



Existence and uniqueness of variational solution to the Neumann problem for the p th sub-Laplacian associated to a system of Hörmander vector fields

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Abstract. We establish the existence and uniqueness of variational solution to the nonlinear Neumann boundary problem for the p th-sub-Laplacian associated to a system of Hörmander vector fields.

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1. Introduction

Two of the fundamental boundary value problems in PDE are the Dirichlet and Neumann problems. They have profound influences in the development of the theory of PDE. Aside from their theoretical applications, they describe many physical models. In the classical setting, these two problems have been advanced to a far extent. We would not attempt to give a bibliography here since we would inadvertently omit some of the important ones. Boundary value problems for Sub-elliptic operators were initiated by the pioneering works of Kohn and Nirenberg, Bony, Gaveau and complemented later by two important works of Jerison. These works established some fundamental aspects of sub-elliptic equations such as the Harnack inequality, smoothness of the Green's function up to the boundary, for boundaries that are nowhere characteristic with respect to the operator [2, 26], explicit formula for the harmonic measure at the center of the Koranyi ball in the Heisenberg group \mathbb{H}^n [18] and the behavior of solutions to the Dirichlet problem near characteristic points

[22, 23]. Of course, we only highlight those that are closely related to our development in this paper. For other early relevant results, the reader can see [1, 14, 29, 30]. These pioneering works subsequently drew a significant attention to boundary value problems in the sub-elliptic setting, see for instance the works [3–5, 7, 9–11, 15, 16, 20, 33] and the references therein. However, none of the works cited above addressed the Neumann problem.

Our purpose is to establish existence and uniqueness of variational solutions of the Neumann problem associated to the non-linear sub-elliptic p th-Laplacian arising from a system of Hörmander vector fields. This is a continuation and extension of the works [28] and [13].

To set the stage we fix an open set $U \subset \mathbb{R}^n$ with $\text{diam}(U) < \infty$ and $X = (X_1, \dots, X_m)$ a system of vector fields with smooth coefficients satisfying Hörmander’s finite rank condition at every point $x \in U$:

$$\text{rankLie}[X_1, \dots, X_m] \equiv n$$

Fix $1 \leq p < \infty$. Let \mathcal{L}_p be the p th-sub-Laplacian associated to the system of vector fields $X = (X_1, \dots, X_m)$ defined on smooth functions u by

$$\mathcal{L}_p u = - \sum_{j=1}^m X_j^* (|Xu|^{p-2} X_j u)$$

on \mathbb{R}^n . In the above the $X_j = \sum_{k=1}^n b_k^{(j)}(x) \frac{\partial}{\partial x_k}$ are smooth vector fields on \mathbb{R}^n (satisfying Hörmander’s finite rank condition though such an assumption is not necessary to introduce the operator) and X_j^* is the formal adjoint of X_j given by $X_j^* g(x) = - \sum_{k=1}^n \frac{\partial}{\partial x_k} (b_k^{(j)}(x) g(x))$. We have also indicated by $|Xu| = \left(\sum_{j=1}^m (X_j u)^2 \right)^{\frac{1}{2}}$. For a smooth (at least C^2) domain, the classical Neumann problem for \mathcal{L}_p consists of seeking a function u that can be differentiated enough times and satisfying

$$\begin{cases} \mathcal{L}_p u = f & \text{in } \Omega, \\ \sum_{j=1}^m |Xu|^{p-2} (X_j \cdot \eta) X_j u = \nu & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where η is the unit outer (Euclidean) normal of $\partial\Omega$, ν and f are continuous functions satisfying the compatibility condition

$$\langle \nu, 1 \rangle = \int_{\Omega} f \, dx. \tag{1.2}$$

When $\nu = g$ for some function g , then (1.2) simply becomes

$$\int_{\partial\Omega} g \, d\mu = \int_{\Omega} f \, dx.$$

In the above, the notation $\langle \cdot, \cdot \rangle$ indicated the pairing between an element $x \in Z$ and $\nu \in Z^*$ for some Banach space Z and its dual Z^* [we make use of an abstract description of the left hand side of (1.2) in preparation for a more general statement later]. For the sake of convenience, we would also indicate the sum $\sum_{j=1}^m (X_j u)(X_j v)$ by the notation $\langle Xu, Xv \rangle$, as long as it is clear from the context.

In the case $p = 2$, the operator \mathcal{L}_p is known as the (real part of the) Kohn-Laplacian. Our results here extend the ones established for $p = 2$ in [13, 28]. In the classical setting, the method of layer potentials has met success in tackling such problems. However, even in the simplest prototype of the Hörmander type vector field, that is, the Heisenberg group, one faces significant difficulties in inverting the corresponding operators on the boundary of the domain (for an appropriate class of functions) due to the presence of characteristic points, see [22]. Besides, layer potentials will not help in dealing with the case $p \neq 2$. In view of this obstacle, we seek a different method to establish the problem of existence of the non-linear Neumann problem in this broad context.

One approach to the problem (1.1) is to broaden the class and the concept of solutions in such a way that existence can be established. One would then establish that solutions are regular in some context. It is our purpose to explore the first aspect in this paper and devote the second one to another occasion [12]. We now introduce variational solutions.

Definition 1.1. Fix $1 \leq p < \infty, 1 \leq q < \infty, f \in L^{q'}(\Omega, dx), \nu \in B_{1-\frac{s}{q}}^q(\partial\Omega, d\mu)^*$ where $1/q + 1/q' = 1$. A function $u \in \mathcal{L}^{1,q}(\Omega, dx)$ is called a variational solution of (1.1) if for any $\phi \in \mathcal{L}^{1,q}(\Omega, dx)$

$$\int_{\Omega} |Xu|^{p-2} \langle Xu, X\phi \rangle dx = \langle \nu, Tr(\phi) \rangle - \int_{\Omega} f \phi dx \tag{1.3}$$

In the above, $\Omega \subset \mathbb{R}^n$ is a domain for which there is a continuous (trace) operator $Tr : \mathcal{L}^{1,q}(\Omega, dx) \rightarrow B_{1-\frac{s}{q}}^q(\partial\Omega, d\mu)$.

In the above, $\mathcal{L}^{1,q}(\Omega, dx)$ denotes the by now standard sub-elliptic Sobolev space associated to the system of vector fields X .

For the definition and some properties of the Besov spaces $B_{1-\frac{s}{q}}^q(\partial\Omega, d\mu)$ we refer the reader to Sect. 2 below. The measure μ is an s -Ahlfors measure (again see Sect. 2 for the definition) supported in $\partial\Omega$ with $0 < s < \min(q, \frac{n+q}{2})$

Remark 1.2. In order for Definition 1.1 to make sense, it is required that the continuous operator Tr exists and this was established in [13, Theorem 10.6] under the assumptions

1. Ω is an extension domain (i.e. $X - (\epsilon, \delta)$ -domain) and
2. μ satisfies the estimate (see definition 2.1)

$$\mu(B(x, r)) \leq M \frac{|B(x, r)|}{r^s}. \tag{1.4}$$

In particular, 1. and 2. are fulfilled when $\Omega = \{\rho < 1\}$ where ρ is the homogenous gauge in a Carnot group of step two and when

$$d\mu = |X\rho| d\sigma$$

where $d\sigma$ is the surface measure on $\partial\Omega$, and $s = 1$ in (1.4), see e.g. [3, 4] and [5]. Therefore, in the sequel we will work under such hypothesis. Whenever an expression involves function values on the boundary, we mean the trace of such function. When the context is clear, we will omit the Tr notation.

- By taking the test function $\phi(x) \equiv 1$ in (1.3), we arrive at the compatibility condition (1.2), which is necessary for the existence of solutions to the above variational Neumann problem.

The close subspace of $\mathcal{L}^{1,q}(\Omega, dx)$ defined by

$$\tilde{\mathcal{L}}^{1,q}(\Omega, dx) = \left\{ f \in \mathcal{L}^{1,q}(\Omega, dx) \mid f_\Omega = \frac{1}{|\Omega|} \int_\Omega f(x) dx = 0 \right\}$$

is an appropriate space to treat the Neumann problem.

Remark 1.3. Due to the Poincaré inequality (see Theorem 2.9 below), an equivalent norm on $\tilde{\mathcal{L}}^{1,q}(\Omega, dx)$ is given by

$$\|f\|_{\tilde{\mathcal{L}}^{1,q}(\Omega)} = \left(\int_\Omega |Xf|^q dx \right)^{\frac{1}{q}}.$$

Our aim is to establish the existence and uniqueness of variational solution in such spaces:

Theorem 1.4. *Let U be as in Theorem 2.8. Fix $1 < p < \infty$ and $1 < q \leq p$. Let $f \in L^{q'}(\Omega, dx)$, $\Omega \subset U \subset \mathbb{R}^n$ be an $X - (\epsilon, \delta)$ domain with $\text{diam}(\Omega) < R_o/2$, $|\Omega| > 0$, μ an upper s -Ahlfors measure and $\nu \in B_{1-s/q}^q(\partial\Omega, d\mu)^*$ satisfying the compatibility condition (1.2). There exists a unique $u \in \tilde{\mathcal{L}}^{1,p}(\Omega, dx)$ that solves the Neumann problem for \mathcal{L}_p in the sense of Definition 1.1. In the above, the parameter R_o is from Definition 2.1 below. Furthermore, the solution u satisfies the estimate*

$$\|Xu\|_{L^p(\Omega, dx)} \leq C (\|\nu\|_{B_{1-s/q}^q(\partial\Omega, d\mu)^*} + \|f\|_{L^{q'}(\Omega, dx)})^{\frac{1}{p-1}}, \tag{1.5}$$

where C depends on Ω and various parameters such as R_o, p, q and so on but does not depend on u, ν, f .

To establish this main result, we show that the functional J_p defined in (3.1) has a unique minimum and it is the variational solution of (1.1). Conversely, variational solutions of (1.1) are minimizers of J_p . The proof of this fact follows the classical approach of the direct method of Calculus of Variations. First, we show that the functional J_p is sequentially lower semi-continuous in the weak topology of $\mathcal{L}^{1,q}(\Omega, dx)$. This is valid for all $1 \leq q < \infty$. We then establish the coercivity of J_p in $\mathcal{L}^{1,q}(\Omega, dx)$. At the moment, this is limited to the range $1 < q \leq p$. A weakness as it may seems in comparison to its Euclidean counterpart. However, in the Euclidean setting, for the case $p = 2$ on Lipschitz domains, similar result only holds for $1 < q < 2 + \epsilon$ where ϵ depends on the domain. The class of domains [the $X - (\epsilon, \delta)$ domains] that we treat here have characteristic points on the boundary. Such singular points act as corners (or perhaps as cusps) of the domain if seen from the Euclidean perspective. It is well-known that in the Euclidean case the inclusion of the classes of domains

$$\text{Lipschitz} \subset \text{NTA} \subset (\epsilon, \delta) \tag{1.6}$$

hold. The so call (ϵ, δ) domains were introduced in [25] and they are the largest class on which Sobolev functions can be extended. It is important to observe

that for a Carnot–Carathéodory space the last inclusion of (1.6) continue to hold, see [6]. A version of Jerison and Kenig’s theory on the boundary behaviour of harmonic functions [24] has been extended to the case of Carnot groups of step two [3] on X – NTA domains. Contrary to the Euclidean case where Lipschitz domains are probably the largest class of domains where a rich theory of boundary value problems can be developed, the analogue notion of Lipschitz domain in the sub-elliptic setting even in the simplest case of the Heisenberg group is almost non-existence, see [3]. Therefore part of our task is to identify a class of domains, as large as possible for which a theory of boundary value problems can be developed. The examples in [6] and [3] are an indication that the class of X – (ϵ, δ) domains treated here is reasonably large. Due to the inclusion (1.6), the domains considered in this paper, when confined to the Euclidean case, is larger than the Lipschitz domains. Hence, it is not surprising if a limitation on the parameter q occurs. However, we have not been able to determine the sharp range at this moment.

We start with some background materials in Sects. 2 and 3 is devoted in establishing our main result. Finally, we like to point out that, to the best of our knowledge, the results here are new even in the classical setting, as far as the class of domains treated is concerned.

2. Preliminaries

We collect previously established results needed for subsequent developments. Some of these results are more general than what is required here. We present the version that is already adapted to our setting and omit the details that are relevant in a broader context.

Let d be the Carnot–Carathéodory distance associated with the system X . It is by now well known that if the system X satisfies Hörmander’s finite rank condition [21], then $d(x, y) < \infty$ for any $x, y \in \mathbb{R}^n$ ([8, 27, 31]) and the metric balls $B(x, r)$ of d satisfy a doubling condition [27]. Denote the Borel measures on the metric space (\mathbb{R}^n, d) by \mathcal{B}_d . For a set $E \subset \mathbb{R}^n$, $|E|$ denotes the Lebesgue measure of E .

Definition 2.1. ([13]) Given $s \geq 0$, a measure $\mu \in \mathcal{B}_d$ will be called an *upper s -Ahlfors measure*, if there exist $M, R_o > 0$, such that for $x \in \mathbb{R}^n$, $0 < r \leq R_o$, one has

$$\mu(B(x, r)) \leq M \frac{|B(x, r)|}{r^s}.$$

We will say that μ is a *lower s -Ahlfors measure*, if for some $M, R_o > 0$ one has instead for x and r as above

$$\mu(B(x, r)) \geq M^{-1} \frac{|B(x, r)|}{r^s}.$$

The (dual of the) following sub-elliptic Besov space will serve as the space for the Neumann datum on the boundary.

Definition 2.2. ([13]) Let $\mu \in \mathcal{B}_d$ having $\text{supp } \mu \subseteq F$, where F is a closed subset of \mathbb{R}^n . For $1 \leq p < \infty$, $0 < \beta < 1$, we introduce the semi-norm

$$\mathcal{N}_\beta^p(f, F, d\mu) = \left\{ \int_F \int_F \left(\frac{|f(x) - f(y)|}{d(x, y)^\beta} \right)^p \frac{d(x, y)^s}{|B(x, d(x, y))|} d\mu(y) d\mu(x) \right\}^{\frac{1}{p}}.$$

The Besov space on F , relative to the measure μ , is defined as

$$B_\beta^p(F, d\mu) = \{f \in L^p(F, d\mu) \mid \mathcal{N}_\beta^p(f, F, d\mu) < \infty\}.$$

If $f \in B_\beta^p(F, d\mu)$, we define the Besov norm of f as

$$\|f\|_{B_\beta^p(F, d\mu)} = \|f\|_{L^p(F, d\mu)} + \mathcal{N}_\beta^p(f, F, d\mu).$$

We note in passing the following result [13, Theorem 11.1], which motivates the use of Besov spaces (since their dual are larger) instead of Lebesgue spaces on the boundary for the Neumann data. However, our main result here does not make use of Theorem 2.3 below. Note also that since we only needed μ to be an upper s -Ahlfors measure, a comparison between Besov and Lebesgue spaces as candidates for the boundary datum also require that we assume μ to be a lower s -Ahlfors measure as well. In the sub-elliptic settings, ample supply of measures that satisfy both an upper and lower s -Ahlfors condition are found in [13] and the references therein.

Theorem 2.3. (Embedding a Besov space into a Lebesgue space) *Given a bounded set $U \subset \mathbb{R}^n$ having characteristic local parameters C_1, R_o , and local homogeneous dimension Q , let $\Omega \subset \bar{\Omega} \subset U$ be an open set with $\text{diam } \Omega < R_o/2$. Let $p \geq 1$, $0 < \beta < 1$. Suppose μ is a lower s -Ahlfors measure with*

$$0 < s \leq n + \beta p, \quad s < Q - \beta p,$$

and such that $\text{supp } \mu = F \subset \Omega$. There exists a continuous embedding

$$B_\beta^p(F, d\mu) \subset L^q(\Omega, d\mu), \quad \text{where } q = p \frac{Q - s}{Q - s - \beta p},$$

and, in fact, for $f \in B_\beta^p(F, d\mu)$ one has

$$\|f\|_{L^q(\Omega, d\mu)} \leq C \left\{ \left(1 + \frac{\text{diam}(\Omega)^\beta}{\mu(F)^{\beta/(Q-s)}} \right) \mathcal{N}_\beta^p(f, F, d\mu) + \frac{1}{\mu(F)^{\beta/(Q-s)}} \|f\|_{L^p(\Omega, d\mu)} \right\},$$

where $C = C(\Omega, C_1, R_o, p, \beta, s, M) > 0$. Furthermore,

$$\begin{aligned} & \left(\int_\Omega |f(x) - f_{\Omega, \mu}|^q d\mu(x) \right)^{\frac{1}{q}} \\ & \leq C \left(\int_F \int_F |f(x) - f(y)|^p \frac{d(x, y)^{s-\beta p}}{|B(x, d(x, y))|} d\mu(y) d\mu(x) \right)^{\frac{1}{p}}, \end{aligned}$$

where $f_{\Omega, \mu}$ denotes the average $\frac{1}{\mu(\Omega)} \int_\Omega f d\mu$.

A wide class of domains to which our results hold is the following.

Definition 2.4. ([13, 28]) An open set $\Omega \subset \mathbb{R}^n$ is called an $X - (\epsilon, \delta)$ -domain if there exist $0 < \delta \leq \infty$, $0 < \epsilon \leq 1$, such that for any pair of points $p, q \in \Omega$, if $d(p, q) \leq \delta$, then one can find a continuous, rectifiable curve $\gamma : [0, T] \rightarrow \Omega$, for which $\gamma(0) = p$, $\gamma(T) = q$, and

$$l(\gamma) \leq \frac{1}{\epsilon} d(p, q), \quad d(z, \partial\Omega) \geq \epsilon \min \{d(p, z), d(z, q)\} \quad \text{for all } z \in \{\gamma\}.$$

The following compact embedding theorem [17, Theorem 1.28] plays an important role in our proof.

Theorem 2.5. (Compact embedding) *Let $\Omega \subset \mathbb{R}^n$ be an $X - (\epsilon, \delta)$ domain with $\text{diam}(\Omega) < \frac{R_0}{2}$. Then, one has the following:*

- (I) *The embedding $BV_X(\Omega, dx) \hookrightarrow L^q(\Omega, dx)$ is compact for any $1 \leq q < \frac{Q}{Q-1}$.*
- (II) *For any $1 \leq p < Q$ the embedding $\mathcal{L}^{1,p}(\Omega, dx) \hookrightarrow L^q(\Omega, dx)$ is compact provided that $1 \leq q < \frac{Qp}{Q-p}$.*
- (III) *For any $Q \leq p < \infty$ and any $1 \leq q < \infty$, the embedding $\mathcal{L}^{1,p}(\Omega, dx) \hookrightarrow L^q(\Omega, dx)$ is compact.*

As an easy consequence of the above theorem we have

Theorem 2.6. *With the same assumptions in Theorem 2.5, for any $1 \leq p < \infty$ the embedding $\mathcal{L}^{1,p}(\Omega, dx) \hookrightarrow L^p(\Omega, dx)$ is compact.*

We can deduce from Theorem 2.6 easily the following

Corollary 2.7. *For any $1 < p < \infty$, if $\{u_h\}_{h=1}^\infty$ is a sequence in $\mathcal{L}^{1,p}(\Omega, dx)$ such that $u_h \rightharpoonup u$ for some $u \in \mathcal{L}^{1,p}(\Omega, dx)$ then $u_h \rightarrow u$ in $L^p(\Omega, dx)$ hence $u_h \rightarrow u$ in $L^1(\Omega, dx)$ also. (We note explicitly the notation \rightharpoonup means weak convergence whereas \rightarrow means convergence in norm).*

The proof of this fact is rather standard but we include it here for the sake of convenience of the reader.

Proof. The weak convergence assumption together with the Banach–Steinhaus (uniform boundedness principle) theorem implies that $\{u_h\}_{h=1}^\infty$ is bounded in $\mathcal{L}^{1,p}(\Omega, dx)$. Now Theorem 2.6 implies that if $\{u_{h_j}\}$ is any sub-sequence of $\{u_h\}$, $\{u_{h_j}\}$ has a sub-sequence we denote it by $\{v_{h_j}\}$ such that $v_{h_j} \rightarrow v \in L^p(\Omega, dx)$ for some function v . Since $u_h \rightharpoonup u$ in $L^p(\Omega, dx)$, we must have $u = v$ and the original sequence $\{u_h\}$ converge in norm to u in $L^p(\Omega, dx)$. \square

The traces of Sobolev functions on the boundary of a domain is a delicate matter. It was the purpose of [13] to develop such a theory in the setting of a Carnot–Carathéodory space. It is also an indispensable tool in dealing with the Neumann problem, even the non-linear version. We thus recall [13, Theorem 10.6]:

Theorem 2.8. (Trace theorem on the boundary) *Let $U \subset \mathbb{R}^n$ be a bounded set with characteristic local parameters C_1, R_o , and let $p > 1$. There is $\sigma = \sigma(X, U) > 0$ such that, if $\Omega \subset U$ is a bounded $X - (\epsilon, \delta)$ -domain with $\text{rad}(\Omega) > 0$, $\text{diam}(\Omega) < \frac{R_o}{2\sigma}$, $\text{dist}(\Omega, \partial U) > R_o$, and μ is an upper s -Ahlfors measure for*

some $0 < s < p$, having $\text{supp } \mu \subseteq \partial\Omega$, then for every $0 < \beta \leq 1 - s/p$ there exist a linear operator

$$\text{Tr} : \mathcal{L}^{1,p}(\Omega, dx) \rightarrow B_\beta^p(\partial\Omega, d\mu),$$

and a constant $C = C(U, p, s, M, \beta, \epsilon, \delta, \text{rad}(\Omega)) > 0$, such that

$$\|\text{Tr } f\|_{B_\beta^p(\partial\Omega, d\mu)} \leq C \|f\|_{\mathcal{L}^{1,p}(\Omega, dx)}. \tag{2.1}$$

Furthermore, if $f \in C^\infty(\bar{\Omega}) \cap \mathcal{L}^{1,p}(\Omega, dx)$, then $\text{Tr } f = f$ on $\partial\Omega$.

Our final pillar is the following result [17, Corollary 1.6] specialized to our setting:

Theorem 2.9. *Let Ω be an $X - (\epsilon, \delta)$ domain, $1 \leq p < Q$. There exists some constant $C = C(\epsilon, \delta, p, n) > 0$ such that for any $1 \leq k \leq \frac{Q}{Q-p}$ and any $u \in \mathcal{L}^{1,p}(\Omega)$*

$$\left(\frac{1}{|\Omega|} \int_\Omega |u - u_\Omega|^{kp} dx \right)^{\frac{1}{kp}} \leq C \text{diam}(\Omega) \left(\frac{1}{|\Omega|} \int_\Omega |Xu|^p dx \right)^{\frac{1}{p}}.$$

3. Existence and uniqueness of solutions to the Neumann problem

Let Ω be an $X - (\epsilon, \delta)$ domain. Consider the functional $J_p : \mathcal{L}^{1,q}(\Omega, dx) \rightarrow \bar{\mathbb{R}}$ given by

$$J_p(u) = \int_\Omega \frac{1}{p} |Xu(x)|^p + f(x)u(x) dx - \langle \nu, \text{Tr}(u) \rangle. \tag{3.1}$$

Note that in a priori, unless $q = p$ or $u \in C^\infty(\bar{\Omega}) \cap \mathcal{L}^{1,q}(\Omega, dx)$, the functional J_p can take on the value of $\pm \infty$ and the assumption that Ω is an $X - (\epsilon, \delta)$ domain is necessary for the trace operator tr (as in Definition 1.1) to be defined. Hence, this assumption is needed in the above definition and throughout the paper.

Lemma 3.1. *Let Ω be an $X - (\epsilon, \delta)$ domain. For any $q \geq 1$ the functional J_p given by (3.1) is convex on $\mathcal{L}^{1,q}(\Omega, dx)$. If in addition we assume that $|\Omega| > 0$ and $p > 1$ then the functional J_p is strictly convex in the subspace $\tilde{\mathcal{L}}^{1,q}(\Omega, dx)$.*

Proof. It suffices to establish the proposition for the non-linear part of J_p , namely

$$I(u) = \frac{1}{p} \int_\Omega |Xu(x)|^p dx. \tag{3.2}$$

It is an elementary fact that the function $g : \mathbb{R}^m \rightarrow \mathbb{R}$ given by $g(z) = |z|^p$ is convex for $1 \leq p < \infty$ (see e.g., [32, Theorem 5.1] and the remark there). This implies that any functional of the form $\mathcal{F}(u) = \int_\Omega g(Xu)(x) dx$ is convex. Hence, I is convex. We turn to the second part of the proposition. For $p > 1$ the function g is strictly convex. Hence for any $0 < t < 1$ and any $u, v \in \tilde{\mathcal{L}}^{1,q}(\Omega, dx)$ we have $g(t(Xu) + (1-t)(Xv)) < tg(Xu) + (1-t)g(Xv)$ unless $Xu = Xv$, that

is, for all $i = 1, \dots, m$, $X_i(u - v) = 0$. To continue, observe that the system of Hörmander vector field $X = (X_1, \dots, X_m)$ satisfy the property that

$$\forall i = 1, \dots, m, \quad \text{for a.e. } x \in \Omega : X_i g(x) = 0 \quad \text{implies} \quad g(x) = \text{constant} \quad \text{for a.e. } x \in \Omega .$$

This imply $u = v + c$ for some constant c . Since $u, v \in \tilde{\mathcal{L}}^{1,p}(\Omega, dx)$, taking into account that $|\Omega| > 0$, we see that c must be zero. \square

Remark 3.2. Two remarks are in order.

- (i) The uniqueness of minimizer would follow from the strict convexity of J_p , that is, minimizer would be unique in the spaces $\tilde{\mathcal{L}}^{1,q}(\Omega, dx)$.
- (ii) Due to the compatibility condition (1.2), we have for any constant $c \in \mathbb{R}$ and any $u \in \mathcal{L}^{1,q}(\Omega, dx)$, $J_p(u+c) = J_p(u)$. Hence, if a (unique) minimizer $u_o \in \tilde{\mathcal{L}}^{1,q}(\Omega, dx)$ exists for J_p , then u_o remains to be a minimizer for J_p in $\mathcal{L}^{1,q}(\Omega, dx)$ since for any $u \in \mathcal{L}^{1,q}(\Omega, dx)$, $J_p(u) = J_p(u - u_o) \geq J_p(u_o)$ and that for any constant $c \in \mathbb{R}$, $u_o + c$ is also a minimizer for J_p in $\mathcal{L}^{1,q}(\Omega, dx)$.

Next, we turn to the (sequential) lower semi-continuity of the functional J_p . The following result (in many equivalent form) from Calculus of Variations is valuable to us and is available from many sources. For the sake of our purpose, we apply the following theorem from [19, Theorem 4.5].

Theorem 3.3. *Let Ω be an open set in \mathbb{R}^n , M a closed set in \mathbb{R}^N , and let $F(x, u, z)$ be a function defined in $\Omega \times M \times \mathbb{R}^l$ such that*

- (i) *F is a Carathéodory function, that is measurable in x for every $(u, z) \in M \times \mathbb{R}^l$ and continuous in (u, z) for almost every $x \in \Omega$.*
- (ii) *$F(x, u, z)$ is convex in z for almost every $x \in \Omega$ and for every $u \in M$.*
- (iii) *$F \geq 0$.*

Let $u_h, u \in L^1(\Omega, M)$, $z_h, z \in L^1(\Omega, \mathbb{R}^l)$ and assume that $u_h \rightarrow u$ and $z_h \rightharpoonup z$ in $L^1_{loc}(\Omega, dx)$. Then,

$$\int_{\Omega} F(x, u, z) dx \leq \liminf_{h \rightarrow \infty} \int_{\Omega} F(x, u_h, z_h) dx$$

Corollary 3.4. *Let Ω be an $X - (\epsilon, \delta)$ domain with $\text{diam}(\Omega) < \frac{R_o}{2}$ (and $f \in L^q(\Omega, dx)$). The functional J_p defined in (3.1) is (sequentially) lower semi-continuous on the spaces $\mathcal{L}^{1,q}(\Omega, dx)$ for any $1 \leq q < \infty$ with respect to the weak topology, that is, if $\{u_h\}_{h=1}^{\infty}$ is a sequence in $\mathcal{L}^{1,q}(\Omega, dx)$ such that $u_h \rightharpoonup u$ for some $u \in \mathcal{L}^{1,q}(\Omega, dx)$ then*

$$J_p(u) \leq \liminf_{h \rightarrow \infty} J_p(u_h) .$$

Proof. By assumption $u \in \mathcal{L}^{1,q}(\Omega, dx)$, $f \in L^q(\Omega, dx)$ and $\nu \in B^q_{1-\frac{s}{q}}(\partial\Omega, d\mu)^*$ the linear part of J is bounded by Hölder’s inequality and the trace inequality (2.1) therefore continuous. The sum of two lower semi-continuous function is lower semi-continuous. Hence, it suffices to establish the lower semi-continuity of the non-linear part of J_p , namely I defined in (3.2). Let $u, \{u_h\}_{h=1}^{\infty}$ be as in the hypothesis. By Corollary 2.7 we have $u_h \rightarrow u$ in $L^1(\Omega, dx)$ and also $X_j u_h \rightharpoonup X_j u$ in $L^q_{loc}(\Omega, dx)$ hence also in $L^1(\Omega, dx)$ for $j = 1, \dots, m$. Now we

apply Theorem 3.3 with $N = 1, M = \mathbb{R}, l = m, z = Xu = (X_1u, \dots, X_mu)$ and $F(x, u, z) = |z|^p$ to reach the conclusion. \square

Lemma 3.5. *Let $1 \leq q < \infty, f \in L^{q'}(\Omega, dx), \Omega$ be $X - (\epsilon, \delta)$ domain. A function $u \in \mathcal{L}^{1,q}(\Omega, dx)$ minimizes the functional J_p given by (3.1) if and only if it is a variational solution to the Neumann problem (1.1) in the sense of Definition 1.1.*

Proof. Let $u \in \mathcal{L}^{1,q}(\Omega, dx)$ be a minimizer of J_p . We have for any $\epsilon > 0$ and any $\phi \in \mathcal{L}^{1,q}(\Omega, dx)$ the function $\psi(\epsilon) = J_p(u + \epsilon\phi)$ reaches a local minimum at $\epsilon = 0$. Therefore $\psi'(0) = 0$. On the other hand, differentiating under the integral sign yields

$$\psi'(\epsilon) = \int_{\Omega} |Xu + \epsilon X\phi|^{p-2} \langle Xu + \epsilon X\phi, X\phi \rangle + f(x)\phi(x) dx - \langle \nu, \phi \rangle.$$

Hence

$$0 = \psi'(0) = \int_{\Omega} |Xu|^{p-2} \langle Xu, X\phi \rangle + f(x)\phi(x) dx - \langle \nu, \phi \rangle,$$

which is (1.3) in Definition 1.1.

We now prove the converse. Let $u \in \mathcal{L}^{1,q}(\Omega, dx)$ be a variational solution in the sense of Definition 1.1. For any $v \in \mathcal{L}^{1,q}(\Omega, dx)$ we let $\phi = u - v \in \mathcal{L}^{1,q}(\Omega, dx)$ in (1.3), adding and subtracting the terms $\int_{\Omega} \frac{1}{p'} |Xu|^p dx, \int_{\Omega} \frac{1}{p} |Xv|^p dx$ we obtain the identity

$$\begin{aligned} J_p(u) &= J_p(v) - \int_{\Omega} \frac{1}{p} |Xv|^p + \frac{1}{p'} |Xu|^p - |Xu|^{p-2} \langle Xu, Xv \rangle dx \\ &= J_p(v) - \int_{\Omega} \frac{1}{p} |Xv|^p - \frac{1}{p} |Xu|^p - |Xu|^{p-2} \langle Xu, Xv - Xu \rangle dx. \end{aligned}$$

To show that the last term in the above is non-negative, we recall that the convexity of the function $g(z) = |z|^p$ for $p \geq 1$ can be rephrase as: for every $z, w \in \mathbb{R}^m, z \neq 0$ then

$$\frac{1}{p} |w|^p \geq \frac{1}{p} |z|^p + |z|^{p-2} \sum_{i=1}^m z_i (w_i - z_i). \tag{3.3}$$

Letting $z_i = X_iu$ and $w_i = X_iv$ in the inequality (3.3) and integrating we have

$$\int_{\Omega} \frac{1}{p} |Xv|^p - \frac{1}{p} |Xu|^p - |Xu|^{p-2} \langle Xu, Xv - Xu \rangle dx \geq 0.$$

Hence, we have $J_p(u) \leq J_p(v)$ and this completes the proof. \square

Our final result leading to the existence of minimizers of J_p [hence, variational solutions to the Neumann problem in $\mathcal{L}^{1,q}(\Omega, dx)$] is the following

Lemma 3.6. *Let Ω be an $X - (\epsilon, \delta)$ domain. For any $1 \leq q \leq p$ and $f \in L^{q'}(\Omega, dx)$, the functional J_p is coercive in $\tilde{\mathcal{L}}^{1,q}(\Omega, dx)$, that is, if $\{u_h\}$ is a sequence in $\tilde{\mathcal{L}}^{1,q}(\Omega, dx)$ such that $\lim_{h \rightarrow \infty} \|u_h\|_{\tilde{\mathcal{L}}^{1,q}(\Omega, dx)} = \lim_{h \rightarrow \infty} \|Xu_h\|_{L^q(\Omega, dx)} = \infty$ then $\lim_{h \rightarrow \infty} J_p(u_h) = \infty$.*

Proof. Observe that due to the Trace Theorem 2.8 and the fact that $f \in L^{q'}(\Omega, dx)$, the linear part of J_p is bounded. Therefore, it suffices to establish the lemma for the non-linear part of J_p , namely I given by (3.2). Let $\{u_h\}$ be a sequence in $\tilde{\mathcal{L}}^{1,q}(\Omega, dx)$ such that $\lim_{h \rightarrow \infty} \|Xu_h\|_{L^q(\Omega, dx)} = \infty$. By Hölder's inequality, we have for $1 \leq q \leq p$ that $\|Xu_h\|_{L^q(\Omega, dx)} \leq |\Omega|^{\frac{1}{q} - \frac{1}{p}} \|Xu_h\|_{L^p(\Omega, dx)}$ hence $\lim_{h \rightarrow \infty} I(u_h) = \lim_{h \rightarrow \infty} \|Xu_h\|_{L^p(\Omega, dx)} = \infty$ as well. \square

We now come to our main result which follows from a standard line of argument of the Calculus of Variations, Lemma 3.6 and Corollary 3.4. We include the proof for the sake of completeness.

Theorem 3.7. *Let $1 < q \leq p$, Ω be an $X - (\epsilon, \delta)$ domain with $\text{diam}(\Omega) < R_o/2$, $|\Omega| > 0$, $f \in L^{q'}(\Omega, dx)$, $\nu \in B_{1-\frac{\epsilon}{q}}^q(\partial\Omega, d\mu)^*$ satisfying the compatibility condition (1.2). The functional J_p given by (3.1) has a unique minimizer in $\mathcal{L}^{1,q}(\Omega, dx)$.*

Proof. Let $l = \inf\{J_p(u) \mid u \in \tilde{\mathcal{L}}^{1,q}(\Omega, dx)\} > -\infty$ since the linear part of J_p is bounded and the non-linear part I given by (3.2) is non-negative. Let $\{u_h\} \subset \tilde{\mathcal{L}}^{1,q}(\Omega, dx)$ be a minimizing sequence, that is $\lim_{h \rightarrow \infty} J_p(u_h) = l$. Clearly, $J_p(u_h)$ is bounded and hence Lemma 3.6 implies that $\{u_h\}$ is a bounded sequence in $\tilde{\mathcal{L}}^{1,q}(\Omega, dx)$. Since for $q > 1$, the balls in $\tilde{\mathcal{L}}^{1,q}(\Omega, dx)$ are weakly compact, $\{u_h\}$ contains a sub-sequence (still denoted by $\{u_h\}$) and $\exists u_o \in \tilde{\mathcal{L}}^{1,q}(\Omega, dx)$ such that $u_h \rightharpoonup u_o$. Corollary 3.4 then imply

$$l \leq J_p(u_o) \leq \liminf_{h \rightarrow \infty} J_p(u_h) = \lim_{h \rightarrow \infty} J_p(u_h) = l,$$

hence $J_p(u_o) = l$. Now from Remark 3.2, u_o remains to be a minimizer of J_p in $\mathcal{L}^{1,q}(\Omega, dx)$. \square

Finally, we now come to the proof of our main result.

Proof of Theorem 1.4. Theorem 3.7 and Lemma 3.5 yield a unique solution $u \in \tilde{\mathcal{L}}^{1,q}(\Omega, dx)$ in the sense of Definition 1.1. To establish the estimate (1.5) we take $\phi = u$ in Definition 1.1 and recalling Remark 1.3 we have

$$\begin{aligned}
 \|Xu\|_{L^p(\Omega, dx)}^p &= \int_{\Omega} |Xu|^{p-2} \langle Xu, Xu \rangle dx \\
 &\leq |\langle \nu, tr(u) \rangle| + \left| \int_{\Omega} f u dx \right| \\
 &\leq \|\nu\|_{B_{1-s/q}^q(\partial\Omega, d\mu)^*} \|tr(u)\|_{B_{1-s/q}^q(\partial\Omega, d\mu)} \\
 &\quad + \|f\|_{L^{q'}(\Omega, dx)} \|u\|_{L^q(\Omega, dx)} \\
 (\text{by Theorems 2.8, 2.9}) &\leq C \|\nu\|_{B_{1-s/q}^q(\partial\Omega, d\mu)^*} \|u\|_{\tilde{\mathcal{L}}^{1,q}(\Omega, dx)} \\
 &\quad + \|f\|_{L^{q'}(\Omega, dx)} \|Xu\|_{L^q(\Omega, dx)} \\
 (\text{by Remark 1.3}) &\leq C (\|\nu\|_{B_{1-s/q}^q(\partial\Omega, d\mu)^*} + \|f\|_{L^{q'}(\Omega, dx)}) \|Xu\|_{L^q(\Omega, dx)} \\
 (\text{since } q \leq p) &\leq C (\|\nu\|_{B_{1-s/q}^q(\partial\Omega, d\mu)^*} + \|f\|_{L^{q'}(\Omega, dx)}) \|Xu\|_{L^p(\Omega, dx)}
 \end{aligned}$$

and therefore

$$\|Xu\|_{L^p(\Omega, dx)} \leq C (\|\nu\|_{B_{1-s/q}^q(\partial\Omega, d\mu)^*} + \|f\|_{L^{q'}(\Omega, dx)})^{\frac{1}{p-1}} .$$

In view of Remark 1.3 again this shows $u \in \tilde{\mathcal{L}}^{1,p}(\Omega, dx)$ and the proof is now complete. □

Remark 3.8. Some intermediate results such as Lemmas 3.1, 3.5 and Corollary 3.4 hold for any $q \geq 1$. The limitation of $1 < q \leq p$ comes in Lemma 3.6 and therefore also in Theorem 1.4. The case for $q < p$ can also be inferred from the case $q = p$ due to the containment of the spaces $B_{1-s/q}^q(\partial\Omega, d\mu)^* \subset B_{1-s/p}^p(\partial\Omega, d\mu)^*$ and $\tilde{\mathcal{L}}^{1,p}(\Omega, dx) \subset \tilde{\mathcal{L}}^{1,q}(\Omega, dx)$.

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