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Zero Lebesgue measure sets as removable sets for degenerate fully nonlinear elliptic PDEs

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Abstract. Our main result in this note can be stated as follows: Assume $E \subset B_1$ and

$$F(D^2u(x), \nabla u(x), u(x), x) \leq \psi(x) \quad \text{in } B_1 \backslash E \tag{0.1}$$

holds in the $C-$viscosity sense where $|E| = 0$ and F is a degenerate elliptic operator. This way, (0.1) holds in the whole unit ball B_1 (i.e, E is removable for (0.1)) provided

$$\mathcal{M}^{-}_{\lambda,\Lambda}(D^2u) - \gamma|\nabla u| \leq f \quad \text{in } B_1 \tag{0.2}$$

where $f \in L^n(B_1)$. Zeroth order term can appear in (0.2) provided u is bounded in B_1. This extends a result due to Caffarelli et al. proven in (Commun Pure Appl Math 66(1):109–143, 2013) where a second order linear uniformly elliptic PDE with bounded RHS appeared in place of (0.2).

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1. Introduction

In this note we consider the following setting. Let u be a continuous function in the unit open ball $B_1 \subset \mathbb{R}^n$. Suppose that $F(D^2u, \nabla u, u, x) \leq \psi(x)$ in $B_1 \backslash E$ in the $C-$viscosity sense where E is some subset of B_1 and F is a degenerate fully nonlinear elliptic operator. An interesting question is to decide under what conditions the PDE hold in the whole B_1. In other words, when the set E is a removable set for the equation above. This is a classical important question in the field of elliptic PDEs and has been studied by many authors throughout the years for several type of equations. For some results about remotion of singularities involving fully nonlinear equations see for instance [1, 12, 14–17].

L. Caffarelli, Y. Li and L. Nirenberg addressed this question in a series of nice articles [7–9]. In fact, in these papers, they also studied many other important questions related to the validity of the comparison principles, strong maximum principle for singular solutions, symmetry of solutions, etc. In [9], they gave an affirmative answer to the removability question mentioned above in the case where E has zero Lebsegue measure and u also satisfies $Lu \leq D$ where D is a positive constant and L is a second order linear uniformly elliptic operator in nondivergence form like

$$Lu(x) := trace(A(x)D^2u(x)) + \langle B(x), \nabla u(x)\rangle + C(x)u(x) \ \text{ in } \ B_1. \tag{1.3}$$

Here, the coefficient matrix $A(x)$ is uniformly elliptic and all other coefficients are bounded and measurable (see Theorem 1.2 in [9]).

The analysis of the particular important case where $\Delta u \leq D$ was done in [7]. The proofs in [7,9] are delicate and rest on a careful perturbation argument for $u - \varphi$ where φ is the touching test function from below in the definition of supersolution. Indeed, the authors there carefully explore the contact set between a perturbation of this function and its convex envelope. The fundamental point there is to prove that the contact set has positive measure. This is the case once precise estimates can be performed inside the contact set. In this note, we extend the removability result on Lebesgue sets of measure zero, i.e, Theorem 1.2 in [9], for the case where the assumption $Lu \leq D$ is replaced by

$$\mathcal{P}^-_{\lambda,\Lambda,\gamma}[u] := \mathcal{M}^-_{\lambda,\Lambda}(D^2u) - \gamma|\nabla u| \leq f \ \text{ in } \ B_1. \tag{1.4}$$

Here the RHS $f \in L^n(B_1)$ and $\mathcal{M}^-_{\lambda,\Lambda}$ is the negative Pucci extremal operator that we will recall in the next section. The equation above is understood to hold in the L^n-viscosity sense. The (possible) unboundedness of the RHS together with the nonlinear character of equation (1.4) above bring the novelty to our result.

We can even allow zeroth order term in the equation (1.4) provided our solution is bounded in B_1. In the approach presented in [7,9], the crucial step towards the estimate of the measure of contact set is the $C^{1,1}_{loc}$ regularity of the convex envelope for a perturbation of $u - \varphi$. In our situation, since the RHS we are dealing with is not L^∞, the convex envelope is no longer $C^{1,1}_{loc}$ (see [2]) and the argument needs to be changed.

The purpose of this short note is to carefully revisit the approach in [7,9] and modify the argument accordingly in order to bypass this difficulty. We do this by using a version of the ABP estimate that appears in [10]. This allows us to extend Theorem 1.2 in [9] and thus bring the removability results in [7,9] to the context of fully nonlinear elliptic equations with measurable ingredients with possibly unbounded coefficients. Finally, we remark that we provide full details and proofs. This makes this note essentially self contained. Now, we state our results. Before, as introduced in [3], we recall the class

$$\overline{S}(\gamma, f) = \Big\{u \in C^0(B_1);\ \mathcal{P}^-_{\lambda,\Lambda,\gamma}[u](x) \leq f(x) \ \text{ in } \ B_1\Big\},$$

where the differential inequality above is considered to hold in the L^n-viscosity sense. As a matter of fact, in this paper, we freely use the concepts of the L^n-viscosity theory as they appear in [10]. For more details on this see next section.

Theorem 1.1. *Let $u \in C^0(B_1)$ be such that $u \in \overline{S}(\gamma, f)$ in B_1 in the L^n-viscosity sense with $f \in L^n(B_1)$. Let F be a degenerate elliptic operator, i.e, F satisfies the conditions (2.6) and (2.7) below. Assume that u is a C-viscosity solution to $F(D^2u, \nabla u, u, x) \leq \psi(x)$ in $B_1 \backslash E$ where $E \subset B_1$ is a subset of zero Lebesgue measure and $\psi : B_1 \to \mathbb{R}$ is an upper-semicontinuous function. Then, E is a removable subset for the equation above, i.e,*

$$F(D^2u, \nabla u, u, x) \leq \psi(x) \quad \text{in} \;\; B_1,$$

in the C-viscosity sense.

From this we obtain the following result which includes fully nonlinear equations with zeroth order term. We highlight that no sign on the function σ below is needed.

Corollary 1.1. *Let $u \in C^0(B_1)$. In the previous Theorem, we can replace the condition $u \in \overline{S}(\gamma; f)$ in B_1 with $f \in L^n(B_1)$ by*

$$\mathcal{M}^{-}_{\lambda,\Lambda}(D^2u) - \gamma|\nabla u(x)| + \sigma(x)u(x) \leq f(x) \quad \text{in } B_1 \tag{1.5}$$

with $f, \sigma \in L^n(B_1)$ provided u is also bounded in B_1.

Remark 1.1. In order to put Theorem 1.1 into perspective, let us consider the following setting. Let $u \in C^0(B_1) \cap W^{2,n}(B_1)$ be a $C-$viscosity solution to $F(D^2u, \nabla u, u, x) \leq 0$ in $B_1 \backslash E$ where $E \subset B_1$ with $|E| = 0$ and F is a degenerate elliptic operator (i.e, F satisfies conditions (2.6) and (2.7)). Once $u \in W^{2,n}(B_1)$, it follows that u is punctually second order differentiable almost everywhere in B_1. This is a classical special case of a result due to Calderón and Zygmund (Theorem 12 in [5]). A short and self contained proof can also be found in Appendix C in [10]. Thus, Lemma 2.1 implies that u satisfies $F(D^2u(x), \nabla u(x), u(x), x) \leq 0$ for almost every $x \in B_1$. Now, it follows as a consequence of the Bony Maximum Principle (see Corollary 3 in [18]) that $F(D^2u, \nabla u, u, x) \leq 0$ in B_1 in the viscosity sense, i.e, E is removable. Observe that a key point in the previous example is the regularity assumed on u, which indeed implies that $D^2u \in L^n(B_1)$. An interesting question one could raise is what can be said about the removability of E in the case D^2u is a (matricial) measure which has a nontrivial singular part with respect to the Lebesgue measure. If not the whole Hessian, at least some operator acting on it (say Δu for instance). There are concave functions for which the Hessian is a (matricial) measure that is singular with respect to the Lebsegue measure. One example is $v(x) = -x_1^+$. In this case, if $F(D^2v, \nabla v, v, x) \leq 0$ in $C-$viscosity sense in $B_1 \backslash E$ with $E \subset B_1$ with $|E| = 0$ then Theorem 1.1 implies that E is removable since $\Delta v \leq 0$ in B_1 (which is a supersolution condition required for the removability). This way, in a certain sense, Theorem 1.1 provides a condition, namely $\mathcal{P}^{-}_{\lambda,\Lambda,\gamma}[u] \leq f \in L^n(B_1)$, that acts as a substitute for the (strong) required regularity assumption $u \in W^{2,n}(B_1)$. This new condition allows eventually the possibility of the Hessian to be a measure with nontrivial singular part with respect to the Lebesgue measure as discussed above. As a matter of fact, we point out here the work of D. Labutin [16,17] where he studied conditions under which $F(D^2u)$ makes sense as a signed Radon Measure, discussing in particular the case where F is an extremal Pucci operator.

2. Structural conditions and theorem

Our goal is to prove a result about removable sets for fully nonlinear equations. For this purpose, it is enough to consider solutions defined only in the open unit ball of $\mathbb{R}^n$ denoted here by B_1. Here, we treat fully nonlinear operators $F(D^2u, \nabla u, u, x)$ and weak solutions $u \in C^0(B_1)$ given in the $C-$viscosity sense. We assume that $F(M, \xi, r, x)$ is a degenerate elliptic operator, i.e.,

$$F \in C^0(\mathcal{S}^{n\times n} \times \mathbb{R}^n \times \mathbb{R} \times B_1) \tag{2.6}$$

and $\forall\ (M, \xi, r, x) \in \mathcal{S}^{n\times n} \times \mathbb{R}^n \times \mathbb{R} \times B_1$ and $\forall N \in \mathcal{S}^{n\times n}_+$

$$F(M + N, \xi, r, x) \geq F(M, \xi, r, x). \tag{2.7}$$

Here, $\mathcal{S}^{n\times n}$ denotes the set of symmetric matrices of order n, and $\mathcal{S}^{n\times n}_+$ the nonnegative ones. We recall that a function $\psi : B_1 \to \mathbb{R}$ defined everywhere is said to be upper-semicontinuous if for each $x_0 \in B_1$

$$\limsup_{x\to x_0} \psi(x) \leq \psi(x_0).$$

We observe that for the theory of fully nonlinear operators, it is required in general the operators $F(x, r, p, M)$ to be proper, i.e, monotone decreasing on the variable r. This is the case, for instance, if one desires the comparison principle to hold (see for instance [10]). We do not need this assumption for our results in this paper. Before we present the proof of the main result, we recall below some definitions and present some basic lemmas.

Definition 2.1. *For $E \subset B_1$ let F be a degenerate fully nonlinear elliptic operator. Let also $u \in C^0(B_1)$ and $\psi : B_1 \to \mathbb{R}$ an upper semicontinuous function. We say that u is a $C-$viscosity solution to $F(D^2u, \nabla u, u, x) \leq \psi(x)$ in $B_1 \backslash E$ if for every $x_0 \in B_1 \backslash E$ for which there exists $\varphi \in C^2(A_{x_0})$ (A_{x_0} a neighbourhood of x_0 inside B_1) such that $\varphi \leq u$ in A_{x_0} and $\varphi(x_0) = u(x_0)$ we have*

$$F(D^2\varphi(x_0), \nabla\varphi(x_0), u(x_0), x_0) \leq \psi(x_0).$$

Remark 2.1. In the definition above, we recover the classical definition of $C-$viscosity supersolution of the equation $F(D^2u(x), \nabla u(x), u(x), x) \leq \psi(x)$ in B_1 for the case where E is the empty set.

We recall the Pucci extremal operators. For $0 < \lambda \leq \Lambda$, $\mathcal{M}^-_{\lambda,\Lambda}, \mathcal{M}^+_{\lambda,\Lambda}$: $\mathcal{S}^{n\times n} \to \mathbb{R}$ are given by

$$\mathcal{M}^-_{\lambda,\Lambda}(M) = \lambda \cdot \sum_{e_i>0} e_i + \Lambda \cdot \sum_{e_i<0} e_i = \lambda \cdot Tr(M^+) - \Lambda \cdot Tr(M^-),$$

$$\mathcal{M}^{+}_{\lambda,\Lambda}(M) = \Lambda \cdot \sum_{e_i>0} e_i + \lambda \cdot \sum_{e_i<0} e_i = \Lambda \cdot Tr(M^{+}) - \lambda \cdot Tr(M^{-}),$$

where e_i are the eigenvalues of M. We also set for $\gamma \geq 0$ the following Pucci operator $\mathcal{P}^{\pm}_{\lambda,\Lambda,\gamma}\colon \mathcal{S}^{n\times n} \times \mathbb{R}^n \times \Omega \to \mathbb{R}$ given by

$$\mathcal{P}^{\pm}_{\lambda,\Lambda,\gamma}(M,p) = \mathcal{M}^{\pm}_{\lambda,\Lambda}(M) \pm \gamma|p|.$$

In order to simplify matters, we also make use of the following notation

$$\mathcal{P}^{\pm}_{\lambda,\Lambda,\gamma}[u](x) := \mathcal{P}^{\pm}_{\lambda,\Lambda,\gamma}\big(D^2u(x), \nabla u(x), u(x)\big).$$

Now we recall the concept of L^n-viscosity supersolutions. There are several equivalent ways to pose it, but here we follow the definition as introduced in [10]. We say that $\mathcal{P}^{-}_{\lambda,\Lambda,\gamma}[u] \leq f$ in B_1 in the L^n-viscosity sense if for any $\phi \in W^{2,n}_{loc}(B_1)$ such that $u - \phi$ has a local mimimum at $x_0 \in B_1$ we have

$$ess \liminf_{x\to x_0}\Big(\mathcal{P}^{-}_{\gamma}[\phi] - f(x)\Big) = ess \liminf_{x\to x_0}\Big(\mathcal{M}^{-}_{\lambda,\Lambda}(D^2\phi(x)) - \gamma|\nabla\phi(x)| - f(x)\Big) \leq 0.$$

Remark 2.2. We recall that $u \in C^0(B_1)$ is said to be punctually second order differentiable at $x_0 \in B_1$ if there exists a paraboloid (second order polynomial) P of the form

$$P(x) = A(x - x_0)\cdot(x - x_0) + B\cdot(x - x_0) + C \quad \forall x \in \mathbb{R}^n$$

where $A \in \mathcal{S}^{n\times n}, B \in \mathbb{R}^n$ and $C \in \mathbb{R}$ such that

$$u(x) = P(x) + o(|x - x_0|^2) \;\text{ as }\; x \to x_0.$$

In this case u is clearly differentiable at x_0 (with $\nabla u(x_0) = B$) and we define

$$D^2u(x_0) := D^2P(x_0).$$

We now state two simple lemmas that are needed in the proof of Theorem 1.1.

Lemma 2.1. *Suppose $u \in C^0(B_1)$ satisfies $F(D^2u, \nabla u, u, x) \leq \psi(x)$ in the $C-$viscosity sense in $B_1\backslash E$ for some subset $E \subset B_1$ where F is a degenerate fully nonlinear elliptic operator, i.e, F satisfies the conditions (2.6) and (2.7) above. Suppose now $x_0 \in B_1\backslash E$ and $v \leq u$ in a neighbourhood A of x_0 with $v(x_0) = u(x_0)$. Additionally, assume that v is punctually second order differentiable at x_0. Then,*

$$F(D^2v(x_0), \nabla v(x_0), v(x_0), x_0) \leq \psi(x_0).$$

Proof. Let P be the second order polynomial given by the definition of punctually second order differentiability of v at x_0. For $\varepsilon > 0$ small enough, we can find $\delta = \delta(\varepsilon) > 0$ small such that $B_\delta(x_0) \subset B_1$ and

$$Q(x) := P(x) - \varepsilon|x - x_0|^2 \leq v \leq u \;\; \forall x \in B_\delta(x_0)$$
$$\text{and} \quad Q(x_0) = v(x_0) = u(x_0).$$

By definition of $C-$viscosity supersolution in $B_1 \backslash E$,

$$\begin{aligned} F(D^2 v(x_0) - 2n\varepsilon I, \nabla v(x_0), v(x_0), x_0) &= F(D^2 P(x_0) - 2\varepsilon n I, \nabla P(x_0), v(x_0), x_0) \\ &= F(D^2 Q(x_0), \nabla Q(x_0), u(x_0), x_0) \\ &\le \psi(x_0) \quad \forall \varepsilon > 0 \text{ small.} \end{aligned}$$

Letting $\varepsilon \to 0$, we finish the proof. □

Lemma 2.2. *Let $u \in \overline{S}(\lambda, \Lambda, \gamma, f)$ in B_1 with $f \in L^n(B_1)$ and $g \in W^{2,n}(B_1)$. Then,*

$$u - g \in \overline{S}(\lambda, \Lambda, \gamma, f - \mathcal{P}^-_{\lambda,\Lambda,\gamma}[g]) \quad \text{in} \quad B_1. \tag{2.8}$$

Moreover,

$$\min\Big\{u - g, 0\Big\} \in \overline{S}(\lambda, \Lambda, \gamma, \Big(f - \mathcal{P}^-_{\lambda,\Lambda,\gamma}[g]\Big)^+\Big) \quad \text{in } B_1. \tag{2.9}$$

Proof. We prove (2.8) first. We start by recalling that functions in $W^{2,n}(B_1)$ are twice differentiable almost everywhere (see Appendix C in [10]). Let $\phi \in W^{2,n}_{loc}(B_1)$ be such that $(u-g) - \phi$ has a local minimum at $x_0 \in B_1$. This way, $u - \phi^*$ has a local minimum at $x_0 \in B_1$ where $\phi^* = g + \phi \in W^{2,n}_{loc}(B_1)$. It is easy to see from the properties of the Pucci extremal operators (see Lemma 2.10 in [4]) that for $M, N \in \mathcal{S}^{n\times n}$ and $p, q \in \mathbb{R}^n$

$$\mathcal{P}^-_{\lambda,\Lambda,\gamma}(M+N, p+q) \le \mathcal{P}^-_{\lambda,\Lambda,\gamma}(M,p) + \mathcal{P}^+_{\lambda,\Lambda,\gamma}(N,q).$$

In particular, if $\phi_1, \phi_2 \in W^{2,n}_{loc}(B_1)$, we have

$$\mathcal{P}^-_{\lambda,\Lambda,\gamma}[\phi_1 + \phi_2](x) \le \mathcal{P}^-_{\lambda,\Lambda,\gamma}[\phi_1] + \mathcal{P}^+_{\lambda,\Lambda,\gamma}[\phi_2] \quad a.e.\ x \ \text{ in } B_1. \tag{2.10}$$

Also,

$$\mathcal{P}^+_{\lambda,\Lambda,\gamma}[-\phi_1](x) = -\mathcal{P}^-_{\lambda,\Lambda,\gamma}[\phi_1] \quad a.e.\ x \ \text{ in } B_1. \tag{2.11}$$

Thus by (2.10) and (2.11), we have for $a.e.$ x in B_1

$$\begin{aligned} &\mathcal{P}^-_{\lambda,\Lambda,\gamma}[\phi](x) - \Big(f(x) - \mathcal{P}^-_{\lambda,\Lambda,\gamma}[g](x)\Big) \\ &\quad = \mathcal{P}^-_{\lambda,\Lambda,\gamma}[\phi^* - g](x) - \Big(f(x) - \mathcal{P}^-_{\lambda,\Lambda,\gamma}[g](x)\Big) \\ &\quad \le \mathcal{P}^-_{\lambda,\Lambda,\gamma}[\phi^*](x) + \mathcal{P}^+_{\lambda,\Lambda,\gamma}[-g](x) - \Big(f(x) - \mathcal{P}^-_{\lambda,\Lambda,\gamma}[g](x)\Big) \\ &\quad = \mathcal{P}^-_{\lambda,\Lambda,\gamma}[\phi^*](x) - f(x) \end{aligned}$$

Thus, once $u \in \overline{S}(\gamma, f)$ in B_1, we obtain

$$\begin{aligned} &ess \liminf_{x\to x_0} \Big(\mathcal{P}^-_{\lambda,\Lambda,\gamma}[\phi](x) - \Big(f(x) - \mathcal{P}^-_{\lambda,\Lambda,\gamma}[g](x)\Big)\Big) \\ &\quad \le ess \liminf_{x\to x_0} \Big(\mathcal{P}^-_{\lambda,\Lambda,\gamma}[\phi^*](x) - f(x)\Big) \\ &\quad \le 0. \end{aligned}$$

This proves (2.8). Now, (2.9) follows from the following more general fact

$$\begin{aligned} u \in \overline{S}(\gamma; f) \ \text{ and } \ v \in \overline{S}(\gamma; g) \ \text{ both in } B_1 \Longrightarrow w \\ := \min\{u, v\} \in \overline{S}(\gamma; \max\{f, g\}) \ \text{ in } B_1. \end{aligned}$$

Indeed, we just take $v = g = 0$ in the implication above. For completeness, we prove the implication above. Let $\phi \in W^{2,n}_{loc}(B_1)$ and suppose $w - \phi$ has a local minimum at $x_0 \in B_1$. If $w(x_0) = u(x_0)$, it is easy to see that $u - \phi$ has a local minimum at x_0. In particular, since

$$\begin{aligned} &ess \liminf_{x \to x_0} \Big(\mathcal{P}^-_{\lambda,\Lambda,\gamma}[\phi](x) - \max\{f(x), g(x)\} \Big) \\ &\leq ess \liminf_{x \to x_0} \Big(\mathcal{P}^-_{\lambda,\Lambda,\gamma}[\phi](x) - f(x) \Big) \leq 0. \end{aligned}$$

On the other hand, if $w(x_0) = v(x_0)$, it follows that $v - \phi$ has a local minimum at x_0 and thus,

$$\begin{aligned} &ess \liminf_{x \to x_0} \Big(\mathcal{P}^-_{\lambda,\Lambda,\gamma}[\phi](x) - \max\{f(x), g(x)\} \Big) \\ &\leq ess \liminf_{x \to x_0} \Big(\mathcal{P}^-_{\lambda,\Lambda,\gamma}[\phi](x) - g(x) \Big) \leq 0. \end{aligned}$$

This proves (2.9) and finishes the proof of the Lemma. □

3. Proof of Theorem 1.1

Proof of Theorem 1.1. By definition, let $x_0 \in B_1$ and $\varphi \in C^2(B_{\delta_0}(x_0))$ satisfying

$$\varphi \leq u \quad \text{in } B_{\delta_0}(x_0) \quad \text{and} \quad \varphi(x_0) = u(x_0) \quad \text{for some } \ 0 < \delta_0 < 1 - |x_0|.$$

We need to prove that

$$F(D^2\varphi(x_0), \nabla\varphi(x_0), u(x_0), x_0) \leq \psi(x_0). \tag{3.12}$$

We only need to consider $x_0 \in E$ and by a suitable translation, we can assume that $x_0 = 0$.
Now, for $0 < \delta < \frac{1}{4}\delta_0$ we define

$$\varphi_\delta(x) := \varphi(x) - \frac{\delta}{2}|x|^2 \quad \text{ for } x \in B_{\delta_0}. \tag{3.13}$$

Observe that

$$u(0) = \varphi(0) = \varphi_\delta(0), \quad u(x) > \varphi_\delta(x) \quad \text{in } B_{\delta_0} \backslash \{0\} \tag{3.14}$$

and

$$u(x) \geq \varphi_\delta(x) + \frac{1}{2}\delta\kappa^2 \quad \text{in } B_{\delta_0} \backslash B_\kappa \ \text{ for } \ \kappa < \delta_0/4. \tag{3.15}$$

Now, we consider $\varepsilon \in (0, \frac{\delta^3}{4})$ and set

$$w_\varepsilon(x) = w_{\varepsilon,\delta}(x) := \min\Big\{(u - \varphi_\delta)(x) - \varepsilon, 0\Big\} \quad \text{for} \quad x \in \overline{B}_{2\delta}. \tag{3.16}$$

Clearly, by (3.16), we conclude that

$$0 \geq w_\varepsilon \in C^0(\overline{B}_{2\delta}). \tag{3.17}$$

Furthermore, by (3.15) and the choice of $\varepsilon > 0$ above

$$u - \varphi_\delta - \varepsilon \geq \frac{1}{4}\delta^3 \ \text{ in } \ \overline{B}_{2\delta} \backslash B_\delta.$$

This way,

$$w_\varepsilon \equiv 0 \quad \text{in} \quad \overline{B}_{2\delta}\backslash B_\delta. \tag{3.18}$$

We now consider the convex envelope of w_ε given by

$$\Gamma_{w_\varepsilon}(x) := \sup\Big\{L(x);\ L \ \text{is affine} \quad \text{and} \quad L \leq w_\varepsilon \ \text{ in } \ \overline{B}_{2\delta}\Big\} \quad \text{for} \ \ x \in \overline{B}_{2\delta}.$$

It follows immediately from (3.17) that

$$\Gamma_{w_\varepsilon} \leq 0 \ \text{ in } \ \overline{B}_{2\delta}. \tag{3.19}$$

Additionally, by (3.14) we conclude

$$\Gamma_{w_\varepsilon}(0) \leq w_\varepsilon(0) = -\varepsilon. \tag{3.20}$$

Since Γ_{w_ε} is a convex function, Γ_{w_ε} is continuous (actually locally Lipschitz inside $B_{2\delta}$). In fact, as in [4], we can easily see that $\Gamma_{w_\varepsilon} \in C^0(\overline{B}_{2\delta})$ and also subharmonic in $B_{2\delta}$ by convexity. We observe that

$$\Big\{x \in \overline{B}_{2\delta};\ \Gamma_{w_\varepsilon}(x) = 0\Big\} \subset \partial B_{2\delta}. \tag{3.21}$$

Indeed, if there exists $z_0 \in B_{2\delta}$ such that $\Gamma_{w_\varepsilon}(z_0) = 0$, the strong comparison principle (for the subharmonic function Γ_{w_ε} and the zero function (harmonic)) applied to $\Gamma_{w_\varepsilon} \leq 0$ in $\overline{B}_{2\delta}$ (recall (3.19)) implies $\Gamma_{w_\varepsilon} \equiv 0$ in $B_{2\delta}$ which is a contradiction to (3.20). This proves (3.21). This information combined together with (3.18) guarantees that

$$\Big\{w_\epsilon = \Gamma_{w_\epsilon}\Big\} := \Big\{x \in B_{2\delta} : w_\epsilon(x) = \Gamma_{w_\epsilon}(x)\Big\} \subset \Big\{x \in B_\delta : w_\epsilon(x) < 0\Big\}. \tag{3.22}$$

Observe also that for $z_0 \in \{w_\epsilon = \Gamma_{w_\epsilon}\}$ we have by (3.14) that $(u - \varphi_\delta)(z_0) \geq 0$. Thus, by the expression given in (3.16), we conclude that

$$0 \geq \Gamma_{w_\varepsilon}(z_0) = (u - \varphi_\delta)(z_0) - \varepsilon \geq -\varepsilon. \tag{3.23}$$

This implies that,

$$|\Gamma_{w_\epsilon}| \leq \varepsilon \quad \text{in} \quad \Big\{w_\epsilon = \Gamma_{w_\epsilon}\Big\}. \tag{3.24}$$

Set

$$\mathcal{D}_\varepsilon := \Big\{x \in B_{2\delta}; \quad \Gamma_{w_\varepsilon} \ \text{ is differentiable at } \ x\Big\}.$$

By convexity, (see for instance Theorem A.1.13 in Appendix of [6]),

$$x \in \mathcal{D}_\varepsilon \Longleftrightarrow \partial\Gamma_{w_\varepsilon}(x) = \big\{\nabla\Gamma_{w_\varepsilon}(x)\big\}.$$

Since Γ_{w_ε} is convex and $\Gamma_{w_\varepsilon} \leq 0$ on $\partial B_{2\delta}$, the gradient estimates for convex functions (Lemma 3.2.1 in [13]) combined together with (3.22) allow us to infer that

$$|\nabla\Gamma_\varepsilon| \leq \frac{\varepsilon}{\delta} \quad \text{in} \quad \Big\{w_\epsilon = \Gamma_{w_\epsilon}\Big\} \cap \mathcal{D}_\varepsilon. \tag{3.25}$$

Furthermore, since $\varphi_\delta \in C^2(\overline{B}_{2\delta}) \subset W^{2,n}(B_{2\delta})$, we have by (2.8) in Lemma 2.2 that

$$u - \varphi_\delta - \varepsilon \in \overline{S}\big(\gamma, f - \mathcal{P}^-_{\lambda,\Lambda,\gamma}[\varphi_\delta]\big) \ \text{ in } \ B_{2\delta}.$$

Moreover, (2.9) also in Lemma 2.2 gives

$$w_\varepsilon = \min\left\{(u-\varphi_\delta)-\epsilon, 0\right\} \in \overline{S}(\gamma,\xi) \quad \text{in } B_{2\delta}, \tag{3.26}$$

where $\xi = \left[f - \mathcal{P}^-_{\lambda,\Lambda,\gamma}[\varphi_\delta]\right]^+ \in L^n(B_{2\delta})$.
Now, observe that by (3.18), we obtain

$$\sup_{\partial B_{2\delta}} (w_\varepsilon)^- = 0.$$

This way, taking into account (3.26) and (3.20), the ABP estimate (Proposition 3.3 in [10]) implies that

$$\varepsilon^n = \left(\left(w_\varepsilon(0)\right)^-\right)^n \le \left(\sup_{B_{2\delta}} (w_\varepsilon)^-\right)^n \le C\cdot \int_{\{w_\epsilon=\Gamma_{w_\epsilon}\}} \left|\xi(x)\right|^n dx$$

where $C = C(\lambda,\Lambda,\gamma,n,\delta) > 0$. This implies that

$$\left|\left\{w_\epsilon = \Gamma_{w_\epsilon}\right\}\right| > 0.$$

By the Aleksandrov Theorem (Theorem 6.9 - section 6.4 in [11]), we know that Γ_ε is punctually second order differentiable in $B_{2\delta}$ except on a set of zero Lebesgue measure. Since E has zero Lebesgue measure, for each $\varepsilon > 0$ small enough, there exists a point x_ε in $\left\{w_\epsilon = \Gamma_{w_\epsilon}\right\}\backslash E$ where Γ_{w_ϵ} is punctually second order differentiable. In particular, $x_\varepsilon \in \mathcal{D}_\varepsilon$. Now, we set the function

$$\Phi_\epsilon(x) := \varphi_\delta(x) + \epsilon + \Gamma_{w_\epsilon} \quad \text{for } x \in B_{2\delta}.$$

Clearly, Φ_ε is punctually second order differentiable at x_ε. Now, by definition in (3.16) we see that

$$u - \varphi_\delta - \varepsilon \ge w_\varepsilon \ge \Gamma_{w_\varepsilon} \quad \text{in } B_{2\delta}.$$

This way, since $x_\varepsilon \in \left\{w_\epsilon = \Gamma_{w_\epsilon}\right\}$, we have by (3.22) (see also (3.23))

$$u \ge \Phi_\varepsilon \quad \text{in } B_{2\delta} \quad \text{and} \quad u(x_\varepsilon) = \Gamma_{w_\varepsilon}(x_\varepsilon) + \varphi_\delta(x_\varepsilon) + \varepsilon = \Phi_\varepsilon(x_\varepsilon). \tag{3.27}$$

Since $x_\varepsilon \notin E$, Lemma 2.1 assures that

$$F(D^2\Phi_\varepsilon(x_\varepsilon), \nabla\Phi_\varepsilon(x_\varepsilon), u(x_\varepsilon), x_\varepsilon) \le \psi(x_\varepsilon). \tag{3.28}$$

On the other hand, since $x_\varepsilon \in \mathcal{D}_\varepsilon \cap \left\{w_\epsilon = \Gamma_{w_\epsilon}\right\}$ (3.24), (3.25) and convexity of Γ_{w_ε} imply

$$|\Gamma_{w_\varepsilon}(x_\varepsilon)| \le \varepsilon, \quad |\nabla\Gamma_{w_\varepsilon}(x_\varepsilon)| \le \frac{\varepsilon}{\delta} \quad \text{and} \quad D^2\Gamma_{w_\varepsilon}(x_\varepsilon) \ge 0. \tag{3.29}$$

We now claim that

$$x_\varepsilon \to 0 \ \text{ as } \ \varepsilon \to 0. \tag{3.30}$$

Indeed, if this is not the case, up to a subsequence (that we still denote by index ε) we can assume that $|x_\varepsilon| \ge \mu > 0$. Now, the equality in (3.27) together with (3.15) and the first estimate in (3.29) yield

$$\frac{1}{2}\delta\mu^2 \le u(x_\varepsilon) - \varphi_\delta(x_\varepsilon) = \Gamma_{w_\varepsilon}(x_\varepsilon) + \varepsilon \le |\Gamma_{w_\varepsilon}(x_\varepsilon)| + \varepsilon \le 2\varepsilon \to 0 \ \text{ as } \ \varepsilon \to 0$$

which is clearly a contradiction. Thus, (3.30) is proven.

Now, by ellipticity and estimates in (3.29), we can use (3.28) to obtain

$$\begin{aligned}\psi(x_\varepsilon) &\geq F(D^2\Phi_\varepsilon(x_\varepsilon), \nabla\Phi_\varepsilon(x_\varepsilon), u(x_\varepsilon), x_\varepsilon) \\ &= F(D^2\varphi_\delta(x_\varepsilon) + D^2\Gamma_{w_\varepsilon}(x_\varepsilon), \nabla\varphi_\delta(x_\varepsilon) + \nabla\Gamma_{w_\varepsilon}(x_\varepsilon), u(x_\varepsilon), x_\varepsilon) \\ &\geq F(D^2\varphi_\delta(x_\varepsilon), \nabla\varphi_\delta(x_\varepsilon) + \nabla\Gamma_{w_\varepsilon}(x_\varepsilon), u(x_\varepsilon), x_\varepsilon).\end{aligned}$$

Recalling that ψ is upper semicontinuous and passing to the limit as $\varepsilon \to 0$ we find

$$\psi(0) \geq \limsup_{\varepsilon\to 0} \psi(x_\varepsilon) \geq F(D^2\varphi(0) - \delta I, \nabla\varphi(0), u(0), 0)$$

Finally, letting $\delta \to 0$, we obtain (3.12) and this finishes the proof of the Theorem. □

Proof of Corollary 1.1

Proof. Indeed, observe that if the equation (1.5) is satisfied then for any $\phi \in W^{2,n}_{loc}(B_1)$ such that $u - \phi$ has a local minimum at $x_0 \in B_1$ we have

$$\begin{aligned}& ess \liminf_{x\to x_0} \Big(\mathcal{M}^-_{\lambda,\Lambda}(D^2\phi(x)) - \gamma|\nabla\phi(x)| - \Big(f(x) + |\sigma(x)| \cdot ||u||_{L^\infty(B_1)}\Big)\Big) \\ &\leq ess \liminf_{x\to x_0} \Big(\mathcal{M}^-_{\lambda,\Lambda}(D^2\phi(x)) - \gamma|\nabla\phi(x)| + \sigma(x)u(x) - f(x)\Big) \\ &\leq 0.\end{aligned}$$

This way, $u \in \overline{S}(\gamma; \overline{f}(x))$ in B_1 where $\overline{f}(x) := f(x) + |\sigma(x)| \cdot ||u||_{L^\infty(B_1)} \in L^n(B_1)$.
Thus, we apply directly Theorem 1.1 and the result follows. □

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