



A Reshetnyak-type lower semicontinuity result for linearised elasto-plasticity coupled with damage in $W^{1,n}$

Vito Crismale  and Gianluca Orlando 

Abstract. In this paper we prove a lower semicontinuity result of Reshetnyak type for a class of functionals which appear in models for small-strain elasto-plasticity coupled with damage. To do so we characterise the limit of measures $\alpha_k Eu_k$ with respect to the weak convergence $\alpha_k \rightharpoonup \alpha$ in $W^{1,n}(\Omega)$ and the weak* convergence $u_k \overset{*}{\rightharpoonup} u$ in $BD(\Omega)$, E denoting the symmetrised gradient. A concentration compactness argument shows that the limit has the form $\alpha Eu + \eta$, with η supported on an at most countable set.

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1. Introduction

In this paper we prove a lower semicontinuity result of Reshetnyak type for a class of functionals which appear in models for small-strain elasto-plasticity coupled with damage. The functionals $\mathcal{H}(\alpha, p)$ that we consider depend on Sobolev functions α , the damage variables, and on bounded Radon measures p , the plastic strains.

In small-strain plasticity, the linearized strain Eu , defined as the symmetric part of the spatial gradient of the displacement $u: \Omega \rightarrow \mathbb{R}^n$, is decomposed as the sum of the elastic strain $e \in L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$, and of the plastic strain $p \in \mathcal{M}_b(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$, i.e., p is a bounded Radon measure with values in the space of symmetric matrices $\mathbb{M}_{\text{sym}}^{n \times n}$. In perfect plasticity (without damage), the energy dissipated in the evolution of the plastic strain is described in terms of the so-called plastic potential, defined in accordance to the theory

of convex functions of measures by

$$\int_{\Omega} H \left(\frac{dp}{d|p|}(x) \right) d|p|(x), \quad \text{for } p \in \mathcal{M}_b(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}).$$

In the formula above, $dp/d|p|$ is the Radon–Nikodym derivative of p with respect to its total variation $|p|$ and H is the support function of a set $K + \mathbb{R}I$, I being the identity matrix and K the convex compact set of the space of $n \times n$ trace-free matrices where the deviatoric part of the stress is constrained to lie. In particular, $H: \mathbb{M}_{\text{sym}}^{n \times n} \rightarrow [0, \infty]$ is convex, lower semicontinuous, and positively 1-homogeneous. We refer to [13] for all the details about the mathematical formulation of small-strain perfect plasticity.

In presence of damage, the constraint set depends on the real-valued damage variable α . Here we assume a multiplicative dependence, that is $K(\alpha) = V(\alpha)K$, with $V: \mathbb{R} \rightarrow [0, \infty)$ lower semicontinuous. In this setting the plastic potential becomes

$$\mathcal{H}(\alpha, p) := \int_{\Omega} V(\alpha(x)) H \left(\frac{dp}{d|p|}(x) \right) d|p|(x). \quad (1.1)$$

The functional above is sequentially lower semicontinuous with respect to the uniform convergence in α and the weak* convergence in p , as a consequence of Reshetnyak's Lower Semicontinuity Theorem (see, e.g., [6, Theorem 2.38]).

The lower semicontinuity of the plastic potential is, in general, a major difficulty in small-strain plasticity when the constraint set depends on an additional variable. For instance, in non-associative plasticity (cf. [7, 14, 20] and the recent [19]) such variable lacks continuity, and Reshetnyak's Theorem cannot be applied directly. The way out consists in replacing the original additional variable by a mollified one.

In gradient damage models, the total energy features a term in $\nabla \alpha$ which provides uniform bounds for α in $W^{1,q}(\Omega)$, for a suitable $q > 1$. When one considers the coupling with plasticity in the case $q > n$, the functional in (1.1) is defined by choosing the continuous representative of α and is sequentially lower semicontinuous with respect to the weak convergence in $W^{1,q}(\Omega)$, in view of the compact embedding of $W^{1,q}(\Omega)$ in $\mathcal{C}(\bar{\Omega})$. In particular, the minimum problems involved in the variational approach to the existence of quasistatic evolutions admit solutions, cf. [8].

However, in many mechanical models [3, 4, 22, 25, 26, 29, 30] the natural space for the damage variable is $W^{1,q}(\Omega)$ for some exponent $1 < q \leq n$, usually the Hilbert space $H^1(\Omega)$. Observe that a function $\alpha \in W^{1,q}(\Omega)$, with $q \leq n$, does not always admit a continuous representative. Nonetheless, the precise representative $\tilde{\alpha}$ of α is defined up to a set of q -capacity zero. In particular, this exceptional set has \mathcal{H}^{n-1} -measure zero and thus it is $|p|$ -negligible. The functional in (1.1) is therefore well-defined upon choosing this precise representative $\tilde{\alpha}$. Here we focus our attention on the critical case $q = n$, which in particular covers two dimensional models with damage in $H^1(\Omega)$.

In this setting we prove the following theorem, which is the main result of this paper.

Theorem 1.1. *Assume that Ω is a bounded, open, Lipschitz set. Let $V: \mathbb{R} \rightarrow [0, \infty]$ and $H: \Omega \times \mathbb{M}_{\text{sym}}^{n \times n} \rightarrow [0, \infty]$ be lower semicontinuous. Assume that H is positively 1-homogeneous and convex in the second variable. Let $\alpha_k, \alpha \in W^{1,n}(\Omega)$, $u_k, u \in BD(\Omega)$, $e_k, e \in L^q(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$ for some $q > 1$, and $p_k, p \in \mathcal{M}_b(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$. Assume that*

$$Eu_k = e_k + p_k \quad \text{in } \mathcal{M}_b(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}), \quad (1.2)$$

$$u_k \xrightarrow{*} u \quad \text{weakly}^* \text{ in } BD(\Omega), \quad (1.3)$$

$$p_k \xrightarrow{*} p \quad \text{weakly}^* \text{ in } \mathcal{M}_b(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}), \quad (1.4)$$

$$e_k \rightharpoonup e \quad \text{weakly in } L^q(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}), \quad (1.5)$$

$$\alpha_k \rightharpoonup \alpha \quad \text{weakly in } W^{1,n}(\Omega). \quad (1.6)$$

Then $Eu = e + p$ and

$$\int_{\Omega} V(\tilde{\alpha}(x)) H\left(x, \frac{dp}{d|p|}(x)\right) d|p|(x) \leq \liminf_{k \rightarrow +\infty} \int_{\Omega} V(\tilde{\alpha}_k(x)) H\left(x, \frac{dp_k}{d|p_k|}(x)\right) d|p_k|(x). \quad (1.7)$$

Theorem 1.1 is essential to prove the existence of the so-called energetic solutions (cf. [27]) for a model which couples small-strain plasticity and damage in $W^{1,n}(\Omega)$ (see Sect. 5). Moreover, it allows us to approximate this model by suitable models for gradient plasticity coupled with damage (see Remark 5.4).

To illustrate the proof of Theorem 1.1, we consider now the simplified case $V(\alpha) = \alpha$, $0 \leq \alpha \leq 1$, and $H(x, \xi) = |\xi|$. The starting point is the following Leibniz formula (Proposition 3.6)

$$\tilde{\alpha}_k Eu_k = E(\alpha_k u_k) - \nabla \alpha_k \odot u_k,$$

where \odot denotes the symmetric tensor product. If the sequence u_k were bounded in $L^\infty(\Omega; \mathbb{R}^n)$, then $\nabla \alpha_k \odot u_k$ would converge weakly in $L^n(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$ to $\nabla \alpha \odot u$, and the formula above would easily imply that $\tilde{\alpha}_k Eu_k \xrightarrow{*} \tilde{\alpha} Eu$. Here, the symmetrised structure of Eu is the main source of difficulty in the problem. Indeed, if $u \in BD(\Omega)$ and ψ is smooth, then, in general, the composite function $\psi(u)$ does not belong to $BD(\Omega)$. This prevents to employ a truncation argument in order to reduce the problem to the case $u \in L^\infty(\Omega; \mathbb{R}^n)$. Conversely, truncation techniques are allowed for instance when plasticity is coupled with damage in the antiplane setting.

In that case, u belongs to $BV(\Omega; \mathbb{R})$ and the lower semicontinuity of the plastic potential is proved via truncation (see [15, Proposition 2.3]). One may extend the argument in [15] to prove Reshetnyak-type lower semicontinuity when $u \in BV(\Omega; \mathbb{R}^m)$, as done in [9, Theorem 3.1] in the context of strain gradient plasticity coupled with damage, where the truncation is carefully chosen. We additionally remark that the techniques mentioned above are also effective in the case where damage belongs to $W^{1,q}(\Omega)$ with $q < n$.

In our setting we are forced to approach the problem differently: we give a precise description of the weak* limit of the sequence $\tilde{\alpha}_k Eu_k$, which may differ from $\tilde{\alpha} Eu$ when the sequence u_k is not bounded in $L^\infty(\Omega; \mathbb{R}^n)$ (cf. Example 3.1). Specifically, a concentration compactness argument in the

spirit of [23] allows us to prove in Theorem 3.2 that $\tilde{\alpha}_k Eu_k \xrightarrow{*} \tilde{\alpha} Eu + \eta$, where η is a measure concentrated on an at most countable set. In particular, $\tilde{\alpha}_k p_k \xrightarrow{*} \tilde{\alpha} p + \eta$. Passing to the total variations, this entails the desired lower semicontinuity since $\tilde{\alpha} p$ and η are mutually singular.

We stress that this type of proof only works in the critical case $\alpha \in W^{1,q}(\Omega)$ with $q = n$. Indeed, Example 3.8 shows that if $q < n$, it may happen that $\tilde{\alpha}_k Eu_k \xrightarrow{*} \tilde{\alpha} Eu + \eta$, where η is not singular with respect to $\tilde{\alpha} Eu$. The case $q < n$ will be the subject of a future investigation. We remark that when $H(\xi) = |\xi|$ and $e_k \rightarrow e$ strongly in $L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$, the plastic potential is lower semicontinuous even in the case $q < n$, as proven in [10, Section 4.6]. Indeed, these conditions on H and e_k allow for a slicing argument as in [16] which reduces the proof to the one-dimensional setting. This technique is however not suited to the case where e_k is only a weakly convergent sequence.

The paper is structured as follows. In Sect. 2 we fix the notation and we collect some preliminary results. Section 3 is devoted to the study of the weak* limit of sequences $\tilde{\alpha}_k Eu_k$: there we provide some explicit examples of concentration effects and we prove that the excess measure in the limit is concentrated on an at most countable set. Section 4 contains the proof of Theorem 1.1. Finally, in Sect. 5 we show applications of Theorem 1.1.

2. Notation and preliminary results

Notation

Throughout the paper we assume that $n \geq 2$. The Lebesgue measure in \mathbb{R}^n is denoted by \mathcal{L}^n , while \mathcal{H}^s is the s -dimensional Hausdorff measure.

The space of $n \times n$ symmetric matrices is denoted by $\mathbb{M}_{\text{sym}}^{n \times n}$; it is endowed with the euclidean scalar product $A : B := \text{tr}(AB^T)$, and the corresponding euclidean norm $|A| := (A : A)^{1/2}$. The symmetrised tensor product $a \odot b$ of two vectors $a, b \in \mathbb{R}^n$ is the symmetric matrix with components $(a_i b_j + a_j b_i)/2$.

Measures

Let Ω be an open set in \mathbb{R}^n . The space of bounded \mathbb{R}^m -valued Radon measures is denoted by $\mathcal{M}_b(\Omega; \mathbb{R}^m)$. This space can be regarded as the dual of the space $\mathcal{C}_0(\Omega; \mathbb{R}^m)$ of \mathbb{R}^m -valued continuous functions on $\bar{\Omega}$ vanishing on $\partial\Omega$. The notion of weak* convergence in $\mathcal{M}_b(\Omega; \mathbb{R}^m)$ refers to this duality. Moreover, we denote by $\mathcal{M}_b^+(\Omega)$ the space of non-negative bounded Radon measures. If $f \in L^1(\Omega; \mathbb{R}^m)$, we shall always identify the bounded Radon measure $f\mathcal{L}^n$ with the function f .

Let us consider a lower semicontinuous function $H : \Omega \times \mathbb{R}^m \rightarrow [0, \infty]$, positively 1-homogeneous and convex in the second variable and let us consider the functional defined in accordance to the theory of convex functions of measures

$$H(\mu) := \int_{\Omega} H \left(x, \frac{d\mu}{d|\mu|}(x) \right) d|\mu|(x), \quad \text{for } \mu \in \mathcal{M}_b(\Omega; \mathbb{R}^m),$$

where $d\mu/d|\mu|$ is the Radon–Nikodym derivative of μ with respect to its total variation $|\mu|$.

Remark 2.1. Let $\mu, \nu \in \mathcal{M}_b(\Omega; \mathbb{R}^m)$. If $|\mu|$ and $|\nu|$ are mutually singular, then $H(\mu + \nu) = H(\mu) + H(\nu)$ (cf. [6, Proposition 2.37]).

We recall the classical Reshetnyak’s Lower Semicontinuity Theorem [31]. For a proof we refer to [6, Theorem 2.38].

Theorem 2.2. (Reshetnyak’s Lower Semicontinuity Theorem) *Let Ω be an open set in \mathbb{R}^n . Let $\mu_k, \mu \in \mathcal{M}_b(\Omega; \mathbb{R}^m)$. If $\mu_k \xrightarrow{*} \mu$ weakly* in $\mathcal{M}_b(\Omega; \mathbb{R}^m)$, then*

$$\int_{\Omega} H\left(x, \frac{d\mu}{d|\mu|}(x)\right) d|\mu|(x) \leq \liminf_{k \rightarrow +\infty} \int_{\Omega} H\left(x, \frac{d\mu_k}{d|\mu_k|}(x)\right) d|\mu_k|(x),$$

for every lower semicontinuous function $H: \Omega \times \mathbb{R}^m \rightarrow [0, \infty]$, positively 1-homogeneous and convex in the second variable.

Functions of bounded deformation

Let Ω be an open set in \mathbb{R}^n . For every $u \in L^1(\Omega; \mathbb{R}^n)$, we denote by Eu the $\mathbb{M}_{\text{sym}}^{n \times n}$ -valued distribution on Ω , whose components are given by $E_{ij}u := \frac{1}{2}(D_j u^i + D_i u^j)$. The space $BD(\Omega)$ of functions of bounded deformation is the space of all $u \in L^1(\Omega; \mathbb{R}^n)$ such that $Eu \in \mathcal{M}_b(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$.

A sequence $(u_k)_k$ converges to u weakly* in $BD(\Omega)$ if and only if $u_k \rightarrow u$ strongly in $L^1(\Omega; \mathbb{R}^n)$ and $Eu_k \xrightarrow{*} Eu$ weakly* in $\mathcal{M}_b(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$. We recall that for every $u \in BD(\Omega)$ the measure Eu vanishes on sets of \mathcal{H}^{n-1} -measure zero.

The two following embedding theorems hold for the space of functions of bounded deformation. We denote by $1^* := \frac{n}{n-1}$ the Sobolev conjugate of 1.

Theorem 2.3. *The space $BD(\mathbb{R}^n)$ is continuously embedded in $L^{1^*}(\mathbb{R}^n; \mathbb{R}^n)$. More precisely, there exists a constant $C_1 = C_1(n) > 0$ such that for every $u \in BD(\mathbb{R}^n)$ we have*

$$\|u\|_{L^{1^*}(\mathbb{R}^n; \mathbb{R}^n)} \leq C_1 |Eu|(\mathbb{R}^n).$$

If Ω is a bounded, open, Lipschitz set, the space $BD(\Omega)$ is continuously embedded in $L^q(\Omega; \mathbb{R}^n)$ for every $1 \leq q \leq 1^*$.

Theorem 2.4. *Let Ω be a bounded, open, Lipschitz set. Then the space $BD(\Omega)$ is compactly embedded in $L^q(\Omega; \mathbb{R}^n)$ for every $1 \leq q < 1^*$.*

We refer to the book [35] for more details on the general properties of functions of bounded deformation and to [5] for their fine properties.

Capacity

For the notion of capacity we refer, e.g., to [18, 21]. We recall here the definition and some properties.

Let $1 \leq q < +\infty$ and let Ω be a bounded, open subset of \mathbb{R}^n . For every subset $B \subset \Omega$, the q -capacity of E in Ω is defined by

$$\text{Cap}_q(E, \Omega) := \inf \left\{ \int_{\Omega} |\nabla v|^q dx : v \in W_0^{1,q}(\Omega), v \geq 1 \text{ a.e. in a neighbourhood of } E \right\}.$$

A set $E \subset \Omega$ has q -capacity zero if $\text{Cap}_q(E, \Omega) = 0$ (actually, the definition does not depend on the open set Ω containing E). A property is said to hold Cap_q -quasi everywhere (abbreviated as Cap_q -q.e.) if it holds for a set of q -capacity zero.

If $1 < q \leq n$ and E has q -capacity zero, then $\mathcal{H}^s(E) = 0$ for every $s > n - q$.

A function $\alpha: \Omega \rightarrow \mathbb{R}$ is Cap_q -quasicontinuous if for every $\varepsilon > 0$ there exists a set $E_\varepsilon \subset \Omega$ with $\text{Cap}_q(E_\varepsilon, \Omega) < \varepsilon$ such that the restriction $\alpha|_{\Omega \setminus E_\varepsilon}$ is continuous. Note that if $q > n$, a function α is Cap_q -quasicontinuous if and only if it is continuous.

Every function $\alpha \in W^{1,q}(\Omega)$ admits a Cap_q -quasicontinuous representative $\tilde{\alpha}$, i.e., a Cap_q -quasicontinuous function $\tilde{\alpha}$ such that $\tilde{\alpha} = \alpha$ \mathcal{L}^n -a.e. in Ω . The Cap_q -quasicontinuous representative is essentially unique, that is, if $\tilde{\beta}$ is another Cap_q -quasicontinuous representative of α , then $\tilde{\beta} = \tilde{\alpha}$ Cap_q -q.e. in Ω . If $\alpha_{k_j} \rightarrow \alpha$ strongly in $W^{1,q}(\Omega)$, then there exists a subsequence k_j such that $\tilde{\alpha}_{k_j} \rightarrow \tilde{\alpha}$ Cap_q -q.e. in Ω (see, e.g., [12, Proposition 1.2]).

3. Concentration phenomena

In the whole section we assume that Ω is a bounded, open, Lipschitz set.

In order to prove the lower semicontinuity result, we shall provide a precise description of the weak* limit of the sequence of measures $\tilde{\alpha}_k E u_k$, for $\alpha_k \rightharpoonup \alpha$ weakly in $W^{1,n}(\Omega)$ and $u_k \overset{*}{\rightharpoonup} u$ weakly* in $BD(\Omega)$. We start by showing that, in general, the sequence $\tilde{\alpha}_k E u_k$ does not converge to $\tilde{\alpha} E u$ weakly* in $\mathcal{M}_b(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$. Indeed concentration phenomena may occur, as the following example shows.

Example 3.1. Let $n = 2$ and let $\Omega = (-1, 1)^2$. We construct here an explicit example of a sequence $(\alpha_k)_k$ in $W^{1,2}(\Omega)$ with $0 \leq \alpha_k \leq 1$ and a sequence $(u_k)_k$ in $BD(\Omega)$ such that

$$\alpha_k \rightarrow 0 \quad \text{weakly in } W^{1,2}(\Omega), \tag{3.1}$$

$$u_k \overset{*}{\rightharpoonup} 0 \quad \text{weakly* in } BD(\Omega), \tag{3.2}$$

but nonetheless

$$\tilde{\alpha}_k E u_k \text{ does not converge to } 0 \text{ weakly* in } \mathcal{M}_b(\Omega; \mathbb{M}_{\text{sym}}^{2 \times 2}). \tag{3.3}$$

Let us define the polygon $P_k = A_k \cup B_k \cup C_k \cup D_k$ as in Fig. 1. Let $A_k := (-\frac{1}{2k}, \frac{1}{2k}) \times (-\frac{1}{k}, 0)$ and $B_k := (-\frac{1}{2k}, \frac{1}{2k}) \times (0, \frac{1}{k})$. Let C_k be the union of the triangle C_k^+ with vertices $(\frac{1}{2k}, 0), (\frac{3}{2k}, 0), (\frac{1}{2k}, \frac{1}{k})$ and of the triangle C_k^- with vertices $(-\frac{1}{2k}, 0), (-\frac{3}{2k}, 0), (-\frac{1}{2k}, \frac{1}{k})$. Let D_k be the union of the triangle D_k^+ with vertices $(\frac{1}{2k}, 0), (\frac{1}{2k}, -\frac{1}{k}), (\frac{3}{2k}, 0)$ and of the triangle D_k^- with vertices $(-\frac{1}{2k}, 0), (-\frac{1}{2k}, -\frac{1}{k}), (-\frac{3}{2k}, 0)$. For k large enough, P_k is contained in Ω .

We define the piecewise affine functions $\alpha_k \in W^{1,\infty}(\Omega)$ in such a way that $\alpha_k(x) = 1$ for every $x \in \partial A_k \cap \partial B_k = (-\frac{1}{2k}, \frac{1}{2k}) \times \{0\}$, $\alpha_k(x) = 0$ for

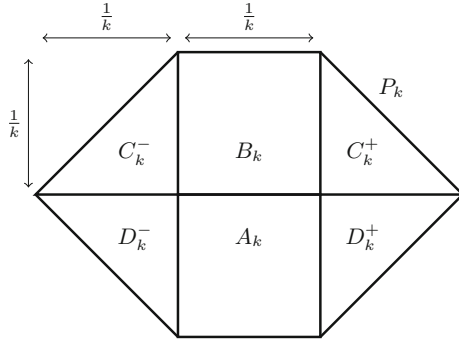


FIGURE 1. Decomposition of the set P_k

every $x \notin P_k$, and α_k is affine on each of the sets which decompose P_k . Notice that $0 \leq \alpha_k \leq 1$ and that

$$\nabla \alpha_k(x) = \begin{cases} ke_2 & \text{if } x \in A_k, \\ -ke_2 & \text{if } x \in B_k, \\ -ke_1 - ke_2 & \text{if } x \in C_k^+, \\ -ke_1 + ke_2 & \text{if } x \in D_k^+, \\ ke_1 - ke_2 & \text{if } x \in C_k^-, \\ ke_1 + ke_2 & \text{if } x \in D_k^-, \end{cases}$$

where $\{e_1, e_2\}$ is the standard basis in \mathbb{R}^2 . In particular $\sup_k \|\nabla \alpha_k\|_{L^2(\Omega; \mathbb{R}^2)} < +\infty$ and $\nabla \alpha_k \rightarrow 0$ strongly in $L^1(\Omega; \mathbb{R}^2)$. Finally, we define $u_k: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $u_k := |\nabla \alpha_k| \mathbb{1}_{A_k} e_1 = k \mathbb{1}_{A_k} e_1$, where $\mathbb{1}_{A_k}$ is the indicator function of the set A_k .

Since

$$\int_{\Omega} |\alpha_k(x)|^2 dx \leq \mathcal{L}^2(P_k) \rightarrow 0, \\ \sup_k \int_{\Omega} |\nabla \alpha_k(x)|^2 dx < +\infty,$$

we deduce (3.1). Moreover

$$\int_{\Omega} |u_k(x)| dx = \int_{A_k} |\nabla \alpha_k(x)| dx \rightarrow 0, \\ \sup_k |Eu_k|(\Omega) \leq C \sup_k [k \mathcal{H}^1(\partial A_k)] < +\infty$$

imply (3.2). In order to prove (3.3), let us fix $\varphi \in \mathcal{C}_0(\Omega; \mathbb{M}_{sym}^{2 \times 2})$. Let us denote the sides of A_k by L_k^i , $i = 1, 2, 3, 4$, L_k^1 being the top side and L_k^3 being the bottom side. Notice that the measure $\tilde{\alpha}_k Eu_k$ is concentrated on $L_k^1 \cup L_k^2 \cup L_k^4$ and that

$$\int_{L_k^2 \cup L_k^4} \tilde{\alpha}_k \varphi : dEu_k = k^2 \int_{-\frac{1}{k}}^0 x_2 \left[\varphi\left(-\frac{1}{2k}, x_2\right) - \varphi\left(\frac{1}{2k}, x_2\right) \right] : e_1 \odot e_1 dx_2 \rightarrow 0,$$

x_2 denoting the second coordinate of x . Therefore, the only contribution to the limit is given by

$$\int_{L_k^1} \tilde{\alpha}_k \varphi : dEu_k = -k \int_{L_k^1} \varphi : (e_1 \odot e_2) \, d\mathcal{H}^1 \rightarrow -\varphi(0) : (e_1 \odot e_2),$$

i.e., $\tilde{\alpha}_k Eu_k \xrightarrow{*} -\delta_0 e_1 \odot e_2$. This proves the claim. The example can be also modified in order to have $\operatorname{div} u_k = 0$. This can be done by suitably extending the vector field u_k in D_k^+ , D_k^- , and $\Omega \setminus P_k$.

In the previous example, the difference between $\tilde{\alpha} Eu = 0$ and the weak* limit of $\tilde{\alpha}_k Eu_k$ is a measure concentrated on a point. Actually, we will show that for every sequence $(\tilde{\alpha}_k Eu_k)_k$ the excess measure in the limit may concentrate on at most countably many points. Specifically, we shall prove the following result.

Theorem 3.2. *Let $\alpha_k, \alpha \in W^{1,n}(\Omega)$ and $u_k, u \in BD(\Omega)$. Assume that*

$$\|\alpha_k\|_{L^\infty(\Omega)} \leq M, \tag{3.4}$$

$$\alpha_k \rightharpoonup \alpha \text{ weakly in } W^{1,n}(\Omega), \tag{3.5}$$

$$u_k \xrightarrow{*} u \text{ weakly* in } BD(\Omega). \tag{3.6}$$

Then, up to a subsequence (which we do not relabel),

$$\tilde{\alpha}_k Eu_k \xrightarrow{*} \tilde{\alpha} Eu + \eta \text{ weakly* in } \mathcal{M}_b(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}),$$

where $\eta \in \mathcal{M}_b(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$ is concentrated on an at most countable set.

Remark 3.3. The symmetrised structure of Eu does not play a major role in the proof of Theorem 3.2. It may be worth noting that the proof can be readily adapted to the case where $u_k, u \in BV(\Omega; \mathbb{R}^m)$ and $u_k \xrightarrow{*} u$ weakly* in $BV(\Omega; \mathbb{R}^m)$. In that case, the conclusion is that $\tilde{\alpha}_k Du_k \xrightarrow{*} \tilde{\alpha} Du + \eta$ weakly* in $\mathcal{M}_b(\Omega; \mathbb{R}^{n \times m})$, where $\eta \in \mathcal{M}_b(\Omega; \mathbb{R}^{n \times m})$ is concentrated on an at most countable set.

The initial step for the proof of Theorem 3.2 is a careful analysis of the limit behaviour of a sequence $(u_k)_k$ converging weakly* in $BD(\Omega)$. The Embedding Theorems for $BD(\Omega)$ (Theorems 2.3, 2.4) do not guarantee that the sequence $(u_k)_k$ converges strongly in $L^{1^*}(\Omega; \mathbb{R}^n)$. Nevertheless, the following concentration compactness argument in the spirit of [23, 24] shows that the lack of compactness of $(u_k)_k$ in $L^{1^*}(\Omega; \mathbb{R}^n)$ is only due to concentration around countably many points. For a proof of the analogous result in the Sobolev case we refer e.g. to [17].

Theorem 3.4. *Let $(u_k)_k$ be a sequence in $BD(\Omega)$. Assume that $u_k \xrightarrow{*} 0$ weakly* in $BD(\Omega)$ and that*

$$|u_k|^{1^*} \xrightarrow{*} \nu \text{ weakly* in } \mathcal{M}_b(\Omega) \tag{3.7}$$

for some non-negative measure $\nu \in \mathcal{M}_b^+(\Omega)$. Then ν is concentrated on an at most countable set, i.e., there exists a countable set $\{x_j\}_j$ of points of Ω such that

$$\nu = \sum_j c_j \delta_{x_j},$$

with $c_j \in (0, +\infty)$.

Proof. Upon extracting a subsequence (which we do not relabel), we suppose that

$$|Eu_k| \overset{*}{\rightharpoonup} \mu \quad \text{weakly* in } \mathcal{M}_b(\Omega) \tag{3.8}$$

for some measure non-negative measure $\mu \in \mathcal{M}_b^+(\Omega)$. Let us define the set

$$D := \{x \in \Omega : \mu(\{x\}) > 0\}.$$

Note that the set D is at most countable, since μ is a finite measure. We claim that ν is concentrated on a subset of D .

We first prove that the measure ν is absolutely continuous with respect to μ . Let us fix a compact set $K \subset \Omega$, and an open set $V \subset \Omega$ such that $K \subset V$. Let us consider a cut-off function $\phi \in C_c^1(\Omega)$ with $0 \leq \phi \leq 1$, $\phi = 1$ on K , $\text{supp}(\phi) \subset V$. The functions ϕu_k have compact support in Ω , they belong to $BD(\mathbb{R}^n)$, and $E(\phi u_k) = \phi Eu_k + \nabla \phi \odot u_k$. By Theorem 2.3, we infer that

$$\left(\int_{\mathbb{R}^n} |\phi u_k|^{1^*} dx \right)^{1/1^*} \leq C_1 |E(\phi u_k)|(\mathbb{R}^n) \leq C_1 \left[\int_{\Omega} |\phi| d|Eu_k| + \int_{\Omega} |\nabla \phi \odot u_k| dx \right].$$

Since $u_k \rightarrow 0$ strongly in $L^1(\Omega; \mathbb{R}^n)$ (Theorem 2.4), we have

$$\int_{\Omega} |\nabla \phi \odot u_k| dx \rightarrow 0$$

as $k \rightarrow +\infty$. Testing (3.7) and (3.8) with the functions $|\phi|^{1^*}$ and $|\phi|$ respectively, we pass to the limit as $k \rightarrow +\infty$ in the inequality above and we get

$$\left(\int_{\Omega} |\phi|^{1^*} d\nu \right)^{1/1^*} \leq C_1 \int_{\Omega} |\phi| d\mu.$$

From the assumptions on ϕ we deduce that

$$(\nu(K))^{1/1^*} \leq C_1 \mu(V).$$

By the arbitrariness of K and V , we have

$$(\nu(B))^{1/1^*} \leq C_1 \mu(B) \tag{3.9}$$

for any Borel set $B \subset \Omega$. Therefore we conclude that ν is absolutely continuous with respect to μ .

By the Radon–Nikodym Theorem

$$\nu = \frac{d\nu}{d\mu} \mu,$$

where $\frac{d\nu}{d\mu}$ is the Radon–Nikodym derivative of ν with respect to μ given by

$$\frac{d\nu}{d\mu}(x) = \lim_{r \rightarrow 0^+} \frac{\nu(B_r(x))}{\mu(B_r(x))} \quad \text{for } \mu\text{-a.e. } x \in \Omega.$$

By (3.9) and the formula above we infer that

$$\frac{d\nu}{d\mu}(x) \leq \limsup_{r \rightarrow 0^+} \left[C_1^{1^*} \mu(B_r(x))^{1^*-1} \right] = 0 \quad \text{for } \mu\text{-a.e. } x \in \Omega \setminus D,$$

i.e., that ν is concentrated on a subset of D . □

The following lemma will be used in the proof of Theorem 3.2 to characterise the limit of the sequence $(\nabla\alpha_k \odot u_k)_k$.

Lemma 3.5. *Let $(g_k)_k$ be a bounded sequence in $L^n(\Omega; \mathbb{R}^n)$ and let $(u_k)_k$ be a sequence in $BD(\Omega)$ such that $u_k \xrightarrow{*} 0$ weakly* in $BD(\Omega)$. Assume that*

$$|g_k \odot u_k| \xrightarrow{*} \nu \quad \text{weakly* in } \mathcal{M}_b(\Omega)$$

for some non-negative measure $\nu \in \mathcal{M}_b^+(\Omega)$. Then ν is concentrated on an at most countable set.

Proof. By Theorem 2.3, the sequence $(|u_k|^{1^*})_k$ is bounded in $L^1(\Omega)$. Upon extracting a subsequence (which we do not relabel), we suppose that

$$|g_k|^n \xrightarrow{*} \nu^g, \quad |u_k|^{1^*} \xrightarrow{*} \nu^u \quad \text{weakly* in } \mathcal{M}_b(\Omega).$$

Let us fix a compact set $K \subset \Omega$, and an open set $V \subset \Omega$ such that $K \subset V$. Let $\phi \in \mathcal{C}_c^1(\Omega)$ be such that $0 \leq \phi \leq 1$, $\phi = 1$ on K , and $\text{supp}(\phi) \subset V$. By Hölder’s Inequality we have

$$\int_{\Omega} \phi^2 |g_k \odot u_k| \, dx \leq C \int_{\Omega} |\phi g_k| |\phi u_k| \, dx \leq C \left(\int_{\Omega} |\phi|^n |g_k|^n \, dx \right)^{1/n} \left(\int_{\Omega} |\phi|^{1^*} |u_k|^{1^*} \, dx \right)^{1/1^*}.$$

Passing to the limit as $k \rightarrow +\infty$ we deduce that

$$\int_{\Omega} \phi^2 \, d\nu \leq C \left(\int_{\Omega} |\phi|^n \, d\nu^g \right)^{1/n} \left(\int_{\Omega} |\phi|^{1^*} \, d\nu^u \right)^{1/1^*}$$

and thus

$$\nu(K) \leq C (\nu^g(V))^{1/n} (\nu^u(V))^{1/1^*}.$$

By the arbitrariness of K and V we conclude that

$$\nu(B) \leq C (\nu^g(B))^{1/n} (\nu^u(B))^{1/1^*}$$

for every Borel set B , and therefore that ν is absolutely continuous with respect to ν^u , which by Theorem 3.4 is concentrated on an at most countable set. □

We shall need the following Leibniz rule formula for the product of Sobolev functions and functions of bounded deformation. We include the proof for the convenience of the reader.

Proposition 3.6. *Let $\alpha \in W^{1,n}(\Omega) \cap L^\infty(\Omega)$, and $u \in BD(\Omega)$. Then $\alpha u \in BD(\Omega)$ and*

$$E(\alpha u) = \tilde{\alpha} E u + \nabla \alpha \odot u \quad \text{in } \mathcal{M}_b(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}). \tag{3.10}$$

Proof. The proof is based on an approximation argument. There exists a sequence of smooth functions $\alpha_k \in C^\infty(\bar{\Omega})$ such that $\alpha_k \rightarrow \alpha$ strongly in $W^{1,n}(\Omega)$ and $\|\alpha_k\|_{L^\infty(\Omega)} \leq \|\alpha\|_{L^\infty(\Omega)}$. It is immediate to prove via integration by parts that

$$E(\alpha_k u) = \alpha_k Eu + \nabla \alpha_k \odot u \quad \text{in } \mathcal{M}_b(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}).$$

In particular, the total variations $|E(\alpha_k u)|$ are bounded, and thus $E(\alpha_k u) \xrightarrow{*} E(\alpha u)$. Moreover, $\nabla \alpha_k \odot u \rightarrow \nabla \alpha \odot u$ strongly in $L^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$. To conclude the proof of (3.10), we simply remark that (up to a subsequence) $\alpha_k \rightarrow \tilde{\alpha}$ Cap_n -q.e. (see, e.g., [12, Proposition 1.2]) and Eu vanishes on sets of n -capacity zero, so that $\alpha_k Eu \xrightarrow{*} \tilde{\alpha} Eu$ in $\mathcal{M}_b(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$ by the Dominated Convergence Theorem. \square

We are now in a position to prove Theorem 3.2.

Proof of Theorem 3.2. By Proposition 3.6 we have

$$\tilde{\alpha}_k Eu_k = E(\alpha_k u_k) - \nabla \alpha_k \odot u_k. \tag{3.11}$$

Notice that

$$E(\alpha_k u_k) \xrightarrow{*} E(\alpha u) \quad \text{weakly* in } \mathcal{M}_b(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}). \tag{3.12}$$

Indeed, by Hölder’s Inequality

$$|E(\alpha_k u_k)|(\Omega) \leq \|\alpha_k\|_{L^\infty(\Omega)} |Eu_k|(\Omega) + C \|\nabla \alpha_k\|_{L^n(\Omega; \mathbb{R}^n)} \|u_k\|_{L^{1^*}(\Omega; \mathbb{R}^n)}.$$

By (3.4)–(3.6) and by Theorem 2.3 the right-hand side in the inequality above is uniformly bounded. Since $\alpha_k u_k \rightarrow \alpha u$ strongly in $L^1(\Omega; \mathbb{R}^n)$, we conclude that (3.12) holds.

We now study the weak* limit of $(\nabla \alpha_k \odot u_k)_k$ in $\mathcal{M}_b(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$. Since $\nabla \alpha_k \rightarrow \nabla \alpha$ weakly in $L^n(\Omega; \mathbb{R}^n)$, we get that

$$\nabla \alpha_k \odot u \rightharpoonup \nabla \alpha \odot u \quad \text{weakly in } L^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}). \tag{3.13}$$

Upon the extraction of a subsequence (that we do not relabel), we can assume that

$$\begin{aligned} \nabla \alpha_k \odot (u - u_k) &\xrightarrow{*} \eta \quad \text{weakly* in } \mathcal{M}_b(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}), \\ |\nabla \alpha_k \odot (u - u_k)| &\xrightarrow{*} \nu \quad \text{weakly* in } \mathcal{M}_b(\Omega). \end{aligned} \tag{3.14}$$

By Lemma 3.5 we have that ν , and *a fortiori* η , is concentrated on an at most countable set.

By (3.13) and (3.14) we get that

$$\nabla \alpha_k \odot u_k \xrightarrow{*} \nabla \alpha \odot u - \eta \tag{3.15}$$

weakly* in $\mathcal{M}_b(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$.

From (3.11), (3.12), (3.15), and Proposition 3.6 we conclude that

$$\tilde{\alpha}_k Eu_k \xrightarrow{*} E(\alpha u) - \nabla \alpha \odot u + \eta = \tilde{\alpha} Eu + \eta$$

weakly* in $\mathcal{M}_b(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$. \square

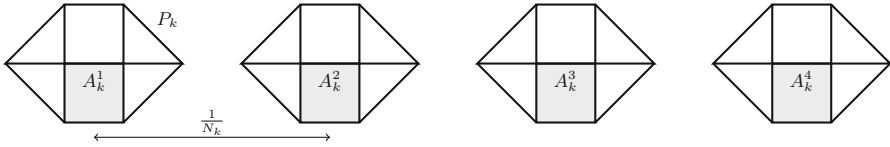


FIGURE 2. The function β_k is supported on the union of the N_k equispaced copies of P_k , while the function u_k is supported on the grey region given by $\bigcup_{j=1}^{N_k} A_k^j$

Remark 3.7. Theorem 3.2 does not hold if $\alpha_k \in W^{1,q}(\Omega)$ with $q < n$. In this case, the difference between $\tilde{\alpha}Eu$ and the weak* limit of $\tilde{\alpha}_k Eu_k$ may be not singular with respect to measures which vanish on sets with Hausdorff dimension strictly less than $n-1$. We provide an example below.

Example 3.8. Let $n = 2$, let $\Omega = (-2, 2)^2$, and let $1 < q < 2$. We provide here an example of a sequence $(\beta_k)_k$ in $W^{1,q}(\Omega)$ with $0 \leq \beta_k \leq 1$ and a sequence $(u_k)_k$ in $BD(\Omega)$ such that $\beta_k \rightharpoonup 0$ weakly in $W^{1,q}(\Omega)$, $u_k \xrightarrow{*} 0$ weakly* in $BD(\Omega)$, and the weak* limit of $\tilde{\beta}_k Eu_k$ is concentrated on a set of Hausdorff dimension 1.

Let $\alpha_k \in W^{1,\infty}(\Omega)$ be the piecewise affine functions supported on the polygons P_k and let A_k be the cubes exhibited in Example 3.1. Let N_k be the integer part of k^{2-q} and let $x_k^j = \left(\frac{j-1}{N_k}, 0\right)$. We define $\beta_k(x) := \sum_{j=1}^{N_k} \alpha_k(x - x_k^j)$ and $u_k := \sum_{j=1}^{N_k} k^{q-1} e_1 \mathbb{1}_{A_k^j}$, where $A_k^j = A_k + x_k^j$. (See Fig. 2.) Notice that $\beta_k \rightarrow 0$ strongly in $L^q(\Omega)$, $u_k \rightarrow 0$ strongly in $L^1(\Omega; \mathbb{R}^2)$, and

$$\int_{\Omega} |\nabla \beta_k|^q dx = \sum_{j=1}^{N_k} \int_{P_k} |\nabla \alpha_k|^q dx \sim N_k \frac{1}{k^2} k^q \sim 1,$$

$$|Eu_k|(\Omega) \leq C \sum_{j=1}^{N_k} k^{q-1} \mathcal{H}^1(\partial A_k) \sim N_k k^{q-1} \frac{1}{k} \sim 1,$$

as $k \rightarrow +\infty$. Thus $\beta_k \rightharpoonup 0$ weakly in $W^{1,q}(\Omega)$ and $u_k \xrightarrow{*} 0$ weakly* in $BD(\Omega)$. With computations similar to those contained in Example 3.1, it is easy to show that only the restriction of $\tilde{\beta}_k Eu_k$ to the top sides of the squares A_k^j gives a contribution to the limit. Hence for every $\varphi \in \mathcal{C}_0(\Omega; \mathbb{M}_{sym}^{2 \times 2})$ we have

$$\int_{\Omega} \tilde{\beta}_k \varphi : dEu_k = - \sum_{j=1}^{N_k} k^{q-1} \int_{-\frac{1}{2k}}^{\frac{1}{2k}} \varphi \left(x_1 + \frac{j-1}{N_k}, 0\right) : e_1 \odot e_2 dx_1$$

$$+ o(1) \rightarrow - \int_{[0,1] \times \{0\}} \varphi : e_1 \odot e_2 d\mathcal{H}^1,$$

i.e., $\tilde{\beta}_k Eu_k \xrightarrow{*} -e_1 \odot e_2 \mathcal{H}^1 \llcorner ([0, 1] \times \{0\})$.

4. Proof of Theorem 1.1

Upon the extraction of a subsequence (that we do not relabel), we assume that the liminf in (1.7) is actually a limit.

We shall prove the theorem supposing that V is a Lipschitz function. Indeed, if this is not the case, we can always find an increasing family of Lipschitz functions $V_h: \mathbb{R} \rightarrow [0, +\infty)$ such that $V = \sup_h V_h$. Then, assuming that (1.7) holds for each V_h , we have

$$\begin{aligned} & \int_{\Omega} V_h(\tilde{\alpha}(x)) H\left(x, \frac{dp}{d|p|}(x)\right) d|p|(x) \\ & \leq \liminf_{k \rightarrow +\infty} \int_{\Omega} V_h(\tilde{\alpha}_k(x)) H\left(x, \frac{dp_k}{d|p_k|}(x)\right) d|p_k|(x) \\ & \leq \liminf_{k \rightarrow +\infty} \int_{\Omega} V(\tilde{\alpha}_k(x)) H\left(x, \frac{dp_k}{d|p_k|}(x)\right) d|p_k|(x), \end{aligned}$$

and by the Monotone Convergence Theorem we deduce (1.7).

Let us define the non-negative functions $\beta_k := V(\alpha_k)$ and $\beta := V(\alpha)$. Since V is Lipschitz and Ω is bounded, the chain rule for Sobolev functions implies that $\beta_k, \beta \in W^{1,n}(\Omega)$. Moreover, it is immediate to see that $\beta_k \rightharpoonup \beta$ weakly in $W^{1,n}(\Omega)$, i.e., the sequence $(\beta_k)_k$ satisfies the same assumptions on the sequence $(\alpha_k)_k$. Moreover, $\beta_k \geq 0$ a.e. in Ω .

Let us prove the theorem under the additional assumption that $\|\beta_k\|_{L^\infty(\Omega)} \leq M$. Notice that $\beta_k \rightarrow \beta$ strongly in $L^2(\Omega)$. Together with (1.5), this implies that $\beta_k e_k \rightharpoonup \beta e$ weakly in $L^1(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$. Hence, by (1.2) and Theorem 3.2, we have (up to a subsequence)

$$\tilde{\beta}_k p_k = \tilde{\beta}_k E u_k - \beta_k e_k \overset{*}{\rightharpoonup} \tilde{\beta} E u - \beta e + \eta = \tilde{\beta} p + \eta \quad \text{weakly* in } \mathcal{M}_b(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}),$$

where the measure $\eta \in \mathcal{M}_b(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$ is concentrated on an at most countable set. Since $|p|$ is concentrated on sets of dimension greater than $n-1$, the measures $|\tilde{\beta} p|$ and $|\eta|$ are mutually singular. By Remark 2.1, by the 1-homogeneity of H , and by Reshetnyak’s Lower Semicontinuity Theorem we infer that

$$\begin{aligned} \int_{\Omega} \tilde{\beta} H\left(x, \frac{dp}{d|p|}\right) d|p| & \leq \int_{\Omega} \tilde{\beta} H\left(x, \frac{dp}{d|p|}\right) d|p| + \int_{\Omega} H\left(x, \frac{d\eta}{d|\eta|}\right) d|\eta| \\ & = \int_{\Omega} H\left(x, \frac{d(\tilde{\beta} p + \eta)}{d|\tilde{\beta} p + \eta|}\right) d|\tilde{\beta} p + \eta| \\ & \leq \liminf_{k \rightarrow +\infty} \int_{\Omega} \tilde{\beta}_k H\left(x, \frac{dp_k}{d|p_k|}\right) d|p_k|. \end{aligned}$$

To remove the assumption that the sequence $(\beta_k)_k$ is bounded in $L^\infty(\Omega)$ we use a truncation argument. For every $M > 0$ we define the functions $\beta_k^M := \beta_k \wedge M$ and $\beta^M := \beta \wedge M$. Since $\beta_k^M \rightharpoonup \beta^M$ weakly in $W^{1,n}(\Omega)$, by the previous step we have

$$\int_{\Omega} \tilde{\beta}^M H\left(x, \frac{dp}{d|p|}\right) d|p| \leq \liminf_{k \rightarrow +\infty} \int_{\Omega} \tilde{\beta}_k H\left(x, \frac{dp_k}{d|p_k|}\right) d|p_k|.$$

We conclude applying the Monotone Convergence Theorem as $M \rightarrow +\infty$.

5. Application to a model for linearised elasto-plasticity coupled with damage

In this section we apply Theorem 1.1 to show the existence of *energetic solutions* (cf. [27]) for a model which couples small-strain plasticity and damage in $W^{1,n}(\Omega)$ (recall $\Omega \subset \mathbb{R}^n$). The mechanical framework for this coupling has been proposed and analysed in [2,3] (for further contribution in this direction see, e.g., [1,32–34]). The existence of quasistatic evolutions has been proven in [8,11] via the *energetic approach* and via *vanishing viscosity*, respectively (see e.g. [28] for details and comparison for the two approaches). The notion of quasistatic evolution we give below is similar to the one in [8]. In that paper, the damage variable belongs to $W^{1,q}(\Omega)$, with $q > n$, and in particular it is continuous.

We assume that Ω is a bounded, open, Lipschitz set with boundary partitioned as $\partial\Omega = \partial_D\Omega \cup \partial_N\Omega \cup N$, with $\partial_D\Omega$ and $\partial_N\Omega$ relatively open, $\partial_D\Omega \cap \partial_N\Omega = \emptyset$, $\mathcal{H}^{n-1}(N) = 0$, and $\partial_D\Omega \neq \emptyset$. Moreover, we assume that the common boundary between $\partial_D\Omega$ and $\partial_N\Omega$ is smooth enough, more precisely that [11, (2.2)] holds; this is only needed to ensure a suitable integration by parts formula in the stress-strain duality. Let $[0, T]$ be the time interval where we study the evolution, and $u_D \in AC([0, T]; H^1(\mathbb{R}^n; \mathbb{R}^n))$ be a prescribed Dirichlet datum for the displacement on $\partial_D\Omega$. For simplicity of notation, both the surface forces on $\partial_N\Omega$ and the volume forces are null.

Let us now briefly recall the energetic and dissipative terms involved in the definition of energetic solutions for the present model, referring to [8] for more details.

The *elastic energy* is defined on $L^1(\Omega; [0, 1]) \times L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$ by

$$\mathcal{Q}(\alpha, e) := \frac{1}{2} \int_{\Omega} \mathbb{C}(\alpha(x)) e(x) : e(x) dx.$$

The *elasticity tensor* $\mathbb{C}(\alpha)$ is a symmetric fourth order tensor for any α , Lipschitz and non-decreasing in α , equicontinuous and equicoercive with respect to α , and it induces a linear map on $\mathbb{M}_{\text{sym}}^{n \times n}$ that preserves the space of symmetric deviatoric matrices $\mathbb{M}_D^{n \times n}$, as well as its orthogonal space $\mathbb{R}I$.

The *plastic potential* is defined on $W^{1,n}(\Omega; [0, 1]) \times \mathcal{M}_b(\Omega \cup \partial_D\Omega; \mathbb{M}_D^{n \times n})$ by

$$\mathcal{H}(\alpha, p) := \int_{\Omega \cup \partial_D\Omega} V(\tilde{\alpha}(x)) H\left(\frac{dp}{d|p|}(x)\right) d|p|(x).$$

We assume that the function $V : [0, 1] \rightarrow [c_1, \infty)$ is Lipschitz and non-decreasing, and that $c_1 > 0$; $H : \mathbb{M}_D^{n \times n} \rightarrow [0, \infty)$ is positively 1-homogeneous and convex, with $r|\xi| \leq H(\xi) \leq R|\xi|$, for some $r > 0$. Notice that every $\alpha \in W^{1,n}(\Omega)$ is well defined in $\bar{\Omega}$ up to a set of n -capacity zero, by considering

any $W^{1,n}$ extension of α to a larger set Ω' . We remark that the hypotheses on H in [8] are slightly more general (see [8, (2.11)]), here we are in the setting of [8, Remark 2.1].

The *plastic dissipation* in a time interval $[s, t]$ is defined for any $\alpha : [s, t] \rightarrow W^{1,n}(\Omega; [0, 1])$ and any $p : [s, t] \rightarrow \mathcal{M}_b(\Omega \cup \partial_D\Omega; \mathbb{M}_D^{n \times n})$ by

$$\mathcal{V}_{\mathcal{H}}(\alpha, p; s, t) := \sup \left\{ \sum_{j=1}^N \mathcal{H}(\alpha(t_j), p(t_j) - p(t_{j-1})) : s = t_0 < t_1 < \dots < t_N = t, N \in \mathbb{N} \right\}. \tag{5.1}$$

Moreover, we consider a non-negative, continuous, and non-increasing function d and we introduce the functional $D : L^1(\Omega; [0, 1]) \rightarrow [0, \infty)$ defined by $D(\alpha) := \int_{\Omega} d(\alpha(x)) \, dx$. This term accounts for the energy dissipated during the damage process. For a given $w \in H^1(\mathbb{R}^n; \mathbb{R}^n)$, the set of *admissible plasticity triples* for w is

$$A(w) := \{(u, e, p) \in BD(\Omega) \times L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}) \times \mathcal{M}_b(\Omega \cup \partial_D\Omega; \mathbb{M}_D^{n \times n}) : Eu = e + p \text{ in } \Omega, p = (w - u) \odot \nu \mathcal{H}^{n-1} \text{ on } \partial_D\Omega\}.$$

We are now ready to give the definition of *energetic solutions* (or *globally stable quasistatic evolutions*) driven by the boundary datum u_D .

Definition 5.1. An *energetic solution* is a function $t \mapsto (\alpha(t), u(t), e(t), p(t))$ from $[0, T]$ into $W^{1,n}(\Omega; [0, 1]) \times BD(\Omega) \times L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}) \times \mathcal{M}_b(\Omega \cup \partial_D\Omega; \mathbb{M}_D^{n \times n})$ such that $(u(t), e(t), p(t)) \in A(u_D(t))$ for every $t \in [0, T]$ and the following conditions are satisfied:

- (QS0) *irreversibility* : $\alpha(t) \leq \alpha(s)$ \mathcal{L}^n -a.e. in Ω for every $0 \leq s \leq t \leq T$;
- (QS1) *global stability*: for any $t \in [0, T]$ and any $\hat{\alpha} \leq \alpha(t)$, $(\hat{u}, \hat{e}, \hat{p}) \in A(u_D(t))$

$$\begin{aligned} \mathcal{Q}(\alpha(t), e(t)) + D(\alpha(t)) + \int_{\Omega} |\nabla \alpha(t; x)|^n \, dx &\leq \mathcal{Q}(\hat{\alpha}, \hat{e}) + D(\hat{\alpha}) \\ &+ \int_{\Omega} |\nabla \hat{\alpha}(x)|^n \, dx + \mathcal{H}(\hat{\alpha}, \hat{p} - p(t)); \end{aligned}$$

- (QS2) *energy balance*: for any $t \in [0, T]$

$$\begin{aligned} \mathcal{Q}(\alpha(t), e(t)) + D(\alpha(t)) + \int_{\Omega} |\nabla \alpha(t; x)|^n \, dx + \mathcal{V}_{\mathcal{H}}(\alpha, p; 0, t) \\ = \mathcal{Q}(\alpha(0), e(0)) + D(\alpha(0)) \\ + \int_{\Omega} |\nabla \alpha(0; x)|^n \, dx + \int_0^t \int_{\Omega} \mathbb{C}(\alpha(s; x)) e(s; x) : E\dot{u}_D(s; x) \, dx \, ds. \end{aligned}$$

Thanks to Theorem 1.1, we can prove the following existence result.

Theorem 5.2. *Let $\alpha_0 \in W^{1,n}(\Omega; [0, 1])$ and $(u_0, e_0, p_0) \in A(u_D(0))$ satisfying the global stability condition (QS1) at the initial time. Then there exists an energetic solution such that $\alpha(0) = \alpha_0$, $u(0) = u_0$, $e(0) = e_0$, $p(0) = p_0$.*

Remark 5.3. The definition of evolutions above differs from the one in [8] not only for the damage regularisation. Indeed, in [8] there is a parameter $\lambda \in [0, 1]$ that accounts for the interplay between damage growth and cumulation of plastic strain, thus for a fatigue phenomenon. We stated, for simplicity of notation, Definition 5.1 only for the case $\lambda = 0$; one can follow the argument in [8] to prove existence of energetic solutions corresponding to any λ . Moreover, in [8] the Dirichlet boundary was the whole $\partial\Omega$. As observed in [8], it is a minor point to consider also external volume and surface forces.

Proof of Theorem 5.2. We can closely follow the proof of [8, Theorem 4.3], based on a time incremental approach, which is by now well consolidated. This consists in solving incremental minimisation problems, to obtain discrete-time evolutions that satisfy a discrete global stability and a discrete energy inequality, then passing these conditions to the limit as the time discretisation step tends to 0, to eventually get the energy balance by the global stability. We need lower semicontinuity of \mathcal{H} in order to prove the existence of minimisers for the incremental minimisation problems, and to show the lower semicontinuity of the plastic dissipation, which is a supremum of suitable plastic potentials, as the time discretisation step tends to 0.

The lower semicontinuity of \mathcal{H} is deduced by Theorem 1.1 in the following way. Let $U \subset \mathbb{R}^n$ be a bounded, open, Lipschitz set such that $U \cap \partial\Omega = \partial_D\Omega$. Let $\Omega^* := \Omega \cup U$ and let us define for any $w \in H^1(\mathbb{R}^n; \mathbb{R}^n)$ and for any $(u, e, p) \in A(w)$

$$u^* := \begin{cases} u & \text{in } \Omega, \\ w & \text{in } \Omega^* \setminus \Omega, \end{cases} \quad e^* := \begin{cases} e & \text{in } \Omega, \\ Ew & \text{in } \Omega^* \setminus \Omega, \end{cases} \quad p^* := \begin{cases} p & \text{in } \bar{\Omega}, \\ 0 & \text{in } \Omega^* \setminus \bar{\Omega}. \end{cases} \tag{5.2}$$

We also consider a continuous extension operator from $W^{1,n}(\Omega)$ to $W^{1,n}(\Omega^*)$ and we associate to any $\alpha \in W^{1,n}(\Omega)$ its extension $\alpha^* \in W^{1,n}(\Omega^*)$. Then

$$\mathcal{H}(\alpha, p) = \int_{\Omega^*} V(\tilde{\alpha}^*(x)) H\left(\frac{dp^*}{d|p^*|}(x)\right) d|p^*|(x).$$

If $\alpha_k \rightharpoonup \alpha$ weakly in $W^{1,n}(\Omega)$, $w_k \rightharpoonup w$ weakly in $H^1(\mathbb{R}^n; \mathbb{R}^n)$, $(u_k, e_k, p_k) \in A(w_k)$, $u_k \xrightarrow{*} u$ weakly* in $BD(\Omega)$, and $e_k \rightharpoonup e$ weakly in $L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$, by Theorem 1.1 we get

$$\mathcal{H}(\alpha, p) \leq \liminf_{k \rightarrow \infty} \mathcal{H}(\alpha_k, p_k).$$

Indeed, by (5.2), we have the convergence $p_k^* \xrightarrow{*} p^*$ weakly* in $\mathcal{M}_b(\Omega^*; \mathbb{M}_D^{n \times n})$ for the extensions.

With the lower semicontinuity property above at hand, one follows the proof of [8, Theorem 4.3] and concludes Theorem 5.2. In particular, the lower semicontinuity of the plastic dissipation $V_{\mathcal{H}}$ as the time discretisation step tends to 0 follows by the definition (5.1) as supremum of a family of plastic potentials. We remark that, as in [8], no continuity in time of α is required: indeed, the monotonicity in time of α guarantees that the supremum in (5.1) is actually a limit as the maximum step of the partition tends to 0 (cf. [8,

Lemma A.1]). This is crucial to deduce the energy balance from the global stability. \square

Remark 5.4. The evolutions in Definition 5.1 can be approximated by suitable evolutions of models for gradient plasticity coupled with damage. Indeed, Theorem 1.1 allows us to extend [9, Theorem 6.1] (proven under the assumption $\alpha \in W^{1,q}(\Omega)$, $q > n$) to the case where $\alpha \in W^{1,n}(\Omega)$. Notice that to follow the lines of the proof of [9, Theorem 6.1] (cf. Step 1), one needs the continuity of the functional \mathcal{H} when $\alpha \in W^{1,n}(\Omega)$ is fixed and $p_k \rightarrow p$ strictly. This property of \mathcal{H} is not difficult to prove.

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Vito Crismale
CMAP, École Polytechnique, UMR CNRS 7641
91128 Palaiseau Cedex
France
e-mail: vito.crismale@polytechnique.edu

Gianluca Orlando
TUM
Boltzmannstrasse 3
85747 Garching bei München
Germany
e-mail: orlando@ma.tum.de

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