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Well-posedness and fast-diffusion limit for a bulk–surface reaction–diffusion system

Stephan Hausberg and Matthias Röger

Abstract. We analyze a certain class of coupled bulk–surface reaction–drift–diffusion systems arising in the modeling of signalling networks in biological cells. The coupling is by a nonlinear Robin-type boundary condition for the bulk variable and a corresponding source term on the cell boundary. For reaction terms with at most linear growth and under different regularity assumptions on the data we prove the existence of weak and classical solutions. In particular, we show that solutions grow at most exponentially with time. Furthermore, we rigorously derive an asymptotic reduction to a non-local reaction–drift–diffusion system on the membrane in the fast-diffusion limit.

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Keywords. Partial differential equations on surfaces, Reaction–diffusion systems, Signaling networks in cells.

1. Introduction

The dynamic distribution of proteins within a cell and on the cell membrane is essential for many biological functions. The spatial localization of GTPase proteins for example plays an important role in cell signaling and the polarization of cells, and hence contributes for instance to cell movement and differentiation [31]. Two particular properties determine the dynamic of GTPase proteins. Firstly, these proteins cycle between an active and an inactive state, and secondly, the inactive form can alternate between a membrane bound and a cytosolic state (where the molecules diffuse in the bulk). In a deterministic mean field type description these characteristics lead to a spatially coupled system of bulk–surface partial differential equations. In this article we investigate the mathematical well-posedness of such systems and rigorously justify an asymptotic reduction of the model.

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The particular class of bulk–surface reaction–diffusion systems that we consider here is a generalization of the GTPase cycle model introduced in [34] and further investigated in [36]. In this model the cell and its outer cell-membrane are characterized by a domain $B \subset \mathbb{R}^3$ and its boundary $\Gamma = \partial B$. We are interested in the evolution of the cytosolic concentration V of the inactive GTPase, of the concentration u of the membrane-bound active, and of the concentration v of the membrane-bound inactive GTPase. While V has its domain of definition in the cell interior, B, the functions u, v live on the cell membrane, Γ. We assume that V can diffuse in the cell interior, and that V and v are converted into each other at the membrane by attachment and detachment. Moreover, u, v can diffuse laterally on the membrane and are transformed into each other by activation and deactivation. We therefore obtain the following coupled system: Given $T > 0$ we consider the time interval $(0, T)$, and look for nonnegative functions $V : B \times (0, T) \to \mathbb{R}$, $u, v : \Gamma \times (0, T) \to \mathbb{R}$, that satisfy the diffusion equation

$$\partial_t V = D\Delta V \tag{1.1}$$

in $B \times (0, T)$, the flux condition

$$-D\nabla V \cdot \nu = q_1(u, v)V - q_2(u, v)v \tag{1.2}$$

on $\Gamma \times (0, T)$, and the reaction–diffusion system

$$\partial_t u = \Delta_\Gamma u + f_1(u, v)v - f_2(u, v)u, \tag{1.3}$$

$$\partial_t v = d\Delta_\Gamma v + (-f_1(u, v)v + f_2(u, v)u + q_1(u, v)V - q_2(u, v)v), \tag{1.4}$$

on $\Gamma \times (0, T)$. In (1.3), (1.4) we denote by Δ_Γ the Laplace–Beltrami operator on the manifold $\Gamma = \partial B$, and we are assuming $d > 0$ and $D > 0$. The constitutive functions f_1, f_2 and q_1, q_2 encode the reaction rates for the activation/deactivation of GTPase on the membrane, and for the exchange of inactive GTPase at the cell membrane. All these constitutive functions are assumed to be nonnegative and uniformly bounded. This in particular yields a quasi-positivity property of the nonlinearities (cf. [33]).

In this paper we generalize the system described above and consider a model with more general and possibly non-smooth drift–diffusion operators on the cell membrane. The diffusion operators are given by mappings $A_1, A_2 : \Gamma \times (0, T) \to \mathbb{R}^{3\times 3}$ and the drift by functions $b_1, b_2 : \Gamma \times (0, T) \to \mathbb{R}^3$, with

$$A_k(y, t) : T_y\Gamma \to T_y\Gamma, \quad b_k(y, t) \in T_y\Gamma \qquad \text{for all } (y, t) \in \Gamma \times (0, T),\ k = 1, 2, \tag{1.5}$$

where $T_y\Gamma$ denotes the tangent plane of Γ in $y \in \Gamma$. We then consider the following bulk–surface reaction–drift–diffusion system:

Problem ***(RDD)****. We are looking for* $V : B\times(0,T)\to\mathbb{R}$*,* $u,v : \Gamma\times(0,T)\to\mathbb{R}$ *such that*

$$\begin{aligned}
\partial_t V &= D\Delta V && \textit{in} \quad B\times(0,T), && (1.6)\\
-D\nabla V\cdot\nu &= q_1(u,v)V - q_2(u,v)v && \textit{on} \quad \Gamma\times(0,T), && (1.7)\\
\partial_t u &= \nabla_\Gamma\cdot(A_1\nabla_\Gamma u) + b_1\cdot\nabla_\Gamma u + f_1(u,v)v \\
&\quad - f_2(u,v)u && \textit{on} \quad \Gamma\times(0,T), && (1.8)\\
\partial_t v &= \nabla_\Gamma\cdot(A_2\nabla_\Gamma v) + b_2\cdot\nabla_\Gamma v \\
&\quad - f_1(u,v)v + f_2(u,v)u + q_1(u,v)V \\
&\quad - q_2(u,v)v && \textit{on} \quad \Gamma\times(0,T), && (1.9)
\end{aligned}$$

and such that the initial conditions

$$V(\cdot,0) = V_0, \quad u(\cdot,0) = u_0, \quad v(\cdot,0) = v_0 \qquad (1.10)$$

are satisfied, where $V_0 : B\to\mathbb{R}$ *and* $u_0, v_0 : \Gamma\to\mathbb{R}$ *are given nonnegative data.*

In (1.8), (1.9) the differential operators ∇_Γ and $(\nabla_\Gamma\cdot)$ refer to the tangential gradient and the tangential divergence on Γ, see also Notation 2.1 below.

The generalized model (RDD) allows to account for heterogeneities of the membrane. In fact, the outer membrane contains several microdomains [10]. Such domains on membranes show distinct physical properties and in particular may lead to different diffusion speeds [6]. The drift-terms are included for mathematical analysis purposes, in particular to prepare the study of problems with evolving membrane shape: parametrizing such an evolution over a fixed manifold even in the case of the simpler system (1.1)–(1.4) results in a system of the type (RDD).

The system (RDD) consists of two parts. Firstly, a reaction–drift–diffusion system on the membrane for the variables u, v, with a V-dependent source term, and secondly, a diffusion equation for V in the interior of the cell with a nonlinear Robin-type boundary condition that depends on u, v. The particular setup of the model guarantees the conservation of the total mass of GTPase in the system,

$$\int_B V(\cdot,t) + \int_\Gamma (u(\cdot,t) + v(\cdot,t)) = \text{const} \ \text{ for all } t\in(0,T). \qquad (1.11)$$

Moreover, due to the quasi-positivity property of the reaction terms, solutions stay nonnegative if we start with nonnegative initial values.

In [36] Rätz and the second author have investigated, by a linearized stability analysis, pattern forming properties of the system (1.1)–(1.4) and of an asymptotic reduction of this system. The goal of this paper is to complement this analysis by addressing the well-posedness of such systems. Coupled bulk–surface systems are not covered by the standard theory of reaction–diffusion systems. Moreover, even for the case of standard reaction–diffusion systems positivity preservation and mass conservation are not sufficient to prevent the blow-up of solutions [33]. For the generalized system (RDD) we prove the unique existence of weak solutions. For more regular data we also deduce the existence of classical solutions. Moreover, we justify an asymptotic reduction to a nonlocal reaction–drift–diffusion system on the membrane in the limit of infinite cytosolic diffusion, $D \to \infty$. For volume reaction–diffusion systems such an asymptotic limit is known as a *shadow system*. Such systems, their rigorous justification, and the relation of the large time behavior of the original and the reduced system have been studied for a long time [19,22,25] and are still an active field of research [21,27,28].

The mathematical analysis of coupled bulk–surface systems of partial differential equations has attracted a lot of attention over the last years, and addresses a variety of different applications for example from cell biology [1,7,8,14,15,24,26,30,32,34,36], thermomechanics [17,29], fluid dynamics [4,39], or ecology [3]. The well-posedness of bulk–surface reaction–diffusion systems with homogeneous diffusion has been recently investigated in [37]. There, under specific conditions on the nonlinearities the global existence of classical solutions has been shown. Under additional restrictions, in [38] the uniform boundedness of solutions has been deduced. In [8], for a linear bulk–surface system, global-in-time existence and rigorous asymptotic limits in various parameter regimes have been proved. A well-posednesss analysis for a bulk diffusion–advection system coupled to a surface reaction–diffusion–sorption system has recently been presented in [4]. For homogeneous diffusion and under suitable conditions on the reaction and bulk–surface exchange global existence of L^p solutions is proved. For a two-variable bulk–surface system and specific nonlinearities, which in particular satisfy a monotonicity condition, the existence of a unique weak solution and exponential convergence to equilibrium via an entropy method is shown in [11]. For linear bulk–surface models with reaction terms satisfying a complex balance condition global existence of weak solutions and convergence in a fast reaction limit has been shown in [12], exponential convergence to equilibrium has been proved by entropy methods in [13].

This paper is partially based on and extends the work in the PhD thesis of the first author [20].

2. Main results

Notation 2.1. *For a set $B \subset \mathbb{R}^3$ we denote by $|B| = \mathcal{L}^3(B)$ the Lebesgue measure. For a surface $\Gamma \subset \mathbb{R}^3$ we denote by $|\Gamma| = \mathcal{H}^2(\Gamma)$ its area (i.e. the 2-dimensional Hausdorff measure) and by $\int_\Gamma \cdot \, d\mathcal{H}^2$ the corresponding surface integral. The differential operators ∇_Γ and $(\nabla_\Gamma \cdot)$ refer to the tangential gradient and the tangential divergence on Γ. For volume integrals over B and surface integrals over Γ we often simplify the notation and suppress the integration variables and measure.*

We use the standard Sobolev spaces $W^{k,p}(B)$ and $H^k(B) = W^{k,2}(B)$ over an open set $B \subset \mathbb{R}^3$. For a at least C^k-regular surface $\Gamma = \partial B$, $k \geq 1$ we consider the Sobolev spaces $W^{k,p}(\Gamma)$ and $H^k(\Gamma) = W^{k,2}(\Gamma)$, see for example [2]. For a Banach space X we denote by X^* its dual and by $\langle \cdot, \cdot \rangle_{X^*,X}$ the duality product. For functions from a real interval $(0,T)$ to X we consider the Bochner spaces $L^p(0,T;X)$ and the Sobolev spaces $H^1(0,T;X)$. We recall the *definition of parabolic Sobolev spaces: for a measurable set $E \subset \mathbb{R}^n$, $T > 0$, $1 \leq p \leq \infty$, $k \in \mathbb{N}$ we let*

$$W_p^{2k,k}(E \times (0,T)) := \left\{ w \in L^p(E \times (0,T)) \,:\, \partial_t^l D^\gamma w \in L^p(E \times (0,T)) \text{ for all } 2l + |\gamma| \leq 2k,\, l \in \mathbb{N}_0, \gamma \in (\mathbb{N}_0)^n \right\}.$$

Concerning classical solution spaces we denote, for an open set $E \subset \mathbb{R}^n$, by $C^{k+\alpha}(\bar{E})$, $k \in \mathbb{N}_0$, $0 \leq \alpha < 1$ the set of functions such that all derivatives up to the order k are α-Hölder continuous on $\bar{E}$. For $\alpha = 1$ we use instead the notation $C^{k,\mathrm{lip}}(\bar{E})$. Concerning functions defined on a space–time domains we let $C^{1,0}(\bar{E} \times [0,T])$ be the space of continuous functions on $\bar{E} \times [0,T]$ with first spatial derivatives in $C^0(\bar{E} \times [0,T])$ and use for $2k \in \mathbb{N}_0$, $0 \leq \alpha < \frac{1}{2}$ the parabolic Hölder spaces

$$C^{2(k+\alpha),k+\alpha}(\bar{E} \times [0,T]) := \left\{ w \in C^\alpha(\bar{E} \times [0,T]) \,:\, \partial_t^l D^\gamma w \in C^\alpha(\bar{E} \times [0,T]) \text{ for all } 2l + |\gamma| \leq 2k,\, l \in \mathbb{N}_0, \gamma \in (\mathbb{N}_0)^n \right\}.$$

An open set B as above has $C^{k+\alpha}$-regular (or $C^{k,\mathrm{lip}}$-regular) boundary, if $\Gamma = \partial B$ can locally be written as the graph of a $C^{k+\alpha}$-regular (or $C^{k,\mathrm{lip}}$-regular) function that separates B from $\mathbb{R}^3 \backslash B$. For B with $C^{2k,\mathrm{lip}}$-regular boundary we define the spaces $C^{2(k+\alpha),k+\alpha}(\Gamma \times [0,T])$ as the space of functions $f : \Gamma \times [0,T] \to \mathbb{R}$ such that $f \circ \varphi \in C^{2(k+\alpha),k+\alpha}(\bar{E} \times [0,T])$ for any $C^{2k,\mathrm{lip}}$-local coordinates $\varphi : U \to \Gamma$ and $E \subset\subset U \subset \mathbb{R}^2$.

Let us state the main assumptions that we impose in the following.

Assumption 2.2. Let $B \subset \mathbb{R}^3$ be an open, bounded, connected set with C^2-regular boundary $\Gamma = \partial B$ and let $T > 0$. Assume that the constitutive functions $f_i, q_i : \mathbb{R}^2 \to \mathbb{R}$ are locally Lipschitz continuous and satisfy

$$0 \leq f_i, q_i \leq K_2 \quad \text{on } \mathbb{R}^2 \quad \text{for } i = 1, 2. \tag{2.1}$$

We further assume that the coefficients of the diffusion and drift operators are measurable functions, satisfy (1.5) and

$$|A_1(y,t)|, |A_2(y,t)|, |b_1(y,t)|, |b_2(y,t)| \leq K_1 \quad \text{for almost all } (y,t) \in \Gamma \times (0,T), \tag{2.2}$$

$$\min_{\xi \in T_y\Gamma, |\xi|=1} \xi \cdot A_k(y,t)\xi \geq k_1 \quad \text{for almost all } (y,t) \in \Gamma \times (0,T),\ k = 1, 2. \tag{2.3}$$

Finally, let nonnegative initial data $V_0 \in L^\infty(B)$ and $v_0, u_0 \in L^\infty(\Gamma)$ be given.

Our first main result is the existence of weak solutions (see Definition 3.1 below) for the general bulk–surface reaction–drift–diffusion system.

Theorem 2.3. *Let Assumption 2.2 hold. Then the system (1.6)–(1.10) has a unique weak solution (V, v, u) with $V \in L^2(0,T;H^1(B)) \cap H^1(0,T;H^1(B)^*)$, $u, v \in L^2(0,T;H^1(\Gamma)) \cap H^1(0,T;H^1(\Gamma)^*)$. Furthermore, solutions are nonnegative and uniformly bounded.*

To prove Theorem 2.3 we characterize the system (1.6)–(1.9) as an evolution equation for a pseudomonotone operator and obtain the existence of weak solutions by the general theory developed in [35]. We also show that solutions depend continuously on the data, see Proposition 3.7. Furthermore, in Theorem 3.6 we prove that the L^∞-norm of the solution grows at most exponentially with time (and therefore remains bounded on bounded time intervals).

In the case of more regular coefficients and data we prove the well-posedness of classical solutions.

Theorem 2.4. *Assume that Γ is $C^{2,\mathrm{lip}}$-regular and that $f_i, q_i \in C^2(\mathbb{R}^2)$, $i = 1,2$. For some $0 < \alpha < 1$ let $A_1, A_2 \in C^{1+\alpha,\frac{1+\alpha}{2}}(\Gamma \times [0,T])$, let $b_1, b_2 \in C^{\alpha,\frac{\alpha}{2}}(\Gamma \times [0,T])$ and assume that $u_0, v_0 \in C^{2+\alpha}(\Gamma)$ and $V_0 \in C^{2+\alpha}(\overline{B})$ are nonnegative. Assume further that the initial data satisfy the compatibility condition*

$$-D\nabla V_0 \cdot \nu = q_1(u_0,v_0)V_0 - q_2(u_0,v_0)v_0 \quad \text{on } \Gamma. \tag{2.4}$$

Then there exists a unique classical solution (V,u,v) of (1.6)–(1.10) with initial data (V_0,u_0,v_0). This solution satisfies $V \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{B} \times [0,T])$ and $u, v \in C^{2+\alpha,1+\frac{\alpha}{2}}(\Gamma \times [0,T])$.

We also prove the continuous dependence of solutions on the initial data, see Proposition 4.4 below. Theorem 2.4 is derived from higher-regularity properties of the weak solutions obtained above.

By a bootstrapping argument we deduce the following result for the original GTPase cycle model from [36]. Here we need to impose that the initial data satisfy compatibility conditions to any order $k \in \mathbb{N}_0$. These are given by the requirement that

$$\frac{d^k}{dt^k}\bigg|_{t=0} [-D\nabla V \cdot \nu - q_1(u,v)V + q_2(u,v)v] = 0 \quad \text{on } \Gamma. \tag{2.5}$$

Using the differential equations (1.6), (1.8), (1.9) one can reformulate (2.5) as a condition on the initial data only, see [23, pp. 319,320].

Theorem 2.5. *Let Assumption 2.2 be satisfied. Let the boundary $\Gamma \subset \mathbb{R}^3$, the nonlinearities f_i, q_i, $i = 1,2$ and the initial data (V_0,u_0,v_0) all be C^∞-regular. Assume that the initial data satisfy the compatibility condition (2.5) on Γ to any order $k \in \mathbb{N}_0$. Then there exists a unique classical solution (V,u,v) of (1.1)–(1.4). The functions V,u,v are C^∞-regular, nonnegative and uniformly bounded.*

Finally, we prove an asymptotic reduction of the system (RDD) to a kind of shadow system in the large-diffusion limit $D \to \infty$. This reduction is motivated by the application to cell signalling and the fact that cytosolic diffusion within the cell is by a factor of hundred larger than the lateral diffusion on the membrane. Formally, in the limit $D \to \infty$ equations (1.8), (1.9) should still hold. Furthermore, the total mass conservation of GTPase (1.11) should determine the spatially constant value of V in the limit. These considerations lead to the following limit problem.

Problem ***(rRDD)***. *Let nonnegative initial data $u_0, v_0 : \Gamma \to \mathbb{R}$ and a constant $\mathfrak{m} \geq 0$, representing the total mass in the system, be given such that*

$$\mathfrak{m} \geq \int_\Gamma \big(u_0 + v_0\big). \tag{2.6}$$

We are looking for functions $u_\infty, v_\infty : \Gamma \times (0,T) \to \mathbb{R}$, $V_\infty : (0,T) \to \mathbb{R}$ such that the reduced system

$$\partial_t u_\infty = \nabla_\Gamma \cdot (A_1 \nabla_\Gamma u_\infty) + b_1 \cdot \nabla_\Gamma u_\infty + f_1(u_\infty, v_\infty) v_\infty - f_2(u_\infty, v_\infty) u_\infty, \tag{2.7}$$

$$\begin{aligned} \partial_t v_\infty = {} & \nabla_\Gamma \cdot (A_2 \nabla_\Gamma v_\infty) + b_2 \cdot \nabla_\Gamma v_\infty - f_1(u_\infty, v_\infty) v_\infty + f_2(u_\infty, v_\infty) u_\infty \\ & + q_1(u_\infty, v_\infty) V_\infty - q_2(u_\infty, v_\infty) v_\infty, \end{aligned} \tag{2.8}$$

coupled to the mass conservation condition

$$|B| V_\infty(t) = \mathfrak{m} - \int_\Gamma \big(u_\infty(\cdot,t) + v_\infty(\cdot,t)\big) \quad \text{for all } t \in (0,T) \tag{2.9}$$

is satisfied, and such that the initial conditions

$$u_\infty(\cdot,0) = u_0, \quad v_\infty(\cdot,0) = v_0 \quad \text{on } \Gamma \tag{2.10}$$

hold.

For given $\mathfrak{m} \geq 0$ the system (rRDD) represents a nonlocal reaction–diffusion system on Γ in the variables u_∞, v_∞. The nonlocality is just the remnant of the spatial coupling in the original system (1.6)–(1.9).

Theorem 2.6. *Let Assumption 2.2 be satisfied. Consider any sequence $D_k \to \infty$ $(k \to \infty)$ and the solutions (V_k, u_k, v_k) of (1.6)–(1.10) with D replaced by D_k and fixed initial data (V_0, u_0, v_0). Then for $k \to \infty$*

$$u_k \to u_\infty, \quad v_k \to v_\infty, \quad V_k \to V_\infty \qquad \text{in } L^2(\Gamma \times (0,T)),$$

where $u_\infty, v_\infty \in L^2(0,T; H^1(\Gamma)) \cap H^1(0,T; H^1(\Gamma)^*)$, $V_\infty \in W^{1,\infty}(0,T)$ and $(V_\infty, u_\infty, v_\infty)$ is the unique weak solution of the nonlocal reaction–diffusion system (2.7)–(2.10) with $\mathfrak{m} := \int_B V_0 + \int_\Gamma \big(u_0 + v_0\big)$.

Let us finally compare our results to related results presented in [4,37,38], and give a brief exposition of the main techniques used in the proof of our results.

In [37,38] bulk–surface systems with general reaction terms and spatially homogeneous diffusion are considered. An arbitrary number of bulk variables and an arbitrary number of surface variables is allowed. The main result in [37] is the global existence of classical solutions and in [38] a uniform L^∞-bound for all times (not only on arbitrary bounded time intervals). Concerning the assumptions on the nonlinearities any polynomial growth is allowed. On the other hand, additional balance conditions are imposed and a certain cancellation of higher-order nonlinearities is required. These conditions are in general not satisfied in the case of the system (1.1)–(1.4). General drift–diffusion operators as in Problem (RDD) and an asymptotic reduction in the case of large bulk diffusion are not considered in [37,38].

Another related analysis has been recently contributed in [4], where a general bulk–surface reaction–diffusion system with homogeneous diffusion and with bulk advection has been considered in a cylindrical domain and for nonlinear sorption and reaction terms. Global existence of $W_p^{2,1}$-regular solutions has been shown, with different techniques and under different conditions compared to our analysis. In particular, in [4] a more general growth condition is imposed, but a monotonicity of the reaction terms and exchange rates that only depend on one bulk and one surface species are assumed. Again, general drift–diffusion operators as in Problem (RDD) and an asymptotic reduction in the case of large bulk diffusion are not considered.

The techniques used in [4,37,38] and in the present paper are different, though similar tools are used in the auxiliary estimates. In our paper we use the theory of evolutions by pseudo-monotone operators and first prove the long-time existence of weak solutions. Using energy methods we then prove uniform L^∞-bounds. These estimates represent the most delicate part of this analysis. We use the uniform maximum bounds to further prove the $W_p^{2,1}$-regularity of solutions, Hölder regularity, and the existence of classical solutions. The linear-growth assumption allows us to use energy methods, which are somehow simpler (and possibly more flexible) as the arguments used in [37,38].

3. Weak solutions of the system (RDD)

We first definition our weak solution concept. We denote by $\langle\cdot,\cdot\rangle_B$ and $\langle\cdot,\cdot\rangle_\Gamma$ the duality pairing in $H^1(B)^* \times H^1(B)$ and $H^1(\Gamma)^* \times H^1(\Gamma)$, respectively.

In the following, we combine the reaction terms and the exchange terms into functions $f, q : \mathbb{R}^2 \to \mathbb{R}$ and let

$$f(u,v) = f_1(u,v)v - f_2(u,v)u, \qquad q(u,v,V) = q_1(u,v)V - q_2(u,v)v. \tag{3.1}$$

Definition 3.1. A triple (V,u,v) with $V \in L^2(0,T;H^1(B)) \cap H^1(0,T;H^1(B)^*)$ and $u,v \in L^2(0,T;H^1(\Gamma)) \cap H^1(0,T;H^1(\Gamma)^*)$ is a weak solution of (1.6)–(1.9) if

$$\int_0^T \left(\langle \partial_t V, \eta_1 \rangle_B + \int_B D\nabla V \cdot \nabla \eta_1 \right) = \int_0^T \int_\Gamma -q(V,u,v)\eta_1, \tag{3.2}$$

$$\int_0^T \left(\langle \partial_t u, \eta_2 \rangle_\Gamma + \int_\Gamma \left(\nabla_\Gamma \eta_2 \cdot A_1 \nabla_\Gamma u - \eta_2 b_1 \cdot \nabla_\Gamma u \right) \right) = \int_0^T \int_\Gamma f(u,v)\eta_2, \tag{3.3}$$

$$\int_0^T \left(\langle \partial_t v, \eta_3 \rangle_\Gamma + \int_\Gamma \left(\nabla_\Gamma \eta_3 \cdot A_2 \nabla_\Gamma v - \eta_3 b_2 \cdot \nabla_\Gamma v \right) \right)$$
$$= \int_0^T \int_\Gamma (q(V,u,v) - f(u,v))\eta_3 \tag{3.4}$$

is satisfied for all $\eta_1 \in L^2(0,T;H^1(B))$, $\eta_2, \eta_3 \in L^2(0,T;H^1(\Gamma))$. A weak solution in addition satisfies (1.10) if

$$\lim_{t \searrow 0} (V,u,v)(t) = (V_0,u_0,v_0) \quad \text{in } L^2(B) \times L^2(\Gamma) \times L^2(\Gamma). \tag{3.5}$$

Note that the regularity required in the weak solution concept implies by [9, Theorem 5.9.3] that V belongs to $C^0([0,T];L^2(B))$ and u,v belong to $C^0([0,T];L^2(\Gamma))$ and that therefore the left-hand side in (3.5) is well-defined.

We reformulate the weak solution concept in a more concise form that allows us to apply the general theory of evolutions by pseudomonotone operators developed in [35]. Therefore, we define the Hilbert space $H := H^1(B)\times H^1(\Gamma)\times H^1(\Gamma)$ and define operators $F_t : H \to H^*$, $t\in(0,T)$ by

$$F_t := F_t^{(1)} + F_t^{(2)}, \tag{3.6}$$

$$\begin{aligned}\langle F_t^{(1)}(w),(\eta)\rangle &:= \int_B \nabla\eta_1\cdot D\nabla w_1 + \int_\Gamma (\nabla_\Gamma\eta_2\cdot A_1(\cdot,t)\nabla_\Gamma w_2\\ &\quad + \nabla_\Gamma\eta_3\cdot A_2(\cdot,t)\nabla_\Gamma w_3),\end{aligned} \tag{3.7}$$

$$\begin{aligned}\langle F_t^{(2)}(w),(\eta)\rangle &:= \int_\Gamma (-\eta_2 b_1(\cdot,t)\cdot\nabla_\Gamma w_2 - \eta_3 b_2(\cdot,t)\cdot\nabla_\Gamma w_3) +\\ &\quad + \int_\Gamma (q(w_1,w_2,w_3)(\eta_1-\eta_3) - f(w_2,w_3)(\eta_2-\eta_3)),\end{aligned} \tag{3.8}$$

for $w=(w_1,w_2,w_3), \eta=(\eta_1,\eta_2,\eta_3)\in H$. We show below in Lemma 3.2 that F_t is well-defined for almost all $t\in(0,T)$ and satisfies an appropriate growth condition. Furthermore, (V,u,v) is a weak solution of (1.6)–(1.9) if and only if $(V,u,v)\in L^2(0,T;H)\cap H^1(0,T;H^*)$ such that for almost all $t\in(0,T)$

$$\partial_t\big(V(t),u(t),v(t)\big) + F_t\big(V(t),u(t),v(t)\big) = 0 \quad \text{in } H^*. \tag{3.9}$$

We first show that the operators F_t are bounded, semi-coercive (i.e. satisfying a Gårding inequality) and pseudo-monotone. It is this step that is specific to the bulk–surface coupling in our particular model, without the presence of the coupling terms f,q the respective properties are well-known, see for example [35, Section 2.4].

Lemma 3.2. *For almost all $t\in(0,T)$ the following properties hold:*

(1) $F_t : H\to H^*$ is well-defined and satisfies $\|F_t(w)\|_{H^*}\le C\cdot(D+K_1+K_2)\|w\|_H$ for all $w\in H$.

(2) $F_t^{(1)}$ is a monotone mapping, $F_t^{(2)}$ is totally continuous (i.e. continuous as a map from H, equipped with the weak topology, to H^*, equipped with the norm topology), and F_t is pseudo-monotone.

(3) F_t is semi-coercive on H: For any $w=(w_1,w_2,w_3)\in H$

$$\begin{aligned}\langle F_t(w,w)\rangle &\ge \frac{D}{2}\int_B |\nabla w_1|^2 + \frac{k_1}{2}\int_\Gamma \big(|\nabla_\Gamma w_2|^2 + |\nabla_\Gamma w_3|^2\big)\\ &\quad - C(K_2,B)\frac{D+1}{D}\int_B w_1^2 - C(K_2,K_1,k_1)\int_\Gamma (w_2^2+w_3^2)\end{aligned} \tag{3.10}$$

holds.

Proof. We fix an arbitrary $t\in(0,T)$ such that $|A_1(\cdot,t)|$, $|A_2(\cdot,t)|$, $|b_1(\cdot,t)|$, $|b_2(\cdot,t)|$ are all uniformly bounded by K_1 almost everywhere on Γ and such that $A_1(\cdot,t)$, $A_2(\cdot,t)$ are elliptic with ellipticity constant k_1 almost everywhere in Γ. To improve readability, we omit in the following the t argument in the coefficients A_i, b_i, $i = 1, 2$.

(1) By Hölder inequality and the continuity of the trace operator from $H^1(B)$ to $L^2(\Gamma)$ we easily deduce that

$$\begin{aligned}\left|\langle F_t(w), \eta\rangle\right| \le\ & D\|\nabla w_1\|_{L^2(B)}\|\nabla\eta_1\|_{L^2(B)}\\ &+ K_1\big(\|\nabla_\Gamma w_2\|_{L^2(\Gamma)}\|\nabla_\Gamma\eta_2\|_{L^2(\Gamma)} + \|\nabla_\Gamma w_3\|_{L^2(\Gamma)}\|\nabla_\Gamma\eta_3\|_{L^2(\Gamma)}\big)\\ &+ K_1\big(\|\nabla_\Gamma w_2\|_{L^2(\Gamma)}\|\eta_2\|_{L^2(\Gamma)} + \|\nabla_\Gamma w_3\|_{L^2(\Gamma)}\|\eta_3\|_{L^2(\Gamma)}\big)\\ &+ 2K_2\big(\|w_1\|_{L^2(\Gamma)} + \|w_2\|_{L^2(\Gamma)} + \|w_3\|_{L^2(\Gamma)}\big)\big(\|\eta_1\|_{L^2(\Gamma)}\\ &+ \|\eta_2\|_{L^2(\Gamma)} + \|\eta_3\|_{L^2(\Gamma)}\big)\\ \le\ & C\cdot(D + K_1 + K_2)\|w\|_H\|\eta\|_H.\end{aligned}$$

This shows the boundedness of $F_t : H \to H^*$.

(2) $F_t^{(1)}$ is clearly monotone. By the compact embedding $H^1(\Gamma) \hookrightarrow L^2(\Gamma)$, the growth bound on f, and the generalized Lebesgue Dominated Convergence Theorem, the mapping $(u, v) \mapsto f(u, v)$ is totally continuous as a map from $H^1(\Gamma)\times H^1(\Gamma)$, equipped with the weak topology, to $H^1(\Gamma)^*$. Similarly, using in addition the compact embedding $H^1(B) \hookrightarrow L^s(\Gamma)$ for any $1 \le s < 4$, we also have that the mapping $(V, u, v) \mapsto q(V, u, v)$ is totally continuous as mapping from $H^1(B) \times H^1(\Gamma) \times H^1(\Gamma)$ to $H^1(\Gamma)^*$ and as a mapping to $H^1(B)^*$. This yields that $F_t^{(2)}$ is totally continuous. By [35, Lemma, 2.11, Corollary 2.12], F_t is pseudo-monotone.

(3) Concerning the coercivity of the operator we compute that

$$\begin{aligned}&\langle F_t(w), w\rangle_{H^*,H}\\ &\quad= \int_B D|\nabla w_1|^2 + \int_\Gamma \left(\nabla_\Gamma w_2\cdot A_1\nabla_\Gamma w_2 + \nabla_\Gamma w_3\cdot A_2\nabla_\Gamma w_3\right)\\ &\qquad+ \int_\Gamma \big(b_1\cdot w_2\nabla_\Gamma w_2 + b_2\cdot w_3\nabla_\Gamma w_3 + \big(q(w_2, w_3, w_1)(w_1 - w_3)\\ &\qquad- f(w_2, w_3)(w_2 - w_3)\big)\big)\\ &\quad\ge \int_B D|\nabla w_1|^2 + \int_\Gamma \left(k_1\left(|\nabla_\Gamma w_2|^2 + |\nabla_\Gamma w_3|^2\right)\right)\\ &\qquad+ \int_\Gamma \big(-K_1\big(|w_2|\,|\nabla_\Gamma w_2| + |w_3|\,|\nabla_\Gamma w_3|\big) + \big(q(w_2, w_3, w_1)(w_1 - w_3)\\ &\qquad- f(w_2, w_3)(w_2 - w_3)\big)\big).\end{aligned} \tag{3.11}$$

To control the last integral on the right-hand side we first use Young's inequality to obtain

$$K_1\big(|w_2|\,|\nabla_\Gamma w_2| + |w_3|\,|\nabla_\Gamma w_3|\big) \le \frac{k_1}{2}\Big(|\nabla_\Gamma w_2|^2 + |\nabla_\Gamma w_3|^2\Big) + \frac{K_1^2}{2k_1}\Big(|w_2|^2 + |w_3|^2\Big).$$

Next, we observe that by (2.1), (3.1)

$$\begin{aligned} & q(w_2,w_3,w_1)(w_1-w_3) - f(w_2,w_3)(w_2-w_3) \\ & \quad = -\big(q_2(w_2,w_3)+q_1(w_2,w_3)\big)w_3w_1 - \big(f_2(w_2,w_3)+f_1(w_2,w_3)\big)w_2w_3 \\ & \qquad + q_1(w_2,w_3)w_1^2 + q_2(w_2,w_3)w_3^2 + f_2(w_2,w_3)w_2^2 + f_1(w_2,w_3)w_3^2 \\ & \quad \geq -2K_2(|w_3||w_1| + |w_2||w_3|) \\ & \quad \geq -K_2\big(w_1^2 + w_2^2 + 2w_3^2\big). \end{aligned}$$

Furthermore, by [18, Theorem 1.5.1.10] we deduce

$$\int_\Gamma w_1^2 \leq \int_B \left(\frac{D}{2}|\nabla w_1|^2 + \frac{C(B)}{D} w_1^2\right). \tag{3.12}$$

The inequalities (3.11)–(3.12) yield that

$$\begin{aligned} & \Big\langle F_t(w_1,w_2,w_3),(w_1,w_2,w_3)\Big\rangle \geq +\frac{D}{2}\int_B |\nabla w_1|^2 \\ & \quad + \frac{k_1}{2}\int_\Gamma \big(|\nabla_\Gamma w_2|^2 + |\nabla_\Gamma w_3|^2\big) \\ & \quad - C(K_2,B)\frac{D+1}{D}\int_B w_1^2 - C(K_2,K_1,k_1)\int_\Gamma (w_2^2+w_3^2). \end{aligned} \tag{3.13}$$

This proves the semi-coercivity of the operator F. □

The existence of weak solutions for the system (RDD) can now be proved by using an implicit time-discretization, as elaborated in [35, Section 8].

Proposition 3.3. *There exists a weak solution (V,u,v) of (1.6)–(1.10) in the sense of Definition 3.1.*

Proof. The conclusion follows by [35, Theorem 8.9]. There the autonomous case of coefficients A_i, b_i, $i=1,2$ independent of t is covered, but the result also holds in the non-autonomous case, see [35, Remark 8.21]. The theorem is here applied to the Gelfand triple (H, H_0, H^*) with $H_0 := L^2(B)\times L^2(\Gamma)\times L^2(\Gamma)$. The required growth condition is satisfied by item (1) in Lemma 3.2, the required pseudo-monotonicity and semi-coercivity was also shown in the same lemma. □

In the following we collect further properties of solutions.

Proposition 3.4. *Assume that (V,u,v) is a weak solution of (1.6)–(1.10). Then there exists a constant $C_5 = C_5(K_1,K_2,k_1,B,T,D)$ such that the following estimate holds,*

$$\begin{aligned} & \|V\|_{L^\infty(0,T;L^2(B))} + \|u\|_{L^\infty(0,T;L^2(\Gamma))} + \|v\|_{L^\infty(0,T;L^2(\Gamma))} \\ & \quad + \sqrt{D}\|\nabla V\|_{L^2(0,T;L^2(B))} + \|\nabla_\Gamma u\|_{L^2(0,T;L^2(\Gamma))} + \|\nabla_\Gamma v\|_{L^2(0,T;L^2(\Gamma))} \\ & \quad + \frac{1}{D}\|\partial_t V\|_{L^2(0,T;H^1(B)^*)} + \|\partial_t u\|_{L^2(0,T;H^1(\Gamma)^*)} + \|\partial_t v\|_{L^2(0,T;H^1(\Gamma)^*)} \\ & \leq C_5\Big(\|V_0\|_{L^2(B)} + \|u_0\|_{L^2(\Gamma)} + \|v_0\|_{L^2(\Gamma)}\Big). \end{aligned} \tag{3.14}$$

holds. The constants C_5 can be chosen non-increasing in D. If the initial data V_0, u_0, v_0 are all nonnegative, then also V, u, v are all nonnegative.

Proof. By (3.9), the semi-coercivity (3.10), and [9, Theorem 5.9.3] we first deduce that in $L^1(0,T)$

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\left(\int_B V^2(\cdot,t) + \int_\Gamma (u^2+v^2)(\cdot,t)\right)\\
&\quad = -\Big\langle F_t\big((V,u,v)(\cdot,t)\big),(V,u,v)(\cdot,t)\Big\rangle\\
&\quad \le -\left(\frac{D}{2}\int_B |\nabla w_1|^2 + \frac{k_1}{2}\int_\Gamma \big(|\nabla_\Gamma w_2|^2 + |\nabla_\Gamma w_3|^2\big)\right)(\cdot,t)\\
&\qquad + C(K_1,K_2,k_1,B)\left(\frac{D+1}{D}\int_B V^2 + \int_\Gamma (u^2+v^2)\right)(\cdot,t).
\end{aligned}$$

Using Gronwalls inequality yields the required bounds on V, u, v and their gradients. The estimates for the time derivatives follows from the bound on F_t proved in Lemma 3.2, (3.9) and the bounds on V, u, v and their gradients. Altogether, (3.14) follows.

For nonnegative initial data we follow the usual technique to prove a maximum principle in spaces of weakly differentiable functions (see for example [5, Theorem 11.9]). For that matter, similarly as above, we control the time derivative of the spatial square integrals of the negative parts V_-, u_-, and v_- of V, u, v by using the quasi-coercivity and, in addition, the quasi-positivity of f, q that implies

$$\begin{aligned}
&- q(u,v,V)(V^- - v^-) + f(u,v)(u^- - v^-)\\
&\qquad \ge\ q_2(u,v)vV^- + q_1(u,v)Vv^- + f_1(u,v)vu^- + f_2(u,v)uv^-\\
&\qquad \ge\ -K_2\big((V^-)^2 + (u^-)^2 + 2(v^-)^2\big).
\end{aligned}$$

Following the estimates above this shows by a Gronwall argument that $V_- = 0$ in $B\times(0,T)$ and $u_- = v_- = 0$ on $\Gamma\times(0,T)$. □

The key step for the higher regularity results below is a uniform maximum bound for the solutions. We fix an arbitrary $\lambda > 0$ (to be chosen below) and derive first suitable estimates for the modified system

$$\partial_t V = D\Delta V - \lambda V, \tag{3.15}$$

$$-D\nabla V\cdot\nu = q(u,v,V), \tag{3.16}$$

$$\partial_t u = \nabla_\Gamma\cdot(A_1\nabla_\Gamma u) + b_1\cdot\nabla_\Gamma u + f(u,v) - \lambda u, \tag{3.17}$$

$$\partial_t v = \nabla_\Gamma\cdot(A_2\nabla_\Gamma v) + b_2\cdot\nabla_\Gamma v - f(u,v) + q(u,v,V) - \lambda v \tag{3.18}$$

on $B\times(0,T)$ and $\Gamma\times(0,T)$, respectively, and subject to prescribed initial data (1.10).

Theorem 3.5. *Let nonnegative initial data V_0, u_0, v_0 be given with*

$$\sup_B V_0 + \sup_\Gamma u_0 + \sup_\Gamma v_0 \ \le\ \Lambda_0 \tag{3.19}$$

for some $\Lambda_0 > 0$. Then there exists $\lambda_0 > 0$ only depending on B, D, K_1, k_1, K_2, such that for any $\lambda > \lambda_0$ and any weak solution of (3.15)–(3.18) with initial data V_0, u_0, v_0

$$\sup_{B\times(0,T)} V + \sup_{\Gamma\times(0,T)} u + \sup_{\Gamma\times(0,T)} v \leq \Lambda_0 \tag{3.20}$$

holds. Moreover, λ_0 can be chosen non-increasing in D.

Proof. Step (1): Let $\kappa > 0$ be arbitrary, to be chosen below. Then there exists a function $\phi \in C^2(\bar{B})$ such that

$$1 \leq \phi \leq 2 \quad \text{and} \quad |\nabla\phi|, |D^2\phi| \leq C(B)\frac{\kappa}{D} \qquad \text{in } \bar{B}, \tag{3.21}$$

$$\phi = 1 \quad \text{and} \quad -D\nabla\phi\cdot\nu = \kappa \qquad \text{on } \Gamma. \tag{3.22}$$

In fact, by [16] there exists $\delta_0 > 0$ such that the signed distance function $\vartheta := \text{dist}(\cdot, B^c) - \text{dist}(\cdot, B)$ to Γ is C^2-regular in the set $\{|\vartheta| < \delta_0\}$. We then set $\delta := \min\{\frac{D}{\kappa}, \delta_0\}$ and define

$$\phi := 1 + \frac{\kappa}{D}\vartheta \quad \text{in } \{|\vartheta| < \frac{\delta}{4}\}.$$

In the set $\{|\vartheta| < \frac{\delta}{4}\}$ we then have

$$|\phi - 1| \leq \frac{\kappa}{D}\frac{\delta}{4} \leq \frac{1}{4}, \qquad \nabla\phi = \frac{\kappa}{D}\nabla\vartheta, \qquad |\nabla\phi| \leq \frac{\kappa}{D}, \quad |D^2\phi| \leq C(B)\frac{\kappa}{D}.$$

Since $\nabla\vartheta = -\nu$ on Γ we obtain (3.22). Moreover, we see that there exists a C^2 regular extension of ϕ on $\bar{B}$ with (3.21).

Step (2): We compute that $\tilde{V} := \phi V$ solves the equation

$$\partial_t\tilde{V} - D\Delta\tilde{V} + 2D\frac{\nabla\phi}{\phi}\cdot\nabla\tilde{V} - \left(2D\frac{|\nabla\phi|^2}{\phi^2} - D\frac{\Delta\phi}{\phi} - \lambda\right)\tilde{V} = 0 \quad \text{in } B\times(0,T) \tag{3.23}$$

and satisfies

$$-D\nabla\tilde{V}\cdot\nu = q(u, v, V) + \kappa V \quad \text{on } \Gamma\times(0,T). \tag{3.24}$$

By (3.21) we deduce that

$$2D\frac{|\nabla\phi|^2}{\phi^2} - D\frac{\Delta\phi}{\phi} \leq C(B)\left(\frac{2\kappa^2}{D} + \kappa\right). \tag{3.25}$$

We assume from now on that $\lambda_0 > 0$ satisfies, for $C(B)$ from (3.25),

$$C(B)\left(\frac{2\kappa^2}{D} + \kappa\right) \leq \frac{1}{2}\lambda_0. \tag{3.26}$$

Step (3): Next we consider an arbitrary $M > 0$ (to be chosen below) and test (3.23) by $(\tilde{V} - M)_+$. Using (3.24) we then obtain, in a weak sense,

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int_B (\tilde{V} - M)_+^2 + \int_B D|\nabla(\tilde{V} - M)_+|^2 \\
&= -2D\int_B (\tilde{V} - M)_+ \frac{\nabla\phi}{\phi}\cdot\nabla(\tilde{V} - M)_+ \\
&\quad + \int_B \Big(2D\frac{|\nabla\phi|^2}{\phi^2} - D\frac{\Delta\phi}{\phi} - \lambda\Big)\tilde{V}(\tilde{V} - M)_+ \\
&\quad - \int_\Gamma \Big(q(u,v,V)(V - M)_+ + \kappa V(V - M)_+\Big) \\
&=: I_B + I_\Gamma.
\end{aligned} \tag{3.27}$$

For the sum of the integrals over B that are collected in the term I_B we further deduce from (3.21) and (3.25), (3.26) that

$$\begin{aligned}
I_B \leq \frac{D}{2}\int_B |\nabla(\tilde{V} - M)_+|^2 + C(B)\frac{\kappa^2}{D}\int_B (\tilde{V} - M)_+^2 \\
-\frac{\lambda}{2}\int_B \Big((\tilde{V} - M)_+^2 + (\tilde{V} - M)_+ M\Big).
\end{aligned}$$

We assume from now on that λ_0 satisfies (3.26) and in addition

$$\lambda_0 \geq 4C(B)\frac{\kappa^2}{D}. \tag{3.28}$$

Then

$$I_B \leq \frac{D}{2}\int_B |\nabla(\tilde{V} - M)_+|^2 - \frac{\lambda}{4}\int_B \Big((\tilde{V} - M)_+^2 + 2(\tilde{V} - M)_+ M\Big). \tag{3.29}$$

We next consider the term I_Γ, representing the integral over Γ on the right-hand side of (3.27). For an arbitrary $m_2 > 0$ we set $\kappa := \max\{1, K_2\frac{m_2}{M}\}$. This implies that

$$\begin{aligned}
I_\Gamma &\leq \int_\Gamma \Big(q_2(u,v)v(V - M)_+ - \kappa V(V - M)_+\Big) \\
&\leq \int_\Gamma \Big(K_2(v - m_2)_+(V - M)_+ + K_2 m_2(V - M)_+ \\
&\quad - \kappa(V - M)_+^2 - \kappa M(V - M)_+\Big) \\
&\leq \int_\Gamma \Big(\frac{K_2^2}{2}(v - m_2)_+^2 - \frac{1}{2}(2\kappa - 1)(V - M)_+^2 \\
&\quad -(M\kappa - K_2 m_2)(V - M)_+\Big) \\
&\leq \int_\Gamma \Big(\frac{K_2^2}{2}(v - m_2)_+^2 - \frac{1}{2}(V - M)_+^2\Big).
\end{aligned} \tag{3.30}$$

We therefore conclude from (3.27), (3.29), (3.30) that

$$\begin{aligned} &\frac{d}{dt}\frac{1}{2}\int_B (\tilde{V}-M)_+^2 + \frac{D}{2}\int_B |\nabla(\tilde{V}-M)_+|^2 \\ &\quad\le -\frac{\lambda}{4}\int_B \Big((\tilde{V}-M)_+^2 + (\tilde{V}-M)_+ 2M\Big) \\ &\qquad + \int_\Gamma \Big(\frac{K_2^2}{4}(v-m_2)_+^2 - \frac{1}{2}(V-M)_+^2\Big). \end{aligned} \tag{3.31}$$

Step (4): We next test for an arbitrary $m_2 > 0$ equation (3.18) with $(v-m_2)_+$ and obtain

$$\begin{aligned} &\frac{d}{dt}\frac{1}{2}\int_\Gamma (v-m_2)_+^2 + k_1\int_\Gamma |\nabla_\Gamma (v-m_2)_+|^2 \\ &\quad\le \frac{d}{dt}\frac{1}{2}\int_\Gamma (v-m_2)_+^2 + \int_\Gamma \nabla_\Gamma (v-m_2)_+ \cdot A_2\nabla_\Gamma (v-m_2)_+ \\ &\quad= \int_\Gamma \Big(b_2\cdot\nabla_\Gamma(v-m_2)_+ - f(u,v) + q(u,v,V) - \lambda v\Big)(v-m_2)_+ \\ &\quad\le \frac{k_1}{2}\int_\Gamma |\nabla_\Gamma(v-m_2)_+|^2 + \int_\Gamma \frac{K_1^2}{2k_1}(v-m_2)_+^2 \\ &\qquad + \int_\Gamma \big(q_1(u,v)V + f_2(u,v)u\big)(v-m_2)_+ \\ &\qquad - \int_\Gamma \Big(\lambda(v-m_2)_+^2 + \lambda m_2(v-m_2)_+\Big). \end{aligned} \tag{3.32}$$

For an arbitrary $m_1 > 0$ we compute for the first integral in the last line

$$\begin{aligned} &\int_\Gamma \big(q_1(u,v)V + f_2(u,v)u\big)(v-m_2)_+ \\ &\quad\le \int_\Gamma K_2\Big(\big((V-M)_+ + (u-m_1)_+\big)(v-m_2)_+ + K_2(M+m_1)(v-m_2)_+\Big) \\ &\quad\le \int_\Gamma \Big(\frac{1}{2}(V-M)_+^2 + \frac{K_2}{2}(u-m_1)_+^2 + \frac{K_2}{2}(K_2+1)(v-m_2)_+^2\Big) \\ &\qquad + \int_\Gamma K_2(M+m_1)(v-m_2)_+. \end{aligned} \tag{3.33}$$

We next prescribe that λ_0 satisfies (3.26), (3.28) and

$$\lambda_0 \;\ge\; \max\Big\{\frac{K_1^2}{k_1} + K_2(K_2+1), K_2\frac{M+m_1}{m_2}\Big\} \tag{3.34}$$

and obtain from (3.32), (3.33) that

$$\begin{aligned} &\frac{d}{dt}\int_\Gamma (v-m_2)_+^2 + \frac{k_1}{2}\int_\Gamma |\nabla_\Gamma(v-m_2)_+|^2 \;\le\; \int_\Gamma \Big(\frac{1}{2}(V-M)_+^2 \\ &\qquad + \frac{K_2}{2}(u-m_1)_+^2 - \frac{\lambda}{2}(v-m_2)_+^2\Big). \end{aligned} \tag{3.35}$$

Step (5): Next, we test for an arbitrary $m_1 > 0$ equation (3.17) with $(u-m_1)_+$. By similar calculations as in the previous step we derive the inequality

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int_\Gamma (u-m_1)_+^2 + \frac{k_1}{2}\int_\Gamma |\nabla_\Gamma (u-m_1)_+|^2 \\
&\quad\le \int_\Gamma \frac{K_1^2}{2k_1}(u-m_1)_+^2 + \int_\Gamma f_1(u,v)v(u-m_1)_+ \\
&\qquad - \int_\Gamma \Big(\lambda(u-m_1)^2 + \lambda m_1 (u-m_1)_+\Big) \\
&\quad\le \int_\Gamma \frac{K_1^2}{2k_1}(u-m_1)_+^2 + \int_\Gamma \bigg(\frac{K_2}{2}(v-m_2)_+^2 + \frac{K_2}{2}(u-m_1)_+^2 \\
&\qquad + K_2 m_2 (u-m_1)_+\bigg) \\
&\qquad - \int_\Gamma \Big(\lambda(u-m_1)^2 + \lambda m_1 (u-m_1)_+\Big).
\end{aligned} \tag{3.36}$$

If we prescribe that λ_0 in addition to (3.26), (3.28), (3.34) satisfies

$$\lambda_0 \ \ge\ K_2 \frac{m_2}{m_1} \tag{3.37}$$

the estimate (3.36) yields

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int_\Gamma (u-m_1)_+^2 + \frac{k_1}{2}\int_\Gamma |\nabla_\Gamma (u-m_1)_+|^2 \ \le\ \int_\Gamma \bigg(\frac{K_2}{2}(v-m_2)_+^2 \\
&\quad -\frac{\lambda}{2}(u-m_1)_+^2\bigg).
\end{aligned} \tag{3.38}$$

Step (6): Summing up the inequalities (3.31), (3.35) and (3.38) we conclude that

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\left(\int_B (V-M)_+^2 + \int_\Gamma (u-m_1)_+^2 + \int_\Gamma (v-m_2)_+^2\right) \\
&\quad \le \int_\Gamma \left(\frac{1}{2}(K_2-\lambda)(u-m_1)^2 + \frac{1}{4}(K_2^2 + 4K_2 - 2\lambda)(v-m_2)_+^2\right) \ \le\ 0,
\end{aligned} \tag{3.39}$$

if in addition to (3.26), (3.28), (3.34), (3.37) λ_0 satisfies

$$\lambda_0 \ \ge\ 2K_2 + \frac{1}{2}K_2^2. \tag{3.40}$$

Hence for $M, m_1, m_2 > 0$ with

$$m_1 \ \ge\ \sup_\Gamma u_0, \quad m_2 \ \ge\ \sup_\Gamma v_0, \quad M \ \ge\ \sup_B V_0 \tag{3.41}$$

and λ_0 complying with all the above constraints, we deduce from (3.39) that

$$\begin{aligned}
(V-M)_+ = 0 \quad &\text{in } B\times(0,T), \qquad (u-m_1)_+ \ =\ (v-m_2)_+ \\
= 0 \quad &\text{in } \Gamma\times(0,T)
\end{aligned} \tag{3.42}$$

hold. We note that λ_0 depends through the conditions (3.26), (3.28), (3.34), (3.37) and (3.40) on B, on D (this dependence can be chosen nonincreasing in

D), on $K_2, \frac{m_2}{M}, K_1, k_1, \frac{M+m_1}{m_2}, \frac{m_2}{m_1}$. In particular, choosing $M = m_1 = m_2 = \Lambda_0$, we see that λ_0 can be chosen independently of Λ_0 and we deduce from (3.42) the claimed maximum bounds. □

The previous theorem yields that any weak solution of the system (RDD) grows at most exponentially with time.

Theorem 3.6. *Let nonnegative initial data* V_0, u_0, v_0 *be given that satisfy* (3.19) *and let* λ_0 *be chosen as in Theorem 3.5. Then for any weak solution of* (1.6)–(1.10)

$$\sup_{B\times(0,T)} V + \sup_{\Gamma\times(0,T)} u + \sup_{\Gamma\times(0,T)} v \le e^{\lambda_0 T}\Lambda_0 \tag{3.43}$$

holds.

Proof. For $\lambda \ge \lambda_0$ consider the functions

$$\begin{aligned} \tilde V(x,t) &= e^{-\lambda t} V(x,t) \qquad \text{for } 0<t<T, x\in B,\\ \tilde u(y,t) &= e^{-\lambda t} u(y,t), \quad \tilde v(y,t) = e^{-\lambda t} v(y,t), \qquad \text{for } 0<t<T, y\in\Gamma. \end{aligned}$$

Then $(\tilde V, \tilde u, \tilde v)$ is a weak solutions of the system (3.15)–(3.18) with initial data V_0, u_0, v_0, where the nonlinearities f, q are replaced by functions $\tilde f, \tilde q$

$$\begin{aligned} \tilde f(t,\tilde u,\tilde v) &= f_1(e^{\lambda t}\tilde u, e^{\lambda t}\tilde v)\tilde v - f_2(e^{\lambda t}\tilde u, e^{\lambda t}\tilde v)\tilde u,\\ \tilde q(t,\tilde u,\tilde v,\tilde V) &= q_1(e^{\lambda t}\tilde u, e^{\lambda t}\tilde v)\tilde V - q_2(e^{\lambda t}\tilde u, e^{\lambda t}\tilde v)\tilde v. \end{aligned}$$

Since $\tilde f, \tilde q$ satisfy the same bounds as required for f, q, Theorem 3.5 (the additional time-dependence of $\tilde f, \tilde q$ does not affect the arguments in the proof) yields that

$$\sup_{B\times(0,T)} \tilde V + \sup_{\Gamma\times(0,T)} \tilde u + \sup_{\Gamma\times(0,T)} \tilde v \le \Lambda_0.$$

Scaling back to the original variables this implies (3.43).

To complete the proof of well-posedness of problem (RDD) we show the uniqueness and the continuous dependence of solutions on the initial data.

Proposition 3.7. *For any* $\Lambda_0 > 0$ *there exists a constant* $C_6 = C_6(B, D, K_1, k_1, K_2, T, \Lambda_0)$ *with the following property: For any two weak solutions* (V, u, v) *and* $(\tilde V, \tilde u, \tilde v)$ *of* (1.6)–(1.10) *with initial data* (V_0, u_0, v_0) *and* $(\tilde V_0, \tilde u_0, \tilde v_0)$*, respectively, that both satisfy the maximum bound* (3.19) *the estimate*

$$\begin{aligned} &\|V-\tilde V\|_{L^2(0,T;H^1(B))\cap L^\infty(0,T;L^2(B))} + \|u-\tilde u\|_{L^2(0,T;H^1(\Gamma))\cap L^\infty(0,T;L^2(\Gamma))}\\ &\qquad + \|v-\tilde v\|_{L^2(0,T;H^1(\Gamma))\cap L^\infty(0,T;L^2(\Gamma))}\\ &\le C_6\Big(\|V_0-\tilde V_0\|_{L^2(B)} + \|u_0-\tilde u_0\|_{L^2(\Gamma)} + \|v_0-\tilde v_0\|_{L^2(\Gamma)}\Big) \end{aligned} \tag{3.44}$$

holds. The constants C_6 *can be chosen non-increasing in* D.

Proof. We consider the difference between the two solutions, the triplet (Z, w, z) with

$$Z := V - \tilde V, \qquad w := u - \tilde u, \qquad z := v - \tilde v.$$

We then have

$$\partial_t Z = D\Delta Z, \tag{3.45}$$

$$-D\nabla Z \cdot \nu = q(u,v,V) - q(\tilde{u},\tilde{v},\tilde{V}), \tag{3.46}$$

$$\partial_t w = \nabla_\Gamma \cdot (A_1 \nabla_\Gamma w) + b_1 \cdot \nabla_\Gamma w + f(u,v) - f(\tilde{u},\tilde{v}), \tag{3.47}$$

$$\begin{aligned} \partial_t z = \nabla_\Gamma \cdot (A_2 \nabla_\Gamma z) + b_2 \cdot \nabla_\Gamma z - f(u,v) + f(\tilde{u},\tilde{v}) \\ + q(u,v,V) - q(\tilde{u},\tilde{v},\tilde{V}). \end{aligned} \tag{3.48}$$

By Theorem 3.6 both weak solutions are uniformly bounded by some constant C_4 depending on $B, D, K_1, k_1, K_2, T, \Lambda_0$ (again, C_4 can be chosen non-increasing in D). Using that both q and f are locally Lipschitz continuous we can therefore follow the calculations in Proposition 3.4 and in the proof of the quasi-coercivity property of F_t to obtain the estimate (3.44).

Proof of Theorem 2.3. Proposition 3.3 gives the existence of a solution, Proposition 3.4 the nonnegativity, Theorem 3.6 the uniform boundedness of solutions, and Proposition 3.7 the uniqueness of solutions and the continuous dependence on the initial data. □

4. Classical solutions of the system (RDD)

In this section we will consider the case of more regular coefficients and prove the existence of classical solutions. With this aim we start from the weak solution obtained in the previous section and first deduce some higher regularity properties.

Proposition 4.1. *Consider initial data (V_0, u_0, v_0) that satisfy (3.19) and let (V, u, v) be the weak solution of (1.6)–(1.10).*

(1) Let Γ be C^2-regular, let $A_1, A_2 \in C^{1,0}(\Gamma \times [0,T])$ and $b_1, b_2 \in C^0(\Gamma \times [0,T])$, fix any $1 < p < \infty$ and assume that $u_0, v_0 \in W^{2,p}(\Gamma)$. Then $u, v \in W_p^{2,1}(\Gamma \times (0,T))$ and there exists C_7, only depending on $B, D, A_1, A_2, b_1, b_2, K_2, T, p$, with

$$\|u,v\|_{W_p^{2,1}(\Gamma\times(0,T))} \le C_7 \left(\Lambda_0 + \|u_0\|_{W^{2-\frac{2}{p},p}(\Gamma)} + \|v_0\|_{W^{2-\frac{2}{p},p}(\Gamma)} \right). \tag{4.1}$$

If in addition $p > 4$ and $0 \le \beta < 1 - \frac{4}{p}$ then there exists a constant C_8 depending only on $B, D, A_1, A_2, b_1, b_2, K_2, T, p$ and β, such that

$$\|u,v\|_{C^{1+\beta,\frac{1+\beta}{2}}(\Gamma\times[0,T])} \le C_8 \left(\Lambda_0 + \|u_0\|_{W^{2-\frac{2}{p},p}(\Gamma)} + \|v_0\|_{W^{2-\frac{2}{p},p}(\Gamma)} \right). \tag{4.2}$$

(2) Let $0 < \alpha < 1$ and assume that Γ is $C^{2,\text{lip}}$-regular, that $A_1, A_2 \in C^{1+\alpha,\frac{1+\alpha}{2}}(\Gamma \times [0,T])$ and $b_1, b_2 \in C^{\alpha,\frac{\alpha}{2}}(\Gamma \times [0,T])$. Furthermore, assume that $V \in C^{\alpha,\frac{\alpha}{2}}(\overline{B} \times [0,T])$ and that $u_0, v_0 \in C^{2+\alpha}(\Gamma)$. Then $u, v \in C^{2+\alpha,1+\frac{\alpha}{2}}(\Gamma \times [0,T])$ holds with

$$\begin{aligned} \|u,v\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\Gamma\times[0,T])} \le C_9 \Big(\|u_0\|_{C^{2+\alpha}(\Gamma)} + \|v_0\|_{C^{2+\alpha}(\Gamma)} \\ + \|V\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{B}\times[0,T])} \Big) \end{aligned} \tag{4.3}$$

for some $C_9 = C_9(B, D, K_1, k_1, \Lambda_0, K_2, T, \alpha)$.

Proof. (1) We divide the proof into several steps.
Step (i): We consider an open subset $W \subset \Gamma$ and a C^2-diffeomorphism $\varphi : \overline{B^2(0,2)} \to W$, where $B^2(0,r) \subset \mathbb{R}^2$ denotes the ball with radius $r > 0$ and center 0. Then $\tilde{u} := u \circ \varphi : B^2(0,2) \to \mathbb{R}$, $\tilde{v} := v \circ \varphi : B^2(0,2) \to \mathbb{R}$ satisfy the system

$$\begin{aligned} \partial_t \tilde{u} &= \nabla \cdot \big(\tilde{A}_1 \nabla \tilde{u}\big) + \tilde{b}_1 \cdot \nabla \tilde{u} + f(u,v) \circ \varphi && \text{in } B^2(0,2) \times (0,T), \quad (4.4)\\ \partial_t \tilde{v} &= \nabla \cdot \big(\tilde{A}_2 \nabla \tilde{v}\big) + \tilde{b}_2 \cdot \nabla \tilde{v} + \big(q(u,v,V) \\ &\quad - f(u,v)\big) \circ \varphi && \text{in } B^2(0,2) \times (0,T), \quad (4.5) \end{aligned}$$

where with $G := D\varphi^T D\varphi$ and denoting by $D\varphi^{-1}(z)$ the inverse of $D\varphi(z) : \mathbb{R}^2 \to T_{\varphi(z)}\Gamma$, for $i = 1, 2$

$$\begin{aligned} \tilde{A}_i &= D\varphi^{-1}(A_i \circ \varphi) D\varphi G^{-1}, \\ \tilde{b}_i &= G^{-1} D\varphi^T (b_i \circ \varphi) - G^{-1}\big(D\varphi^{-1}(A_i \circ \varphi) D\varphi\big)^T \sqrt{\det G}\, \nabla\Big(\frac{1}{\sqrt{\det G}}\Big). \end{aligned}$$

By our assumptions on the data $\tilde{A}_i \in C^{1,0}(B^2(0,2) \times [0,T])$ and $\tilde{b}_i \in C^0(B^2 (0,2) \times [0,T])$ hold for $i = 1, 2$.
Step (ii): We prove that $\tilde{u}, \tilde{v}$ are $W_2^{2,1}$-regular in $B^2(0,1)$ and belong to $L^4(0,T; W^{1,4}(B^2(0,1)))$. In order to apply results for parabolic equations with (zero) Dirichlet boundary data we first multiply $\tilde{v}$ with a cut-off function. Therefore, consider an arbitrary $\psi \in C_c^2(B^2(0,2))$ with $0 \le \psi \le 1$ and $\psi = 1$ in $\overline{B^2(0,1)}$. We find from (4.5) that for $w := \psi \tilde{v}$

$$\partial_t w - \nabla \cdot \big(\tilde{A}_2 \nabla w\big) - \tilde{b}_2 \cdot \nabla w = F \quad \text{in } B^2(0,2) \times (0,T), \tag{4.6}$$

where

$$F := -\tilde{v}\big((\nabla \cdot \tilde{A}_2)\nabla\psi + \tilde{A}_2 : D^2\psi + \tilde{b}_2 \cdot \nabla\psi\big) - 2\nabla\tilde{v} \cdot \tilde{A}_2 \nabla\psi + 4\big(q(u,v,V) - f(u,v)\big) \circ \varphi.$$

By the L^∞-bounds proved in Theorem 3.6 the first and the third term are controlled. However, due to the localization the second term includes $\nabla\tilde{v}$ that is only controlled by the uniform energy bounds (3.14). Therefore, $F \in L^2(B^2(0,2) \times (0,T))$ holds with

$$\begin{aligned} \|F\|_{L^2(B^2(0,2)\times(0,T)} &\le \|\big(q(u,v,V) - f(u,v)\big) \circ \varphi\|_{L^2(B^2(0,2)\times(0,T))} \\ &\quad + C(\varphi, A_2, b_2)\|\tilde{v}\|_{L^2(0,T;H^1(B^2(0,2))} \\ &\le C(\varphi, A_2, b_2, K_2)\big(\|V\|_{L^2(\Gamma\times(0,T))} + \|u\|_{L^2(\Gamma\times(0,T))} \\ &\quad + \|v\|_{L^2(0,T;H^1(\Gamma))}\big) \\ &\le C(T, K_2, \varphi, A_2, b_2, C_5, \Gamma)\Lambda_0. \end{aligned} \tag{4.7}$$

By [23, Theorem III.6.1, Remark III.6.3 and (6.10)] we deduce that $w \in W_2^{2,1}(B^2(0,2)\times(0,T))$, with

$$\begin{aligned}\|w\|_{W_2^{2,1}(B^2(0,2)\times(0,T))} &\le C(T,\varphi,A_2,b_2)\big(\|F\|_{L^2(B^2(0,2)\times(0,T))}\\ &\quad + \|w(\cdot,0)\|_{H^1(B^2(0,2))}\big)\\ &\le C(T,K_2,\varphi,A_2,b_2,C_5,\Gamma)\big(\Lambda_0+\|v_0\|_{H^1(\Gamma)}\big),\end{aligned}$$

where we have used (4.7) in the last step. This yields $\tilde v \in W_2^{2,1}(B^2(0,1)\times(0,T))$ and, by [23, Lemma II.3.3] we have $\tilde v \in L^4(0,T;W^{1,4}(B^2(0,1)))$, with

$$\begin{aligned}\|\tilde v\|_{L^4(0,T;W^{1,4}(B^2(0,1)))} &\le C(T)\|\tilde v\|_{W_2^{2,1}(B^2(0,1)\times(0,T))}\\ &\le C(T)\|w\|_{W_2^{2,1}(B^2(0,2)\times(0,T))}\\ &\le C(T,K_2,\varphi,A_2,b_2,C_5,\Gamma)\big(\Lambda_0+\|v_0\|_{H^1(\Gamma)}\big). \end{aligned}\tag{4.8}$$

Arguing in the same way, we deduce the corresponding regularity properties and the corresponding estimate also for $\tilde u$.

Step (iii): Using that $v=\tilde v\circ\varphi^{-1}$, $u=\tilde u\circ\varphi^{-1}$, the previous estimates show that $u,v\in L^4(0,T;W^{1,4}(\varphi(B^2(0,1)))$. Since Γ is compact there exists a finite family $\{\varphi_i\}_{i=1}^N$ of local parametrizations as above such that Γ is covered by $\bigcup_{i=1}^N \varphi_i(B^2(0,1))$. We then deduce from (4.8) and the corresponding estimate for $\tilde u$ that

$$\begin{aligned}&\|u,v\|_{L^4(0,T;W^{1,4}(\Gamma))}\\ &\quad\le C(T,K_2,\Gamma,A_1,A_2,b_1,b_2,C_5)\big(\Lambda_0+\|u_0\|_{H^1(\Gamma)}+\|v_0\|_{H^1(\Gamma)}\big).\end{aligned}\tag{4.9}$$

Step (iv): We again consider (4.6) for an arbitrary $\psi\in C_c^2(B^2(0,2))$ with $0\le\psi\le 1$ and $\psi=1$ in $\overline{B^2(0,1)}$. By (4.9) we have $F\in L^4(B^2(0,2)\times(0,T))$, with

$$\begin{aligned}&\|F\|_{L^4(B^2(0,2)\times(0,T))}\\ &\quad\le \|\big(q(u,v,V)-f(u,v)\big)\circ\varphi\|_{L^4(B^2(0,2)\times(0,T))}\\ &\qquad + C(\varphi,A_2,b_2)\|\tilde v\|_{L^4(0,T;W^{1,4}(B^2(0,2)))}\\ &\quad\le C(K_2,\varphi,A_2,b_2)\big(\|V\|_{L^4(\Gamma\times(0,T))}+\|u\|_{L^4(\Gamma\times(0,T))}+\|v\|_{L^2(0,T;W^{1,4}(\Gamma))}\big)\\ &\quad\le C(T,K_2,\Gamma,A_1,A_2,b_1,b_2,C_5)\big(\Lambda_0+\|u_0\|_{H^1(\Gamma)}+\|v_0\|_{H^1(\Gamma)}\big).\end{aligned}$$

From maximal regularity results, see [23, Theorem IV.9.1], we conclude that $w\in W_4^{2,1}(B^2(0,2)\times(0,T))$, with

$$\begin{aligned}\|w\|_{W_4^{2,1}(B^2(0,2)\times(0,T))} &\le C(A_2,b_2,\varphi,T)\big(\|F\|_{L^4(B^2(0,2)\times(0,T))}\\ &\quad + \|w(\cdot,0)\|_{W^{\frac32,4}(B^2(0,2))}\big)\\ &\le C(T,K_2,\Gamma,A_1,A_2,b_1,b_2,C_5)\big(\Lambda_0\\ &\quad + \|u_0\|_{H^1(\Gamma)}+\|v_0\|_{W^{\frac32,4}(\Gamma)}\big).\end{aligned}$$

This in particular implies $\tilde{v} \in W^{2,1}_4(B^2(0,1) \times (0,T))$. The embedding [23, Lemma II.3.3] yields $\tilde{v} \in L^p(0,T;W^{1,p}(B^2(0,1)))$ for any $1 \le p < \infty$, with

$$\begin{aligned}\|\tilde{v}\|_{L^p(0,T;W^{1,p}(B^2(0,1)))} &\le C(p,T)\|w\|_{W^{2,1}_4(B^2(0,2)\times(0,T))}\\ &\le C(p,T,K_2,\Gamma,A_1,A_2,b_1,b_2,C_5)\big(\Lambda_0 + \|u_0\|_{H^1(\Gamma)}\\ &\quad + \|v_0\|_{W^{\frac{3}{2},4}(\Gamma)}\big).\end{aligned}$$

Step (v): By the corresponding estimate for $\tilde{u}$, and the same arguments as in Step (iii) we deduce that for any $1 < p < \infty$

$$\begin{aligned}\|u,v\|_{L^p(0,T;W^{1,p}(\Gamma))} \le C(p,T,K_2,\Gamma,A_1,A_2,b_1,b_2,C_5)\big(\Lambda_0\\ +\|u_0\|_{W^{\frac{3}{2},4}(\Gamma)} + \|v_0\|_{W^{\frac{3}{2},4}(\Gamma)}\big).\end{aligned} \tag{4.10}$$

Step (vi): Repeating the procedure once more, we now conclude that the right-hand side F in (4.6) belongs to $L^p(B^2(0,2)\times(0,T))$ for any $1 \le p < \infty$, with

$$\begin{aligned}&\|F\|_{L^p(B^2(0,2)\times(0,T)}\\ &\quad\le \|\big(q(u,v,V) - f(u,v)\big)\circ\varphi\|_{L^p(B^2(0,2)\times(0,T))}\\ &\qquad + C(\varphi,A_2,b_2)\|\tilde{v}\|_{L^p(0,T;W^{1,p}(B^2(0,2)))}\\ &\quad\le C(K_2,\varphi,A_2,b_2,p)\big(\|V\|_{L^p(\Gamma\times(0,T))} + \|u\|_{L^p(\Gamma\times(0,T))} + \|v\|_{L^p(0,T;W^{1,p}(\Gamma))}\big)\\ &\quad\le C(p,T,K_2,\Gamma,A_1,A_2,b_1,b_2,C_5)\left(\Lambda_0 + \|u_0\|_{W^{\frac{3}{2},4}(\Gamma)} + \|v_0\|_{W^{\frac{3}{2},4}(\Gamma)}\right),\end{aligned}$$

where we have used (4.10) and Theorem 3.6. By maximal $L^p(L^p)$ regularity [23, Theorem IV.9.1] we deduce that $w \in W^{2,1}_p(B^2(0,2)\times(0,T))$ and $\tilde{v} \in W^{2,1}_p(B^2(0,1)\times(0,T))$ for any $1 < p < \infty$, with an estimate

$$\begin{aligned}&\|\tilde{v}\|_{W^{2,1}_p(B^2(0,1)\times(0,T))}\\ &\quad\le \|w\|_{W^{2,1}_p(B^2(0,1)\times(0,T))}\\ &\quad\le C(p,T,\varphi,A_2,b_2)\left(\|F\|_{L^p(B^2(0,1)\times(0,T)} + \|w(\cdot,0)\|_{W^{2-\frac{2}{p},p}(B^2(0,1))}\right)\\ &\quad\le C(p,T,K_2,\Gamma,A_1,A_2,b_1,b_2,C_5)\left(\Lambda_0 + \|u_0\|_{W^{\frac{3}{2},4}(\Gamma)} + \|v_0\|_{W^{2-\frac{2}{p},p}(\Gamma)}\right).\end{aligned}$$

Step (vii): Analogously, we derive a corresponding estimate for $\tilde{u}$. Covering Γ with a suitable collection of local parametrizations, as above, we conclude that (4.1) holds.

Step (viii): For any $4 \le p < \infty$ the embedding into Hölder spaces [23, Lemma II.3.3] and (4.1) imply that $u, v \in C^{1+\beta,\frac{1+\beta}{2}}(\Gamma\times[0,T])$ for any $0 < \beta < 1 - \frac{4}{p}$ and that the estimate (4.2) holds.

(2) We again consider a local parametrization and the solution w of (4.6), for $\psi \in C^\infty_c(B^2(0,2))$ with $0 \le \psi \le 1$ and $\psi = 1$ on $\overline{B^2(0,1)}$. By our assumptions on the data, $\tilde{A}_2 \in C^{1+\alpha,\frac{1+\alpha}{2}}(\Gamma\times[0,T])$ and $\tilde{b}_2 \in C^{\alpha,\frac{\alpha}{2}}(\Gamma\times[0,T])$ hold. From (4.2) we deduce that F in (4.6) belongs to $C^{\alpha,\frac{\alpha}{2}}(\overline{B^2(0,2)}\times[0,T])$. Applying [23, Theorem IV.5.2] we obtain that $w \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{B^2(0,2)}\times[0,T])$

holds, with

$$\|w\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{B^2(0,2)}\times[0,T])} \leq C(\alpha,T)\Big(\|F\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{B^2(0,2)}\times[0,T])} + \|w(\cdot,0)\|_{C^{2+\alpha}(\overline{B^2(0,2)})}\Big).$$

This further implies $\tilde{v} \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{B^2(0,1)} \times [0,T])$ and a bound

$$\begin{aligned}
&\|\tilde{v}\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{B^2(0,1)}\times[0,T])}\\
&\leq C(\alpha,\Gamma,T,\Lambda_0)\Big(\|V\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{B}\times[0,T])} + \|v\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\Gamma\times[0,T])}\\
&\quad + \|u\|_{C^{\alpha,\frac{\alpha}{2}}(\Gamma\times[0,T])} + \|v_0\|_{C^{2+\alpha}(\Gamma)}\Big)\\
&\leq C(\alpha,\Gamma,T,\Lambda_0,C_8)\Big(\Lambda_0 + \|V\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{B}\times[0,T])} + \|u_0\|_{C^{2+\alpha}(\Gamma)} + \|v_0\|_{C^{2+\alpha}(\Gamma)}\Big),
\end{aligned}$$

where we have used that, by (4.2), $\|u,v\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\Gamma\times[0,T])}$ is controlled by Λ_0 and the initial data u_0, v_0. Covering Γ by a suitable family of parametrizations we deduce the estimate (4.3) for v and, by similar arguments, for u. □

Proposition 4.2. *Consider the weak solution (V,u,v) of (1.6)–(1.10) with initial data (V_0,u_0,v_0) that satisfy (3.19) and the compatibility condition (2.4). Let $0<\alpha<1$ and assume that Γ is $C^{2,\mathrm{lip}}$-regular, that $u,v \in C^{1+\alpha,\frac{1+\alpha}{2}}(\Gamma\times[0,T])$ and that $V_0 \in C^{2+\alpha}(\overline{B})$. Then $V \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{B}\times[0,T])$ holds with*

$$\begin{aligned}
&\|V\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{B}\times[0,T])}\\
&\quad\leq C_{10}\Big(\|u\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\Gamma\times[0,T])} + \|v\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\Gamma\times[0,T])}\\
&\qquad + \|V_0\|_{C^{2+\alpha}(\overline{B})}\Big)
\end{aligned} \tag{4.11}$$

for some $C_{10} = C_{10}\big(B,D,K_2,T,\alpha,\|u,v\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\Gamma\times[0,T])}\big)$.

Proof. We observe that V is a weak solution of the parabolic equation with Robin boundary condition

$$\begin{aligned}
\partial_t V &= D\Delta V && \text{in } B\times(0,T),\\
D\nabla V\cdot\nu + dV &= g && \text{on } \Gamma\times(0,T),
\end{aligned}$$

where $d = q_1(u,v)$ and $g = q_2(u,v)v$ are considered as given functions on $\Gamma\times[0,T]$. By our assumptions we have $d,g \in C^{1+\alpha,\frac{1+\alpha}{2}}(\Gamma\times[0,T])$ and we deduce from [23, Theorem IV.5.3] that $V \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{B}\times[0,T])$ holds with

$$\begin{aligned}
&\|V\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{B}\times[0,T])}\\
&\quad\leq C_\alpha\big(B,T,\|d\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\Gamma\times[0,T])},D\big)\Big(\|g\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\Gamma\times[0,T])} + \|V_0\|_{C^{2+\alpha}(\overline{B})}\Big).
\end{aligned}$$

This proves (4.11). □

Proposition 4.3. *Let the assumptions in Theorem 2.4 be satisfied and let $\Lambda_1 > 0$ be given with*

$$\|V_0\|_{C^{2+\alpha}(\overline{B})} + \|u_0\|_{C^{2+\alpha}(\Gamma)} + \|v_0\|_{C^{2+\alpha}(\Gamma)} \leq \Lambda_1. \tag{4.12}$$

Consider the weak solution (V,u,v) *of* (1.6)–(1.10)*. Then* $V\in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{B}\times[0,T])$ *and* $u,v\in C^{2+\alpha,1+\frac{\alpha}{2}}(\Gamma\times[0,T])$ *hold, and* (V,u,v) *is a classical solution of Problem (RDD). Moreover, there exists a constant* $C_{11}=C_{11}(B,D,K_1,k_1,K_2,\Lambda_1,T,\alpha)$ *such that*

$$\|V\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{B}\times[0,T])}+\|u,v\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\Gamma\times[0,T])} \leq C_{11}\big(\|V_0\|_{C^{2+\alpha}(\overline{B})}+\|u_0\|_{C^{2+\alpha}(\Gamma)}+\|v_0\|_{C^{2+\alpha}(\Gamma)}\big).$$

Proof. We let $\Lambda_0:=\|V_0\|_{L^\infty(B)}+\|u_0\|_{L^\infty(\Gamma)}+\|v_0\|_{L^\infty(\Gamma)}$ and note that $\Lambda_0\leq\Lambda_1$. Applying first Proposition 4.1, (4.2) we obtain $u,v\in C^{1+\alpha,\frac{1+\alpha}{2}}(\Gamma\times[0,T])$ with

$$\|u,v\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\Gamma\times[0,T])}\leq C_8\Big(\Lambda_0+\|u_0\|_{C^{2+\alpha}(\Gamma\times[0,T])}+\|v_0\|_{C^{2+\alpha}(\Gamma\times[0,T])}\Big). \tag{4.13}$$

Then Proposition 4.2 and (4.13) imply that $V\in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{B}\times[0,T])$ with

$$\|V\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{B}\times[0,T])} \leq C(C_{10})\Big(\|V_0\|_{C^{2+\alpha}(\overline{B})}+\|u_0\|_{C^{2+\alpha}(\Gamma\times[0,T])}+\|v_0\|_{C^{2+\alpha}(\Gamma\times[0,T])}\Big).$$

Finally, Proposition 4.1, (4.3) implies $u,v\in C^{2+\alpha,1+\frac{\alpha}{2}}(\Gamma\times[0,T])$ and

$$\|u,v\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\Gamma\times[0,T])} \leq C(C_9,C_{10})\big(\|V_0\|_{C^{2+\alpha}(\overline{B})}+\|u_0\|_{C^{2+\alpha}(\Gamma)}+\|v_0\|_{C^{2+\alpha}(\Gamma)}\big).$$

By (4.13) the $\|u,v\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\Gamma\times[0,T])}$ dependence of C_{10} can be replaced by a dependence on Λ_1.

By standard arguments the $C^{2+\alpha,1+\frac{\alpha}{2}}$-regular weak solution (V,u,v) is in fact a classical solution of the system (RDD). □

Proposition 4.4. *For any* $\Lambda_1>0$ *there exists a constant* $C_{12}=C_{12}(B,D,K_1,k_1,f,q,T,\alpha,\Lambda_1)$ *with the following property: Assume that* A_1,A_2,b_1,b_2 *and* f,q *satisfy the assumptions of Theorem* 2.4*. Let two tuples of initial data* (V_0,u_0,v_0) *and* $(\tilde{V}_0,\tilde{u}_0,\tilde{v}_0)$ *are given that both satisfy the assumptions of Theorem* 2.4*, and both satisfy* (4.12)*.*

Then the solutions (V,u,v) *and* $(\tilde{V},\tilde{u},\tilde{v})$ *of system (RDD), with initial data* (V_0,u_0,v_0) *and* $(\tilde{V}_0,\tilde{u}_0,\tilde{v}_0)$*, respectively, satisfy the estimate*

$$\|V-\tilde{V}\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{B}\times[0,T])}+\|u-\tilde{u}\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\Gamma\times[0,T])} +\|v-\tilde{v}\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\Gamma\times[0,T])} \leq C_{12}\Big(\|V_0-\tilde{V}_0\|_{C^{2+\alpha}(\bar{B})}+\|u_0-\tilde{u}_0\|_{C^{2+\alpha}(\Gamma)}+\|v_0-\tilde{v}_0\|_{C^{2+\alpha}(\Gamma)}\Big). \tag{4.14}$$

Proof. As in Proposition 3.7 we consider the differences

$$Z:=V-\tilde{V},\qquad w:=u-\tilde{u},\qquad z:=v-\tilde{v}$$

and observe that they satisfy (3.45)–(3.48). Using the C^2-regularity of f_1,f_2,q_1,q_2 we show as in Proposition 4.1 that for some constant C_9' independent of (Z,w,z)

$$\|w,z\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\Gamma\times[0,T])} \leq C_9'\Big(\|w_0\|_{C^{2+\alpha}(\Gamma)}+\|z_0\|_{C^{2+\alpha}(\Gamma)}+\|Z\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{B}\times[0,T])}\Big) \tag{4.15}$$

holds. Using similar arguments as in Proposition 4.2 in the first inequality and similar arguments as in Proposition 4.1 in the second inequality we obtain

$$\begin{aligned} &\|Z\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{B}\times[0,T])} \\ &\quad\leq C_{10}'\Big(\|w\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\Gamma\times[0,T])}+\|z\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\Gamma\times[0,T])}+\|Z_0\|_{C^{2+\alpha}(\overline{B})}\Big) \\ &\quad\leq C_{10}'(1+2C_9')\Big(\|w_0\|_{C^{2+\alpha}(\Gamma)}+\|z_0\|_{C^{2+\alpha}(\Gamma)}+\|Z\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{B}\times[0,T])} \\ &\qquad+\|Z_0\|_{C^{2+\alpha}(\overline{B})}\Big). \end{aligned} \tag{4.16}$$

By Ehrling's Lemma [35] and by Proposition 3.7 we finally deduce

$$\begin{aligned} &C_{10}'(1+2C_9')\|Z\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{B}\times[0,T])} \\ &\quad\leq \frac{1}{2}\|Z\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{B}\times[0,T])}+C(C_9',C_{10}')\|Z\|_{L^2(B\times(0,T))} \\ &\quad\leq \frac{1}{2}\|Z\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{B}\times[0,T])} \\ &\qquad+C(C_9',C_{10}',C_6)\Big(\|Z_0\|_{L^2(B\times(0,T))}+\|w_0\|_{L^2(\Gamma\times(0,T))} \\ &\qquad+\|z_0\|_{L^2(\Gamma\times(0,T))}\Big). \end{aligned}$$

This inequality, together with (4.15), (4.16) proves (4.14). □

Proof of Theorem 2.4. The claim follows by Theorem 2.3 on the existence of weak solutions, and Propositions 4.3, 4.4. □

Proof of Theorem 2.5. This follows by bootstrapping and the arguments used in Propositions 4.1 and 4.2. □

5. Asymptotic reduction in the infinite cytosolic diffusion limit

We finally study the asymptotic limit of solutions to (1.6)–(1.10) for $D \to \infty$. As a first step in the proof of Theorem 2.6 we show the convergence to solutions of the reduced system (rRDD) .

Proposition 5.1. *Consider any sequence $D_k \to \infty$ $(k \to \infty)$ and the solutions (V_k, u_k, v_k) of (1.6)–(1.10) with D replaced by D_k and fixed initial data (V_0, u_0, v_0) that satisfy (3.19). Then there exists a subsequence $k \to \infty$ (not relabeled) and a solution $(V_\infty, u_\infty, v_\infty)$ of (2.7)–(2.10) with $\mathfrak{m} = \int_B V_0 + \int_\Gamma \big(u_0 + v_0\big)$, such that $u_\infty, v_\infty \in L^2(0,T;H^1(\Gamma)) \cap H^1(0,T;H^1(\Gamma)^*)$, $V_\infty \in W^{1,\infty}(0,T)$ and*

$$u_k \to u_\infty, \quad v_k \to v_\infty, \quad V_k \to V_\infty \qquad \text{in } L^2(\Gamma\times(0,T)).$$

Proof. By Proposition 3.4 and Theorem 3.6 the sequences $(u_k)_k$ and $(v_k)_k$ are uniformly bounded in $L^\infty(\Gamma\times(0,T)) \cap L^2(0,T;H^1(\Gamma)) \cap H^1(0,T;H^1(\Gamma)^*)$. Furthermore, $(V_k)_k$ is uniformly bounded in $L^\infty(B\times(0,T)) \cap L^2(0,T;H^1(B))$

with a uniform bound on $\sqrt{D_k}\|\nabla V_k\|_{L^2(B\times(0,T))}$. We claim that there exist $u_\infty, v_\infty \in L^2(0,T;H^1(\Gamma))\cap H^1(0,T;H^1(\Gamma)^*)$ and $V_\infty \in L^2(0,T;H^1(B))$ such that for a subsequence $k\to\infty$ (not relabled)

$$V_k \rightharpoonup V_\infty \quad \text{weakly in } L^2(0,T;H^1(B)), \tag{5.1}$$

$$u_k \rightharpoonup u_\infty \quad \text{weakly in } L^2(0,T;H^1(\Gamma))\cap H^1(0,T;H^1(\Gamma)^*), \tag{5.2}$$

$$u_k \to u_\infty \quad \text{in } L^2(0,T;L^2(\Gamma)), \tag{5.3}$$

$$v_k \rightharpoonup v_\infty \quad \text{weakly in } L^2(0,T;H^1(\Gamma))\cap H^1(0,T;H^1(\Gamma)^*), \tag{5.4}$$

$$v_k \to v_\infty \quad \text{in } L^2(0,T;L^2(\Gamma)). \tag{5.5}$$

The compactness of $(V_k)_k$ in $L^2(0,T;H^1(B))$ and of $(u_k)_k, (v_k)_k$ in $L^2(0,T;H^1(\Gamma))$ and in $H^1(0,T;H^1(\Gamma)^*)$ follow from the estimates in Proposition 3.4 and the weak precompactness of bounded sets in reflexive Banach spaces. The energy bounds (3.14) and the Aubin–Lions Lemma furthermore imply (5.3), (5.5).

As another consequence of (3.14) we have

$$\int_0^T\int_B |\nabla V_\infty|^2 \le \liminf_{k\to\infty}\int_0^T\int_B |\nabla V_k|^2 = 0,$$

and for almost all $t\in(0,T)$ the functions $V_\infty(\cdot,t)$ are constant almost everywhere in B.

Next we observe from (1.6), (1.7) that in a weak sense

$$\frac{d}{dt}\int_B V_k = \int_\Gamma\Big(-q_1(u_k,v_k)V_k + q_2(u_k,v_k)v_k\Big)$$

holds. By Theorem 3.6 the right hand side is uniformly bounded and we obtain that $\int_B V_k$, as a function of time, is uniformly bounded in $W^{1,\infty}(0,T)$ and hence converges to some limit $w\in W^{1,\infty}(0,T)$ strongly in $C^\alpha([0,T])$ for any $0\le\alpha<1$. Furthermore, by (5.1) for any $\eta\in L^2(0,T)$

$$\begin{aligned}&\int_0^T \eta(t)\big(w(t) - V_\infty(t)|B|\big)\,dt\\ &\quad= \lim_{k\to\infty}\int_0^T\eta(t)\int_B V_k(x,t)\,dx\,dt - \lim_{k\to\infty}\int_0^T\int_B\eta(t)V_k(x,t)\,dx\,dt = 0,\end{aligned}$$

which shows that $w = V_\infty|B|$ almost everywhere in $(0,T)$. In particular we have $V_\infty\in W^{1,\infty}(0,T)$ and

$$V_\infty|B| = \lim_{k\to\infty}\int_B V_k(x,\cdot)\,dx \quad \text{in } C^\alpha([0,T]),$$

for all $0\le\alpha<1$. Analogue arguments yield that $t\mapsto\int_\Gamma u_\infty(\cdot,t)$ and $t\mapsto\int_\Gamma v_\infty(\cdot,t)$ belong to $W^{1,\infty}(0,T)$.

Letting $k\to\infty$ in the mass conservation property (1.11) implies (2.9).

In order to show that $(u_\infty, v_\infty, V_\infty)$ solves (2.7), (2.8) we first show that V_∞ is also the limit of the traces of V_k on Γ. In fact, we have by the continuity of the embedding $H^1(B) \hookrightarrow L^2(\Gamma)$ and the Poincaré inequality for functions in $H^1(B)$ with vanishing mean that

$$\int_0^T \Big\|V_k(\cdot,t) - \frac{1}{|B|}\int_B V_k(\cdot,t)\Big\|^2_{L^2(\Gamma)}\,dt \le C\int_0^T \|\nabla V_k(\cdot,t)\|^2_{L^2(B)}\,dt \to 0$$

and conclude that

$$V_k \to V_\infty \quad \text{in } L^2(\Gamma\times(0,T)). \tag{5.6}$$

From (5.3), (5.5), and (5.6) we deduce that (possibly passing to another subsequence)

$$q(u_k,v_k,V_k) \to q(u,v,V) \quad \text{pointwise almost everywhere in } \Gamma\times(0,T).$$

Moreover, the estimate $|q(u_k,v_k,V_k)| \le K_2\big(|V_k|+|v_k|\big)$ gives an $L^2(\Gamma\times(0,T))$ convergent sequence of majorants and from the generalized Lebesgue Convergence Theorem we deduce that

$$q(u_k,v_k,V_k) \to q(u,v,V) \quad \text{in } L^2(\Gamma\times(0,T)).$$

Similar arguments show that $f(u_k,v_k) \to f(u,v)$ in $L^2(\Gamma\times(0,T))$.

Using in addition (5.2)–(5.5) we therefore can pass to the limit in the weak formulation of equations (1.8), (1.9) and deduce that (2.7), (2.8) are satisfied in a weak sense.

We finally prove that the prescribed initial data are attained. From [9, Theorem 5.9.3] we deduce that $u_\infty, v_\infty \in C^0([0,T];L^2(\Gamma))$. Next, for any $\eta \in C_c^\infty([0,T)\times\Gamma)$ a partial integration, (1.10) and (5.2), (5.3) imply that

$$\begin{aligned}
&-\int_\Gamma \eta(x,0)u_0(x)\,d\mathcal{H}^2(x)\\
&= \lim_{k\to\infty}\int_0^T \Big(\langle\partial_t u_k(\cdot,t),\eta(\cdot,t)\rangle_{H^1(\Gamma),H^1(\Gamma)^*} + \int_\Gamma u_k(x,t)\partial_t\eta(x,t)\,d\mathcal{H}^2(x)\Big)\,dt\\
&= \int_0^T \Big(\langle\partial_t u_\infty(\cdot,t),\eta(\cdot,t)\rangle_{H^1(\Gamma),H^1(\Gamma)^*} + \int_\Gamma u_\infty(x,t)\partial_t\eta(x,t)\,d\mathcal{H}^2(x)\Big)\,dt\\
&= -\int_\Gamma \eta(x,0)u_\infty(x,0)\,d\mathcal{H}^2(x),
\end{aligned}$$

which shows that $u_\infty(\cdot,0) = u_0$ holds. The proof that also $v_\infty(\cdot,0) = v_0$ holds is analogue. □

It remains to prove the uniqueness of solutions to the reduced system (rRDD). We first prove the nonnegativity and uniform boundedness for any weak solution of this system. We use the same methods as in Proposition 3.4 and in Theorems 3.5, 3.6 but now applied to (2.7), (2.8) and the following initial value problem for V_∞,

$$\begin{aligned}
\frac{d}{dt}V_\infty &= \frac{d}{dt}\frac{1}{|B|}\int_\Gamma -(u_\infty+v_\infty)\\
&= -\frac{1}{|B|}\Big(\int_\Gamma q_1(u_\infty,v_\infty)\Big)V_\infty + \frac{1}{|B|}\int_\Gamma q_2(u_\infty,v_\infty)v_\infty, &(5.7)\\
V_\infty(0) &= \frac{1}{|B|}\mathfrak{m} - \int_\Gamma (u_0+v_0). &(5.8)
\end{aligned}$$

We start with the nonnegativity of solutions.

Proposition 5.2. *Let nonnegative initial data* (u_0, v_0) *and a constant* $\mathfrak{m} \geq 0$ *be given that satisfies* (2.6)*. Then any weak solution* $(u_\infty, v_\infty, V_\infty)$ *of* (2.7)–(2.10) *is nonnegative.*

Proof. To prove the nonnegativity we follow the proof of Proposition 3.4 and test (2.7) with $(u_\infty)_-$, (2.8) with $(v_\infty)_-$ and (5.7) with $(V_\infty)_-$, where for convenience we drop the index ∞ in the following. This gives

$$\begin{aligned}\frac{d}{dt}\int_\Gamma \frac{1}{2}\left(u_-^2 + v_-^2\right) =& -\int_\Gamma \left(\nabla_\Gamma u_- \cdot A_1 \nabla_\Gamma u_- + \nabla_\Gamma v_- \cdot A_2 \nabla_\Gamma v_-\right)\\ &+ \int_\Gamma \big(b_1 \cdot u_- \nabla_\Gamma u_- + b_2 \cdot v_- \nabla_\Gamma v_- - q(u,v,V) v_-\\ &- f(u,v)(u_- - v_-)\big)\\ \leq& -\frac{k_1}{2}\int_\Gamma \left(|\nabla_\Gamma u_-|^2 + |\nabla_\Gamma v_-|^2\right) + C(K_1,k_1)\int_\Gamma \left(u_-^2 + v_-^2\right)\\ &+ \int_\Gamma \left(q_1(u,v)V_- v_- + \big(f_1(u,v) + f_2(u,v)\big)u_- v_-\right)\\ \leq& K_2|\Gamma|V_-^2 + C(K_1,k_1,K_2)\int_\Gamma \left(u_-^2 + v_-^2\right) \end{aligned} \tag{5.9}$$

and

$$\begin{aligned}\frac{d}{dt}\frac{|B|}{2}V_-^2 = V_- \frac{d}{dt}\int_B V_- =& \int_\Gamma V_-\big(-q_1(u,v)V_- - q_2(u,v)v\big)\\ \leq& \int_\Gamma q_2(u,v)V_- v_- \leq \frac{K_2|\Gamma|}{2}V_-^2 + \frac{K_2}{2}\int_\Gamma v_-^2.\end{aligned}$$

Adding this inequality to (5.9) we deduce from Gronwalls inequality that u, v, V remain nonnegative. □

By similar arguments we obtain, as in the fully coupled system, that any weak solution of the reduced system is uniformly bounded.

Proposition 5.3. *Let nonnegative initial data* u_0, v_0 *and a constant* $\mathfrak{m} \geq 0$ *be given that satisfies* (2.6)*. Further assume that for some* $\Lambda_0 > 0$

$$\frac{1}{|B|}\mathfrak{m} + \sup_\Gamma u_0 + \sup_\Gamma v_0 \leq \Lambda_0.$$

Then there exists $\lambda_0 > 0$ *only depending on* B, K_1, k_1, K_2*, such that for any weak solution* $(u_\infty, v_\infty, V_\infty)$ *of* (2.7)–(2.10)

$$\sup_{(0,T)} V_\infty + \sup_{\Gamma\times(0,T)} u_\infty + \sup_{\Gamma\times(0,T)} v_\infty \leq e^{\lambda_0 T}\Lambda_0 \tag{5.10}$$

holds.

Proof. For convenience, we again drop the index ∞. We follow the proof of Theorems 3.5, only the treatment of the V equation is different. For $\lambda > 0$, to be fixed below, we consider the modified system (3.17), (3.18) coupled with the following initial value problem

$$\frac{d}{dt}V = -\Big(\lambda + \frac{1}{|B|}\int_\Gamma q_1(u,v)\Big)V + \frac{1}{|B|}\int_\Gamma q_2(u,v)v \quad \text{in } (0,T), \tag{5.11}$$

$$V(0) = \frac{1}{|B|}\mathfrak{m} - \int_\Gamma (u_0+v_0). \tag{5.12}$$

We fix arbitrary $M, m_2 > 0$ (to be chosen below) and deduce from (5.11) that

$$\begin{aligned}
\frac{|B|}{2}\frac{d}{dt}(V-M)_+^2 &\leq -\lambda|B|V(V-M)_+ + \int_\Gamma K_2 v(V-M)_+ \\
&\leq -\lambda|B|V(V-M)_+ + K_2\int_\Gamma \Big((v-m_2)_+(V-M)_+ \\
&\quad + m_2(V-M)_+\Big) \\
&\leq \Big(-\lambda|B| + \frac{1}{2}|\Gamma|\Big)(V-M)_+^2 + \int_\Gamma \frac{K_2^2}{2}(v-m_2)_+^2 \\
&\quad - \Big(\lambda|B|M - K_2 m_2|\Gamma|\Big)(V-M)_+ \\
&\leq -\frac{1}{2}|\Gamma|(V-M)_+^2 + \int_\Gamma \frac{K_2^2}{2}(v-m_2)_+^2
\end{aligned} \tag{5.13}$$

for all $\lambda \geq \lambda_0$, if λ_0 satisfies

$$\lambda_0|B| \geq |\Gamma| \quad \text{and} \quad \lambda_0|B|M \geq K_2 m_2|\Gamma|. \tag{5.14}$$

We next test for an arbitrary $m_1, m_2 \geq 0$ equation (3.17) with $(u-m_1)_+$ and equation (3.18) with $(v-m_2)_+$. We consider λ_0 that, in addition to (5.14), satisfies the conditions (3.34) and (3.37). Then, we follow the proof of Theorem 3.5 and deduce that for any $\lambda \geq \lambda_0$ the estimates (3.35) and (3.38) hold. Summing up the inequalities (5.13), (3.35) and (3.38) we conclude as in the proof of Theorem 3.5 that for $M = m_1 = m_2 = \Lambda_0$ and λ_0 sufficiently large (where λ_0 can be chosen independently of Λ_0) the functions V, u, v stay below Λ_0. As in the proof of Theorem 3.6, this implies the maximum bound (5.10).

□

Proposition 5.4. *For any $\Lambda_0 > 0$ there exists a constant $C_{13} = C_{13}(B, K_1, k_1, K_2, T, \Lambda_0)$ with the following property: Consider initial data (u_0, v_0) and $(\tilde{u}_0, \tilde{v}_0)$ and any $\mathfrak{m}, \tilde{\mathfrak{m}}$ that satisfy (2.6) and*

$$0 \leq \frac{1}{|B|}\mathfrak{m} + u_0 + v_0 \leq \Lambda_0, \qquad 0 \leq \frac{1}{|B|}\tilde{\mathfrak{m}} + \tilde{u}_0 + \tilde{v}_0 \leq \Lambda_0.$$

Let $(V_\infty, u_\infty, v_\infty)$ and $(\tilde{V}_\infty, \tilde{u}_\infty, \tilde{v}_\infty)$ be weak solutions of (2.7)–(2.9) with initial data (u_0, v_0) and $(\tilde{u}_0, \tilde{v}_0)$, and with total mass $\mathfrak{m}$ and $\tilde{\mathfrak{m}}$, respectively. Then

$$\begin{aligned}
&\|V_\infty - \tilde{V}_\infty\|_{L^\infty(0,T)} + \|u_\infty - \tilde{u}_\infty\|_{L^2(0,T;H^1(\Gamma))\cap L^\infty(0,T;L^2(\Gamma))} \\
&\qquad + \|v_\infty - \tilde{v}_\infty\|_{L^2(0,T;H^1(\Gamma))\cap L^\infty(0,T;L^2(\Gamma))} \\
&\quad \le C_{13}\Big(|\mathfrak{m} - \tilde{\mathfrak{m}}| + \|u_0 - \tilde{u}_0\|_{L^2(\Gamma)} + \|v_0 - \tilde{v}_0\|_{L^2(\Gamma)}\Big)
\end{aligned}$$

holds.

Proof. Using similar modifications as above we can adopt the proof of Proposition 3.7, where we instead of (1.6), (1.7) use (5.7). □

Proof of Theorem 2.6. The assertions follow immediately from Propositions 5.1–5.4. □

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Stephan Hausberg and Matthias Röger
Fakultät für Mathematik
Technische Universität Dortmund
Vogelpothsweg 87
44227 Dortmund
Germany
e-mail: matthias.roeger@tu-dortmund.de

Stephan Hausberg
e-mail: stephan.hausberg@math.tu-dortmund.de

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