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Liouville-type theorems with finite Morse index for semilinear Δ_λ-Laplace operators

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Abstract. In this paper we study solutions, possibly unbounded and sign-changing, of the following equation

$$-\Delta_\lambda u = |x|_\lambda^a |u|^{p-1}u \quad \text{in } \mathbb{R}^n,$$

where $n \geq 1$, $p > 1$, $a \geq 0$ and Δ_λ is a strongly degenerate elliptic operator, the functions $\lambda = (\lambda_1, \ldots, \lambda_k) : \mathbb{R}^n \to \mathbb{R}^k$ satisfies some certain conditions, and $|.|_\lambda$ the homogeneous norm associated to the Δ_λ-Laplacian. We prove various Liouville-type theorems for smooth solutions under the assumption that they are stable or stable outside a compact set of $\mathbb{R}^n$. First, we establish the standard integral estimates via stability property to derive the nonexistence results for stable solutions. Next, by mean of the Pohozaev identity, we deduce the Liouville-type theorem for solutions stable outside a compact set.

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1. Introduction and main results

In the last decades, the nonnegative solutions of the semilinear elliptic equation

$$-\Delta u = |u|^{p-1}u \quad \text{in } \mathbb{R}^n,\ p > 1, \tag{1.1}$$

has been studied by Gidas and Spruck [5]. They proved that if $1 < p < \frac{n+2}{n-2}$, then the above equation only has the trivial solution $u \equiv 0$ and this result is optimal. In an elegant paper, Farina [3] proved that nontrivial finite Morse index solutions (whether positive or sign changing) to (1.1) exists if and only if $p \geq p_c(n)$ and $n \geq 11$, or $p = \frac{n+2}{n-2}$ and $n \geq 3$, where $p_c(n)$ is the so-called Joseph-Lundgren exponent. The study of stable solutions in the Hénon type elliptic equation: $-\Delta u = |x|^a|u|^{p-1}u$, in $\mathbb{R}^n$, $p > 1$ and $a > -2$ has been studied recently, Wang and Ye [14] gave a complete classification of stable weak solutions and those of finite Morse index solutions.

In the past years, the Liouville property has been refined considerably and emerged as one of the most powerful tools in the study of initial and boundary value problems for nonlinear PDEs. It turns out that one can obtain from Liouville-type theorems a variety of results on qualitative properties of solutions such as universal, pointwise, a priori estimates of local solutions; universal and singularity estimates; decay estimates; blow-up rate of solutions of nonstationary problems, etc., see [12] and references therein.

Liouville-type theorems for degenerate elliptic equations have been attracted the interest of many mathematicians. The classical Liouville theorem was generalized to p-harmonic functions on the whole space $\mathbb{R}^n$ and on exterior domains by Serrin and Zou [13], see also [2] for related results. The Liouville theorems for some linear degenerate elliptic operators such as X-elliptic operators, Kohn–Laplacian (and more general sublaplacian on Carnot groups) and degenerate Ornstein–Uhlenbeck operators were proved in [7–9].

More recently, Yu [15] studied the equation

$$-L_\alpha u = f(u) \quad \text{in} \quad \mathbb{R}^{n_1} \times \mathbb{R}^{n_2},$$

where $L_\alpha u = \Delta_x u + (1+\alpha)^2 |x|^{2\alpha} \Delta_y u$, $\alpha > 0$ and $Q = n_1 + (1+\alpha)n_2$ is the *homogeneous dimension* of the space. Under some assumptions on the nonlinear term f , he showed that the above equation possesses no positive solutions and the main technique used is the moving plane method in the integral form.

In this paper, we are concerned with the Liouville-type theorems for the following equation

$$-\Delta_\lambda u = |x|_\lambda^a |u|^{p-1} u, \quad \text{in } \mathbb{R}^n := \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \ldots \times \mathbb{R}^{n_k}, \tag{1.2}$$

where $n \geq 1$, $a \geq 0$, $p > 1$,

$$\Delta_\lambda = \lambda_1^2 \Delta_{x^{(1)}} + \ldots + \lambda_k^2 \Delta_{x^{(k)}}, \quad |x|_\lambda := \left(\sum_{j=1}^{k} \prod_{i \neq j} \lambda_i^2(x) \epsilon_j^2 |x^{(j)}|^2 \right)^{\frac{1}{2\sigma}},$$

$\sigma = 1 + \sum_{i=1}^{k} (\epsilon_i - 1)$, $1 \leq \epsilon_1 \leq \ldots \leq \epsilon_k$, $x = (x^{(1)}, \ldots, x^{(k)}) \in \mathbb{R}^n$. Here the functions $\lambda_i : \mathbb{R}^n \to \mathbb{R}$ are continuous, strictly positive and of class C^1 outside the coordinate hyperplanes, i.e. $\lambda_i > 0$, $i = 1, \ldots, k$ in $\mathbb{R}^n \setminus \prod$, where $\prod = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \prod_{i=1}^n x_i = 0\}$, and $\Delta_{x^{(i)}}$ denotes the classical Laplacian in $\mathbb{R}^{n_i}$, $i = 1, \ldots, k$. As in [4] we assume that λ_i satisfy the following properties:

- (H_1) $\lambda_1(x) = 1$, $\lambda_i(x) = \lambda_i(x^{(1)}, \ldots, x^{(i-1)})$, $i = 2, \ldots, k$.
- (H_2) For every $x \in \mathbb{R}^n$, $\lambda_i(x) = \lambda_i(x^*)$, $i = 1, \ldots, k$, where $x^* = (|x^{(1)}|, \ldots, |x^{(k)}|)$ if $x = (x^{(1)}, \ldots, x^{(k)})$.
- (H_3) There exists a group of dilations $\{\delta_t\}_{t>0}$

$$\delta_t : \mathbb{R}^n \to \mathbb{R}^n, \ \delta_t(x) = \delta_t(x^{(1)}, \ldots, x^{(k)}) = (t^{\epsilon_1} x^{(1)}, \ldots, t^{\epsilon_k} x^{(k)}),$$

where $1 \leq \epsilon_1 \leq \epsilon_2 \leq \ldots \leq \epsilon_k$, such that λ_i is δ_t-homogeneous of degree $\epsilon_i - 1$, i.e.

$$\lambda_i(\delta_t(x)) = t^{\epsilon_i - 1}\lambda_i(x), \ \ \forall\, x \in \mathbb{R}^n, \ t > 0, \ i = 1, \ldots, k.$$

This implies that the operator Δ_λ is δ_t-homogeneous of degree two, i.e.

$$\Delta_\lambda(u(\delta_t(x))) = t^2(\Delta_\lambda u)(\delta_t(x)), \ \ \forall\, u \in C^\infty(\mathbb{R}^n).$$

We denote by Q the *homogeneous dimension* of $\mathbb{R}^n$ with respect to the group of dilations $\{\delta_t\}_{t>0}$, i.e.

$$Q := \epsilon_1 n_1 + \epsilon_2 n_2 + \ldots + \epsilon_k n_k.$$

The Δ_λ-Laplace operator was first introduced by Franchi and Lanconelli [4], and recently reconsidered in [6] under an additional assumption that the operator is homogeneous of degree two with respect to a group dilation in $\mathbb{R}^n$. It was proved in [1], that the autonomous case, i.e. $a = 0$, (1.2) has no positive classical solution if $1 < p \leq \frac{Q}{Q-2}$, with $Q = \epsilon_1 + \epsilon_2 + \ldots + \epsilon_n$, ($n_i = 1$, $i = 1, \ldots, n$).

The Δ_λ-operator contains many degenerate elliptic operators. We now give some examples of Δ_λ-Laplace operators (see also [10]). We use the following notation: we split $\mathbb{R}^n$ as follows $\mathbb{R}^n = \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_k}$ and write

$$x = (x^{(1)}, \ldots, x^{(k)}), \ x^{(i)} = (x_1^{(i)}, \ldots, x_{n_i}^{(i)}) \in \mathbb{R}^{n_i},$$
$$|x^{(i)}|^2 = \sum_{j=1}^{n_i} |x_j^{(i)}|^2, \quad i = 1, 2, \ldots, k.$$

We denote the classical Laplace operator in $\in \mathbb{R}^{n_i}$ by

$$\Delta_{x^{(i)}} = \sum_{j=1}^{n_i} \partial^2_{x_j^{(i)}}.$$

Example 1. Let α be a real positive constant and $k = 2$. We consider the Grushin-type operator

$$\Delta_\lambda = \Delta_x + |x|^{2\alpha}\Delta_y,$$

where $\lambda = (\lambda_1, \lambda_2)$ with

$$\lambda_1(x) = 1, \quad \lambda_2(x) = |x^{(1)}|^\alpha, \quad x \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}.$$

Our group of dilations is

$$\delta_t(x) = \delta_t(x^{(1)}, x^{(2)}) = (tx^{(1)}, t^{\alpha+1}x^{(2)}),$$

and the homogenous dimension with respect to $(\delta_t)_{t>0}$ is $Q = n_1 + (\alpha+1)n_2$.

Example 2. Given a multi-index $\alpha = (\alpha_1, \ldots, \alpha_{k-1})$, $\alpha_j \geq 1$, $j = 1, \ldots, k-1$, define

$$\Delta_\alpha := \Delta_{x^{(1)}} + |x^{(1)}|^{2\alpha_1}\Delta_{x^{(2)}} + \cdots + |x^{(k-1)}|^{2\alpha_{k-1}}\Delta_{x^{(k)}}.$$

Then $\Delta_\alpha = \Delta_\lambda$ with $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\lambda_i = |x^{(i-1)}|^{\alpha_{i-1}}$, $i = 1, \ldots, k$. Here we agree to let $|x^{(0)}|^{\alpha_0} = 1$. A group of dilations for which λ satisfies (H_3) is given by

$$\delta_t : \mathbb{R}^n \to \mathbb{R}^n,\ \delta_t(x) = \delta_t(x^{(1)}, \ldots, x^{(k)}) = (t^{\epsilon_1} x^{(1)}, \ldots, t^{\epsilon_k} x^{(k)}),$$

with $\epsilon_1 = 1$ and $\epsilon_i = \alpha_{i-1}\epsilon_{i-1} + 1$, $i = 2, \ldots, k$. In particular, if $\alpha_1 = \ldots = \alpha_{k-1} = 1$, the operator Δ_α and the dilation δ_t becomes, respectively

$$\Delta_\alpha = \Delta_{x^{(1)}} + |x^{(1)}|^2 \Delta_{x^{(2)}} + \ldots + |x^{(k-1)}|^2 \Delta_{x^{(k)}},$$

and

$$\delta_t(x) = (tx^{(1)}, t^2 x^{(2)}, \ldots, t^k x^{(k)}).$$

Example 3. Let α, β and γ be positive real constants. For the operator

$$\Delta_\lambda = \Delta_{x^{(1)}} + |x^{(1)}|^{2\alpha} \Delta_{x^{(2)}} + |x^{(1)}|^{2\beta} |x^{(2)}|^{2\gamma} \Delta_{x^{(3)}},$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ with

$$\lambda_1(x) = 1, \quad \lambda_2(x) = |x^{(1)}|^\alpha, \quad \lambda_3(x) = |x^{(1)}|^\beta |x^{(2)}|^\gamma, \quad x \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3},$$

we find the group of dilations

$$\delta_t(x) = \delta_t(x^{(1)}, x^{(2)}, x^{(3)}) = (tx^{(1)}, t^{\alpha+1} x^{(2)}, t^{\beta+(\alpha+1)\gamma+1} x^{(3)}).$$

The aim of the present paper was to establish the Liouville-type theorems with finite Morse index for the equation (1.2). In order to state our results we need the following:

Definition 1.1. We say that a solution u of (1.2) belonging to $C^2(\mathbb{R}^n)$

- is stable, if

$$Q_u(\psi) := \int_{\mathbb{R}^n} |\nabla_\lambda \psi|^2 - p \int_{\mathbb{R}^n} |x|_\lambda^a |u|^{p-1} \psi^2 \geq 0\ , \quad \forall\, \psi \in C_c^1(\mathbb{R}^n),$$

where $\nabla_\lambda = (\lambda_1 \nabla_{x^{(1)}}, \ldots, \lambda_k \nabla_{x^{(k)}})$.

- has Morse index equal to $K \geq 1$ if K is the maximal dimension of a subspace X_K of $C_c^1(\mathbb{R}^n)$ such that $Q_u(\psi) < 0$ for any $\psi \in X_K \backslash \{0\}$.
- is stable outside a compact set $\mathcal{K} \subset \mathbb{R}^n$ if $Q_u(\psi) \geq 0$ for any $\psi \in C_c^1(\mathbb{R}^n \backslash \mathcal{K})$.

Remark 1.1. a) Clearly, a solution stable if and only if its Morse index is equal to zero. **b)** It is well know that any finite Morse index solution u is stable outside a compact set $\mathcal{K} \subset \mathbb{R}^n$. Indeed, there exists $m_0 \geq 1$ and $X_{m_0} := \mathrm{Span}\{\phi_1, \ldots, \phi_{m_0}\} \subset C_c^1(\mathbb{R}^n)$ such that $Q_u(\phi) < 0$ for any $\phi \in X_{m_0} \backslash \{0\}$. Hence, $Q_u(\psi) \geq 0$ for every $\psi \in C_c^1(\mathbb{R}^n \backslash \mathcal{K})$, where $\mathcal{K} := \cup_{j=1}^{m_0} supp(\phi_j)$.

In the following, we state Liouville-type results for solutions $u \in C^2(\mathbb{R}^n)$ of (1.2). In what follows, we divide our study to stable solutions and solutions which are stable outside a compact set.

1.1. Stable solutions

To state the following result we need to introduce some notation. We set $\Gamma_M(p) = 2p - 1 + 2\sqrt{p(p-1)}$ and denote by $\Omega_R = B_1(0, R^{\epsilon_1}) \times B_2(0, R^{\epsilon_2}) \times \ldots \times B_k(0, R^{\epsilon_k})$, where $B_i(0, R^{\epsilon_i}) \subset \mathbb{R}^{n_i}$, $i = 1, \ldots, k$, the Euclidean balls of center 0 and radius R^{ϵ_i}.

Proposition 1.1. *Let $u \in C^2(\mathbb{R}^n)$ be a stable solution of (1.2). Then, for any $\gamma \in [1, \Gamma_M(p))$, there exists a positive constant C independent of R, such that*

$$\int_{\Omega_R} \left(|x|_\lambda^a |u|^{p+\gamma} + |\nabla_\lambda (|u|^{\frac{\gamma-1}{2}} u)|^2 \right) dx \leq C R^{Q - \frac{2(p+\gamma)+(\gamma+1)a}{p-1}}, \quad \text{for all } R > 0. \tag{1.3}$$

Proposition 1.1 provides an important estimate on the integrability of u and $\nabla_\lambda u$. As we will see, our nonexistence results will follow by showing that the right-hand side of (1.3) vanishes under the right assumptions on p when $R \to +\infty$. More precisely, as a corollary of Proposition 1.1, we can state our first Liouville type theorem.

Theorem 1.1. *Let $u \in C^2(\mathbb{R}^n)$ be a stable solution of (1.2) with,*

$$p_c(Q,a) = \begin{cases} +\infty & \text{if } Q \leq 10 + 4a, \\ \frac{(Q-2)^2 - 2(a+2)(a+Q) + 2\sqrt{(a+2)^3(a+2Q-2)}}{(Q-2)(Q-4a-10)} & \text{if } Q > 10 + 4a. \end{cases}$$

Then $u \equiv 0$.

1.2. Solutions which are stable outside a compact set

In this subsection we prove some integral identities extending to the Δ_λ setting the classical Pohozaev identity for semilinear Poisson equation [11]. Pohozaev identity has been extended by several authors to general elliptic equations and systems, both in Riemannian and sub-Riemannian context, see, e.g., [6] and the references therein. To prove our identities we closely follow the original procedure of Pohozaev, just replacing the vector field $P = \sum_{i=1}^n x_i \partial_{x_i}$ in [11], page 1410], by

$$T = \sum_{i=1}^k \epsilon_i x^{(i)} \nabla_{x^{(i)}},$$

in [6], page 4642], the generator of the group of dilation $(\delta_t)_{t\geq 0}$ in (H_3)(we say that T generates $(\delta_t)_{t\geq 0}$ since a function u is δ_t-homogeneous of degree m if and only if $Tu = mu$).

Proposition 1.2. *Let $u \in C^2(\mathbb{R}^n)$ be a solution of (1.2) and $\phi \in C_c^1(\Omega_R)$. If $T(|x|_\lambda) = |x|_\lambda$, then*

$$\begin{aligned} &\int_{\Omega_R} \left[\frac{Q-2}{2} |\nabla_\lambda u|^2 - \frac{Q+a}{p+1} |x|_\lambda^a |u|^{p+1} \right] \phi \\ &= \int_{\Omega_R} \left[\nabla_\lambda u \nabla_\lambda \phi T(u) + \left[-\frac{1}{2} |\nabla_\lambda u|^2 + \frac{|x|_\lambda^a}{p+1} |u|^{p+1} \right] T(\phi) \right]. \end{aligned} \tag{1.4}$$

Thanks to Proposition 1.2, we derive

Theorem 1.2. *Let $u \in C^2(\mathbb{R}^n)$ be a solution of (1.2) which is stable outside a compact set of $\mathbb{R}^n$, with*

$$p_s(Q,a) = \begin{cases} +\infty & \text{if} \quad Q \leq 2, \\ \frac{Q+2+2a}{Q-2} & \text{if} \quad Q > 2. \end{cases}$$

If $T(|x|_\lambda) = |x|_\lambda$, then $u \equiv 0$.

2. Example which satisfies $T(|x|_\lambda) = |x|_\lambda$

The degenerate elliptic operators we consider are of the form

$$\Delta_\lambda = \lambda_1^2 \Delta_{x^{(1)}} + \ldots + \lambda_k^2 \Delta_{x^{(k)}}.$$

We denote by $|x^{(j)}|$ the euclidean norm of $x^{(j)} \in \mathbb{R}^{n_j}$ and assume the functions λ_i are of the form

$$\lambda_i(x) = \prod_{j=1}^{k} |x^{(j)}|^{\alpha_{ij}}, \quad i = 1, \ldots, k, \tag{2.1}$$

such that
1) $\alpha_{ij} \geq 0$ for $i = 2, \ldots, k$, $j = 1, \ldots, i-1$.
2) $\alpha_{ij} = 0$ for $j \geq i$.
3) $\sum_{l=1}^{k} \epsilon_l \alpha_{jl} = \epsilon_j - 1$, $j = 1, \ldots, k$ with $1 = \epsilon_1 \leq \epsilon_2 \leq \ldots \leq \epsilon_k$.
Clearly, λ_i is δ_t-homogeneous of degree $\epsilon_i - 1$ with respect to a group of dilations $\{\delta_t\}_{t>0}$

$$\delta_t : \mathbb{R}^n \to \mathbb{R}^n, \; \delta_t(x) = \delta_t(x^{(1)}, \ldots, x^{(k)}) = (t^{\epsilon_1} x^{(1)}, \ldots, t^{\epsilon_k} x^{(k)}).$$

Now, using the relation $\sum_{l=1}^{k} \epsilon_l \alpha_{jl} = \epsilon_j - 1$, we get $T(|x|_\lambda) = |x|_\lambda$ is satisfied.

This paper is organized as follows. In section 3, we give the proof of Proposition 1.1 and Theorem 1.1. Section 4 is devoted to the proof of Proposition 1.2 and Theorem 1.2.

3. The Liouville theorem for stable solutions: proof of Theorem 1.1

In this section we prove all the results concerning the classification of stable solutions, i.e., Proposition 1.1 and Theorem 1.1. First, to prove Proposition 1.1, we need the following technical Lemma.

Let $R > 0$, $\Omega_{2R} = B_1(0, 2R^{\epsilon_1}) \times B_2(0, 2R^{\epsilon_2}) \times \ldots \times B_k(0, 2R^{\epsilon_k})$, where $B_i(0, 2R^{\epsilon_i}) \subset \mathbb{R}^{n_i}$, $i = 1, \ldots, k$, and consider k functions $\psi_{1,R}, \ldots, \psi_{k,R}$ such that

$$\psi_{1,R}(r^{(1)}) = \psi_1\left(\frac{r^{(1)}}{R^{\epsilon_1}}\right), \; \ldots, \; \psi_{k,R}(r^{(k)}) = \psi_k\left(\frac{r^{(k)}}{R^{\epsilon_k}}\right),$$

with $\psi_{1,R},\ \dots\ \psi_{k,R} \in C_c^{\infty}([0,+\infty)),\ 0 \le \psi_{1,R},\ \dots\ \psi_{k,R} \le 1,$

$$\psi_i(t) = \begin{cases} 1 & \text{in } [0,\ 1], \\ 0 & \text{in } [2,\ +\infty), \end{cases}$$

and for some constant $C > 0$ and $\psi_{1,R},\ \dots\ \psi_{k,R}$ satisfy

$$|\nabla_{x^{(1)}}\psi_{1,R}| \le CR^{-\epsilon_1},\ \dots,\ |\nabla_{x^{(k)}}\psi_{k,R}| \le CR^{-\epsilon_k},$$
$$|\Delta_{x^{(1)}}\psi_{1,R}| \le CR^{-2\epsilon_1},\ \dots,\ |\Delta_{x^{(k)}}\psi_{k,R}| \le CR^{-2\epsilon_k},$$

where $r^{(i)} = |x^{(i)}|$, $i = 1, \dots, k$.

Lemma 3.1. (1) *There exists a constant $C > 0$ independent of R such that*

a) $|\lambda_i(x)| \le CR^{\epsilon_i - 1},\ \forall\, x \in \Omega_{2R},\ i = 1, \dots, k.$

b) $|\nabla_\lambda \psi_R|^2 + |\Delta_\lambda \psi_R| \le CR^{-2}$, *where* $\psi_R = \prod_{i=1}^k \psi_{i,R}$.

(2) *The homogeneous norm, $|.|_\lambda$, is δ_t-homogeneous of degree one, i.e.*

$$|\delta_t(x)|_\lambda = t|x|_\lambda,\ \ \forall\, x \in \mathbb{R}^n,\ t > 0.$$

(3) *There exists a constant $C > 0$ independent of R such that*

$$|x|_\lambda \le CR,\ \forall\, x \in \Omega_{2R}.$$

Proof. Proof of (1) a). For any $x = (x^{(1)}, \dots, x^{(k)}) \in \Omega_{2R}$, we have $x^{(i)} \in B_i(0, 2R^{\epsilon_i})$, $i = 1, \dots, k$, this implies $\frac{|x^{(i)}|}{R^{\epsilon_i}} \le 2$, $i = 1, \dots, k$. Therefore, if we write

$$x = (x^{(1)}, \dots, x^{(k)}) = \left(R^{\epsilon_1} \times \frac{x^{(1)}}{R^{\epsilon_1}}, \dots, R^{\epsilon_k} \times \frac{x^{(k)}}{R^{\epsilon_k}}\right),$$

and let $y = (y^{(1)}, \dots, y^{(k)}) = \left(\frac{x^{(1)}}{R^{\epsilon_1}}, \dots, \frac{x^{(k)}}{R^{\epsilon_k}}\right)$, then $y \in \overline{\Omega_2}$. Hence by assumption (H_3) made on functions λ_i, we get

$$\begin{aligned} \lambda_i(x) &= \lambda_i(R^{\epsilon_1}y^{(1)}, \dots, R^{\epsilon_k}y^{(k)}) \\ &= R^{\epsilon_i - 1}\lambda_i(y^{(1)}, \dots, y^{(k)}) \\ &= R^{\epsilon_i - 1}\lambda_i(y). \end{aligned} \tag{3.1}$$

Moreover, since λ_i, $i = 1, \dots, k$ are continuous, then

$$|\lambda_i(y)| \le C,\ \ \forall\, y \in \overline{\Omega_2}. \tag{3.2}$$

Therefore, from (3.1) and (3.2), we obtain

$$|\lambda_i(x)| \le CR^{\epsilon_i - 1},\ \forall\, x \in \Omega_{2R},\ i = 1, \dots, k.$$

Proof of (1) b). Using assumption (H_2) made on functions λ_i, $i = 1, \dots, k$, with $r = (r^{(1)}, \dots, r^{(k)}) = (|x^{(1)}|, \dots, |x^{(k)}|)$, we have

$$\lambda_1(r) = 1,\ \lambda_i(r) = \lambda_i(r^{(1)}, \dots, r^{(i-1)}),\ \forall\, i = 2, \dots, k.$$

If we denote by $\psi_R = \prod_{i=1}^k \psi_{i,R}$, we get

$$\begin{aligned} \nabla_\lambda \psi_R &= (\lambda_1(r)\nabla_{x^{(1)}}\psi_R,\ \dots,\ \lambda_k(r)\nabla_{x^{(k)}}\psi_R) \\ &= \left(\lambda_1(r)\nabla_{x^{(1)}}\psi_{1,R}\prod_{i=2}^k \psi_{i,R}, \dots, \lambda_k(r)\nabla_{x^{(k)}}\psi_{k,R}\prod_{i=1}^{k-1}\psi_{i,R}\right), \end{aligned}$$

and

$$\begin{aligned}\Delta_\lambda \psi_R &= \lambda_1^2(r)\Delta_{x^{(1)}}\psi_R + \ldots + \lambda_k^2(r)\Delta_{x^{(k)}}\psi_R \\ &= \lambda_1^2(r)\Delta_{x^{(1)}}\psi_{1,R}\prod_{i=2}^{k}\psi_{i,R} + \ldots + \lambda_k^2(r)\Delta_{x^{(k)}}\psi_{k,R}\prod_{i=1}^{k-1}\psi_{i,R}.\end{aligned}$$

Since $|\lambda_i(r)| = |\lambda_i(x)| \leq CR^{\epsilon_i - 1}$, $\forall\, x \in \Omega_{2R}$, $i = 1, \ldots, k$, then there exists a constant $C > 0$ independent of R such that

$$|\nabla_\lambda \psi_R|^2 \leq CR^{-2} \text{ and } |\Delta_\lambda \psi_R| \leq CR^{-2}.$$

Proof of (2). Let $x \in \mathbb{R}^n$. The homogeneity of the functions λ_i implies that

$$\begin{aligned}|\delta_t(x)|_\lambda : &= \left(\sum_{j=1}^{k}\prod_{i\neq j}(\lambda_i(\delta_t(x)))^2\epsilon_j^2|t^{\epsilon_j}x^{(j)}|^2\right)^{\frac{1}{2(1+\sum_{i=1}^{k}(\epsilon_i-1))}} \\ &= \left(\sum_{j=1}^{k}\prod_{i\neq j}t^{2\epsilon_j}t^{2(\epsilon_i-1)}(\lambda_i(x))^2\epsilon_j^2|x^{(j)}|^2\right)^{\frac{1}{2(1+\sum_{i=1}^{k}(\epsilon_i-1))}} \\ &= \left(t^{2(1+\sum_{i=1}^{k}(\epsilon_i-1))}\sum_{j=1}^{k}\prod_{i\neq j}(\lambda_i(x))^2\epsilon_j^2|x^{(j)}|^2\right)^{\frac{1}{2(1+\sum_{i=1}^{k}(\epsilon_i-1))}} \\ &= t|x|_\lambda \end{aligned} \tag{3.3}$$

Proof of (3). For any $x = (x^{(1)}, \ldots, x^{(k)}) \in \Omega_{2R}$, we have $x^{(i)} \in B_i(0, 2R^{\epsilon_i})$, $i = 1, \ldots, k$, this implies $\frac{|x^{(i)}|}{R^{\epsilon_i}} \leq 2$, $i = 1, \ldots, k$. Therefore, if we write

$$x = (x^{(1)}, \ldots, x^{(k)}) = \left(R^{\epsilon_1} \times \frac{x^{(1)}}{R^{\epsilon_1}}, \ldots, R^{\epsilon_k} \times \frac{x^{(k)}}{R^{\epsilon_k}}\right),$$

and let $y = (y^{(1)}, \ldots, y^{(k)}) = \left(\frac{x^{(1)}}{R^{\epsilon_1}}, \ldots, \frac{x^{(k)}}{R^{\epsilon_k}}\right)$, then $y \in \overline{\Omega_2(0)}$.
Using (3.3), we get

$$\begin{aligned}|x|_\lambda &= |(R^{\epsilon_1}y^{(1)}, \ldots, R^{\epsilon_k}y^{(k)})|_\lambda \\ &= R|(y^{(1)}, \ldots, y^{(k)})|_\lambda \\ &= R|y|_\lambda.\end{aligned}$$

Since $|\lambda_i(y)| \leq C$, $\forall\, y \in \overline{\Omega_2}$, $i = 1, \ldots, k$, then there exists a constant $C > 0$ independent of R such that

$$|x|_\lambda \leq CR, \quad \forall\, x \in \Omega_{2R}.$$

This completes the proof of Lemma 3.1. □

Proof of Proposition 1.1. The proof follows the main lines of the demonstration of proposition 4 in [3], with more modifications. We split the proof into four steps:

Step 1. For any $\phi \in C_c^2(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} |\nabla_\lambda(|u|^{\frac{\gamma-1}{2}}u)|^2\phi^2 dx = \frac{(\gamma+1)^2}{4\gamma}\int_{\mathbb{R}^n} |x|_\lambda^a |u|^{p+\gamma}\phi^2 dx + \frac{\gamma+1}{4\gamma}\int_{\mathbb{R}^n} |u|^{\gamma+1}\Delta_\lambda(\phi^2)dx. \tag{3.4}$$

Multiply equation (1.2) by $|u|^{\gamma-1}u\phi^2$ and integrate by parts to find

$$\gamma\int_{\mathbb{R}^n} |\nabla_\lambda u|^2|u|^{\gamma-1}\phi^2 dx + \int_{\mathbb{R}^n} \nabla_\lambda u \nabla_\lambda(\phi^2)|u|^{\gamma-1}u\, dx = \int_{\mathbb{R}^n} |x|_\lambda^a|u|^{p+\gamma}\phi^2 dx,$$

therefore

$$\begin{aligned}\int_{\mathbb{R}^n} |x|_\lambda^a|u|^{p+\gamma}\phi^2 dx &= \frac{4\gamma}{(\gamma+1)^2}\int_{\mathbb{R}^n} |\nabla_\lambda(|u|^{\frac{\gamma-1}{2}}u)|^2\phi^2 dx \\ &\quad + \frac{1}{\gamma+1}\int_{\mathbb{R}^n} \nabla_\lambda(|u|^{\gamma+1})\nabla_\lambda(\phi^2)dx \\ &= \frac{4\gamma}{(\gamma+1)^2}\int_{\mathbb{R}^n} |\nabla_\lambda(|u|^{\frac{\gamma-1}{2}}u)|^2\phi^2 dx \\ &\quad - \frac{1}{\gamma+1}\int_{\mathbb{R}^n} |u|^{\gamma+1}\Delta_\lambda(\phi^2)dx.\end{aligned}$$

Identity (3.4) then follows by multiplying the latter identity by the factor $\frac{(\gamma+1)^2}{4\gamma}$.

Step 2. For any $\phi \in C_c^2(\mathbb{R}^n)$ we have

$$\begin{aligned}&\left(p - \frac{(\gamma+1)^2}{4\gamma}\right)\int_{\mathbb{R}^n} |x|_\lambda^a|u|^{p+\gamma}\phi^2 dx \\ &\le \int_{\mathbb{R}^n} |u|^{\gamma+1}\left[|\nabla_\lambda\phi|^2 + \left(\frac{\gamma+1}{4\gamma} - \frac{1}{2}\right)\Delta_\lambda(\phi^2)\right]dx.\end{aligned} \tag{3.5}$$

The function $|u|^{\frac{\gamma-1}{2}}u\phi$ belongs to $C_c^1(\mathbb{R}^n)$, and thus it can be used as a test function in the quadratic form Q_u. Hence, the stability assumption on u gives

$$p\int_{\mathbb{R}^n} |x|_\lambda^a|u|^{p+\gamma}\phi^2 dx \le \int_{\mathbb{R}^n} |\nabla_\lambda(|u|^{\frac{\gamma-1}{2}}u\phi)|^2 dx. \tag{3.6}$$

A direct calculation shows that the right hand side of (3.6) equals to

$$\begin{aligned}&\int_{\mathbb{R}^n}\left[|u|^{\gamma+1}|\nabla_\lambda\phi|^2 + |\nabla_\lambda(|u|^{\frac{\gamma-1}{2}}u)|^2\phi^2 + \frac{1}{2}\nabla_\lambda\phi^2\nabla_\lambda(|u|^{\gamma+1})\right]dx \\ &= \int_{\mathbb{R}^n} |u|^{\gamma+1}\left[|\nabla_\lambda\phi|^2 - \frac{1}{2}\Delta_\lambda(\phi^2)\right]dx + \int_{\mathbb{R}^n} |\nabla_\lambda(|u|^{\frac{\gamma-1}{2}}u)|^2\phi^2 dx.\end{aligned} \tag{3.7}$$

From (3.6) and (3.7), we obtain that

$$\begin{aligned}&p\int_{\mathbb{R}^n} |x|_\lambda^a|u|^{p+\gamma}\phi^2 dx \\ &\le \int_{\mathbb{R}^n} |u|^{\gamma+1}\left[|\nabla_\lambda\phi|^2 - \frac{1}{2}\Delta_\lambda(\phi^2)\right]dx + \int_{\mathbb{R}^n} |\nabla_\lambda(|u|^{\frac{\gamma-1}{2}}u)|^2\phi^2 dx.\end{aligned} \tag{3.8}$$

Putting this back into (3.4) gives

$$\left(p-\frac{(\gamma+1)^2}{4\gamma}\right)\int_{\mathbb{R}^n}|x|_\lambda^a|u|^{p+\gamma}\phi^2dx$$
$$\leq\int_{\mathbb{R}^n}|u|^{\gamma+1}\left[|\nabla_\lambda\phi|^2+\left(\frac{\gamma+1}{4\gamma}-\frac{1}{2}\right)\Delta_\lambda(\phi^2)\right]dx.$$

Step 3. For any $\gamma\in[1,\,\Gamma_M(p))$ and any integer $m\geq\max\{\frac{p+\gamma}{p-1},\ 2\}$ there exists a constant $C(p,m,\gamma)>0$ depending only on p, m and γ

$$\int_{\mathbb{R}^n}|x|_\lambda^a|u|^{p+\gamma}\psi_R^{2m}dx$$
$$\leq C(p,m,\gamma)\int_{\mathbb{R}^n}|x|_\lambda^{\frac{-(\gamma+1)a}{p-1}}\left(|\nabla_\lambda\psi_R|^2+|\psi_R||\Delta_\lambda\psi_R|\right)^{\frac{p+\gamma}{p-1}}dx, \tag{3.9}$$

$$\int_{\mathbb{R}^n}|\nabla_\lambda(|u|^{\frac{\gamma-1}{2}}u)|^2\psi_R^{2m}dx$$
$$\leq C(p,m,\gamma)\int_{\mathbb{R}^n}|x|_\lambda^{\frac{-(\gamma+1)a}{p-1}}\left(|\nabla_\lambda\psi_R|^2+|\psi_R||\Delta_\lambda\psi_R|\right)^{\frac{p+\gamma}{p-1}}dx, \tag{3.10}$$

where $\psi_R=\prod_{i=1}^k\psi_{i,R}$. Moreover, the constant $C(p,m,\gamma)$ can be explicitly computed. From (3.5), we obtain that

$$\alpha\int_{\mathbb{R}^n}|x|_\lambda^a|u|^{p+\gamma}\phi^2dx\leq\int_{\mathbb{R}^n}|u|^{\gamma+1}|\nabla_\lambda\phi|^2+\beta\int_{\mathbb{R}^n}|u|^{\gamma+1}\Delta_\lambda\phi dx. \tag{3.11}$$

where we have set $\alpha=\left(p-\frac{(\gamma+1)^2}{4\gamma}\right)$ and $\beta=\frac{1-\gamma}{4\gamma}$. Notice that $\alpha>0$ and $\beta<0$, since $p>1$ and $\gamma\in[1,\,\Gamma_M(p))$.

Now, we set $\phi=\psi_R^m$. The function ϕ belongs to $C_c^2(\mathbb{R}^n)$, since $m\geq2$ and m is an integer, hence it can be used in (3.11). A direct computation gives

$$\alpha\int_{\mathbb{R}^n}|x|_\lambda^a|u|^{p+\gamma}\psi_R^{2m}dx$$
$$\leq\int_{\mathbb{R}^n}|u|^{\gamma+1}\psi_R^{2m-2}\left(m^2|\nabla_\lambda\psi_R|^2\right.$$
$$\left.+\beta m(m-1)|\nabla_\lambda\psi_R|^2+\beta m\psi_R\Delta_\lambda\psi_R\right)dx, \tag{3.12}$$

hence

$$\int_{\mathbb{R}^n}|x|_\lambda^a|u|^{p+\gamma}\psi_R^{2m}dx$$
$$\leq C_1\int_{\mathbb{R}^n}|u|^{\gamma+1}\psi_R^{2m-2}\left(|\nabla_\lambda\psi_R|^2+|\psi_R||\Delta_\lambda\psi_R|\right)dx, \tag{3.13}$$

with $C_1=\frac{m^2+\beta m(m-1)}{\alpha}>-\frac{\beta m}{\alpha}\geq0$.

An application of Young's inequality yields

$$\int_{\mathbb{R}^n}|x|_\lambda^a|u|^{p+\gamma}\psi_R^{2m}\leq C_1\int_{\mathbb{R}^n}|u|^{\gamma+1}\psi_R^{2m-2}\left(|\nabla_\lambda\psi_R|^2+|\psi_R||\Delta_\lambda\psi_R|\right)dx$$
$$=C_1\int_{\mathbb{R}^n}|x|_\lambda^{\frac{(\gamma+1)a}{p+\gamma}}|u|^{\gamma+1}\psi_R^{2m-2}|x|_\lambda^{\frac{-(\gamma+1)a}{p+\gamma}}$$

$$
\begin{aligned}
&\left(|\nabla_\lambda \psi_R|^2 + |\psi_R||\Delta_\lambda \psi_R|\right) dx \\
&\le \frac{\gamma+1}{p+\gamma}\int_{\mathbb{R}^n} |x|_\lambda^a |u|^{p+\gamma}\psi_R^{(2m-2)\frac{p+\gamma}{\gamma+1}} \\
&\quad + \frac{(p-1)\,C_1}{p+\gamma}\int_{\mathbb{R}^n} |x|_\lambda^{\frac{-(\gamma+1)a}{p-1}} \left(|\nabla_\lambda \psi_R|^2 + |\psi_R||\Delta_\lambda \psi_R|\right)^{\frac{p+\gamma}{p-1}}.
\end{aligned}
\tag{3.14}
$$

At this point we notice that $m \ge \max\{\frac{p+\gamma}{p-1},\ 2\}$ implies $(2m-2)\frac{p+\gamma}{p-1} \ge 2m$ and thus $\psi_R^{(2m-2)\frac{p+\gamma}{\gamma+1}} \le \psi_R^{2m}$ in $\mathbb{R}^n$, since $0 \le \psi_R \le 1$ everywhere in $\mathbb{R}^n$.

Therefore, we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^n} |x|_\lambda^a |u|^{p+\gamma}\psi_R^{2m} \le \frac{\gamma+1}{p+\gamma}\int_{\mathbb{R}^n} |x|_\lambda^a |u|^{p+\gamma}\psi_R^{2m} \\
+ \frac{(p-1)\,C_1}{p+\gamma}\int_{\mathbb{R}^n} |x|_\lambda^{\frac{-(\gamma+1)a}{p-1}} \left(|\nabla_\lambda \psi_R|^2 + |\psi_R||\Delta_\lambda \psi_R|\right)^{\frac{p+\gamma}{p-1}}.
\end{aligned}
$$

The latter immediately implies

$$
\int_{\mathbb{R}^n} |x|_\lambda^a |u|^{p+\gamma}\psi_R^{2m} dx \le C_1 \int_{\mathbb{R}^n} |x|_\lambda^{\frac{-(\gamma+1)a}{p-1}} \left(|\nabla_\lambda \psi_R|^2 + |\psi_R||\Delta_\lambda \psi_R|\right)^{\frac{p+\gamma}{p-1}} dx,
\tag{3.15}
$$

which proves inequality (3.9) with $C(p,m,\gamma) = C_1$.

To prove (3.10), we combine (3.4) and (3.5). This leads to

$$
\int_{\mathbb{R}^n} |\nabla_\lambda(|u|^{\frac{\gamma-1}{2}}u)|^2\phi^2 dx \le A\int_{\mathbb{R}^n} |u|^{\gamma+1}|\nabla_\lambda \phi|^2 dx + B\int_{\mathbb{R}^n} |u|^{\gamma+1}\phi\Delta_\lambda\phi dx,
$$

where $A = \frac{(\gamma+1)^2}{4\gamma\alpha} + \frac{(\gamma+1)}{2\gamma} > 0$ and $B = \frac{\beta(\gamma+1)^2}{4\gamma\alpha} + \frac{(\gamma+1)}{2\gamma} \in \mathbb{R}$.

Now, we insert the test function $\phi = \psi_R^m$ in the latter inequality to find,

$$
\begin{aligned}
\int_{\mathbb{R}^n} |\nabla_\lambda(|u|^{\frac{\gamma-1}{2}}u)|^2\psi_R^{2m} dx \le \int_{\mathbb{R}^n} |u|^{\gamma+1}\psi_R^{2m-2}\big(Am^2|\nabla_\lambda\psi_R|^2 \\
+ Bm(m-1)|\nabla_\lambda\psi_R|^2 + Bm\psi_R\Delta_\lambda\psi_R\big)\, dx,
\end{aligned}
$$

and hence

$$
\int_{\mathbb{R}^n} |\nabla_\lambda(|u|^{\frac{\gamma-1}{2}}u)|^2\psi_R^{2m} dx \le C_2\int_{\mathbb{R}^n} |u|^{\gamma+1}\psi_R^{2m-2}\left(|\nabla_\lambda\psi_R|^2 + |\psi_R||\Delta_\lambda\psi_R|\right) dx,
\tag{3.16}
$$

with $C_2 = \max\{Am^2 + Bm(m-1),\ |B|m\} > 0$. Using Hölder's inequality in (3.16) yields

$$
\begin{aligned}
&\int_{\mathbb{R}^n} |\nabla_\lambda(|u|^{\frac{\gamma-1}{2}}u)|^2\psi_R^{2m} \\
&\le C_2\left(\int_{\mathbb{R}^n} |x|_\lambda^a |u|^{p+\gamma}\psi_R^{(2m-2)\frac{p+\gamma}{\gamma+1}}\right)^{\frac{\gamma+1}{p+\gamma}} \\
&\quad \times \left(\int_{\mathbb{R}^n} |x|_\lambda^{\frac{-(\gamma+1)a}{p-1}} \left(|\nabla_\lambda \psi_R|^2 + |\psi_R||\Delta_\lambda \psi_R|\right)^{\frac{p+\gamma}{p-1}}\right)^{\frac{p-1}{p+\gamma}}
\end{aligned}
$$

$$\leq C_2 \left(\int_{\mathbb{R}^n} |x|_\lambda^a |u|^{p+\gamma} \psi_R^{2m} \right)^{\frac{\gamma+1}{p+\gamma}}$$
$$\times \left(\int_{\mathbb{R}^n} |x|_\lambda^{\frac{-(\gamma+1)a}{p-1}} \left(|\nabla_\lambda \psi_R|^2 + |\psi_R||\Delta_\lambda \psi_R| \right)^{\frac{p+\gamma}{p-1}} \right)^{\frac{p-1}{p+\gamma}}.$$

Finally, inserting (3.15) into the latter we obtain

$$\int_{\mathbb{R}^n} |\nabla_\lambda (|u|^{\frac{\gamma-1}{2}} u)|^2 \psi_R^{2m} dx$$
$$\leq C_2 C_1^{\frac{1+\gamma}{p-1}} \int_{\mathbb{R}^n} |x|_\lambda^{\frac{-(\gamma+1)a}{p-1}} \left(|\nabla_\lambda \psi_R|^2 + |\psi_R||\Delta_\lambda \psi_R| \right)^{\frac{p+\gamma}{p-1}} dx,$$

which gives the desired inequality (3.10).

Step 4. For any $\gamma \in [1, \Gamma_M(p))$, there exists a constant $C > 0$ independent of R such that

$$\int_{\Omega_R} \left(|x|_\lambda^a |u|^{p+\gamma} + |\nabla_\lambda (|u|^{\frac{\gamma-1}{2}} u)|^2 \right) dx \leq C R^{Q - \frac{2(p+\gamma)+(\gamma+1)a}{p-1}}, \quad \forall\, R > 0. \tag{3.17}$$

The proof of (3.17) follows immediately by adding inequality (3.9) to inequality (3.10) and using Lemma 3.1. □

Proof of Theorem 1.1. By Proposition 1.1, there exists a positive constant C independent of R such that

$$\int_{\Omega_R} |x|_\lambda^a |u|^{p+\gamma} \leq C R^{Q - \frac{2(p+\gamma)+a(\gamma+1)}{p-1}}. \tag{3.18}$$

Then it suffices to show that we can always choose a $\gamma \in [1, \Gamma_M(p))$, such that $Q - \frac{2(p+\gamma)+a(\gamma+1)}{p-1} < 0$. Therefore, by letting $R \to +\infty$ in (3.18), we deduce that

$$\int_{\mathbb{R}^n} |x|_\lambda^a |u|^{p+\gamma} = 0,$$

which implies that $u \equiv 0$ in $\mathbb{R}^n$.

Next, we claim that, under the assumptions on the exponent p assumed in Theorem 1.1, we can always choose $\gamma \in [1, \Gamma_M(p))$ such that

$$Q - \frac{2(p+\gamma)+a(\gamma+1)}{p-1} < 0. \tag{3.19}$$

As in [3], we consider separately the case $Q \leq 10+4a$ and the case $Q > 10+4a$.
Case 1. $Q \leq 10+4a$ and $p > 1$. In this case we have

$$2(p + \Gamma_M(p)) + a(\Gamma_M(p) + 1) >$$
$$2(3p - 1 + 2(p-1)) + a(2p + 2(p-1)) > (10+4a)(p-1)$$

and therefore

$$Q - \frac{2(p+\Gamma_M(p)) + a(\Gamma_M(p)+1)}{p-1} < Q - (10+4a) \leq 0. \tag{3.20}$$

The latter inequality and the continuity of the function $x \mapsto Q - \frac{2(p+x)+a(x+1)}{p-1}$ immediately imply the existence of $\gamma \in [1, \Gamma_M(p))$ satisfying (3.19).

Case 2. $Q > 10 + 4a$ and $1 < p < p_c(Q, a)$. In this case we consider the real-valued function $x \mapsto g(x) := \frac{2(x+\Gamma_M(x))+a(\Gamma(x)+1)}{x-1}$ on $(1, +\infty)$. Since g is strictly decreasing function satisfying $\lim_{x\to 1^+} g(x) = +\infty$ and $\lim_{x\to+\infty} g(x) = 10 + 4a$, there exists a unique $p_0 > 1$ such that $Q = g(p_0)$. We claim that $p_0 = p_c(Q, a)$. Indeed,

$$
\begin{aligned}
&Q = g(p) \;\Leftrightarrow\; (Q-2)(p-1) - (4+2a)p = (4+2a)\sqrt{p(p-1)} \\
&\Leftrightarrow\; (Q-10-4a)(Q-2)p^2 + (-2(Q-2)^2 + 4(a+2)(Q+a))p + (Q-2)^2 = 0,
\end{aligned}
$$

which implies that

$$
(Q-10-4a)(Q-2)p_0^2 + (-2(Q-2)^2 + 4(a+2)(Q+a))p_0 + (Q-2)^2 = 0, \tag{3.21}
$$

and

$$
(Q-2)(p_0-1) - (4+2a)p_0 > (4+2a)(p_0-1). \tag{3.22}
$$

The roots of (3.21)

$$
p_1 = \frac{(Q-2)^2 - 2(a+2)(a+Q) + 2\sqrt{(a+2)^3(a+2Q-2)}}{(Q-2)(Q-4a-10)} = p_c(Q, a), \tag{3.23}
$$

$$
p_2 = \frac{(Q-2)^2 - 2(a+2)(a+Q) - 2\sqrt{(a+2)^3(a+2Q-2)}}{(Q-2)(Q-4a-10)} < p_0, \tag{3.24}
$$

while (3.22) easily implies $p_0 > \frac{Q-6-2a}{Q-4a-10} > p_2$. This proves that $p_0 = p_1$. Hence

$$
p_c(Q, a) = \frac{(Q-2)^2 - 2(a+2)(a+Q) + 2\sqrt{(a+2)^3(a+2Q-2)}}{(Q-2)(Q-4a-10)}
$$

as claimed. Since we have just proven that $g(p_c(Q, a)) = Q$ and g is a strictly decreasing function, it follows that

$$
\forall\, 1 < p < p_c(Q, a),\; Q < g(p). \tag{3.25}
$$

Now we can conclude as in the first case, i.e, the continuity of $x \mapsto Q - \frac{2(p+x)+a(x+1)}{p-1}$ immediately implies the existence of $\gamma \in [1, \Gamma_M(p))$ satisfying (3.19). □

4. The Liouville theorem for solutions which are stable outside a compact set of $\mathbb{R}^n$: proof of Theorem 1.2

In this section, we prove Proposition 1.2 and Theorem 1.2.

Proof of Proposition 1.2. Let $u \in C^2(\mathbb{R}^n)$ be a solution of (1.2) and $\phi \in C_c^1(\Omega_R)$. Multiplying equation (1.2) by $T(u)\phi$ and integrating by parts in Ω_R, we obtain

$$\begin{aligned}
-\int_{\Omega_R} \Delta_\lambda u T(u)\phi dx &= -\int_{\Omega_R} \Delta_\lambda u\, \epsilon_j\, x^{(j)}\, \nabla_{x^{(j)}} u\, \phi dx \\
&= \int_{\Omega_R} \lambda_i^2\, \nabla_{x^{(i)}} u\, \nabla_{x^{(i)}} \left(\epsilon_j x^{(j)} \nabla_{x^{(j)}} u \phi\right) dx \\
&= \int_{\Omega_R} \lambda_i^2\, \nabla_{x^{(i)}} u\, \epsilon_j \delta_{ij}\, \nabla_{x^{(j)}} u\, \phi dx \\
&\quad + \int_{\Omega_R} \lambda_i^2\, \nabla_{x^{(i)}} u\, \epsilon_j\, x^{(j)}\, \nabla_{x^{(i)}} \left(\nabla_{x^{(j)}} u\right)\, \phi dx \\
&\quad + \int_{\Omega_R} \lambda_i^2\, \nabla_{x^{(i)}} u\, \epsilon_j\, x^{(j)}\, \nabla_{x^{(j)}} u\, \nabla_{x^{(i)}} \phi dx \\
&:= I_1 + I_2 + I_3, \qquad (4.1)
\end{aligned}$$

Here and in the sequel, we use the Einstein summation convention: an index occurring twice in a product is to be summed from 1 up to the space dimension.

Obviously

$$\begin{aligned}
I_1 : &= \int_{\Omega_R} \lambda_i^2 \nabla_{x^{(i)}} u\, \epsilon_j \delta_{ij}\, \nabla_{x^{(j)}} u\, \phi dx \\
&= \int_{\Omega_R} \lambda_i^2 \left|\nabla_{x^{(i)}} u\right|^2 \epsilon_i \phi dx. \qquad (4.2)
\end{aligned}$$

Moreover, an integration by parts in I_2 gives

$$\begin{aligned}
I_2 : &= \int_{\Omega_R} \lambda_i^2 \nabla_{x^{(i)}} u \epsilon_j x^{(j)} \nabla_{x^{(i)}} \left(\nabla_{x^{(j)}} u\right) \phi dx \\
&= -\int_{\Omega_R} \nabla_{x^{(j)}} (\lambda_i^2) \left|\nabla_{x^{(i)}} u\right|^2 \epsilon_j x^{(j)} \phi dx - I_2 - \int_{\Omega_R} \lambda_i^2 \left|\nabla_{x^{(i)}} u\right|^2 \epsilon_j n_j \phi dx \\
&\quad - \int_{\Omega_R} \lambda_i^2 \left|\nabla_{x^{(i)}} u\right|^2 \epsilon_j x^{(j)} \nabla_{x^{(j)}} \phi dx \\
&= -2\int_{\Omega_R} \lambda_i \left|\nabla_{x^{(i)}} u\right|^2 T(\lambda_i) \phi dx - I_2 - Q\int_{\Omega_R} |\nabla_\lambda u|^2 \phi dx \\
&\quad - \int_{\Omega_R} |\nabla_\lambda u|^2 T(\phi) dx.
\end{aligned}$$

Since λ_i is δ_t-homogeneous of degree $\epsilon_i - 1$, then $T(\lambda_i) = (\epsilon_i - 1)\lambda_i$. Hence

$$\begin{aligned}
I_2 &= -2\int_{\Omega_R} (\epsilon_i - 1)\lambda_i^2 \left|\nabla_{x^{(i)}} u\right|^2 \phi dx - I_2 - Q\int_{\Omega_R} |\nabla_\lambda u|^2 \phi dx \\
&\quad - \int_{\Omega_R} |\nabla_\lambda u|^2 T(\phi) dx \\
&= (2 - Q)\int_{\Omega_R} |\nabla_\lambda u|^2 \phi dx - 2I_1 - I_2 - \int_{\Omega_R} |\nabla_\lambda u|^2 T(\phi) dx.
\end{aligned}$$

Then

$$I_2 = \frac{2-Q}{2}\int_{\Omega_R} |\nabla_\lambda u|^2 \phi dx - I_1 - \frac{1}{2}\int_{\Omega_R} |\nabla_\lambda u|^2 T(\phi) dx. \tag{4.3}$$

It is easily seen that

$$\begin{aligned} I_3 :&= \int_{\Omega_R} \lambda_i^2 \nabla_{x^{(i)}} u \epsilon_j x^{(j)} \nabla_{x^{(j)}} u \nabla_{x^{(i)}} \phi dx \\ &= \int_{\Omega_R} \nabla_\lambda u \nabla_\lambda \phi T(u) dx. \end{aligned} \tag{4.4}$$

Hence, by (4.1),

$$\begin{aligned} -\int_{\Omega_R} \Delta_\lambda u T(u) \phi dx = \frac{2-Q}{2} & \int_{\Omega_R} |\nabla_\lambda u|^2 \phi dx \\ & -\frac{1}{2}\int_{\Omega_R} |\nabla_\lambda u|^2 T(\phi) dx + \int_{\Omega_R} \nabla_\lambda u \nabla_\lambda \phi T(u) dx. \end{aligned} \tag{4.5}$$

On the other hand, an integration by parts gives

$$\begin{aligned} \int_{\Omega_R} |x|_\lambda^a |u|^{p-1} u T(u) \phi dx &= \frac{1}{p+1}\int_{\Omega_R} |x|_\lambda^a \nabla_{x^{(j)}} (|u|^{p+1}) \epsilon_j x^{(j)} \phi dx \\ &= -\frac{Q}{p+1}\int_{\Omega_R} |x|_\lambda^a |u|^{p+1} \phi \\ &\quad -\frac{a}{p+1}\int_{\Omega_R} |x|_\lambda^{a-1} |u|^{p+1} T(|x|_\lambda) \phi \\ &\quad -\frac{1}{p+1}\int_{\Omega_R} |x|_\lambda^a |u|^{p+1} T(\phi) dx. \end{aligned}$$

If $T(|x|_\lambda) = |x|_\lambda$, then

$$\begin{aligned} \int_{\Omega_R} |x|_\lambda^a |u|^{p-1} u T(u) \phi dx &= \frac{1}{p+1}\int_{\Omega_R} |x|_\lambda^a \nabla_{x^{(j)}} (|u|^{p+1}) \epsilon_j x^{(j)} \phi dx \\ &= -\frac{Q+a}{p+1}\int_{\Omega_R} |x|_\lambda^a |u|^{p+1} \phi \\ &\quad -\frac{1}{p+1}\int_{\Omega_R} |x|_\lambda^a |u|^{p+1} T(\phi) dx. \end{aligned} \tag{4.6}$$

Clearly (1.4) follows directly from (4.5) and (4.6). $\square$

Proof of Theorem 1.2. Let u be a solution of (1.2) which is stable outside a compact set. We begin defining some smooth compactly supported functions which will be used several times in the sequel. More precisely, for $R_* > 0$, we choose a function $\zeta_{i,R} \in C_c^2(\mathbb{R}^{n_i})$, $i = 1, \ldots, k$, $0 \le \zeta_{i,R} \le 1$, everywhere on $\mathbb{R}^{n_i}$ and

$$\begin{cases} \zeta_{i,R}(x^{(i)}) = 0 & \text{if} \quad |x^{(i)}| < R_* + 1 \quad \text{or} \quad |x^{(i)}| > 2R^{\epsilon_i}, \\ \zeta_{i,R}(x^{(i)}) = 1 & \text{if} \;\; R_* + 2 < |x^{(i)}| < R^{\epsilon_i}, \\ |\nabla_{x^{(i)}} \zeta_{i,R}|^2 + |\Delta_{x^{(i)}} \zeta_{i,R}| \le C R^{-2\epsilon_i} & \text{for} \quad \{R^{\epsilon_i} < |x^{(i)}| < 2R^{\epsilon_i}\}. \end{cases}$$

The rest of the proof splits into several steps.

Step 1. Let $p>1$. There exists $R_*>0$ such that for every $\gamma\in[1,\Gamma_M(p))$ and every $R^{\epsilon_i}>R_*+2$, we have

$$\int_{\Sigma_0(R)}\left(|x|_\lambda^a|u|^{p+\gamma}+|\nabla_\lambda(|u|^{\frac{\gamma-1}{2}}u)|^2\right)dx\le C_{R_*}+CR^{Q-\frac{2(p+\gamma)+(\gamma+1)a}{p-1}},\qquad(4.7)$$

where $\Sigma_0(R)=\Omega_R\setminus[B_1(0,R_*+2)\times\ldots\times B_k(0,R_*+2)]$, C_{R_*} and C are positive constants depending on p, γ, R_* but not on R.

Since u is stable outside a compact set of $\mathbb{R}^n$, there exists $R_*>0$ such that, similar to that of Proposition 1.1 we derive

$$\begin{aligned}&\int_{\Sigma_0(R)}\left(|x|_\lambda^a|u|^{p+\gamma}+|\nabla_\lambda(|u|^{\frac{\gamma-1}{2}}u)|^2\right)dx\\&\le C(p,m,\gamma)\int_{\mathbb{R}^n}|x|_\lambda^{\frac{-a(\gamma+1)}{p-1}}\left(|\nabla_\lambda\zeta_R|^2+|\zeta_R||\Delta_\lambda\zeta_R|\right)^{\frac{p+\gamma}{p-1}}dx\\&\le C_{R_*}+CR^{Q-\frac{2(p+\gamma)+(\gamma+1)a}{p-1}},\end{aligned}$$

where $\zeta_R=\prod_{i=1}^n\zeta_{i,R}$. Hence, the desired integral estimate (4.7) follows.

Step 2. If $Q=2$ and $1<p<+\infty$ or $Q\ge3$ and $1<p<\frac{Q+2+2a}{Q-2}$, then $u\equiv0$.

By choosing $\gamma=1$ and using *Step* 1, we get $|x|_\lambda^{\frac{a}{p+1}}u\in L^{p+1}(\mathbb{R}^n)$ and $|\nabla_\lambda u|\in L^2(\mathbb{R}^n)$ for $1<p<p_s(Q,a)$.

Take $\phi=\psi_R=\prod_{i=1}^k\psi_{i,R}$ in (1.4) where $\psi_{i,R}$ defined as above. Since $|x|_\lambda^{\frac{a}{p+1}}u\in L^{p+1}(\mathbb{R}^n)$ and $|\nabla_\lambda u|\in L^2(\mathbb{R}^n)$, then

$$\begin{aligned}&\int_{\Sigma_R}|\nabla_\lambda u|^2dx\to0,\ \text{as } R\to+\infty\\&\text{and}\quad\int_{\Sigma_R}|x|_\lambda^a|u|^{p+1}dx\to0,\ \text{as } R\to+\infty,\end{aligned}\qquad(4.8)$$

where $\Sigma_R=\Omega_{2R}\setminus\Omega_R$.

Recalling that λ_i and $\lambda_i\nabla_{x^{(i)}}u$ are δ_t-homogeneous of degree ϵ_i-1 and one respectively. Then, since T generates $(\delta_t)_{t\ge0}$, we have

$$T(\lambda_i)=(\epsilon_i-1)\lambda_i\quad\text{and}\quad T(\lambda_i\nabla_{x^{(i)}}u)=\lambda_i\nabla_{x^{(i)}}u.\qquad(4.9)$$

Integrating by parts and using (4.9), we derive

$$\begin{aligned}\int_{\Omega_{2R}}\nabla_\lambda u\nabla_\lambda\psi_RT(u)&=\int_{\Omega_{2R}}\lambda_i\nabla_{x^{(i)}}u\lambda_i\nabla_{x^{(i)}}\psi_R\epsilon_jx^{(j)}\nabla_{x^{(j)}}u\\&=-\int_{\Omega_{2R}}T(\lambda_i\nabla_{x^{(i)}}u)\lambda_i\nabla_{x^{(i)}}\psi_R\,u-\int_{\Omega_{2R}}\lambda_i\nabla_{x^{(i)}}uT(\lambda_i)\nabla_{x^{(i)}}\psi_R\,u\\&\quad-\int_{\Omega_{2R}}\lambda_i^2\nabla_{x^{(i)}}uT(\nabla_{x^{(i)}}\psi_R)\,u-\ Q\int_{\Omega_{2R}}\nabla_\lambda u\nabla_\lambda\psi_R\,u\\&=-(Q+1)\int_{\Omega_{2R}}\nabla_\lambda u\nabla_\lambda\psi_R\,u-\int_{\Omega_{2R}}(\epsilon_i-1)\lambda_i^2\nabla_{x^{(i)}}u\nabla_{x^{(i)}}\psi_R\,u\end{aligned}$$

$$
\begin{aligned}
&-\int_{\Omega_{2R}} \lambda_i^2 \nabla_{x^{(i)}} u T(\nabla_{x^{(i)}} \psi_R)\, u \\
=&\ \frac{Q+1}{2} \int_{\Omega_{2R}} u^2 \Delta_\lambda \psi_R + \int_{\Omega_{2R}} \frac{\epsilon_i - 1}{2} u^2 \lambda_i^2 \Delta_{x^{(i)}} \psi_R \\
&+ \frac{1}{2} \int_{\Omega_{2R}} u^2 \lambda_i^2 \nabla_{x^{(i)}} [T(\nabla_{x^{(i)}} \psi_R)] \qquad (4.10)
\end{aligned}
$$

By Lemma 3.1, (4.10) and using Hölder's inequality, we obtain

$$
\begin{aligned}
\left| \int_{\Omega_{2R}} \nabla_\lambda u \nabla_\lambda \psi_R T(u) \right| &\leq \frac{C}{R^{-2}} \int_{\Sigma_R} u^2 \\
&= \frac{C}{R^{-2}} \int_{\Sigma_R} |x|_\lambda^{\frac{-2a}{p+1}} |x|_\lambda^{\frac{2a}{p+1}} u^2 \\
&\leq C R^{(Q - \frac{2a}{p-1}) \frac{p-1}{p+1} - 2} \left(\int_{\Sigma_R} |x|_\lambda^a |u|^{p+1} \right)^{\frac{2}{p+1}}.
\end{aligned} \tag{4.11}
$$

Similarly, we get

$$
\left| \int_{\Omega_{2R}} \left[-\frac{1}{2} |\nabla_\lambda u|^2 + \frac{|x|_\lambda^a}{p+1} |u|^{p+1} \right] T(\psi_R) \right| \leq C \int_{\Sigma_R} \left(|\nabla_\lambda u|^2 + |x|_\lambda^a |u|^{p+1} \right). \tag{4.12}
$$

From (4.8), (4.11) and (4.12), we obtain

$$
\lim_{R \to +\infty} \left| \int_{\Omega_{2R}} \left(\nabla_\lambda u \nabla_\lambda \psi_R T(u) + \left[-\frac{1}{2} |\nabla_\lambda u|^2 + \frac{|x|_\lambda^a}{p+1} |u|^{p+1} \right] T(\psi_R) \right) \right| = 0.
$$

As a consequence, (1.4) becomes

$$
\frac{Q-2}{2} \int_{\mathbb{R}^n} |\nabla_\lambda u|^2 dx - \frac{Q+a}{p+1} \int_{\mathbb{R}^n} |x|_\lambda^a |u|^{p+1} dx = 0. \tag{4.13}
$$

On the other hand, multiplying equation (1.2) by $u\psi_R$ and integrating by parts yields

$$
\int_{\mathbb{R}^n} |\nabla_\lambda u|^2 \psi_R dx - \int_{\mathbb{R}^n} |x|_\lambda^a |u|^{p+1} \psi_R dx = \frac{1}{2} \int_{\mathbb{R}^n} u^2 \Delta_\lambda \psi_R dx.
$$

Since $1 < p < p_s(Q, a)$, we get

$$
\begin{aligned}
\left| \int_{\mathbb{R}^n} u^2 \Delta_\lambda \psi_R dx \right| &\leq \left(\int_{\mathbb{R}^n} |x|_\lambda^a |u|^{p+1} dx \right)^{\frac{2}{p+1}} \left(\int_{\Sigma_R} |x|_\lambda^{\frac{-2a}{p-1}} |\Delta_\lambda \psi_R|^{\frac{p+1}{p-1}} dx \right)^{\frac{p-1}{p+1}} \\
&\leq C R^{Q \frac{p-1}{p+1} - 2 - \frac{2a}{p+1}} \to 0 \text{ as } R \to +\infty.
\end{aligned}
$$

Then

$$
\int_{\mathbb{R}^n} |\nabla_\lambda u|^2 dx = \int_{\mathbb{R}^n} |x|_\lambda^a |u|^{p+1} dx. \tag{4.14}
$$

To complete the proof we combine (4.13) and (4.14) to get

$$
\left(\frac{Q-2}{2} - \frac{Q+a}{p+1} \right) \int_{\mathbb{R}^n} |x|_\lambda^a |u|^{p+1} dx = 0,
$$

but $\frac{Q-2}{2} - \frac{Q+a}{p+1} \neq 0$, since p is subcritical, hence u must be identically zero, as claimed. □

Compliance with ethical standards

Conflict of interest The author declare that he has no competing interests.

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