\sigma}}\big),$$

where $c = -\cos(\frac{\pi}{1+\alpha}), \kappa = \frac{1}{1+\alpha} \in (\frac{1}{2},1)$ and $\tau > 0$. Following the approach from the previous proof we may conclude an exponential type decay estimate

$$\begin{aligned}
&\big\|F^{-1}_{\xi\to x}\big(\langle\xi\rangle^2_{m,\sigma}e^{-ct\langle\xi\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle\xi\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\big)\big\|_{L^p}\\
&\quad+\big\|F^{-1}_{\xi\to x}\big(\langle\xi\rangle^2_{m,\sigma}e^{-\tau t\langle\xi\rangle^{2\kappa}_{m,\sigma}}\big)\big\|_{L^p} \lesssim e^{-Ct}
\end{aligned}$$

with a suitable positive constant $C = C(m,\alpha)$, and for $p \in [1,\infty]$ and $t \geq 0$. Let us make some comments to the third oscillating integral. Following the same steps of treatment of the previous proof we may conclude

$$\big\|F^{-1}_{\xi\to x}\big(\langle\xi\rangle^2_{m,\sigma} l_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\big)\big\|_{L^p} \lesssim (1+t)^{-(1+\alpha)}$$

for $p \in [1,\infty]$ and $t \geq 0$. Indeed, we use the formula

$$\langle\xi\rangle^2_{m,\sigma} l_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma}) \sim \int_0^\infty \langle\xi\rangle^2_{m,\sigma} \frac{\exp\big(-t\langle\xi\rangle^{\frac{2}{1+\alpha}}_{m,\sigma} s^{\frac{1}{1+\alpha}}\big)}{s^2+2s\cos((1+\alpha)\pi)+1}\,ds.$$

Taking account of the definition of modified Bessel functions we get

$$\begin{aligned}
&F^{-1}_{\xi\to x}\Big(\langle\xi\rangle^2_{m,\sigma} l_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\Big)\\
&\quad= \int_0^\infty \bigg(\int_0^\infty \langle r\rangle^2_{m,\sigma} \frac{\exp\Big(-t\langle r\rangle^{\frac{2}{1+\alpha}}_m s^{\frac{1}{1+\alpha}}\Big)}{s^2+2s\cos((1+\alpha)\pi)+1} r^{n-1}\tilde{J}_{\frac{n}{2}-1}(r|x|)\,dr\bigg)\,ds\\
&\quad= \int_0^\infty \frac{1}{s^2+2s\cos((1+\alpha)\pi)+1}\\
&\qquad\times\bigg(\int_0^\infty \langle r\rangle^2_{m,\sigma}\exp\Big(-t\langle r\rangle^{\frac{2}{1+\alpha}}_m s^{\frac{1}{1+\alpha}}\Big) r^{n-1}\tilde{J}_{\frac{n}{2}-1}(r|x|)\,dr\bigg)\,ds\\
&\quad= \int_0^\infty \frac{1}{s^2+2s\cos((1+\alpha)\pi)+1}\\
&\qquad\times\Big(F^{-1}_{\xi\to x}\big(\langle\xi\rangle^2_{m,\sigma} e^{-s^{\frac{1}{1+\alpha}} t\langle\xi\rangle^{\frac{2}{1+\alpha}}_{m,\sigma}}\big)(t,x)\Big)\,ds.
\end{aligned}$$

The estimate

$$\big\|F^{-1}_{\xi\to x}\big(\langle\xi\rangle^2_{m,\sigma} e^{-s^{\frac{1}{1+\alpha}} t\langle\xi\rangle^{\frac{2}{1+\alpha}}_{m,\sigma}}\big)(t,\cdot)\big\|_{L^p} \lesssim e^{-\frac{1}{2}s^{\frac{1}{1+\alpha}} t m^{\frac{2}{1+\alpha}}}$$

implies

$$\begin{aligned}
&\big\|F^{-1}_{\xi\to x}\big(\langle\xi\rangle^2_{m,\sigma} l_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\big)(t,\cdot)\big\|_{L^p}\\
&\quad\lesssim \int_0^\infty \frac{1}{s^2+2s\cos((1+\alpha)\pi)+1}\big\|F^{-1}_{\xi\to x}\big(e^{-s^{\frac{1}{1+\alpha}} t\langle\xi\rangle^{\frac{2}{1+\alpha}}_{m,\sigma}}\big)(t,\cdot)\|_{L^p}\,ds\\
&\quad\lesssim \int_0^\infty \frac{e^{-\frac{1}{2}s^{\frac{1}{1+\alpha}} t m^{\frac{2}{1+\alpha}}}}{s^2+2s\cos((1+\alpha)\pi)+1}\,ds.
\end{aligned}$$

As in the previous proof we conclude the desired estimate. □

5. $L^r - L^q$ estimates for the formal solutions from Sect. 3

5.1. Models without any mass term

Proposition 5.1. *Let $u_0 \in L^r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $n \geq 1$, $r \geq 1$ and $\alpha \in (0,1)$. Then the function*

$$u = u(t,x) = \big(G^0_{\alpha,\sigma}(t) * u_0\big)(t,x)$$

satisfies the following $L^m - L^q$ estimates:

$$\|u(t,\cdot)\|_{L^q} \lesssim t^{-\frac{n(1+\alpha)}{2\sigma}(\frac{1}{m}-\frac{1}{q})}\|u_0\|_{L^m} \tag{5.1}$$

for all $r \le m \le q \le \infty$ provided that $n(\frac{1}{m}-\frac{1}{q}) < 2\sigma$.
Let $u_0 \in H^\gamma_r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $n \ge 1$, $1 < r < \infty$, $\gamma \ge 0$, and $\alpha \in (0,1)$. Then the function

$$u = u(t,x) = \big(G^0_{\alpha,\sigma}(t) * u_0\big)(t,x)$$

satisfies the following estimates:

$$\|u(t,\cdot)\|_{H^\gamma_r} \lesssim \|u_0\|_{H^\gamma_r} \text{ and } \|u(t,\cdot)\|_{\dot{H}^\gamma_r} \lesssim \|u_0\|_{\dot{H}^\gamma_r}. \tag{5.2}$$

Proof. The inequality (5.1) follows from Young's inequality and Proposition 4.1. Applying these tools to the relation

$$|D|^\gamma(G^0_{\alpha,\sigma}(t) * u_0)(t,x) = \big(F^{-1}_{\xi\to x}\big(E_{\alpha+1}\big(-t^{\alpha+1}|\xi|^{2\sigma}\big)\big) * |D|^\gamma u_0\big)(t,x)$$

implies the inequality (5.2). This completes the proof. □

Proposition 5.2. *Let $u_0 \in L^r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $n \ge 1$, $r \ge 1$ and $\alpha \in (0,1)$. Then the function*

$$u = u(t,x) = \big(G^0_{\alpha,\sigma}(t) * u_0\big)(t,x)$$

satisfies the following estimate for any fixed $\delta > 0$ small enough:

$$\|u(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\beta^{r,\delta}_{\alpha,q,\sigma}}(\|u_0\|_{L^r} + \|u_0\|_{L^q}) \quad \text{for all} \quad q \in [r,\infty], \tag{5.3}$$

where

$$\beta^{r,\delta}_{\alpha,q,\sigma} := \min\Big\{\frac{n(1+\alpha)}{2\sigma}\Big(\frac{1}{r}-\frac{1}{q}\Big); 1-\delta\Big\}.$$

Proof. To get (5.3) we use ideas of D'Abbicco (cf. with [3]). For $t \in (0,1]$ we set $m = q$ in (5.1) to get the $L^q - L^q$ estimate

$$\|u(t,\cdot)\|_{L^q} \lesssim \|u_0\|_{L^q}.$$

For $t \ge 1$ we choose $m = r$ in (5.1) if $n(\frac{1}{r}-\frac{1}{q}) < 2\sigma$. Otherwise, in (5.1) the parameter m is chosen as the solution to

$$\frac{n(1+\alpha)}{2\sigma}\Big(\frac{1}{m}-\frac{1}{q}\Big) = 1-\delta$$

with a fixed sufficiently small positive δ. In this way, we may conclude the $L^r - L^q$ estimate

$$\|u(t,\cdot)\|_{L^q} \lesssim t^{-\beta^{r,\delta}_{\alpha,q,\sigma}}\|u_0\|_{L^r}.$$

Gluing both estimates together we derive the desired estimate (5.3). □

Remark 5.3. The last two statements are valid for $r = 1$, too, in contrary to the paper [3]. In this paper the authors use estimates in scales of Morrey spaces from the paper [1], where $r = 1$ is excluded. For this reason the positive parameter δ appears in Proposition 1.1.

The statements of the Propositions 5.1 and 5.2 allow to conclude the following result.

Corollary 5.4. *Let* $u_0 \in L^r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $n \geq 1$, $r \geq 1$ *and* $\alpha \in (0,1)$. *Then the function*

$$u = u(t,x) = \big(G^0_{\alpha,\sigma}(t) * u_0\big)(t,x)$$

belongs to

$$L^\infty\big((0,T), L^r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big) \text{ for all } T > 0.$$

Let $u_0 \in H^\gamma_r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $n \geq 1$, $1 < r < \infty$, $\gamma \geq 0$ *and* $\alpha \in (0,1)$. *Then the function*

$$u = u(t,x) = \big(G^0_{\alpha,\sigma}(t) * u_0\big)(t,x)$$

belongs to

$$L^\infty\big((0,T), H^\gamma_r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big) \text{ for all } T > 0.$$

The next result contains even the continuity property with respect to the time variable.

Proposition 5.5. *Let* $u_0 \in L^r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $n \geq 1$, $r \geq 1$ *and* $\alpha \in (0,1)$. *Then the function*

$$u = u(t,x) = \big(G^0_{\alpha,\sigma}(t) * u_0\big)(t,x)$$

belongs to

$$C\big([0,\infty), L^r(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)\big) \text{ for all } q \in [r,\infty).$$

Let $u_0 \in H^\gamma_r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $n \geq 1$, $1 < r < \infty$, $\gamma \geq 0$ *and* $\alpha \in (0,1)$. *Then the function*

$$u = u(t,x) = \big(G^0_{\alpha,\sigma}(t) * u_0\big)(t,x)$$

belongs to

$$C\big([0,\infty), H^\gamma_r(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)\big) \text{ for all } q \in [r,\infty).$$

Proof. The second statement follows immediately from the first statement by using only the higher regularity H^γ_r instead of L^r. The first statement follows from Proposition 7.9 of the Appendix. □

5.2. Models with a mass term

Proposition 5.6. *Let* $u_0 \in L^r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $n \geq 1$, $r \geq 1$ *and* $\alpha \in (0,1)$. *Then the function*

$$u = u(t,x) = \big(G^m_{\alpha,\sigma}(t) * u_0\big)(t,x)$$

belongs to

$$C\big([0,\infty), L^r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big)$$

and satisfies the following $L^r - L^q$ *estimates:*

$$\|u(t,\cdot)\|_{L^q} \lesssim (1+t)^{-(1+\alpha)} \|u_0\|_{L^r} \tag{5.4}$$

for all $1 \leq r \leq q \leq \infty$.
Let $u_0 \in H^\gamma_r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $n \geq 1$, $1 \leq r < \infty$, $\gamma \geq 0$ *and* $\alpha \in (0,1)$. *Then the function*

$$u = u(t,x) = \big(G^m_{\alpha,\sigma}(t) * u_0\big)(t,x)$$

belongs to

$$C\big([0,\infty), H^\gamma_r(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)\big)$$

and satisfies the following estimates:

$$\|u(t,\cdot)\|_{H^\gamma_r}\lesssim (1+t)^{-(1+\alpha)}\|u_0\|_{H^\gamma_r}\ \ \text{and}\ \ \|u(t,\cdot)\|_{\dot H^\gamma_r}\lesssim (1+t)^{-(1+\alpha)}\|u_0\|_{\dot H^\gamma_r}. \tag{5.5}$$

Proof. The proof follows immediately from (3.1), (3.2), Proposition 4.2 and Lemma 7.10. To verify the last inequality we use

$$|D|^\gamma(G^m_{\alpha,\sigma}(t)\ast u_0)(t,x)=\big(F^{-1}_{\xi\to x}\big(E_{\alpha+1}\big(-t^{\alpha+1}\langle\xi\rangle^2_{m,\sigma}\big)\big)\ast|D|^\gamma u_0\big)(t,x).$$

The continuity of solutions follows from (3.4) and Proposition 4.3. This completes the proof. □

6. Proofs of the main results

6.1. Proof of Theorem 2.1

For any $n\ge\frac{2\sigma r}{1+\alpha}$ and sufficiently small $\delta\in(0,1)$ there exists a parameter $\overline{q}=\overline{q}(\delta)\in(r,\infty)$ such that

$$\frac{n(1+\alpha)}{2\sigma}\Big(\frac{1}{r}-\frac{1}{\overline{q}}\Big)=1-\delta. \tag{6.1}$$

We define the space

$$X(T):=L^\infty\big((0,T),L^r(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)\big)$$

with the norm

$$\|u\|_{X(T)}:=\operatorname{esssup}_{t\in(0,T)}\big\{(1+t)^{-\lambda}\|u(t,\cdot)\|_{L^r} +(1+t)^{1-\delta-\lambda}\big(\|u(t,\cdot)\|_{L^{\overline{q}}}+\|u(t,\cdot)\|_{L^\infty}\big)\big\}.$$

For any $u\in X(T)$ we consider for $m=0$ the operator

$$P:X(T)\longrightarrow X(T),\quad Pu:=(G^0_{\alpha,\sigma}(t)\ast u_0)(t,x)+N^0_{\alpha,\sigma}(u)(t,x).$$

We shall prove that

$$\|Pu\|_{X(T)}\lesssim\|u_0\|_{L^r\cap L^\infty}+\|u\|^p_{X(T)}, \tag{6.2}$$

$$\|Pu-Pv\|_{X(T)}\lesssim\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big). \tag{6.3}$$

For the proof of (6.2), after taking into consideration the estimates (5.3), we have

$$\begin{aligned}\|G^0_{\alpha,\sigma}(t)\ast u_0\|_{X(T)}&=\operatorname{esssup}_{t\in(0,T)}\big\{(1+t)^{-\lambda}\|(G^0_{\alpha,\sigma}(t)\ast u_0)(t,\cdot)\|_{L^r}\\&\quad+(1+t)^{1-\delta-\lambda}\big(\|(G^0_{\alpha,\sigma}(t)\ast u_0)(t,\cdot)\|_{L^{\overline{q}}}+\|(G^0_{\alpha,\sigma}(t)\ast u_0)(t,\cdot)\|_{L^\infty}\big)\big\}\\&\lesssim\|u_0\|_{L^r\cap L^\infty}.\end{aligned}$$

It remains to prove that $\|N^0_{\alpha,\sigma}(u)\|_{X(T)}\lesssim\|u\|^p_{X(T)}$. If $u\in X(T)$, then we derive by interpolation the following estimate:

$$\|u(t,\cdot)\|_{L^q}\lesssim(1+t)^{-\beta^{r,\delta}_{\alpha,q,\sigma}+\lambda}\|u\|_{X(T)}\quad\text{for all}\quad q\in[r,\infty]. \tag{6.4}$$

Consequently,

$$\||u(t,\cdot)|^p\|_{L^q} \lesssim \|u(t,\cdot)\|^p_{L^{pq}} \lesssim (1+t)^{-p(\beta^{r,\delta}_{\alpha,pq,\sigma}-\lambda)}\|u\|_{X(T)}$$
$$\lesssim (1+t)^{-p(\beta^{r,\delta}_{\alpha,p,\sigma}-\lambda)}\|u\|_{X(T)} \tag{6.5}$$

for any $q \in [r,\infty]$ and due to $\beta^{r,\delta}_{\alpha,pq,\sigma} \geq \beta^{r,\delta}_{\alpha,p,\sigma}$. Thanks to (5.3) and (6.5) we can estimate

$$\|N^0_{\alpha,\sigma}(u)(t,\cdot)\|_{L^q} \lesssim \|u\|_{X(T)} I_q(t) \quad \text{for all} \quad t \in [0,T] \text{ and } q \in [r,\infty], \tag{6.6}$$

where

$$I_q(t) = \int_0^t (1+t-\tau)^{-\beta^{r,\delta}_{\alpha,q,\sigma}} \int_0^\tau (\tau-s)^{\alpha-1}(1+s)^{-p(\beta^{r,\delta}_{\alpha,p,\sigma}-\lambda)}\, ds\, d\tau.$$

We are interested to estimate the function $I_q(t)$ in (6.6). For this we apply Lemma 7.11. We notice that $p(\beta^{r,\delta}_{\alpha,p,\sigma}-\lambda) > 1$ if and only if

$$p > p_{\alpha,\lambda,\sigma,r,\delta}(n) := \max\left\{p^r_{\alpha,\lambda,\sigma}(n); \frac{1}{1-\delta-\lambda}\right\}.$$

Consequently, by using Lemma 7.11 we may estimate $I_q(t)$ as follows:

$$I_q(t) \lesssim \int_0^t (1+t-\tau)^{-\beta^{r,\delta}_{\alpha,q,\sigma}}(1+\tau)^{\alpha-1}\, d\tau \lesssim (1+t)^{-\beta^{r,\delta}_{\alpha,q,\sigma}+\alpha} \lesssim (1+t)^{-\beta^{r,\delta}_{\alpha,q,\sigma}+\lambda},$$

thanks to the fact that $\beta^{r,\delta}_{\alpha,q,\sigma} \in (0,1-\delta]$ and $\alpha \in (0,1)$. Therefore (6.5) gives

$$\|N^0_{\alpha,\sigma}(u)\|_{X(T)} \lesssim \|u\|^p_{X(T)}.$$

Finally, it remains to show (6.3). Let $q \in [r,\infty]$. By Hölder's inequality, for $u, v \in X(T)$, and if p' denotes the conjugate to p, then we have

$$\begin{aligned}
&\||u(s,\cdot)|^p - |v(s,\cdot)|^p\|_{L^q} \\
&\quad\lesssim \left(\int_{\mathbb{R}^n} |u(s,x)-v(s,x)|^q \left(|u(s,x)|^{p-1}+|v(s,x)|^{p-1}\right)^q dx\right)^{\frac{1}{q}} \\
&\quad\lesssim \left(\int_{\mathbb{R}^n} |u(s,x)-v(s,x)|^{pq}\, dx\right)^{\frac{1}{pq}} \left(\int_{\mathbb{R}^n}\left(|u(s,x)|^{p-1}+|v(s,x)|^{p-1}\right)^{qp'} dx\right)^{\frac{1}{qp'}} \\
&\quad\lesssim \|u(s,\cdot)-v(s,\cdot)\|_{L^{pq}} \||u(s,\cdot)|^{p-1}+|v(s,\cdot)|^{p-1}\|_{L^{qp'}} \\
&\quad\lesssim \|u(s,\cdot)-v(s,\cdot)\|_{L^{pq}} \left(\||u(s,\cdot)|^{p-1}\|_{L^{qp'}} + \||v(s,\cdot)|^{p-1}\|_{L^{qp'}}\right) \\
&\quad\lesssim \|u(s,\cdot)-v(s,\cdot)\|_{L^{pq}} \left(\|u(s,\cdot)\|^{p-1}_{L^{qp'(p-1)}} + \|v(s,\cdot)\|^{p-1}_{L^{qp'(p-1)}}\right) \\
&\quad\lesssim \|u(s,\cdot)-v(s,\cdot)\|_{L^{pq}} \left(\|u(s,\cdot)\|^{p-1}_{L^{pq}} + \|v(s,\cdot)\|^{p-1}_{L^{pq}}\right) \\
&\quad\lesssim (1+s)^{-p(\beta^{r,\delta}_{\alpha,p,\sigma}-\lambda)}\|u-v\|_{X(T)}\left(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\right).
\end{aligned}$$

Hence,

$$\begin{aligned}\|N^0_{\alpha,\sigma}(u)(t,\cdot)-N^0_{\alpha,\sigma}(v)(t,\cdot)\|_{L^q} &\lesssim I_q(t)\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big)\\ &\lesssim (1+t)^{-\beta^{r,\delta}_{\alpha,q,\sigma}+\lambda}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big) \quad \text{for all} \quad t\in[0,T].\end{aligned}$$

We deduce that

$$\begin{aligned}\|Pu-Pv\|_{X(T)} &= \|N^0_{\alpha,\sigma}(u)-N^0_{\alpha,\sigma}(v)\|_{X(T)}\\ &\lesssim \|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big).\end{aligned}$$

Notice that $p>p_{\alpha,\lambda,\sigma,r,\delta}$ for all $\delta>0$ if and only if $p>p_{\alpha,\lambda,\sigma,r}$.

Remark 6.1. All the estimates (6.2) and (6.3) are uniformly with respect to $T\in(0,\infty)$ if $p>p_{\alpha,\lambda,\sigma,r}(n)$.

From (6.2) it follows that P maps $X(T)$ into itself for all T and for small data. By standard contraction arguments (see [5]) the estimates (6.2) and (6.3) lead to the existence of a unique solution to $u=Pu$ and, consequently, to (1.3) with $m=0$, that is, the solution of (1.3) with $m=0$ satisfies (5.3). Since all constants are independent of T we let T tend to ∞ and we obtain a global (in time) existence result for small data solutions to (1.3).

Finally, let us discuss the continuity of the solution with respect to t. The solution satisfies the operator equation

$$u(t)=G^0_{\alpha,\sigma}(t)*u_0+N^0_{\alpha,\sigma}(u)(t).$$

The above estimates for $N^0_{\alpha,\sigma}(u)$ and the integral term $\int_0^t$ in $N^0_{\alpha,\sigma}(u)$ imply for all $T>0$

$$\begin{aligned}&N^0_{\alpha,\sigma}(u)\in C\big([0,T],L^r(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)\big)\\ &\quad\text{with} \lim_{t\to+0}\|N^0_{\alpha,\sigma}(u)(t,\cdot)\|_{L^r\cap L^\infty}=0.\end{aligned} \tag{6.7}$$

Proposition 5.5 gives

$$G^0_{\alpha,\sigma}(t)*u_0\in C\big([0,T],L^r(\mathbb{R}^n)\cap L^q(\mathbb{R}^n)\big) \text{ for all } q\in[r,\infty). \tag{6.8}$$

Consequently,

$$u\in C\big([0,\infty),L^r(\mathbb{R}^n)\cap L^q(\mathbb{R}^n)\big) \text{ for all } q\in[r,\infty)$$

what we wanted to have.

If the data are large, then instead we get for $p>1$ the estimates

$$\begin{aligned}&\|Pu\|_{X(T)}\le C\|u_0\|_{L^r\cap L^\infty}+C(T)\|u\|^p_{X(T)},\\ &\|Pu-Pv\|_{X(T)}\le C(T)\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big),\end{aligned}$$

where $C(T)$ tends to 0 for $T\to+0$. For this reason we can have for general (large) data a local (in time) existence result of weak solutions only. The proof is complete.

6.2. Proof of Theorem 2.2

If $1 \le n < \frac{2\sigma r}{1+\alpha}$, then for all $q \in [r,\infty]$ we obtain

$$\frac{n(1+\alpha)}{2\sigma}\Big(\frac{1}{r}-\frac{1}{q}\Big) < 1 - \frac{n(1+\alpha)}{2\sigma q} \le 1.$$

Hence, we can choose a positive δ such that there does not exist any $\overline{q} \in [r,\infty]$ which satisfies (6.1). For this reason,

$$\beta^{r,\delta}_{\alpha,q,\sigma} = \beta^{r}_{\alpha,q,\sigma} := \frac{n(1+\alpha)}{2\sigma}\Big(\frac{1}{r}-\frac{1}{q}\Big).$$

We define the space

$$X(T) := L^\infty\big((0,T), L^r(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)\big)$$

with the norm

$$\|u\|_{X(T)} := \text{esssup}_{t\in(0,T)}\big\{(1+t)^{-\lambda}\|u(t,\cdot)\|_{L^r} + (1+t)^{\beta^r_{\alpha,\infty,\sigma}-\lambda}\|u(t,\cdot)\|_{L^\infty}\big\},$$

where $\beta^r_{\alpha,\infty,\sigma} = \frac{n(1+\alpha)}{2\sigma r}$. For any $u \in X(T)$, we consider for $m = 0$ the operator

$$P : X(T) \longrightarrow X(T), \quad Pu := (G^0_{\alpha,\sigma}(t) * u_0)(t,x) + N^0_{\alpha,\sigma}(u)(t,x).$$

We shall prove that

$$\|Pu\|_{X(T)} \lesssim \|u_0\|_{L^r\cap L^\infty} + \|u\|^p_{X(T)}, \tag{6.9}$$

$$\|Pu - Pv\|_{X(T)} \lesssim \|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big). \tag{6.10}$$

For the proof of (6.9), after taking into consideration the estimates (5.3), we have

$$\begin{aligned}
&\|G^0_{\alpha,\sigma}(t) * u_0\|_{X(T)} \\
&\quad= \text{esssup}_{t\in(0,T)}\big\{(1+t)^{-\lambda}\|(G^0_{\alpha,\sigma}(t) * u_0)(t,\cdot)\|_{L^r} \\
&\qquad+ (1+t)^{\beta^r_{\alpha,\infty,\sigma}-\lambda}\|(G^0_{\alpha,\sigma}(t) * u_0)(t,\cdot)\|_{L^\infty}\big\} \\
&\quad\lesssim \|u_0\|_{L^r\cap L^\infty}.
\end{aligned}$$

It remains to prove that $\|N^0_{\alpha,\sigma}(u)\|_{X(T)} \lesssim \|u\|^p_{X(T)}$. If $u \in X(T)$, then we derive by interpolation the following estimate:

$$\|u(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\beta^r_{\alpha,q,\sigma}+\lambda}\|u\|_{X(T)} \quad \text{for all} \quad q \in [r,\infty]. \tag{6.11}$$

Consequently,

$$\begin{aligned}
\||u(t,\cdot)|^p\|_{L^q} &\lesssim \|u(t,\cdot)\|^p_{L^{pq}} \lesssim (1+t)^{-p(\beta^r_{\alpha,pq,\sigma}-\lambda)}\|u\|_{X(T)} \\
&\lesssim (1+t)^{-p(\beta^r_{\alpha,p,\sigma}-\lambda)}\|u\|_{X(T)}
\end{aligned} \tag{6.12}$$

for any $q \in [r,\infty]$ and due to $\beta^r_{\alpha,pq,\sigma} \ge \beta^r_{\alpha,p,\sigma}$. Thanks to (5.3) and (6.12) we can estimate

$$\|N^0_{\alpha,\sigma}(u)(t,\cdot)\|_{L^q} \lesssim \|u\|_{X(T)} I_q(t) \quad \text{for all} \quad t \in [0,T] \text{ and } q \in [r,\infty], \tag{6.13}$$

where

$$I_q(t) = \int_0^t (1+t-\tau)^{-\beta^r_{\alpha,q,\sigma}} \int_0^\tau (\tau-s)^{\alpha-1}(1+s)^{-p(\beta^r_{\alpha,p,\sigma}-\lambda)}\, ds\, d\tau.$$

We notice that $p(\beta^r_{\alpha,p,\sigma}-\lambda)>1$ if and only if

$$p > p^r_{\alpha,\lambda,\sigma}(n) := 1+\frac{n(r-1)(1+\alpha)+2\sigma r(1+\lambda)}{(n-2\sigma r)(1+\alpha)+2\sigma r(1+\alpha-\lambda)}$$

under the assumptions $1\le\sigma<\frac{\alpha+1}{2\lambda}$ and $1\le r<\frac{\alpha+1}{2\sigma\lambda}$. Consequently, by using Lemma 7.11 we may estimate as follows:

$$I_q(t)\lesssim\int_0^t(1+t-\tau)^{-\beta^r_{\alpha,q,\sigma}}(1+\tau)^{\alpha-1}\,d\tau\lesssim(1+t)^{-\beta^r_{\alpha,q,\sigma}+\lambda},$$

thanks to the fact that $\beta^r_{\alpha,q,\sigma}\in(0,1)$ and $\alpha\in(0,1)$. Therefore, (6.12) gives

$$\|N_{\alpha,\sigma}(u)\|_{X(T)}\lesssim\|u\|^p_{X(T)}.$$

The proof of (6.10) is similar to the proof of (6.3) of Theorem 2.1. Then we may conclude a uniquely determined solution

$$u\in L^\infty\big((0,T),L^r(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)\big)\ \text{ for all }\ T>0.$$

As at the end of the proof of Theorem 2.1 we verify that the solution u belongs even to

$$C\big([0,\infty),L^r(\mathbb{R}^n)\cap L^q(\mathbb{R}^n)\big)\ \text{ for all }\ q\in[r,\infty).$$

The proof is complete.

6.3. Proof of Theorem 2.3

We define the solution space

$$X(T):=L^\infty\big((0,T),H^\gamma_r(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)\big)$$

with the norm

$$\|u\|_{X(T)}:=\operatorname{esssup}_{t\in(0,T)}\big\{(1+t)^{-\lambda}\|u(t,\cdot)\|_{H^\gamma_r}\\ +(1+t)^{1-\delta-\lambda}\big(\|u(t,\cdot)\|_{L^{\overline{q}}}+\|u(t,\cdot)\|_{L^\infty}\big)\big\},$$

where $\overline{q}$ is defined as in Sect. 6.2. For any $u\in X(T)$, we consider for $m=0$ the operator

$$P:X(T)\longrightarrow X(T),\quad Pu:=(G^0_{\alpha,\sigma}(t)\ast u_0)(t,x)+N^0_{\alpha,\sigma}(u)(t,x).$$

We shall prove that

$$\|Pu\|_{X(T)}\lesssim\|u_0\|_{H^\gamma_r\cap L^\infty}+\|u\|^p_{X(T)},\tag{6.14}$$

$$\|Pu-Pv\|_{X(T)}\lesssim\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big).\tag{6.15}$$

For the proof of (6.14), after taking account of the estimates (5.3) and (5.2) we have

$$\|G^0_{\alpha,\sigma}(t)\ast u_0\|_{X(T)}\\ =\operatorname{esssup}_{0\le t\le T}\big\{(1+t)^{-\lambda}\|(G^0_{\alpha,\sigma}(t)\ast u_0)(t,\cdot)\|_{H^\gamma_r}\\ +(1+t)^{1-\delta-\lambda}\big(\|(G^0_{\alpha,\sigma}(t)\ast u_0)(t,\cdot)\|_{L^{\overline{q}}}+\|(G^0_{\alpha,\sigma}(t)\ast u_0)(t,\cdot)\|_{L^\infty}\big)\big\}\\ \lesssim\|u_0\|_{H^\gamma_r\cap L^\infty}.$$

It remains to prove for $m = 0$ that $\|N^0_{\alpha,\sigma}(u)\|_{X(T)} \lesssim \|u\|^p_{X(T)}$. If $u \in X(T)$, then we derive by interpolation the following estimate:

$$\|u(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\beta^{r,\delta}_{\alpha,q,\sigma}+\lambda}\|u\|_{X(T)} \quad \text{for all} \quad q \in [r,\infty]. \tag{6.16}$$

Moreover, we have

$$\|u(t,\cdot)\|_{\dot H^\gamma_r} \lesssim (1+t)^{\lambda}\|u\|_{X(T)}. \tag{6.17}$$

As in Sect. 6.1 we deduce

$$\|N^0_{\alpha,\sigma}(u)(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\beta^{r,\delta}_{q,\sigma}+\lambda}\|u\|_{X(T)} \quad \text{for all} \quad t \in [0,T] \text{ and } q \in [r,\infty],$$

if and only if

$$p > p_{\alpha,\lambda,\sigma,r,\delta} := \max\left\{p^r_{\alpha,\lambda,\sigma}(n); \frac{1}{1-\delta-\lambda}\right\}.$$

Now let us turn to the desired estimate of the norm $\|N^m_{\alpha,\sigma}(u)(t,\cdot)\|_{\dot H^\gamma_r}$. We need to estimate the norm $\||u(t,\cdot)|^p\|_{\dot H^\gamma_r}$. Applying Proposition 7.6, with $p > \max\{2;\gamma\}$, we obtain

$$\begin{aligned}\||u(t,\cdot)|^p\|_{\dot H^\gamma_r} &\lesssim \|u(t,\cdot)\|_{\dot H^\gamma_r}\|u(t,\cdot)\|^{p-1}_{L^\infty}\\ &\lesssim (1+t)^{\lambda}\|u\|_{X(T)}(1+t)^{(p-1)(\lambda+\delta-1)}\|u\|^{p-1}_{X(T)}\\ &\lesssim (1+t)^{\lambda-(p-1)(1-\delta-\lambda)}\|u\|^p_{X(T)}\\ &\lesssim (1+t)^{-((p-1)(1-\delta-\lambda)-\lambda)}\|u\|^p_{X(T)}.\end{aligned} \tag{6.18}$$

Then

$$\|N^0_{\alpha,\sigma}(u)(t,\cdot)\|_{\dot H^\gamma_r} \lesssim \|u\|_{X(T)} I_r(t) \quad \text{for all} \quad t \in [0,T], \tag{6.19}$$

where

$$I_r(t) = \int_0^t\int_0^\tau (\tau-s)^{\alpha-1}(1+s)^{-((p-1)(1-\delta-\lambda)-\lambda)}\,ds\,d\tau. \tag{6.20}$$

If

$$p > \max\{p_0(\lambda,\delta);\gamma\},$$

where $p_0(\lambda,\delta) = 1 + \frac{1+\lambda}{1-\delta-\lambda}$, then

$$I_r(t) \lesssim (1+t)^{\alpha} \lesssim (1+t)^{\lambda}.$$

We remark that $p_0(\lambda,\delta) > \frac{1}{1-\delta-\lambda}$ and also $p_0(\lambda,\delta) > 2$. Then we deduce that

$$\|N^0_{\alpha,\sigma}(u)\|_{X(T)} \lesssim \|u\|^p_{X(T)}$$

if and only if

$$p > p^1_{\alpha,\lambda,\sigma,r,\gamma,\delta} := \max\{p^r_{\alpha,\lambda,\sigma}(n); p_0(\lambda,\delta);\gamma\}.$$

Finally, we have to show (6.15). From Sect. 6.1 we get for $m = 0$ the estimate

$$\begin{aligned}&\|N^0_{\alpha,\sigma}(u)(t,\cdot) - N^0_{\alpha,\sigma}(v)(t,\cdot)\|_{L^q}\\ &\quad\lesssim (1+t)^{-\beta^{r,\delta}_{\alpha,q,\sigma}+\lambda}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big)\end{aligned}$$

for all $t \in [0,T]$ and $q \in [r,\infty]$. It remains to prove

$$\begin{aligned}&\|N^0_{\alpha,\sigma}(u)(t,\cdot) - N^0_{\alpha,\sigma}(v)(t,\cdot)\|_{\dot{H}^\gamma_r}\\ &\quad\lesssim (1+t)^\lambda \|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big) \quad \text{for all} \quad t \in [0,T].\end{aligned}$$

From the above considerations it is sufficient to prove that

$$\begin{aligned}&\||u(s,\cdot)|^p - |v(s,\cdot)|^p\|_{\dot{H}^\gamma_r}\\ &\quad\lesssim (1+s)^{\lambda+(p-1)(\lambda+\delta-1)}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big).\end{aligned}$$

By using the integral representation

$$|u(s,\cdot)|^p - |v(s,\cdot)|^p = p\int_0^1 (u(s,\cdot)-v(s,\cdot))Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\,d\omega, \quad (6.21)$$

where $Q(u) = u|u|^{p-2}$, we obtain

$$\begin{aligned}&\||u(s,\cdot)|^p - |v(s,\cdot)|^p\|_{\dot{H}^\gamma_r}\\ &\quad\lesssim \textstyle\int_0^1 \||D|^\gamma\big((u(s,\cdot)-v(s,\cdot))Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\big)\|_{L^r}\,d\omega.\end{aligned} \quad (6.22)$$

Applying the fractional Leibniz formula from Proposition 7.8 to estimate a product in $\dot{H}^\gamma_r(\mathbb{R}^n)$ we get

$$\begin{aligned}&\||u(s,\cdot)|^p - |v(s,\cdot)|^p\|_{\dot{H}^\gamma_r}\\
&\quad\lesssim \int_0^1 \||D|^\gamma\big(u(s,\cdot)-v(s,\cdot)\big)\|_{L^r}\|Q(\omega u(s,x)+(1-\omega)v(s,x))\|_{L^\infty}\,d\omega\\
&\qquad+ \int_0^1 \|u(s,\cdot)-v(s,\cdot)\|_{L^\infty}\||D|^\gamma Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\|_{L^r}\,d\omega\\
&\quad\lesssim \||D|^\gamma\big(u(s,\cdot)-v(s,\cdot)\big)\|_{L^r}\big(\|u(s,\cdot)\|^{p-1}_{L^\infty} + \|v(s,\cdot)\|^{p-1}_{L^\infty}\big)\\
&\qquad+ \|u(s,\cdot)-v(s,\cdot)\|_{L^\infty}\int_0^1 \||D|^\gamma Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\|_{L^r}\,d\omega\\
&\quad\lesssim \|u(s,\cdot)-v(s,\cdot)\|_{\dot{H}^\gamma_r}\big(\|u(s,\cdot)\|^{p-1}_{L^\infty} + \|v(s,\cdot)\|^{p-1}_{L^\infty}\big)\\
&\qquad+ \|u(s,\cdot)-v(s,\cdot)\|_{L^\infty}\int_0^1 \||D|^\gamma Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\|_{L^r}\,d\omega\\
&\quad\lesssim (1+s)^{\lambda+(p-1)(\lambda+\delta-1)}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big)\\
&\qquad+ (1+s)^{\lambda+\delta-1}\|u-v\|_{X(T)}\int_0^1 \||D|^\gamma Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\|_{L^r}\,d\omega.\end{aligned}$$

We apply again Proposition 7.6 to estimate the term inside of the integral. In this way we obtain

$$\begin{aligned}&\int_0^1 \||D|^\gamma Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\|_{L^r}d\omega\\
&\quad\lesssim \int_0^1 \||D|^\gamma(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\|_{L^r}\\
&\qquad\times \|\omega u(s,\cdot)+(1-\omega)v(s,\cdot)\|^{p-2}_{L^\infty}\,d\omega\end{aligned}$$

$$\begin{aligned}
&\lesssim \int_0^1 (1+s)^{\lambda}\|\omega u+(1-\omega)v\|_{X(T)} \\
&\qquad \times (1+s)^{(p-2)(\lambda+\delta-1)}\|\omega u+(1-\omega)v\|^{p-2}_{X(T)} d\omega \\
&\lesssim \int_0^1 (1+s)^{\lambda+(p-2)(\lambda+\delta-1)}\|\omega u+(1-\omega)v\|^{p-1}_{X(T)}\, d\omega \\
&\lesssim (1+s)^{\lambda+(p-2)(\lambda+\delta-1)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big).
\end{aligned}$$

Then

$$\||u(s,\cdot)|^p-|v(s,\cdot)|^p\|_{\dot H^\gamma_r} \lesssim (1+s)^{\lambda+(p-1)(\lambda+\delta-1)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big).$$

Hence,

$$\begin{aligned}
&\|N^0_{\alpha,\sigma}(u)(t,\cdot)-N^0_{\alpha,\sigma}(v)(t,\cdot)\|_{\dot H^\gamma_r} \\
&\qquad \lesssim (1+t)^{\lambda}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big) \quad \text{for all} \quad t\in[0,T].
\end{aligned}$$

We deduce that

$$\begin{aligned}
\|Pu-Pv\|_{X(T)} &= \|N^0_{\alpha,\sigma}(u)-N^0_{\alpha,\sigma}(v)\|_{X(T)} \\
&\lesssim \|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big).
\end{aligned}$$

Notice that $p>p^1_{\alpha,\lambda,\sigma,r,\gamma,\delta}$ for all $\delta>0$ if and only if $p>p_{\alpha,\lambda,\sigma,r,\gamma}$. Then we may conclude a uniquely determined solution

$$u\in L^\infty\big((0,T),H^\gamma_r(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)\big) \text{ for all } T>0.$$

As at the end of the proof of Theorem 2.1 we verify that the solution u belongs even to

$$C\big([0,\infty),H^\gamma_r(\mathbb{R}^n)\cap L^q(\mathbb{R}^n)\big) \text{ for all } q\in[r,\infty).$$

If the data are large, then instead we get for $p>2$ the estimates

$$\begin{aligned}
&\|Pu\|_{X(T)} \le C\|u_0\|_{H^\gamma_r\cap L^\infty}+C(T)\|u\|^p_{X(T)}, \\
&\|Pu-Pv\|_{X(T)} \le C(T)\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big),
\end{aligned}$$

where $C(T)$ tends to 0 for $T\to+0$. For this reason we can have for general (large) data a local (in time) existence result of weak solutions only. This completes the proof.

6.4. Proof of Theorem 2.4

We define the solution space

$$X(T):=L^\infty\big((0,T),H^\gamma_r(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)\big)$$

with the norm

$$\|u\|_{X(T)}:=\operatorname{esssup}_{t\in(0,T)}\big\{(1+t)^{-\lambda}\|u(t,\cdot)\|_{H^\gamma_r}+(1+t)^{\beta^r_{\alpha,\infty,\sigma}-\lambda}\|u(t,\cdot)\|_{L^\infty}\big\},$$

where $\beta^r_{\alpha,\infty,\sigma}$ is defined as in Sect. 6.2. For any $u\in X(T)$, we consider for $m=0$ the operator

$$P:X(T)\longrightarrow X(T), \quad Pu:=(G^0_{\alpha,\sigma}(t)\ast u_0)(t,x)+N^0_{\alpha,\sigma}(u)(t,x).$$

We shall prove that

$$\|Pu\|_{X(T)} \lesssim \|u_0\|_{H^\gamma_r \cap L^\infty} + \|u\|^p_{X(T)}, \tag{6.23}$$

$$\|Pu - Pv\|_{X(T)} \lesssim \|u - v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big). \tag{6.24}$$

For the proof of (6.23), after taking account of the estimates (5.3) and (5.2) we have

$$\begin{aligned}
&\|G^0_{\alpha,\sigma}(t) * u_0\|_{X(T)} \\
&\quad = \text{esssup}_{t\in(0,T)}\big\{(1+t)^{-\lambda}\|(G^0_{\alpha,\sigma}(t) * u_0)(t,\cdot)\|_{H^\gamma_r} \\
&\qquad + (1+t)^{\beta^r_{\alpha,\infty,\sigma}-\lambda}\|(G^0_{\alpha,\sigma}(t) * u_0)(t,\cdot)\|_{L^\infty}\big)\big\} \\
&\quad \lesssim \|u_0\|_{H^\gamma_r \cap L^\infty}.
\end{aligned}$$

It remains to prove for $m = 0$ that $\|N^0_{\alpha,\sigma}(u)\|_{X(T)} \lesssim \|u\|^p_{X(T)}$. If $u \in X(T)$, then we derive by interpolation the following estimate:

$$\|u(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\beta^r_{\alpha,q,\sigma}+\lambda}\|u\|_{X(T)} \quad \text{for all} \quad q \in [r,\infty]. \tag{6.25}$$

Moreover, we have

$$\|u(t,\cdot)\|_{\dot H^\gamma_r} \lesssim (1+t)^{\lambda}\|u\|_{X(T)}. \tag{6.26}$$

As in Sect. 6.2 we deduce

$$\|N^0_{\alpha,\sigma}(u)(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\beta^r_{\alpha,q,\sigma}+\lambda}\|u\|_{X(T)} \quad \text{for all } t \in [0,T] \text{ and } q \in [r,\infty],$$

if and only if

$$p > p^r_{\alpha,\lambda,\sigma}(n).$$

Now let us turn to the desired estimate of the norm $\|N^0_{\alpha,\sigma}(u)(t,\cdot)\|_{\dot H^\gamma_r}$. We need to estimate the norm $\||u(t,\cdot)|^p\|_{\dot H^\gamma_r}$. Applying Proposition 7.6 with $p > \max\{2;\gamma\}$ we obtain

$$\begin{aligned}
\||u(t,\cdot)|^p\|_{\dot H^\gamma_r} &\lesssim \|u(t,\cdot)\|_{\dot H^\gamma_r}\|u(t,\cdot)\|^{p-1}_{L^\infty} \\
&\lesssim (1+t)^{\lambda}\|u\|_{X(T)}(1+t)^{(p-1)(-\beta^r_{\alpha,\infty,\sigma}+\lambda)}\|u\|^{p-1}_{X(T)} \\
&\lesssim (1+t)^{\lambda-(p-1)(\beta^r_{\alpha,\infty,\sigma}-\lambda)}\|u\|^p_{X(T)} \\
&\lesssim (1+t)^{-((p-1)(\beta^r_{\alpha,\infty,\sigma}-\lambda)-\lambda)}\|u\|^p_{X(T)}.
\end{aligned} \tag{6.27}$$

Then

$$\|N^0_{\alpha,\sigma}(u)(t,\cdot)\|_{\dot H^\gamma_r} \lesssim \|u\|_{X(T)} I_r(t) \quad \text{for all} \quad t \in [0,T], \tag{6.28}$$

where

$$I_r(t) = \int_0^t\int_0^\tau (\tau - s)^{\alpha-1}(1+s)^{-((p-1)(\beta^r_{\alpha,\infty,\sigma}-\lambda)-\lambda)}\,ds\,d\tau. \tag{6.29}$$

Notice that $(p-1)(\beta^r_{\alpha,\infty,\sigma} - \lambda) - \lambda > 1$ if and only if

$$p > p^1_{\alpha;\lambda;\sigma;r}(n) = 1 + \frac{2\sigma r(1+\lambda)}{(n-2\sigma r)(1+\alpha) + 2\sigma r(1+\alpha-\lambda))}$$

under the assumptions $1 \le \sigma < \frac{\alpha+1}{2\lambda}$ and $1 < r < \frac{\alpha+1}{2\sigma\lambda}$. If

$$p > p^1_{\alpha;\lambda;\sigma;r}(n),$$

then

$$I_r(t) \lesssim (1+t)^{\lambda}.$$

We remark that $p^r_{\alpha,\lambda,\sigma}(n) \geq p^1_{\alpha;\lambda;\sigma;r}(n) > 2$. Then we deduce that

$$\|N^0_{\alpha,\sigma}(u)\|_{X(T)} \lesssim \|u\|^p_{X(T)}$$

if and only if

$$p > \max\{p^r_{\alpha,\lambda,\sigma}(n); \gamma\}.$$

Finally, we have to show (6.15). From Sect. 6.1 we get for $m=0$ the estimate

$$\begin{aligned}&\|N^0_{\alpha,\sigma}(u)(t,\cdot) - N^0_{\alpha,\sigma}(v)(t,\cdot)\|_{L^q}\\ &\quad\lesssim (1+t)^{-\beta^r_{\alpha,q,\sigma}+\lambda}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big)\end{aligned}$$

for all $t \in [0,T]$ and $q \in [r,\infty]$. It remains to prove

$$\begin{aligned}&\|N^0_{\alpha,\sigma}(u)(t,\cdot) - N^0_{\alpha,\sigma}(v)(t,\cdot)\|_{\dot{H}^\gamma_r}\\ &\quad\lesssim (1+t)^{\lambda}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big) \quad \text{for all} \quad t \in [0,T].\end{aligned}$$

From the above considerations it is sufficient to prove that

$$\begin{aligned}&\||u(s,\cdot)|^p - |v(s,\cdot)|^p\|_{\dot{H}^\gamma_r}\\ &\quad\lesssim (1+s)^{\lambda+(p-1)(-\beta^r_{\alpha,\infty,\sigma}+\lambda)}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big).\end{aligned}$$

By using the integral representation

$$|u(s,\cdot)|^p - |v(s,\cdot)|^p = p\int_0^1 (u(s,\cdot)-v(s,\cdot))Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\,d\omega, \tag{6.30}$$

where $Q(u) = u|u|^{p-2}$, we obtain

$$\begin{aligned}&\||u(s,\cdot)|^p - |v(s,\cdot)|^p\|_{\dot{H}^\gamma_r}\\ &\quad\lesssim \int_0^1 \||D|^\gamma\big((u(s,\cdot)-v(s,\cdot))Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\big)\|_{L^r}\,d\omega.\end{aligned} \tag{6.31}$$

Applying the fractional Leibniz formula from Proposition 7.8 to estimate a product in $\dot{H}^\gamma_r(\mathbb{R}^n)$ we get

$$\begin{aligned}&\||u(s,\cdot)|^p - |v(s,\cdot)|^p\|_{\dot{H}^\gamma_r}\\ &\quad\lesssim \int_0^1 \||D|^\gamma\big(u(s,\cdot)-v(s,\cdot)\big)\|_{L^r}\|Q(\omega u(s,x)+(1-\omega)v(s,x))\|_{L^\infty}\,d\omega\\ &\qquad+\int_0^1 \|u(s,\cdot)-v(s,\cdot)\|_{L^\infty}\||D|^\gamma Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\|_{L^r}\,d\omega\\ &\quad\lesssim \||D|^\gamma\big(u(s,\cdot)-v(s,\cdot)\big)\|_{L^r}\big(\|u(s,\cdot)\|^{p-1}_{L^\infty} + \|v(s,\cdot)\|^{p-1}_{L^\infty}\big)\\ &\qquad+\|u(s,\cdot)-v(s,\cdot)\|_{L^\infty}\int_0^1 \||D|^\gamma Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\|_{L^r}\,d\omega\\ &\quad\lesssim \|u(s,\cdot)-v(s,\cdot)\|_{\dot{H}^\gamma_r}\big(\|u(s,\cdot)\|^{p-1}_{L^\infty} + \|v(s,\cdot)\|^{p-1}_{L^\infty}\big)\\ &\qquad+\|u(s,\cdot)-v(s,\cdot)\|_{L^\infty}\int_0^1 \||D|^\gamma Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\|_{L^r}\,d\omega\end{aligned}$$

$$\lesssim (1+s)^{\lambda+(p-1)(-\beta^r_{\alpha,\infty,\sigma}+\lambda)}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big)$$
$$+ (1+s)^{-\beta^r_{\alpha,\infty,\sigma}+\lambda}\|u-v\|_{X(T)}\int_0^1\||D|^\gamma Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\|_{L^r}\,d\omega.$$

We apply again Proposition 7.6 to estimate the term inside of the integral. In this way we obtain

$$\begin{aligned}\int_0^1 &\||D|^\gamma Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\|_{L^r}d\omega\\ &\lesssim \int_0^1 \||D|^\gamma(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\|_{L^r}\\ &\qquad\times\|\omega u(s,\cdot)+(1-\omega)v(s,\cdot)\|^{p-2}_{L^\infty}\,d\omega\\ &\lesssim \int_0^1 (1+s)^\lambda\|\omega u+(1-\omega)v\|_{X(T)}\\ &\qquad\times(1+s)^{(p-2)(-\beta^r_{\alpha,\infty,\sigma}+\lambda)}\|\omega u+(1-\omega)v\|^{p-2}_{X(T)}d\omega\\ &\lesssim \int_0^1 (1+s)^{\lambda+(p-2)(-\beta^r_{\alpha,\infty,\sigma}+\lambda)}\|\omega u+(1-\omega)v\|^{p-1}_{X(T)}\,d\omega\\ &\lesssim (1+s)^{\lambda+(p-2)(-\beta^r_{\alpha,\infty,\sigma}+\lambda)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big).\end{aligned}$$

Then

$$\||u(s,\cdot)|^p-|v(s,\cdot)|^p\|_{\dot H^\gamma_r}\lesssim (1+s)^{\lambda+(p-1)(-\beta^r_{\alpha,\infty,\sigma}+\lambda)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big).$$

Hence,

$$\begin{aligned}&\|N^0_{\alpha,\sigma}(u)(t,\cdot)-N^0_{\alpha,\sigma}(v)(t,\cdot)\|_{\dot H^\gamma_r}\\ &\quad\lesssim (1+t)^\lambda\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big)\quad\text{for all}\quad t\in[0,T].\end{aligned}$$

We deduce that

$$\begin{aligned}\|Pu-Pv\|_{X(T)}&=\|N^0_{\alpha,\sigma}(u)-N^0_{\alpha,\sigma}(v)\|_{X(T)}\\ &\lesssim\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big).\end{aligned}$$

Summarizing we may conclude a uniquely determined solution

$$u\in L^\infty\big((0,T),H^\gamma_r(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)\big)\ \text{for all}\ T>0.$$

As at the end of the proof of Theorem 2.1 we verify that the solution u belongs even to

$$C\big([0,\infty),H^\gamma_r(\mathbb{R}^n)\cap L^q(\mathbb{R}^n)\big)\ \text{for all}\ q\in[r,\infty).$$

If the data are large, then instead we get for $p>2$ the estimates

$$\begin{aligned}&\|Pu\|_{X(T)}\le C\|u_0\|_{H^\gamma_r\cap L^\infty}+C(T)\|u\|^p_{X(T)},\\ &\|Pu-Pv\|_{X(T)}\le C(T)\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big),\end{aligned}$$

where $C(T)$ tends to 0 for $T\to+0$. For this reason we can have for general (large) data a local (in time) existence result of weak solutions only.
This completes the proof.

6.5. Proof of Theorem 2.5

We recall that the solution of (1.3) is given by

$$u(t,x) = (G^m_{\alpha,\sigma}(t) * u_0)(t,x) + N^m_{\alpha,\sigma}(u)(t,x).$$

Let $T > 0$. We define the space

$$X(T) := C\big([0,T]; L^r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big)$$

with the norm

$$\|u\|_{X(T)} := \sup_{0\le t\le T} \big\{(1+t)^{1-\alpha}\big(\|u(t,\cdot)\|_{L^r} + \|u(t,\cdot)\|_{L^\infty}\big)\big\}.$$

For any $u \in X(T)$ we consider the operator

$$P : X(T) \to X(T), \quad Pu := (G^m_{\alpha,\sigma}(t) * u_0)(t,x) + N^m_{\alpha,\sigma}(u)(t,x).$$

We shall prove that

$$\|Pu\|_{X(T)} \lesssim \|u_0\|_{L^r\cap L^\infty} + \|u\|^p_{X(T)}, \tag{6.32}$$

$$\|Pu - Pv\|_{X(T)} \lesssim \|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big). \tag{6.33}$$

After proving (6.32) and (6.33) we may conclude the global (in time) result of small data solutions in Theorem 2.5. Due to Proposition 5.6 we know that

$$G^m_{\alpha,\sigma}(t) * u_0 \in C\big([0,\infty), L^r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big).$$

By using (5.4) we have

$$\begin{aligned}
&\|G^m_{\alpha,\sigma}(t) * u_0\|_{X(T)} \\
&\quad= \sup_{0\le t\le T} \big\{(1+t)^{1-\alpha}\big(\|(G^m_{\alpha,\sigma}(t) * u_0)(t,\cdot)\|_{L^r} + \|(G^m_{\alpha,\sigma}(t) * u_0)(t,\cdot)\|_{L^\infty}\big)\big\} \\
&\quad\lesssim \sup_{0\le t\le T} \big\{(1+t)^{1-\alpha}(1+t)^{-(1+\alpha)}\big\}\|u_0\|_{L^r\cap L^\infty} \\
&\quad\lesssim \sup_{t\ge 0} \big\{(1+t)^{1-\alpha}(1+t)^{-(1+\alpha)}\big\}\|u_0\|_{L^q\cap L^\infty} \lesssim \|u_0\|_{L^r\cap L^\infty}.
\end{aligned} \tag{6.34}$$

It remains to prove $\|N^m_{\alpha,\sigma}u\|_{X(T)} \lesssim \|u\|^p_{X(T)}$. If $u \in X(T)$, then by interpolation we derive

$$\|u(t,\cdot)\|_{L^q} \lesssim (1+t)^{\alpha-1}\|u\|_{X(T)} \quad \text{for all} \quad t \in [0,T] \quad \text{and} \quad q \in [r,\infty].$$

On the other hand, we have

$$\begin{aligned}
\||u(t,\cdot)|^p\|_{L^q} \le \|u(t,\cdot)\|^p_{L^{pq}} \lesssim (1+t)^{-p(1-\alpha)}\|u\|^p_{X(T)} \\
\text{for all} \quad t \in [0,T] \quad \text{and} \quad q \in [r,\infty].
\end{aligned} \tag{6.35}$$

Thanks to (5.4) and (6.35) we may derive the estimate

$$\|N^m_{\alpha,\sigma}u(t,\cdot)\|_{L^q} \lesssim \|u\|^p_{X(T)} I(t) \quad \text{for all} \quad t \in [0,T] \quad \text{and} \quad q \in [r,\infty], \text{ where}$$

$$I(t) = \int_0^t (1+t-\tau)^{-(1+\alpha)} \int_0^\tau (\tau-s)^{\alpha-1}(1+s)^{-p(1-\alpha)}\, ds\, d\tau. \tag{6.36}$$

We are interested to estimate the right-hand side of (6.36). For this we need the Lemma 7.11. We put

$$\omega(\tau)=\int_0^\tau(\tau-s)^{\alpha-1}(1+s)^{-p(1-\alpha)}\,ds.$$

Thanks to Lemma 7.11 we obtain

$$\omega(\tau)\lesssim\begin{cases}(1+\tau)^{\alpha-1} & \text{if}\quad p>\frac{1}{1-\alpha},\\ (1+\tau)^{\alpha-1}\ln(2+\tau) & \text{if}\quad p=\frac{1}{1-\alpha},\\ (1+\tau)^{\alpha-p(1-\alpha)} & \text{if}\quad p<\frac{1}{1-\alpha}.\end{cases}\tag{6.37}$$

If we assume that $p>\frac{1}{1-\alpha}$, then we obtain $\omega(\tau)\lesssim(1+\tau)^{\alpha-1}$.

Hence,

$$I(t)\lesssim\int_0^t(1+t-\tau)^{-(1+\alpha)}\omega(\tau)\,d\tau\lesssim\int_0^t(1+t-\tau)^{-(1+\alpha)}(1+\tau)^{\alpha-1}\,d\tau.\tag{6.38}$$

Once more we apply Lemma 7.11 to (6.38) to obtain $I(t)\lesssim(1+t)^{\alpha-1}$. Hence, $\|N^m_{\alpha,\sigma}u\|_{X(T)}\lesssim\|u\|^p_{X(T)}$. Finally, it remains to show (6.33). Let $r\in[q,\infty]$. By Hölder's inequality, for $u,v\in X(T)$, and if p' denotes the conjugate to p, then we have

$$\begin{aligned}
&\||u(s,\cdot)|^p-|v(s,\cdot)|^p\|_{L^q}\\
&\quad\lesssim\left(\int_{\mathbb{R}^n}|u(s,x)-v(s,x)|^q\Big(|u(s,x)|^{p-1}+|v(s,x)|^{p-1}\Big)^q\,dx\right)^{\frac{1}{q}}\\
&\quad\lesssim\left(\int_{\mathbb{R}^n}|u(s,x)-v(s,x)|^{pq}\,dx\right)^{\frac{1}{pq}}\left(\int_{\mathbb{R}^n}\Big(|u(s,x)|^{p-1}+|v(s,x)|^{p-1}\Big)^{qp'}dx\right)^{\frac{1}{qp'}}\\
&\quad\lesssim\|u(s,\cdot)-v(s,\cdot)\|_{L^{pq}}\||u(s,\cdot)|^{p-1}+|v(s,\cdot)|^{p-1}\|_{L^{qp'}}\\
&\quad\lesssim\|u(s,\cdot)-v(s,\cdot)\|_{L^{pq}}\big(\||u(s,\cdot)|^{p-1}\|_{L^{qp'}}+\||v(s,\cdot)|^{p-1}\|_{L^{qp'}}\big)\\
&\quad\lesssim\|u(s,\cdot)-v(s,\cdot)\|_{L^{pq}}\big(\|u(s,\cdot)\|^{p-1}_{L^{qp'(p-1)}}+\|v(s,\cdot)\|^{p-1}_{L^{qp'(p-1)}}\big)\\
&\quad\lesssim\|u(s,\cdot)-v(s,\cdot)\|_{L^{pq}}\big(\|u(s,\cdot)\|^{p-1}_{L^{pq}}+\|v(s,\cdot)\|^{p-1}_{L^{pq}}\big)\\
&\quad\lesssim(1+s)^{-p(1-\alpha)}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big).
\end{aligned}$$

Hence,

$$\begin{aligned}
\|N^m_{\alpha,\sigma}(u)(t,\cdot)-N^m_{\alpha,\sigma}(v)(t,\cdot)\|_{L^q}&\lesssim I(t)\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big)\\
&\lesssim(1+t)^{\alpha-1}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big)\quad\text{for all}\quad t\in[0,T].
\end{aligned}$$

We deduce that

$$\begin{aligned}
\|Pu-Pv\|_{X(T)}&=\|N^m_{\alpha,\sigma}(u)-N^m_{\alpha,\sigma}(v)\|_{X(T)}\\
&\lesssim\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big).
\end{aligned}$$

Remark 6.2. All estimates (6.32) and (6.33) are uniformly with respect to $T\in(0,\infty)$ if $p>\frac{1}{1-\alpha}$.

From (6.32) it follows that P maps $X(T)$ into itself for all T and for small data. By standard contraction arguments (see [5]) the estimates (6.32) and (6.33) lead to the existence of a unique solution to $u = Pu$ and, consequently, to (1.3), that is, the solution of (1.3) satisfies (6.34). Since all constants are independent of T we let T tend to ∞ and we obtain a global (in time) existence result for small data solutions to (1.3).
If the data are large, then instead we get for $p > 1$ the estimates

$$\|Pu\|_{X(T)} \leq C\|u_0\|_{L^r \cap L^\infty} + C(T)\|u\|^p_{X(T)},$$
$$\|Pu - Pv\|_{X(T)} \leq C(T)\|u - v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big),$$

where $C(T)$ tends to 0 for $T \to +0$. For this reason we can have for general (large) data a local (in time) existence result of weak solutions only. This completes the proof.

6.6. Proof of Theorem 2.6

Let $T > 0$. We define the space

$$X(T) := C\big([0, T], H^\gamma_r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big)$$

with the norm

$$\|u\|_{X(T)} := \sup_{0\leq t\leq T} \big\{(1 + t)^{1-\alpha}\big(\|u(t, \cdot)\|_{H^\gamma_r} + \|u(t, \cdot)\|_{L^\infty}\big)\big\}.$$

For any $u \in X(T)$ we consider the operator

$$P : X(T) \to X(T), \quad Pu := (G^m_{\alpha,\sigma}(t) * u_0)(t, x) + N^m_{\alpha,\sigma}(u)(t, x).$$

We shall prove that

$$\|Pu\|_{X(T)} \lesssim \|u_0\|_{H^\gamma_r \cap L^\infty} + \|u\|^p_{X(T)}, \tag{6.39}$$

$$\|Pu - Pv\|_{X(T)} \lesssim \|u - v\|_{X(T)}\left(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\right). \tag{6.40}$$

After proving (6.39) and (6.40) we may conclude the global (in time) existence result of small data solutions in Theorem 2.6. Due to Proposition 5.6 we know that

$$G^m_{\alpha,\sigma}(t) * u_0 \in C\big([0, \infty), H^\gamma_r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big).$$

By using (5.4) we have

$$\begin{aligned}
&\|G^m_{\alpha,\sigma}(t) * u_0\|_{X(T)} \\
&\quad = \sup_{0\leq t\leq T} \big\{(1 + t)^{1-\alpha}\big(\|(G^m_{\alpha,\sigma}(t) * u_0)(t, \cdot)\|_{H^\gamma_r} + \|(G^m_{\alpha,\sigma}(t) * u_0)(t, \cdot)\|_{L^\infty}\big)\big\} \\
&\quad \lesssim \sup_{0\leq t\leq T} \big\{(1 + t)^{1-\alpha}(1 + t)^{-(1+\alpha)}\big\}\|u_0\|_{H^\gamma_r} \\
&\quad \lesssim \sup_{t\geq 0} \big\{(1 + t)^{1-\alpha}(1 + t)^{-(1+\alpha)}\big\}\|u_0\|_{H^\gamma_r} \lesssim \|u_0\|_{H^\gamma_r \cap L^\infty}.
\end{aligned} \tag{6.41}$$

It remains to prove $\|N^m_{\alpha,\sigma}u\|_{X(T)} \lesssim \|u\|^p_{X(T)}$. If $u \in X(T)$, then we derive

$$\|u(t, \cdot)\|_{H^\gamma_r \cap L^\infty} \lesssim (1 + t)^{\alpha-1}\|u\|_{X(T)}.$$

On the other hand, applying Proposition 7.5 with $p > \max\{2;\gamma\}$ we obtain

$$\begin{aligned}\||u(t,\cdot)|^p\|_{H^\gamma_r} &\lesssim \|u(t,\cdot)\|_{H^\gamma_r}\|u(t,\cdot)\|^{p-1}_{L^\infty}\\ &\lesssim (1+t)^{\alpha-1}\|u\|_{X(T)}(1+t)^{(p-1)(\alpha-1)}\|u\|^{p-1}_{X(T)}\\ &\lesssim (1+t)^{-p(1-\alpha)}\|u\|^p_{X(T)}. \end{aligned} \tag{6.42}$$

Moreover, we have

$$\||u(t,\cdot)|^p\|_{L^\infty} \lesssim (\|u(t,\cdot)\|_{L^\infty})^p \lesssim (1+t)^{-p(1-\alpha)}\|u\|^p_{X(T)}. \tag{6.43}$$

Thanks to (5.5), (6.42) and (6.43) we may derive the estimates

$$\begin{aligned}\|N^m_{\alpha,\sigma}u(t,\cdot)\|_{H^\gamma_r} &\lesssim \|u\|^p_{X(T)}I(t) \quad \text{for all} \quad t\in[0,T],\\ \|N^m_{\alpha,\sigma}u(t,\cdot)\|_{L^\infty} &\lesssim \|u\|^p_{X(T)}I(t) \quad \text{for all} \quad t\in[0,T],\end{aligned}$$

where $I(t)$ is as in (6.36). We recall that we obtain $I(t) \lesssim (1+t)^{\alpha-1}$ for $p > \frac{1}{1-\alpha}$. Hence, $\|N^m_{\alpha,\sigma}u\|_{X(T)} \lesssim \|u\|^p_{X(T)}$. Finally, it remains to show (6.40). We have

$$\begin{aligned}\||u(s,\cdot)|^p - |v(s,\cdot)|^p\|_{L^\infty} &\lesssim \|u(s,\cdot)-v(s,\cdot)\|_{L^\infty}\big(\|u(s,\cdot)\|^{p-1}_{L^\infty} + \|v(s,\cdot)\|^{p-1}_{L^\infty}\big)\\ &\lesssim (1+s)^{-p(1-\alpha)}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big).\end{aligned}$$

Hence,

$$\begin{aligned}\|N^m_{\alpha,\sigma}(u)(t,\cdot) - N^m_{\alpha,\sigma}(v)(t,\cdot)\|_{L^\infty} &\lesssim I(t)\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big)\\ &\lesssim (1+t)^{\alpha-1}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big) \quad \text{for all} \quad t\in[0,T].\end{aligned}$$

It remains to prove

$$\begin{aligned}\|N^m_{\alpha,\sigma}(u)(t,\cdot) - N^m_{\alpha,\sigma}(v)(t,\cdot)\|_{H^\gamma_r} &\lesssim I(t)\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big)\\ &\lesssim (1+t)^{\alpha-1}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big) \quad \text{for all} \quad t\in[0,T].\end{aligned}$$

We have

$$\begin{aligned}&\|N^m_{\alpha,\sigma}(u)(t,\cdot) - N^m_{\alpha,\sigma}(v)(t,\cdot)\|_{H^\gamma_r}\\ &\quad\approx \|N^m_{\alpha,\sigma}(u)(t,\cdot) - N^m_{\alpha,\sigma}(v)(t,\cdot)\|_{L^r} + \|N^m_{\alpha,\sigma}(u)(t,\cdot) - N^m_{\alpha,\sigma}(v)(t,\cdot)\|_{\dot H^\gamma_r}.\end{aligned}$$

Here $f \approx g$ means that $g \lesssim f \lesssim g$. As above we have

$$\begin{aligned}&\|N^m_{\alpha,\sigma}(u)(t,\cdot) - N^m_{\alpha,\sigma}(v)(t,\cdot)\|_{L^r}\\ &\qquad\lesssim (1+t)^{\alpha-1}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big) \quad \text{for all} \quad t\in[0,T].\end{aligned}$$

It remains to prove

$$\begin{aligned}&\|N^m_{\alpha,\sigma}(u)(t,\cdot) - N^m_{\alpha,\sigma}(v)(t,\cdot)\|_{\dot H^\gamma_r}\\ &\qquad\lesssim (1+t)^{\alpha-1}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big) \quad \text{for all} \quad t\in[0,T],\end{aligned}$$

that is, it is sufficient to prove that

$$\||u(s,\cdot)|^p - |v(s,\cdot)|^p\|_{\dot H^\gamma_r} \lesssim (1+s)^{-p(1-\alpha)}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big).$$

By using the integral representation

$$|u(s,\cdot)|^p - |v(s,\cdot)|^p = p\int_0^1 (u(s,\cdot)-v(s,\cdot))Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\,d\omega,$$

where $Q(u)=u|u|^{p-2}$, we obtain

$$\begin{aligned}&\||u(s,\cdot)|^p - |v(s,\cdot)|^p\|_{\dot H^\gamma_r}\\ &\qquad \lesssim \int_0^1 \||D|^\gamma\big((u(s,\cdot)-v(s,\cdot))Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\big)\|_{L^r}\,d\omega.\end{aligned}$$

Applying the fractional Leibniz formula from Proposition 7.8 to estimate a product in $\dot H^\gamma_r$ we get

$$\begin{aligned}&\||u(s,\cdot)|^p - |v(s,\cdot)|^p\|_{\dot H^\gamma_r}\\
&\quad\lesssim \int_0^1 \||D|^\gamma\big(u(s,\cdot)-v(s,\cdot)\big)\|_{L^r}\|Q(\omega u(s,x)+(1-\omega)v(s,x))\|_{L^\infty}\,d\omega\\
&\qquad+\int_0^1 \|u(s,\cdot)-v(s,\cdot)\|_{L^\infty}\||D|^\gamma Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\|_{L^r}\,d\omega\\
&\quad\lesssim \||D|^\gamma\big(u(s,\cdot)-v(s,\cdot)\big)\|_{L^r}\big(\|u(s,\cdot)\|^{p-1}_{L^\infty}+\|v(s,\cdot)\|^{p-1}_{L^\infty}\big)\\
&\qquad+\|u(s,\cdot)-v(s,\cdot)\|_{L^\infty}\int_0^1 \||D|^\gamma Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\|_{L^r}\,d\omega\\
&\quad\lesssim \|u(s,\cdot)-v(s,\cdot)\|_{\dot H^\gamma_r}\big(\|u(s,\cdot)\|^{p-1}_{L^\infty}+\|v(s,\cdot)\|^{p-1}_{L^\infty}\big)\\
&\qquad+\|u(s,\cdot)-v(s,\cdot)\|_{L^\infty}\int_0^1 \||D|^\gamma Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\|_{L^r}\,d\omega\\
&\quad\lesssim (1+s)^{-p(1-\alpha)}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big)\\
&\qquad+(1+s)^{-(1-\alpha)}\|u-v\|_{X(T)}\int_0^1 \||D|^\gamma Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\|_{L^r}\,d\omega.\end{aligned}$$

We apply again the Proposition 7.6 to estimate the term in the integral. In this way we may conclude

$$\begin{aligned}&\int_0^1 \||D|^\gamma Q(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\|_{L^r}\,d\omega\\
&\qquad\lesssim \int_0^1 \||D|^\gamma(\omega u(s,\cdot)+(1-\omega)v(s,\cdot))\|_{L^r}\\
&\qquad\qquad\times\|\omega u(s,\cdot)+(1-\omega)v(s,\cdot)\|^{p-2}_{L^\infty}\,d\omega\\
&\qquad\lesssim \int_0^1 (1+s)^{-(1-\alpha)}\|\omega u+(1-\omega)v\|_{X(T)}\\
&\qquad\qquad\times(1+s)^{-(p-2)(1-\alpha)}\|\omega u+(1-\omega)v\|^{p-2}_{X(T)}\,d\omega\\
&\qquad\lesssim \int_0^1 (1+s)^{-(p-1)(1-\alpha)}\|\omega u+(1-\omega)v\|^{p-1}_{X(T)}\,d\omega\\
&\qquad\lesssim (1+s)^{-(p-1)(1-\alpha)}\big(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\big).\end{aligned}$$

Then

$$\||u(s,\cdot)|^p - |v(s,\cdot)|^p\|_{\dot{H}^\gamma_r} \lesssim (1+s)^{-p(1-\alpha)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big).$$

Hence,

$$\begin{aligned}&\|N^m_{\alpha,\sigma}(u)(t,\cdot) - N^m_{\alpha,\sigma}(v)(t,\cdot)\|_{\dot{H}^\gamma_r}\\ &\quad\lesssim (1+t)^{\alpha-1}\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big) \quad \text{for all} \quad t\in[0,T].\end{aligned}$$

We deduce that

$$\begin{aligned}\|Pu - Pv\|_{X(T)} &= \|N^m_{\alpha,\sigma}(u) - N^m_{\alpha,\sigma}(v)\|_{X(T)}\\ &\lesssim \|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\big).\end{aligned}$$

Remark 6.3. All estimates (6.39) and (6.40) are uniformly with respect to $T \in (0,\infty)$ if $p > \max\{2; \gamma; \frac{1}{1-\alpha}\}$.

From (6.39) it follows that P maps $X(T)$ into itself for all T and for small data. By standard contraction arguments (see [5]) the estimates (6.39) and (6.40) lead to the existence of a unique solution to $u = Pu$ and, consequently, to (1.3), that is, the solution of (1.3) satisfies the desired decay estimate. Since all constants are independent of T, after letting T tend to ∞ we obtain a global (in time) existence result for small data solutions to (1.3). If the data are large, then instead we get for $p > 2$ the estimates

$$\begin{aligned}&\|Pu\|_{X(T)} \le C\|u_0\|_{H^\gamma_r\cap L^\infty} + C(T)\|u\|^p_{X(T)},\\ &\|Pu - Pv\|_{X(T)} \le C(T)\|u-v\|_{X(T)}\big(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(X)}\big),\end{aligned}$$

where $C(T)$ tends to 0 for $T \to +0$. For this reason we can have for general (large) data a local (in time) existence result of weak solutions only. By the same argument as above we obtain the desired results. The proof is complete.

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7. Appendix

7.1. Modified Bessel functions

Here we review basic properties of modified Bessel functions. These properties can be found in [8] and [14].

Definition 7.1. The Bessel function J_μ of first kind and of order $\mu \in \mathbb{R}$ is defined by

$$J_\mu(s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\mu+1)} \Big(\frac{s}{2}\Big)^{2k+\mu},$$

where μ is not allowed to be a negative integer. The modified Bessel function $\tilde{J}_\mu(s)$ is defined by $\tilde{J}_\mu(s) := \frac{J_\mu(s)}{s^\mu}$.

Lemma 7.2. *Let $f \in L^p(\mathbb{R}^n)$, $p \in [1,2]$, be a radial function. Then the inverse Fourier transform is also a radial function and it satisfies*

$$F^{-1}(f)(x) = \int_0^\infty g(r) r^{n-1} \tilde{J}_{\frac{n}{2}-1}(r|x|)\, dr, \quad g(|x|) := f(x).$$

Lemma 7.3. *Assume that μ is not a negative integer. Then the following rules hold:*

1. $s d_s \tilde{J}_\mu(s) = \tilde{J}_{\mu-1}(s) - 2\mu \tilde{J}_\mu(s)$,
2. $d_s \tilde{J}_\mu(s) = -s \tilde{J}_{\mu+1}(s)$,
3. $\tilde{J}_{-1/2}(s) = \sqrt{\frac{2}{\pi}} \cos(s)$,
4. *we have the relations*

$$|\tilde{J}_\mu(s)| \le C e^{\pi|\Im\mu|} \quad \text{if} \quad |s| \le 1,$$
$$J_\mu(s) = C s^{-\frac{1}{2}} \cos\Big(s - \frac{\mu}{2}\pi - \frac{\pi}{4}\Big) + O(|s|^{-\frac{3}{2}}) \quad \text{if} \quad |s| \ge 1,$$

5. $\tilde{J}_{\mu+1}(r|x|) = -\frac{1}{r|x|^2} \partial_r \tilde{J}_\mu(r|x|), \quad r \ne 0, x \ne 0.$

7.2. Mittag-Leffler function

Here we review basic properties of Mittag-Leffler functions. These properties can be found in [6] and [9]. The Mittag-Leffler function E_β allows the following implicit definition:

$$\frac{\lambda}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} E_\beta(\lambda s^\beta)\, ds = E_\beta(\lambda t^\beta) - 1. \tag{7.1}$$

The Mittag-Leffler function $E_\beta(-t^\beta \langle\xi\rangle^2_{m,\sigma})$ with

$$E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k+1)}, \quad \beta \in \mathbb{C} \quad \text{with} \quad \Re\beta > 0,$$

may be written in the following form:

$$E_\beta(-t^\beta \langle\xi\rangle^2_{m,\sigma}) = \frac{1}{\beta}\Big(\exp\big(a_\beta(t^{\frac{\beta}{2}} \langle\xi\rangle_{m,\sigma})\big) + \exp\big(b_\beta(t^{\frac{\beta}{2}} \langle\xi\rangle_{m,\sigma})\big)\Big)$$
$$+ l_\beta(t^{\frac{\beta}{2}} \langle\xi\rangle_{m,\sigma}),$$

where

$$a_\beta(y) = y^{\frac{2}{\beta}} \exp(\tfrac{\pi i}{\beta}) \ \text{ for } \ y \geq 0,$$
$$b_\beta(y) = y^{\frac{2}{\beta}} \exp(-\tfrac{\pi i}{\beta}) \ \text{ for } \ y \geq 0,$$
$$l_\beta(y) = \begin{cases} \frac{\sin(\beta\pi)}{\pi} \displaystyle\int_0^\infty \frac{y^2 s^{\beta-1}\exp(-s)}{s^{2\beta} + 2y^2 s^\beta \cos(\beta\pi) + y^4}\, ds \\ \quad = \frac{\sin(\beta\pi)}{\beta\pi} \displaystyle\int_0^\infty \frac{\exp(-y^{\frac{2}{\beta}} s^{\frac{1}{\beta}})}{s^2 + 2s\cos(\beta\pi) + 1}\, ds \ \text{ for } \ y > 0, \\ 1 - \frac{2}{\beta} \ \text{ for } \ y = 0. \end{cases}$$

Here $\beta = 1 + \alpha$. The proof can be found in the paper [6].

Remark 7.4. We have also the relation

$$\begin{aligned} &\exp\big(a_\beta(t^{\frac{\beta}{2}}\langle\xi\rangle_{m,\sigma})\big) + \exp\big(b_\beta(t^{\frac{\beta}{2}}\langle\xi\rangle_{m,\sigma})\big) \\ &= 2e^{t\langle\xi\rangle_{m,\sigma}^{\frac{2}{1+\alpha}}\cos(\frac{\pi}{1+\alpha})} \cos\Big(t\langle\xi\rangle_{m,\sigma}^{\frac{2}{1+\alpha}} \sin\Big(\frac{\pi}{1+\alpha}\Big)\Big) \\ &= 2e^{-ct\langle\xi\rangle_{m,\sigma}^{\frac{2}{1+\alpha}}} \cos\Big(t\langle\xi\rangle_{m,\sigma}^{\frac{2}{1+\alpha}}\sqrt{1-c^2}\Big), \ \text{ where } \ c = -\cos\Big(\frac{\pi}{1+\alpha}\Big). \end{aligned}$$

7.3. Results from Harmonic Analysis

We recall some results from Harmonic Analysis (cf. with [11]).

Proposition 7.5. *Let $r \in (1,\infty), p > 1$ and $\sigma \in (0,p)$. Let $Q(u)$ denote one of the functions $|u|^p, \pm u|u|^{p-1}$. Then the following inequality holds:*

$$\|Q(u)\|_{H^\sigma_r} \lesssim \|u\|_{H^\sigma_r} \|u\|_{L^\infty}^{p-1}$$

for any $u \in H^\sigma_r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Here we use for $\gamma \geq 0$ and $1 < q < \infty$ the fractional Sobolev spaces or Bessel potential spaces

$$H^\gamma_q(\mathbb{R}^n) := \big\{f \in S'(\mathbb{R}^n) : \|f\|_{H^\gamma_q} := \|F^{-1}(\langle\xi\rangle^\gamma F(f))\|_{L^q} < \infty\big\}.$$

Moreover, $\langle D\rangle^\gamma$ stands for the pseudo-differential operator with symbol $\langle\xi\rangle^\gamma$ and it is defined by $\langle D\rangle^\gamma u = F^{-1}(\langle\xi\rangle^\gamma F(u))$.

Proof. This result is a special case of the following more general inequality for Triebel-Lizorkin spaces $F^\sigma_{r,q}$:

$$\|Q(u)\|_{F^\sigma_{r,q}} \lesssim \|u\|_{F^\sigma_{r,q}} \|u\|_{L^\infty}^{p-1} \quad \text{for any } \ u \in F^\sigma_{r,q} \cap L^\infty,$$

where $q > 0$, whose proof may be found in [13, Theorem 1 in Section 5.4.3]. □

Proposition 7.6. *Let $r \in (1,\infty), p > 1$ and $\sigma \in (0,p)$. Let $Q(u)$ denote one of the functions $|u|^p, \pm u|u|^{p-1}$. Then the following inequality holds:*

$$\|Q(u)\|_{\dot H^\sigma_r} \lesssim \|u\|_{\dot H^\sigma_r} \|u\|_{L^\infty}^{p-1}$$

for any $u \in \dot H^\sigma_r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, where

$$\dot H^\gamma_q(\mathbb{R}^n) := \{f \in S'(\mathbb{R}^n) : \|f\|_{\dot H^\gamma_q} := \|F^{-1}(|\xi|^\gamma F(f))\|_{L^q} < \infty\}.$$

Here $|D|^\gamma$ stands for the pseudo-differential operator with symbol $|\xi|^\gamma$ and it is defined by $|D|^\gamma u = F^{-1}(|\xi|^\gamma F(u))$.

Proof. We will use a homogeneity argument. For any positive λ we define $u_\lambda(x) = u(\lambda x)$. Applying Proposition 7.5 to u_λ we get

$$\|Q(u_\lambda)\|_{H^\sigma_r} \lesssim \|u_\lambda\|_{H^\sigma_r} \|u_\lambda\|^{p-1}_{L^\infty}. \tag{7.2}$$

Since for $r \in (1, \infty)$ we have the decomposition

$$\|v\|_{H^\sigma_r} \approx \|v\|_{\dot{H}^\sigma_r} + \|v\|_{L^r} \qquad \text{for any} \quad v \in H^\sigma_r$$

and the scaling properties

$$\|u_\lambda\|_{\dot{H}^\sigma_r} = \lambda^{\sigma - \frac{n}{r}} \|u\|_{\dot{H}^\sigma_r}, \; \|u_\lambda\|_{L^r} = \lambda^{-\frac{n}{r}} \|u\|_{L^r} \text{ and } \|u_\lambda\|_{L^\infty} = \|u\|_{L^\infty}$$

diving both sides of (7.2) by $\lambda^{\sigma - \frac{n}{r}}$ and taking the limit as $\lambda \to \infty$ we obtain the desired inequality. □

Proposition 7.7. *Let $r \in (1, \infty)$ and $\sigma > 0$. Then the following inequality holds:*

$$\|uv\|_{H^\sigma_r} \lesssim \|u\|_{H^\sigma_r} \|v\|_{L^\infty} + \|u\|_{L^\infty} \|v\|_{H^\sigma_r}$$

for any $u, v \in H^\sigma_r \cap L^\infty$.

Proof. The result that we want to prove is a special case of the following inequality for Triebel-Lizorkin spaces $F^\sigma_{r,q}$:

$$\|uv\|_{F^\sigma_{r,q}} \lesssim \|u\|_{F^\sigma_{r,q}} \|v\|_{L^\infty} + \|u\|_{L^\infty} \|v\|_{F^\sigma_{r,q}}$$

for any $u, v \in F^\sigma_{r,q} \cap L^\infty$, where $q > 0$, whose proof can be found in [13, Theorem 2 in Section 4.6.4]. □

Finally let us state the corresponding inequality in homogeneous spaces $\dot{H}^\sigma_r$. For the proof it is possible to follow the same strategy as in the proof of Proposition 7.6.

Proposition 7.8. *(Fractional Leibniz formula) Let $r \in (1, \infty)$ and $\sigma > 0$. Then the following inequality holds:*

$$\|uv\|_{\dot{H}^\sigma_r} \lesssim \|u\|_{\dot{H}^\sigma_r} \|v\|_{L^\infty} + \|u\|_{L^\infty} \|v\|_{\dot{H}^\sigma_r}$$

for any $u, v \in \dot{H}^\sigma_r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

The following result was proposed and proved by Marcello D'Abbicco (University of Bari) and already used in a special case in [4]. We present the proof to make the paper more self-contained.

Proposition 7.9. *Let $u_0 \in L^r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $n \geq 1$, $r \geq 1$ and $\alpha \in (0, 1)$. Then the function*

$$u = u(t, x) = \big(G^0_{\alpha,\sigma}(t) * u_0\big)(t, x)$$

belongs to

$$C\big([0, \infty), L^r(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)\big) \; \textit{for all } q \in [r, \infty).$$

Proof. Due to (3.2) we have

$$G^0_{\alpha,\sigma}(t,x)=\int_{\mathbb{R}^n} e^{ix\cdot\xi}E_{\alpha+1}\big(-t^{\alpha+1}|\xi|^{2\sigma}\big)\,d\xi.$$

The estimate (4.1) from Proposition 4.1 implies $G^0_{\alpha,\sigma}(t,\cdot)\in L^1(\mathbb{R}^n)$ for all $t>0$. Moreover, $G^0_{\alpha,\sigma}(t,\cdot)$ has the following scale-invariant property:

$$G^0_{\alpha,\sigma}(t,x)=t^{-n\beta}G^0_{\alpha,\sigma}(1,t^{-\beta}x) \ \text{ with } \ \beta=\frac{\alpha+1}{2\sigma}. \tag{7.3}$$

Consequently, we conclude for all $t>0$ the relations

$$\|G^0_{\alpha,\sigma}(t,\cdot)\|_{L^1}=\|G^0_{\alpha,\sigma}(1,\cdot)\|_{L^1} \tag{7.4}$$

and

$$\int_{\mathbb{R}^n} G^0_{\alpha,\sigma}(t,x)dx=\int_{\mathbb{R}^n} G^0_{\alpha,\sigma}(1,x)dx=1. \tag{7.5}$$

Let us choose a positive zero sequence $\{t_l\}_l$. We want to prove for a given $g\in L^p(\mathbb{R}^n)$, $p\in[1,\infty)$, that the sequence $\{T_l*g\}_l$ tends to g, where $T_l(\cdot):=G^0_{\alpha,\sigma}(t_l,\cdot)$. We have $\lim_{l\to\infty}T_l=\delta_0$ in the distributional sense. Hence, $\lim_{l\to\infty}T_l*g=g$ in distributional sense, too. But, this implies the desired relation $\int_{\mathbb{R}^n}G^0_{\alpha,\sigma}(t,x)dx=1$. Otherwise, if we would have for $t>0$ the relation

$$\int_{\mathbb{R}^n} G^0_{\alpha,\sigma}(t,x)dx=\int_{\mathbb{R}^n} G^0_{\alpha,\sigma}(1,x)dx=M\in\mathbb{C},$$

then we might conclude $\lim_{l\to\infty}T_l*g=Mg$ in the distributional sense, in contradiction to $\lim_{l\to\infty}T_l=\delta_0$ in the distributional sense.
The scale-invariant property (7.3) implies for all positive δ

$$\int_{|x|\geq\delta}|T_l(x)|dx\to 0 \ \text{ for } \ l\to\infty. \tag{7.6}$$

Indeed, the relation (7.6) holds after taking account of

$$\begin{aligned}\int_{|x|\geq\delta}|T_l(x)|dx&=t_l^{-n\beta}\int_{|x|\geq\delta}|G^0_{\alpha,\sigma}(1,t_l^{-\beta}x)|dx\\&=\int_{|y|\geq t_l^{-\beta}\delta}|G^0_{\alpha,\sigma}(1,y)|dy\to 0.\end{aligned}$$

Let us choose a function g having the additional regularity $g\in C(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$. We prove that the sequence $\{(T_l*g)(x)\}_l$ tends to $g(x)$ for all $x\in\mathbb{R}^n$. Using (7.5) we obtain

$$(T_l*g)(x)-g(x)=\int_{\mathbb{R}^n}(g(x-y)-g(x))T_l(y)dy.$$

For a fixed positive ε we choose $\kappa=\kappa(\varepsilon,x)$ such that $|g(x-y)-g(x)|<\varepsilon$ for $|y|<\kappa$. Then,

$$\begin{aligned}|(T_l*g)(x)-g(x)|&\leq\varepsilon\int_{|y|\leq\kappa}|T_l(y)|dy+2\|g\|_{L^\infty}\int_{|y|\geq\kappa}|T_l(y)|dy\\&\leq\varepsilon(\|G^0_{\alpha,\sigma}(1,\cdot)\|_{L^1}+2\|g\|_{L^\infty})\end{aligned}$$

for sufficiently large $l = l(\kappa)$. This implies the desired relation $\lim_{l\to\infty}(T_l * g)(x) = g(x)$ for all $x \in \mathbb{R}^n$.
Applying Hölder's inequality gives

$$\begin{aligned} |(T_l * g)(x) - g(x)| &\le \|(g(x-\cdot) - g(x))T_l(\cdot)\|_{L^1} \\ &\le \||g(x-\cdot) - g(x)|^p T_l(\cdot)\|_{L^1}^{\frac{1}{p}} \|T_l(\cdot)\|_{L^1}^{\frac{1}{p'}}, \end{aligned}$$

where p' is the conjugate exponent to p. From this estimate it follows

$$\begin{aligned} \|(T_l * g - g)(\cdot)\|_{L^p}^p &\le c_p \int_{\mathbb{R}^n_y} |T_l(y)| \Big(\int_{\mathbb{R}^n_x} |g(x-y) - g(x)|^p dx \Big) dy \\ &= c_p \int_{\mathbb{R}^n_y} |T_l(y)| \varphi(-y) dy = c_p(|T_l| * \varphi)(0), \end{aligned}$$

where we introduced

$$\varphi(-y) := \int_{\mathbb{R}^n_x} |g(x-y) - g(x)|^p dx.$$

The function $\varphi = \varphi(-y)$ is bounded and continuous. Consequently, we get $\lim_{l\to\infty} \|(T_l * g - g)(\cdot)\|_{L^p} = 0$ what we wanted to have for all bounded and continuous functions $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Taking the set $C_0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ of such functions with compact support, then a density argument in $L^p(\mathbb{R}^n)$ completes the proof. □

7.4. Inequalities

First we recall Young's inequality.

Lemma 7.10. *Let $u \in L^p(\mathbb{R}^n)$ and $v \in L^r(\mathbb{R}^n)$ with $1 \le p, r \le \infty$. Then $u * v \in L^q(\mathbb{R}^n)$, where $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$ and*

$$\|u * v\|_{L^q} \lesssim \|u\|_{L^p} \|v\|_{L^r}.$$

Finally, we recall the following lemma from [2].

Lemma 7.11. *Suppose that $\theta \in [0,1), a \ge 0$ and $b \ge 0$. Then there exists a constant $C = C(a,b,\theta) > 0$ such that for all $t > 0$ the following estimate holds:*

$$\begin{aligned} &\int_0^t (t-\tau)^{-\theta}(1+t-\tau)^{-a}(1+\tau)^{-b}\, d\tau \\ &\quad \le \begin{cases} C(1+t)^{-\min\{a+\theta;b\}} & \text{if} \quad \max\{a+\theta;b\} > 1, \\ C(1+t)^{-\min\{a+\theta;b\}} \ln(2+t) & \text{if} \quad \max\{a+\theta;b\} = 1, \\ C(1+t)^{1-a-\theta-b} & \text{if} \quad \max\{a+\theta;b\} < 1. \end{cases} \end{aligned} \tag{7.7}$$

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