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Virial functional and dynamics for nonlinear Schrödinger equations of local interactions

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Abstract. Our aim is to verify that the functional in the virial identity classifies the dynamics for nonlinear Schrödinger equations of local interactions. In particular, we give a condition under that there exist stable ground states. Our proof of this stability result is based on the ideas in Colin (Ann Inst H Pincaré 23:753–764, 2006) and Shatah (Math Phys 91:313–327, 1983). However, we emphasize that our argument does not use the strict convexity of the $\dot{H}^1$-norm of ground state with respect to ω: a key lemma is Lemma 4.8 below. Furthermore, we discuss the limiting profile of ground states (see Theorem 4.4).

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1. Introduction

In this paper, we consider the following nonlinear Schrödinger equation:

$$i\frac{\partial \psi}{\partial t}(x,t) + \Delta\psi(x,t) + f(\psi(x,t)) = 0, \qquad (x,t) \in \mathbb{R}^d \times \mathbb{R}, \tag{NLS}$$

where ψ is a complex-valued function of space-time $\mathbb{R}^d \times \mathbb{R}$ with $d \geq 1$, Δ is the Laplace operator on $\mathbb{R}^d$, and f is a complex-valued function on $\mathbb{C}$ satisfying the following conditions:

(N1) f is "super-linear around the origin" in the sense that $f(z) = o(|z|)$ as $|z| \to 0$. In particular, $f(0) = 0$.

(N2) f is continuously differentiable in the real-sense. Furthermore, there exist positive numbers p, q and C_1 such that $2 \leq p+1 \leq q+1 \leq 2^*$ and for any $z \in \mathbb{C}$,

$$\left|\frac{\partial f}{\partial z}(z)\right| + \left|\frac{\partial f}{\partial \bar{z}}(z)\right| \leq C_1(|z|^{p-1} + |z|^{q-1}),$$

where

$$2^* := \begin{cases} \infty & \text{if } d = 1, 2, \\ \frac{2d}{d-2} & \text{if } d \geq 3. \end{cases}$$

(N3) There exists a real-valued function F on $\mathbb{C}$ such that $F \in C^2(\mathbb{C}, \mathbb{R})$ in the real-sense, $F(0) = 0$, and for any $z \in \mathbb{C}$,

$$2\frac{\partial F}{\partial \overline{z}}(z) = f(z).$$

(N4) $\Im[\overline{z} f(z)] = 0$ for any $z \in \mathbb{C}$.

We introduce a functional $\mathcal{K}$ as

$$\mathcal{K}(u) := \|\nabla u\|_{L^2}^2 - \frac{d}{2}\int_{\mathbb{R}^d} G(u(x))\, dx, \tag{1.1}$$

where

$$G(z) := \bar{z} f(z) - 2F(z). \tag{1.2}$$

In the study of behavior of solutions to the Eq. (NLS), the following identity, called the virial identity, plays an important role:

$$\frac{d^2}{dt^2}\int_{\mathbb{R}^d} |x|^2 |\psi(x,t)|^2\, dx = 8\mathcal{K}(\psi(t)). \tag{1.3}$$

Indeed, employing this identity, Glassey proved that some solutions blow up in finite time (see [13]). Moreover, recently, the importance of the virial identity has been recognized in the scattering theory (see [1,10,16,17]).

The aim of this paper is to investigate the dynamics for (NLS) under a certain condition for the functional $\mathcal{K}$, rather than the nonlinearity f. In particular, we want to clarify that the properties of $\mathcal{K}$ determine the dynamics of the Eq. (NLS) (see Corollary 2.2, Corollary 3.1 and Theorem 4.2).

Now, let us recall basic results for the Eq. (NLS). It is well known that the Cauchy problem for the Eq. (NLS) is locally well-posed in $H^1(\mathbb{R}^d)$ under the conditions (N1) and (N2) (see, e.g., [6,7,15]). Moreover, the Eq. (NLS) is governed by the Hamiltonian under the condition (N3). Indeed, the Hamiltonian $\mathcal{H}\colon H^1(\mathbb{R}^d) \longrightarrow \mathbb{R}$ is given by

$$\mathcal{H}(u) := \frac{1}{2}\|\nabla u\|_{L^2}^2 - \int_{\mathbb{R}^d} F(u(x))\, dx, \tag{1.4}$$

and conserved in time for any H^1-solution. The condition (N3) also gives us the conservation of the momentum

$$\Im \int_{\mathbb{R}^d} \overline{u(x)} \nabla u(x)\, dx. \tag{1.5}$$

Under the condition (N4), we have the conservation of the mass

$$\mathcal{M}(u) := \frac{1}{2}\|u\|_{L^2}^2. \tag{1.6}$$

In addition to these functionals, we also introduce the action $\mathcal{S}_\omega$:

$$\mathcal{S}_\omega := \omega\mathcal{M} + \mathcal{H}. \tag{1.7}$$

Obviously, the action of a solution is conserved in time.

Next, we recall fundamental notions for solutions. Let ψ be a solution to (NLS) with the maximal existence interval $(T_{\min}, T_{\max})$. Then, we say that ψ scatters to a free solution in $H^1(\mathbb{R}^d)$ forward in time if $T_{\max} = \infty$ and there exists $\phi \in H^1(\mathbb{R}^d)$ such that

$$\lim_{t\to\infty} \|\psi(t) - e^{it\Delta}\phi\|_{H^1} = 0. \tag{1.8}$$

On the other hand, we say that ψ blows up (grows up, respectively) forward in time if $T_{\max} < \infty$ ($T_{\max} = \infty$, respectively) and

$$\limsup_{t\to T_{\max}} \|\nabla\psi(t)\|_{L^2} = \infty. \tag{1.9}$$

The corresponding notions for the backward in time are defined similarly. In addition, we say that ψ is a standing wave of frequency ω, if ψ is of the form $\psi(x,t) = e^{it\omega}u(x)$: it is easy to see that u must satisfy the equation

$$\omega u - \Delta u - f(u) = 0. \tag{1.10}$$

Note here that

$$\mathcal{S}_\omega'(u)\phi = \langle \omega u - \Delta u - f(u), \phi\rangle_{H^{-1},H^1} \tag{1.11}$$

for any $\phi \in H^1(\mathbb{R}^d)$. Thus, the existence of solution to (1.10) is equivalent to the existence of critical point of the action $\mathcal{S}_\omega$. We want to take this opportunity to remember the notions of ground state and its stability. A function Q_ω is called a ground state of (1.10) if it is a solution to (1.10) and satisfies

$$\mathcal{S}_\omega(Q_\omega) = \min\left\{\mathcal{S}_\omega(u) \colon u \text{ is a non-trivial } H^1\text{-solution to (1.10)}\right\}. \tag{1.12}$$

For each $\omega > 0$, we use the symbol $\mathcal{G}_\omega$ to denote the set of all ground states to (1.10). The set $\mathcal{G}_\omega$ is said to be stable if for any $\varepsilon > 0$, there exists $\delta > 0$ such that if a function $\psi_0 \in H^1(\mathbb{R}^d)$ satisfies

$$\inf_{Q\in\mathcal{G}_\omega} \|\psi_0 - Q\|_{H^1} < \delta, \tag{1.13}$$

then the solution ψ to (NLS) with $\psi(0) = \psi_0$ obeys

$$\sup_{t\in I_{\max}} \inf_{Q\in\mathcal{G}_\omega} \|\psi(t) - Q\|_{H^1} < \varepsilon, \tag{1.14}$$

where $I_{\max}$ is the maximal interval where ψ exists.

Now, we shall give an outline of this paper. In Sect. 2 below, we consider the case where the functional $\mathcal{K}$ is non-negative:

$$\mathcal{K}(u) \geq 0 \qquad \text{for any } u \in H^1(\mathbb{R}^d). \tag{K0}$$

An example of nonlinearity for which (K0) holds is

$$f(z) = -|z|^{p-1}z - |z|^{q-1}z. \tag{1.15}$$

We will see that all solutions to (NLS) is uniformly bounded in $H^1(\mathbb{R}^d)$ under the condition (K0) (see Corollary 2.1 below). Furthermore, we discuss the global well-posedness and the scattering problem under some additional assumptions.

In Sect. 3, we consider the following condition: for any non-trivial function $u \in H^1(\mathbb{R}^d)$, there exists a unique number $\lambda(u) > 0$ such that

$$\mathcal{K}\left(T_\lambda u\right) \begin{cases} > 0 & \text{if } 0 < \lambda < \lambda(u), \\ = 0 & \text{if } \lambda = \lambda(u), \\ < 0 & \text{if } \lambda(u) < \lambda, \end{cases} \tag{K1}$$

where

$$T_\lambda u(x) := \lambda^{\frac{d}{2}} u(\lambda x). \tag{1.16}$$

In particular, the condition (K1) implies that

$$\lim_{s\to\infty} G(s) > 0. \tag{1.17}$$

Typical examples of such nonlinearity are

$$f(z) = -|z|^{p-1}z + |z|^{q-1}z, \tag{1.18}$$

$$f(z) = |z|^{p-1}z + |z|^{q-1}z, \tag{1.19}$$

where $2+\frac{4}{d} < p+1 < q+1 < 2^*$. In [2], the condition (K1) was already discussed and the existence of ground state to the Eq. (1.10) is proved. Furthermore, it is shown that for any solution ψ below "ground state threshold", if $\mathcal{K}(\psi(0)) > 0$, then ψ scatters to a free solution both forward and backward in time; and if $\mathcal{K}(\psi(0)) < 0$, then ψ blows up or grows up both forward and backward in time. Recently, we became aware that some condition in [2] is redundant. Hence, we give a slight refinement of the result in [2] (see Corollary 3.1 below).

Note here that the condition (K0) or (K1) rules out the nonlinearity

$$f(z) = |z|^{p-1}z - |z|^{q-1}z, \tag{1.20}$$

where $1 < p < q$. Thus, in Sect. 4, we consider nonlinearities including (1.20). Precisely, we assume that there exists $0 < s_1 \leq s_2$ such that

$$G(s) \begin{cases} > 0 & \text{if } 0 < s < s_1, \\ < 0 & \text{if } s_2 < s. \end{cases} \tag{K2}$$

Note here that if $2 + \frac{4}{d} < p + 1$ in (N2), then (K2) implies that there exists a function $u_0 \in H^1(\mathbb{R}^d)$ such that $\mathcal{K}\left(T_\lambda u_0\right)$ changes its sign at least twice as a function of λ. Under this condition (K2), we will see that all solutions to (NLS) exist globally in time (see Proposition 4.1 below). Moreover, for $d \geq 3$ and $\omega > 0$, we discuss the existence and the stability of ground state (see Theorems 4.1 and 4.2). Here, we emphasize that our argument for the stability does not need any "strict convexity" with respect to ω: a key lemma is Lemma 4.8 below. Furthermore, we discuss a limiting profile of ground states as frequencies ω tends to ω_* (see Theorem 4.4).

2. (NLS) under the condition (K0)

In this section, we assume (K0), so that the virial functional $\mathcal{K}$ is non-negative. Then, we see that

$$0 \leq \lambda^d \mathcal{K}(u(\lambda\cdot)) = \lambda^2 \|\nabla u\|_{L^2}^2 - \frac{d}{2}\int_{\mathbb{R}^d} G(u)\, dx \tag{2.1}$$

for all $u \in H^1(\mathbb{R}^d)$ and all $\lambda > 0$. Furthermore, taking $\lambda \to 0$, we find that

$$\int_{\mathbb{R}^d} G(u)\, dx \le 0 \tag{2.2}$$

for all $u \in H^1(\mathbb{R}^d)$. Thus, we could conclude:

Lemma 2.1. *Let $d \ge 1$, and assume the conditions* (N1) *through* (N4)*. Then, the condition* (K0) *is equivalent to*

$$G(z) \le 0 \tag{2.3}$$

for any $z \in \mathbb{C}$.

Proof of Lemma 2.1. Obviously, the condition (2.3) implies (K0). The opposite claim would be immediate from (2.2). However, for the sake of completeness, we give a proof. Assume (K0) in addition to (N1) through (N4). Furthermore, suppose for contradiction that (2.3) failed. Then, we could take $z_0 \in \mathbb{C}$ such that

$$G(z_0) > 0. \tag{2.4}$$

Let n be a positive integer, and define the complex-valued function u_n by

$$u_n(x) := \begin{cases} z_0 & \text{if } |x| \le n, \\ (n+1-|x|)z_0 & \text{if } n < |x| \le n+1, \\ 0 & \text{if } n+1 < |x|. \end{cases} \tag{2.5}$$

We consider $\mathcal{K}(u_n)$. We decompose $\mathcal{K}(u_n) = I_n - J_n$, where

$$I_n := \int_{n \le |x| < n+1} |\nabla u_n(x)|^2\, dx - d \int_{n \le |x| < n+1} G(u_n(x))\, dx, \tag{2.6}$$

$$J_n := d \int_{|x| \le n} G(u_n(x))\, dx. \tag{2.7}$$

It is easy to verify that

$$|I_n| \le \int_{n \le |x| \le n+1} |z_0|^2\, dx + d \int_{n \le |x| \le n+1} \sup_{|z| \le |z_0|} |G(z)|\, dx \lesssim n^{d-1}. \tag{2.8}$$

On the other hand, it follows from (2.4) that

$$J_n = d \int_{|x| \le n} G(z_0)\, dx \gtrsim G(z_0) n^d. \tag{2.9}$$

Hence, we see from (2.8) and (2.9) that for any sufficiently large n, $|I_n| < J_n$, so that $\mathcal{K}(u_n) < 0$. However, this contradicts the condition (K0). Thus, we have proved that (2.3) holds. □

In addition to Lemma 2.1, we have:

Lemma 2.2. *Let $d \ge 1$. Assume the conditions* (N1) *through* (N4)*. Then, the condition* (K0) *implies that $F(z) \le 0$ for any $z \in \mathbb{C}$.*

Proof of Lemma 2.2. Since

$$\frac{\partial}{\partial r}\frac{F(r)}{r^2} = \frac{G(r)}{r^3} \tag{2.10}$$

for any $r > 0$, Lemma 2.1 shows that $r^{-2}F(r)$ is non-increasing on $(0, \infty)$. Furthermore, it follows from Lemma A.1 that $F(z) = F(|z|)$. Hence, we find from (A.8) in Lemma A.2 that $F(z) \le 0$ for any $z \in \mathbb{C}$. □

Using Lemma 2.2 and the conservation laws of energy (1.4) and mass (1.6), we can obtain the uniform boundedness of solutions in $H^1(\mathbb{R}^d)$:

Proposition 2.1. *Let $d \ge 1$. Assume the conditions* (N1) *through* (N4) *and* (K0)*. Then, any solution ψ to* (NLS) *satisfies*

$$\sup_{t \in I_{\max}(\psi)} \|\psi(t)\|_{H^1}^2 \le \mathcal{M}(\psi(0)) + \mathcal{H}(\psi(0)). \tag{2.11}$$

Next, we discuss the global well-posedness and the scattering problem. Let us begin with reminding readers the following result by Nakanishi [19]:

Theorem 2.1. *Let $d \ge 1$. Assume the conditions* (N1) *through* (N4)*. Furthermore, assume $2 + \frac{4}{d} < p + 1 \le q + 1 < 2^*$ in* (N2) *and*

$$\frac{d}{ds}\frac{F(s)}{s^2} = \frac{G(s)}{s^3} \le 0 \tag{2.12}$$

for any $s > 0$. Then, any solution to (NLS) *exists globally in time and scatters to a free solution in $H^1(\mathbb{R}^d)$ both forward and backward in time.*

The condition (2.12) in Theorem 2.1 is equivalent to (K0) (see Lemma 2.1). Hence, we can rephrase Theorem 2.1 in terms of the virial functional $\mathcal{K}$:

Corollary 2.2. *Let $d \ge 1$. Assume* (N1) *through* (N4) *and* (K0)*. Furthermore, assume $2 + \frac{4}{d} < p + 1 \le q + 1 < 2^*$ in* (N2)*. Then, any solution to* (NLS) *scatters to a free solution in $H^1(\mathbb{R}^d)$ both forward and backward in time.*

Remark 2.1. In Corollary 2.2, we could include the case $q + 1 = 2^*$ for radial solutions, using the argument similar to [21] and assuming that

$$\left|\frac{\partial f}{\partial z}(z) - \frac{\partial f}{\partial z}(w)\right| + \left|\frac{\partial f}{\partial \bar{z}}(z) - \frac{\partial f}{\partial \bar{z}}(w)\right| \le C_f \left(|z - w|^{p-1} + |z - w|^{\frac{4}{d-2}}\right).$$

3. (NLS) under the condition (K1)

In this section, we consider the Eq. (NLS) under the condition (K1). Our aim here is to give a slight refinement of the result in [2].

Let us begin with a brief introduction to [2]. In that paper, the following condition, in addition to (N1) through (N4) and (K1), is assumed:

$$\lim_{s \to \infty} \frac{F(s)}{s^{2+\frac{4}{d}}} = \infty. \tag{3.1}$$

Then, for any $\omega > 0$, the existence of ground state to the Eq. (1.10) is proved via the following variational problem

$$m_\omega := \inf\left\{\mathcal{S}_\omega(u) \colon u \in H^1(\mathbb{R}^d) \setminus \{0\},\ \mathcal{K}(u) = 0\right\}. \tag{3.2}$$

Furthermore, it is proved that: if a solution to (NLS) starts from the set

$$A_{\omega,+} := \left\{ u \in H^1(\mathbb{R}^d) \colon \mathcal{S}_\omega(u) < m_\omega,\ \mathcal{K}(u) > 0 \right\}, \tag{3.3}$$

then it scatters to a free solution; and if a solution starts from

$$A_{\omega,-} := \left\{ u \in H^1(\mathbb{R}^d) \colon \mathcal{S}_\omega(u) < m_\omega,\ \mathcal{K}(u) < 0 \right\}, \tag{3.4}$$

then it blows up or grows up. Note here that

$$\left\{ u \in H^1(\mathbb{R}^d) \colon \mathcal{S}_\omega(u) < m_\omega \right\} = A_{\omega,+} \cup A_{\omega,-} \cup \{0\}. \tag{3.5}$$

We became aware that we do not need to assume the condition (3.1). Indeed, we have:

Lemma 3.1. *Let $d \geq 1$. Assume the conditions* (N1) *through* (N4) *and* (K1)*. Furthermore assume that $1 < p$ in* (N2)*. Then,* (3.1) *holds.*

Proof of Lemma 3.1. Let $u \in H^1(\mathbb{R}^d) \setminus \{0\}$. Then, we shall show that

$$\lim_{t\to\infty} t^{-2-\frac{4}{d}} \int_{\mathbb{R}^d} G(tu(x))\, dx = \infty. \tag{3.6}$$

We see from the assumption (K1) that for any $a > 0$, there exists $\lambda(au) > 0$ such that for any $\lambda \geq \lambda(au)$,

$$\mathcal{K}(T_\lambda(au)) = a^2\lambda^2 \|\nabla u\|_{L^2}^2 - \lambda^{-d}\frac{d}{2}\int_{\mathbb{R}^d} G(\lambda^{\frac{d}{2}} au(x))\, dx \leq 0. \tag{3.7}$$

Hence, we find that for any $\lambda \geq \lambda(au)$,

$$a^{-\frac{4}{d}} \|\nabla u\|_{L^2}^2 \leq (\lambda^{\frac{d}{2}} a)^{-2-\frac{4}{d}} \frac{d}{2}\int_{\mathbb{R}^d} G(\lambda^{\frac{d}{2}} au(x))\, dx. \tag{3.8}$$

Since a is arbitrary, we obtain (3.6).

Next, let $t > 0$, and use Lemma A.2 to obtain

$$\frac{F(t)}{t^{2+\frac{4}{d}}} = t^{-\frac{4}{d}} \int_0^t \frac{G(s)}{s^3}\, ds = t^{-\frac{4}{d}} \int_0^t \frac{G_+(s)}{s^3} ds - t^{-\frac{4}{d}} \int_0^t \frac{G_+(s) - G(s)}{s^3}\, ds, \tag{3.9}$$

where $G_+(s) := \max\{G(s), 0\}$. Note here that the assumption (K1) implies that there exists $s_+ > 0$ such that

$$0 \leq \inf_{s \geq s_+} G(s). \tag{3.10}$$

Moreover, it follows from $G(s) = G_+(s)$ for $s \geq s_+$, (A.1) and (A.2) with $p + 1 > 2$ that

$$\int_0^t \frac{G_+(s) - G(s)}{s^3}\, ds \leq \int_0^{s_+} \frac{G_+(s) - G(s)}{s^3}\, ds \leq C(s_+) \tag{3.11}$$

for all $t \geq 0$, where $C(s_+) > 0$ is some constant depending only on s_+, p, q and the constant C_1 in (N2). Using (3.9), (3.11), substitution of the variables $s = t(1 + r)^{-\frac{d+1}{2}}$ and $G \leq G_+$, we see that

$$
\begin{aligned}
\frac{F(t)}{t^{2+\frac{4}{d}}} &\geq t^{-\frac{4}{d}} \int_0^t \frac{G_+(s)}{s^3} ds - C(s_+) t^{-\frac{4}{d}} \\
&\geq \frac{d+1}{2} t^{-2-\frac{4}{d}} \int_0^\infty G_+\big(t(1+r)^{-\frac{d+1}{2}}\big)\, r^{d-1} dr - C(s_+) t^{-\frac{4}{d}} \\
&\gtrsim \frac{d+1}{2} t^{-2-\frac{4}{d}} \int_{\mathbb{R}^d} G\big(t(1+|x|)^{-\frac{d+1}{2}}\big)\, dx - C(s_+) t^{-\frac{4}{d}}.
\end{aligned} \tag{3.12}
$$

Since $u(x) := (1+|x|)^{-\frac{d+1}{2}} \in H^1(\mathbb{R}^d) \setminus \{0\}$, we find from (3.12) and (3.6) that (3.1) holds. □

Now, we restate the result in [2] without assuming the condition (3.1):

Corollary 3.1. *Let $d \geq 1$. Assume the conditions* (N1) *through* (N4) *and* (K1). *Furthermore, assume $2 + \frac{4}{d} < p + 1 < q + 1 < 2^*$ in* (N2). *Then, for any $\omega > 0$, there exists a ground state Q_ω to the Eq.* (1.10) *such that $\mathcal{S}_\omega(Q_\omega) = m_\omega$. Furthermore, if $\psi_0 \in A_{\omega,+}$, then the corresponding solution ψ scatters to a free solution in $H^1(\mathbb{R}^d)$ both forward and backward in time; and if $\psi_0 \in A_{\omega,-}$, then the corresponding solution ψ blows up or grows up both forward and backward in time.*

4. (NLS) under the condition (K2)

In this section, in addition to (N1) through (N4), we always assume (K2). Our aim here is to prove the existence and stability of ground state to the Eq. (1.10). Moreover, we discuss a limiting profile of ground sate under an additional condition which still includes the nonlinearity (1.20) (see Theorem 4.4).

4.1. Statement of main results

In order to state our main results (Theorems 4.1 and 4.2), we need some preparations. Let us begin with the following easy fact:

Lemma 4.1. *Let $d \geq 1$. Assume* (N1) *through* (N4) *and* (K2). *Then, $F(s)/s^2$ is strictly increasing on $[0, s_1)$, positive on $(0, s_1)$, and strictly decreasing on (s_2, ∞). In particular, the function $F(s)/s^2$ takes the positive maximum on $[0, \infty)$.*

Proof of Lemma 4.1. The claim follows from the condition (K2), (A.7) and (A.8) in Lemma A.2. □

Using this lemma, we immediately obtain the uniform boundedness in $H^1(\mathbb{R}^d)$:

Proposition 4.1. *Let $d \geq 1$. Assume* (N1) *through* (N4) *and* (K2). *Then, any solution ψ to* (NLS) *satisfies*

$$
\sup_{t \in I_{\max}(\psi)} \|\psi(t)\|_{H^1}^2 \leq 2\Big(1 + \max_{s \geq 0} \frac{F(s)}{s^2}\Big) \mathcal{M}(\psi(0)) + 2\mathcal{H}(\psi(0)). \tag{4.1}
$$

In particular, if $q + 1 < 2^*$ in (N2), *then any solution exists globally in time.*

Proof of Proposition 4.1. We see from Lemmas A.1 and 4.1 that

$$\frac{1}{2}\|\nabla\psi(t)\|_{L^2}^2 = \mathcal{H}(\psi(t)) + \int_{\mathbb{R}^d} F(|\psi(t)|)\,dx \le \mathcal{H}(\psi(t)) + \max_{s\ge 0}\frac{F(s)}{s^2}\|\psi(t)\|_{L^2}^2. \tag{4.2}$$

Thus, the desired result follows from the mass and the Hamiltonian conservation laws. □

In order to describe our main results, we need more preparations. Assume $d \ge 3$. Then, any solution Q_ω to the Eq. (1.10) necessarily satisfies the following "Pohozaev identity" (see [4]):

$$\frac{1}{2^*}\|\nabla Q_\omega\|_{L^2}^2 = -\omega\mathcal{M}(Q_\omega) + \int_{\mathbb{R}^d} F(Q_\omega)\,dx = -\int_{\mathbb{R}^d}\left\{\frac{\omega}{2}|Q_\omega|^2 - F(Q_\omega)\right\}dx. \tag{4.3}$$

From the point of view of (4.3), we introduce

$$\omega_* := \sup\left\{\, \omega > 0\colon \inf_{s\ge 0}\left[\omega s^2 - 2F(s)\right] < 0 \,\right\}. \tag{4.4}$$

Since $\max_{s\ge 0} F(s)/s^2$ is finite and positive (see Lemma 4.1), the following lemma tells us that ω_* is finite and positive:

Lemma 4.2. *Let $d \ge 1$. Assume* (N1) *through* (N4) *and* (K2)*. Then,*

$$\omega_* = 2\max_{s\ge 0}\frac{F(s)}{s^2}. \tag{4.5}$$

Proof of Lemma 4.2. It follows from Lemma 4.1 that there exists $s_{\max} \in [s_1, s_2]$ such that $F(s_{\max})/s_{\max}^2 = \max_{s\ge 0} F(s)/s^2$. Thus, if

$$\omega < 2\max_{s\ge 0}\frac{F(s)}{s^2} = 2\frac{F(s_{\max})}{s_{\max}^2}, \tag{4.6}$$

then we have

$$\inf_{s\ge 0}[\omega s^2 - 2F(s)] \le s_{\max}^2\left\{\omega - 2\frac{F(s_{\max})}{s_{\max}^2}\right\} < 0. \tag{4.7}$$

On the other hand, if

$$\omega > 2\max\frac{F(s)}{s^2} = 2\frac{F(s_{\max})}{s_{\max}^2}, \tag{4.8}$$

then we have

$$\omega s^2 - 2F(s) > 2s^2\left\{\max_{s\ge 0}\frac{F(s)}{s^2} - \frac{F(s)}{s^2}\right\} \ge 0 \tag{4.9}$$

for any $s > 0$. Hence, we find that (4.5) holds. □

Now, we are in a position to state our main results:

Theorem 4.1. *Assume $d \ge 3$,* (N1) *through* (N4) *and* (K2)*. Then, for any $\omega \in (0, \omega_*)$, there exists a ground state Q_ω to the Eq.* (1.10)*, that is $\mathcal{G}_\omega \ne \emptyset$.*

Remark 4.1. See [14] for the existence of ground state in the dimension 2.

Theorem 4.2. *Assume $d \geq 3$,* (N1) *through* (N4) *and* (K2). *Then, there exists a sequence $\{w_n\}$ in $(0, \omega_*)$ such that $\lim_{n\to\infty} \omega_n = \omega_*$ and the set $\mathcal{G}_{\omega_n}$ is stable for all n.*

Remark 4.2. If the ground state is unique up to the phase shifts and the space-translations, then Theorem 4.2 corresponds with the orbital stability of the standing wave. It is known that the uniqueness holds for the nonlinearity (1.20) (see [20]).

We give proofs of Theorems 4.1 and 4.2 in Sect. 4.2 and Sect. 4.3, respectively. Moreover, we discuss a limiting profile of ground state in Sect. 4.4.

4.2. Existence of ground state

In this section, we shall prove Theorem 4.1. From the point of view of (4.3), we introduce the Pohozaev functional $\mathcal{P}_\omega$ to be that for all $u \in H^1(\mathbb{R}^d)$,

$$\begin{aligned}\mathcal{P}_\omega(u) &:= \frac{1}{2^*}\|\nabla u\|_{L^2}^2 + \frac{1}{2}\int_{\mathbb{R}^d} \left\{\omega|u|^2 - 2F(u)\right\} dx \\ &= \frac{1}{2^*}\|\nabla u\|_{L^2}^2 + \omega\mathcal{M}(u) - \int_{\mathbb{R}^d} F(u)\, dx.\end{aligned} \tag{4.10}$$

Note here that

$$\begin{aligned}\mathcal{P}_\omega(u(\lambda\cdot)) &= \lambda^{-d}\left[\frac{\lambda^2}{2^*}\|\nabla u\|_{L^2}^2 + \frac{1}{2}\int_{\mathbb{R}^d}\left\{\omega|u|^2 - 2F(u)\right\} dx\right] \\ &= \lambda^{-d}\left[\frac{\lambda^2-1}{2^*}\|\nabla u\|_{L^2}^2 + \mathcal{P}_\omega(u)\right]\end{aligned} \tag{4.11}$$

for any $\lambda > 0$ and any $u \in H^1(\mathbb{R}^d)$. Then, we consider the following variational problem:

$$d(\omega) := \inf\left\{\mathcal{S}_\omega(u) \colon u \in H^1(\mathbb{R}^d) \setminus \{0\},\ \mathcal{P}_\omega(u) = 0\right\}. \tag{4.12}$$

A significance of the variational problem (4.12) is as follows:

Proposition 4.2. *Assume $d \geq 3$,* (N1) *through* (N4) *and* (K2). *Then, any minimizer of the variational problem* (4.12) *becomes a ground state to the Eq.* (1.10).

Thus, Theorem 4.1 follows from the following proposition:

Proposition 4.3. *Assume $d \geq 3$,* (N1) *through* (N4) *and* (K2). *Then, for any $\omega \in (0, \omega_*)$, there exists a non-trivial function $Q_\omega \in H^1(\mathbb{R}^d)$ such that $\mathcal{P}_\omega(Q_\omega) = 0$ and $d(\omega) = \mathcal{S}_\omega(Q_\omega)$, that is, Q_ω is a minimizer for* (4.12).

Before proceeding to the proofs, we give a result immediately follows from these propositions:

Corollary 4.3. *Assume $d \geq 3$,* (N1) *through* (N4) *and* (K2). *Then, any ground state to the Eq.* (1.10) *is a minimizer of the variational problem* (4.12).

Proof of Corollary 4.3. Let Q_ω be a ground state to (1.10). Since Proposition 4.2 shows that any minimizer is a solution to (1.10), we have $d(\omega) \geq \mathcal{S}_\omega(Q_\omega)$. Furthermore, it follows from the Pohozaev identity (4.3) that $\mathcal{P}_\omega(Q_\omega) = 0$. Then, we also have $d(\omega) \leq \mathcal{S}_\omega(Q_\omega)$. Hence, Q_ω is a minimizer. □

We will give proofs of Propositions 4.2 and 4.3 after some preparations. First, we note that

$$\mathcal{S}_\omega(u) - \mathcal{P}_\omega(u) = \frac{1}{d}\|\nabla u\|_{L^2}^2 \tag{4.13}$$

for any function $u \in H^1(\mathbb{R}^d)$. Considering this relation, we introduce a variational value

$$\widetilde{d}(\omega) := \inf\left\{\frac{1}{d}\|\nabla u\|_{L^2}^2 : u \in H^1(\mathbb{R}^d)\setminus\{0\},\ \mathcal{P}_\omega(u) \le 0\right\}. \tag{4.14}$$

Lemma 4.3. *Assume* $d \ge 3$, (N1) *through* (N4), *and* (K2). *Then, for each* $\omega > 0$,

$$\widetilde{d}(\omega) = d(\omega) > 0. \tag{4.15}$$

Proof of Lemma 4.3. Using (4.13), we can verify that $\widetilde{d}(\omega) \le d(\omega)$. Let $u \in H^1(\mathbb{R}^d)\setminus\{0\}$ be an arbitrary function with $\mathcal{P}_\omega(u) \le 0$. Then, we see from (4.11) that there exists $\lambda_0(u) \ge 1$ such that

$$\mathcal{P}_\omega(u(\lambda_0\cdot)) = 0. \tag{4.16}$$

Thus, it follows from the definition of $d(\omega)$, (4.13) and $\lambda_0 \ge 1$ that

$$d(\omega) \le \mathcal{S}_\omega(u(\lambda_0\cdot)) = \frac{1}{d}\|\nabla(u(\lambda_0\cdot))\|_{L^2}^2 \le \lambda_0^{-(d-2)}\frac{1}{d}\|\nabla u\|_{L^2}^2 \le \frac{1}{d}\|\nabla u\|_{L^2}^2, \tag{4.17}$$

which implies $d(\omega) \le \widetilde{d}(\omega)$. Hence, $d(\omega) = \widetilde{d}(\omega)$. It remains to show that $\widetilde{d}(\omega) > 0$. Let $u \in H^1(\mathbb{R}^d)\setminus\{0\}$ be a function with $\mathcal{P}_\omega(u) \le 0$. Then, we see from (A.8) in Lemma A.2, (A.2) and Sobolev's embedding that

$$\frac{1}{2^*}\|\nabla u\|_{L^2}^2 + \frac{\omega}{2}\|u\|_{L^2}^2 \le \int_{\mathbb{R}^d} F(|u|)\,dx \le \frac{\omega}{4}\|u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^{2^*}, \tag{4.18}$$

where $C > 0$ is some constant depending on ω, but not on u, so that

$$\frac{1}{2^*} \le C\|\nabla u\|_{L^2}^{2^*-2}. \tag{4.19}$$

This implies that $\widetilde{d}(\omega) > 0$. □

Now, we are in a position to prove Proposition 4.2:

Proof of Proposition 4.2. Since any solution to the Eq. (1.10) satisfies the Pohozaev identity (4.3), it suffices to show that any minimizer becomes a solution to (1.10). Suppose for contradiction that the claim was false, and let Q_ω be a minimizer. Then, we can take a function $u_0 \in H^1(\mathbb{R}^d)$ such that

$$\mathcal{S}'_\omega(Q_\omega)u_0 = \langle \omega Q_\omega - \Delta Q_\omega - f(Q_\omega), u_0\rangle_{H^{-1},H^1} = -1. \tag{4.20}$$

For a given $\lambda > 0$, we put $Q_{\omega,\lambda}(x) := Q_\omega(\lambda x)$. Then, it follows from $\mathcal{P}_\omega(Q_\omega) = 0$ that

$$\frac{d}{d\lambda}\mathcal{S}_\omega(Q_{\omega,\lambda}) = -\lambda^{-(d+1)}(\lambda^2-1)\frac{d-2}{2}\|\nabla Q_\omega\|_{L^2}^2 \begin{cases} > 0 & \text{if } 0 < \lambda < 1, \\ = 0 & \text{if } \lambda = 1, \\ < 0 & \text{if } 1 < \lambda, \end{cases} \tag{4.21}$$

Hence, we find that

$$\mathcal{S}_\omega(Q_\omega) = \max_{\lambda>0} \mathcal{S}_\omega(Q_{\omega,\lambda}). \tag{4.22}$$

Moreover, since $\mathcal{S}_\omega$ is of class C^1, we can take $\delta_0 \in (0, \frac{1}{2})$ such that if $|\lambda - 1| < \delta_0$,

$$-\frac{3}{2} \le \mathcal{S}'_\omega(Q_{\omega,\lambda})u_0 \le -\frac{1}{2}. \tag{4.23}$$

Let η denote the even function on $\mathbb{R}$ such that

$$\eta(r) = \begin{cases} -r+1 & \text{if } 0 \le r < 1, \\ 0 & \text{if } 1 \le r. \end{cases} \tag{4.24}$$

Moreover, let $\delta \in (0, \delta_0)$ be a constant to be chosen later, and put

$$U_{\omega,\lambda,\delta} := Q_{\omega,\lambda} + \delta\eta\Big(\frac{\lambda-1}{\delta}\Big)u_0. \tag{4.25}$$

Then, we see from $\mathcal{P}_\omega(Q_\omega) = 0$ that

$$\mathcal{P}_\omega(U_{\omega,1-\delta,\delta}) = \mathcal{P}_\omega(Q_{\omega,1-\delta}) = \{(1-\delta)^{2-d} - (1-\delta)^{-d}\}\frac{1}{2^*}\|\nabla Q_\omega\|_{L^2}^2 < 0, \tag{4.26}$$

$$\mathcal{P}_\omega(U_{\omega,1+\delta,\delta}) = \mathcal{P}_\omega(Q_{\omega,1+\delta}) = \{(1+\delta)^{2-d} - (1+\delta)^{-d}\}\frac{1}{2^*}\|\nabla Q_\omega\|_{L^2}^2 > 0. \tag{4.27}$$

Thus, we can take $\lambda_0 \in (-\delta, \delta)$ such that $\mathcal{P}_\omega(U_{\omega,\lambda_0,\delta}) = 0$. Hence, we see from the definition of $d(\omega)$, (4.23), elementary calculations and (4.22) that

$$\begin{aligned} d(\omega) &\le \mathcal{S}_\omega(U_{\omega,\lambda_0,\delta}) \\ &= \mathcal{S}_\omega(Q_{\omega,\lambda_0}) + \mathcal{S}'_\omega(Q_{\omega,\lambda_0})\delta\eta\Big(\frac{\lambda_0-1}{\delta}\Big)u_0 + o(\|U_{\omega,\lambda_0,\delta} - Q_{\omega,\lambda_0}\|_{H^1}) \\ &\le \mathcal{S}_\omega(Q_{\omega,\lambda_0}) - \frac{1}{2}\delta\eta\Big(\frac{\lambda_0-1}{\delta}\Big) + o\Big(\delta\eta\Big(\frac{\lambda_0-1}{\delta}\Big)\Big) \\ &< \mathcal{S}_\omega(Q_\omega) - \frac{1}{4}\delta\eta\Big(\frac{\lambda_0-1}{\delta}\Big) \\ &< \mathcal{S}_\omega(Q_\omega) = d(\omega). \end{aligned} \tag{4.28}$$

However, this is a contradiction. Hence, the claim of Proposition 4.2 is true. □

Next, we give a proof of Proposition 4.3. Considering a minimizing sequence of the variational problem (4.12), we see that Proposition 4.3 immediately follows from the following lemma:

Lemma 4.4. *Assume $d \ge 3$,* (N1) *through* (N4) *and* (K2). *Let $\omega > 0$, and let $\{u_n\}$ be a sequence in $H^1(\mathbb{R}^d)$ such that*

$$\lim_{n\to\infty} \mathcal{S}_\omega(u_n) = d(\omega), \tag{4.29}$$

$$\lim_{n\to\infty} \mathcal{P}_\omega(u_n) = 0. \tag{4.30}$$

Then, there exists a subsequence of $\{u_n\}$ *(still denoted by the same symbol), a non-trivial function* $u_\infty \in H^1(\mathbb{R}^d)$ *and sequence* $\{y_n\}$ *in* $\mathbb{R}^d$ *such that*

$$\lim_{n\to\infty} u_n(\cdot + y_n) = u_\infty \qquad \text{strongly in } H^1(\mathbb{R}^d). \tag{4.31}$$

In particular, u_∞ *is a minimizer of the variational problem* (4.12).

Remark 4.3. We will also use this lemma in the proof of Theorem 4.2 (see Sect. 4.3). This is the reason why we consider the condition (4.30), rather than $\mathcal{P}(u_n) = 0$ for all n.

Proof of Lemma 4.4. We see from (4.13) and the assumptions (4.29) and (4.30) that

$$\lim_{n\to\infty} \frac{1}{d}\|\nabla u_n\|_{L^2}^2 = \lim_{n\to\infty} \mathcal{S}_\omega(u_n) - \lim_{n\to\infty} \mathcal{P}_\omega(u_n) = d(\omega). \tag{4.32}$$

Thus,

$$\frac{1}{10}d(\omega) \le \frac{1}{d}\|\nabla u_n\|_{L^2}^2 \le 2d(\omega) \tag{4.33}$$

for any sufficiently large n.

Fix $1 < q_0 < \frac{d+2}{d-2}$. Then, it follows from (A.8) in Lemmas A.2 and 4.1 (or Lemma 4.2) that there exists $C_0 > 0$ such that for any $z \in \mathbb{Z}$,

$$F(z) \le \frac{\omega}{4}|z|^2 + C_0|z|^{q_0+1}. \tag{4.34}$$

Recall that $d(\omega) > 0$ (see Lemma 4.3). Hence, we see from the assumption (4.30), (4.34) and the Gagliardo-Nirenberg inequality that for any sufficiently large n,

$$\begin{aligned} \mathcal{P}_\omega(u_n) &\ge \frac{1}{2^*}\|\nabla u_n\|_{L^2}^2 + \frac{\omega}{2}\|u_n\|_{L^2}^2 - \frac{\omega}{4}\|u_n\|_{L^2}^2 - C_0\|u_n\|_{L^{q_0+1}}^{q_0+1} \\ &\ge \frac{\omega}{4}\|u_n\|_{L^2}^2 - C\|u_n\|_{L^2}^{q_0+1-\frac{d(q_0-1)}{2}}\|\nabla u_n\|_{L^2}^{\frac{d(q_0-1)}{2}}, \end{aligned} \tag{4.35}$$

where C is some positive constant independent of n. Since $q_0+1-\frac{d(q_0-1)}{2} < 2$, this inequality (4.35) together with the assumption (4.30) and (4.33) implies that

$$\|u_n\|_{L^2}^2 \lesssim 1, \tag{4.36}$$

where the implicit constant may depend on ω, but is independent of n. Moreover, we find from the first line in (4.35) and (4.33) that

$$\mathcal{P}_\omega(u_n) \gtrsim d(\omega) - C\|u_n\|_{L^{q_0+1}}^{q_0+1}, \tag{4.37}$$

which together with the assumption (4.30) yields

$$d(\omega) \lesssim \|u_n\|_{L^{q_0+1}}^{q_0+1} \tag{4.38}$$

for any sufficiently large n. Thus, Lemma B.1 can apply to $\{u_n\}$ with $p_1 = 2$, $p_2 = q_0 + 1$ and $p_3 = 2^*$: there exist constants $C > 0$, $\eta > 0$ such that

$$|\{x \in \mathbb{R}^d \colon |u_n(x)| > \eta\}| \ge C \tag{4.39}$$

for any sufficiently large n. Since $\{u_n\}$ is bounded in $H^1(\mathbb{R}^d)$, it follows from Lemma B.2 that we can extract a subsequence of $\{u_n\}$ (still denoted by the same symbol), a non-trivial function u_∞ and a sequence $\{y_n\}$ in $\mathbb{R}^d$ such that

$$\lim_{n\to\infty} u_n(\cdot + y_n) = u_\infty \qquad \text{weakly in } H^1(\mathbb{R}^d). \tag{4.40}$$

Furthermore, we see from Lemma B.3 that

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \left| |\nabla u_n|^2 - |\nabla\{u_n - u_\infty\}|^2 - |\nabla u_\infty|^2 \right| dx = 0, \tag{4.41}$$

$$\lim_{n\to\infty} \left\{ \mathcal{P}_\omega(u_n) - \mathcal{P}_\omega(u_n - u_\infty) - \mathcal{P}_\omega(u_\infty) \right\} = 0. \tag{4.42}$$

Suppose here that $\mathcal{P}_\omega(u_\infty) > 0$. Then, we see from (4.42) and the assumption (4.30) that

$$\lim_{n\to\infty} \mathcal{P}_\omega(u_n - u_\infty) = -\mathcal{P}_\omega(u_\infty) < 0. \tag{4.43}$$

Thus, it follows from the definition of $\widetilde{d}(\omega)$ and $\widetilde{d}(\omega) = d(\omega)$ (see Lemma 4.3) that

$$d(\omega) \le \frac{1}{d}\|\nabla\{u_n - u_\infty\}\|_{L^2}^2 \tag{4.44}$$

for any sufficiently large n. This together with (4.41) and (4.32) shows that

$$\begin{aligned} \frac{1}{d}\|\nabla u_\infty\|_{L^2}^2 &= \frac{1}{d}\lim_{n\to\infty} \left\{ \|\nabla u_n\|_{L^2}^2 - \|\nabla u_n(\cdot + y_n) - \nabla u_\infty\|_{L^2}^2 \right\} \\ &\le d(\omega) - d(\omega) = 0. \end{aligned} \tag{4.45}$$

However, this contradicts the non-triviality of u_∞. Hence, $\mathcal{P}_\omega(u_\infty) \le 0$. Then, we see from the definition of $\widetilde{d}(\omega)$, $\widetilde{d}(\omega) = d(\omega)$, $\|\nabla u_\infty\|_{L^2} \le \liminf_{n\to\infty} \|\nabla u_n\|_{L^2}$ (see (4.40)) and $\mathcal{P}_\omega(u_\infty) \le 0$ that

$$d(\omega) \le \frac{1}{d}\|\nabla u_\infty\|_{L^2}^2 \le d(\omega), \tag{4.46}$$

so that

$$\frac{1}{d}\lim_{n\to\infty} \|\nabla u_n\|_{L^2}^2 = d(\omega) = \frac{1}{d}\|\nabla u_\infty\|_{L^2}^2. \tag{4.47}$$

This together with (4.40) implies that

$$\lim_{n\to\infty} u_n(\cdot + y_n) = u_\infty \qquad \text{strongly in } \dot{H}^1(\mathbb{R}^d). \tag{4.48}$$

We also find from (4.36) and (4.48) that for any $r \in (2, 2^*]$,

$$\lim_{n\to\infty} u_n(\cdot + y_n) = u_\infty \qquad \text{strongly in } L^r(\mathbb{R}^d). \tag{4.49}$$

Moreover, since $\mathcal{P}_\omega(u_\infty) \le 0$, we see from (4.11) that there exists $\lambda_\infty \ge 1$ such that

$$\mathcal{P}_\omega(u_\infty(\lambda_\infty \cdot)) = 0. \tag{4.50}$$

Thus, it follows from the definition of $d(\omega)$, (4.13) and (4.47) that

$$d(\omega) \le \mathcal{S}_\omega(u_\infty(\lambda_\infty \cdot)) = \frac{1}{d}\|\nabla\{u_\infty(\lambda_\infty \cdot)\}\|_{L^2}^2 = \lambda_\infty^{2-d} d(\omega), \tag{4.51}$$

so that $\lambda_\infty = 1$. Hence, $\mathcal{P}_\omega(u_\infty) = 0$, which together with (4.42) and the assumption (4.30) yields that

$$\lim_{n\to\infty} \mathcal{P}_\omega(u_n - u_\infty) = 0. \tag{4.52}$$

Furthermore, this together with (4.47) and (4.49) gives us the convergence in $L^2(\mathbb{R}^d)$. □

4.3. Stability of standing wave

In this section, we shall prove Theorem 4.2. To this end, we introduce sets

$$\mathcal{A}_{\omega,+} := \left\{ u \in H^1(\mathbb{R}^d) \colon \mathcal{S}_\omega(u) < d(\omega),\ \frac{1}{d}\|\nabla u\|_{L^2}^2 < d(\omega) \right\}, \tag{4.53}$$

$$\mathcal{A}_{\omega,-} := \left\{ u \in H^1(\mathbb{R}^d) \colon \mathcal{S}_\omega(u) < d(\omega),\ \frac{1}{d}\|\nabla u\|_{L^2}^2 > d(\omega) \right\}. \tag{4.54}$$

Lemma 4.5. *Assume $d \ge 3$,* (N1) *through* (N4) *and* (K2). *Then, for any $\omega \in (0, \omega_*)$, the sets $\mathcal{A}_{\omega,+}$ and $\mathcal{A}_{\omega,-}$ are invariant under the flow defined by the Eq.* (NLS).

Proof of Lemma 4.5. We shall show the invariance of $\mathcal{A}_{\omega,+}$. Let $\psi_0 \in \mathcal{A}_{\omega,+}$, and let ψ be a solution to (NLS) with $\psi(0) = \psi_0$. Then, it follows from the conservation law of the action that

$$\mathcal{S}_\omega(\psi(t)) < d(\omega) \tag{4.55}$$

for all $t \in \mathbb{R}$. Moreover, it follows from the identity (4.13) that

$$\frac{1}{d}\|\nabla \psi(t)\|_{L^2}^2 = \mathcal{S}_\omega(\psi(t)) - \mathcal{P}_\omega(\psi(t)) < d(\omega) - \mathcal{P}_\omega(\psi(t)) \tag{4.56}$$

for all $t \in \mathbb{R}$. Thus, it suffices to show that $\mathcal{P}_\omega(\psi(t)) \ge 0$ for all $t \in \mathbb{R}$. Since $\frac{1}{d}\|\nabla \psi_0\|_{L^2}^2 < d(\omega)$, we see from the definition of $\widetilde{d}(\omega)$ and $d(\omega) = \widetilde{d}(\omega)$ (see Lemma 4.3) that $\mathcal{P}_\omega(\psi_0) > 0$. Suppose for contradiction that there exists $t_- \in \mathbb{R}$ such that $\mathcal{P}_\omega(\psi(t_-)) < 0$. Then, the continuity of solutions in time implies that $\mathcal{P}_\omega(\psi(t_*)) = 0$ for some t_* between 0 and t_-. Furthermore, we see from the definition of $d(\omega)$ that

$$\mathcal{S}_\omega(\psi(t_*)) \ge d(\omega). \tag{4.57}$$

However, this contradicts (4.55). Similarly, we can prove the invariance of $\mathcal{A}_{\omega,-}$. □

Lemma 4.6. *Assume $d \ge 3$,* (N1) *through* (N4) *and* (K2). *Then, we have*

$$\lim_{\omega \to \omega_*} d(\omega) = \infty. \tag{4.58}$$

Proof of Lemma 4.6. Put

$$c(\omega) := \sup_{s>0} \frac{F(s) - \omega \frac{s^2}{2}}{s^{2^*}}. \tag{4.59}$$

Then, we see from (A.8) in Lemma A.2 and Lemma 4.2 that $\lim_{\omega \to \omega_*} c(\omega) = 0$. Furthermore, we see that

$$\frac{1}{2^*}\|\nabla u\|_{L^2}^2 \le \int_{\mathbb{R}^d} \{F(u) - \frac{\omega}{2}|u|^2\}\, dx \lesssim c(\omega)\|\nabla u\|_{L^2}^{2+\frac{4}{d-2}} \tag{4.60}$$

for all $u \in H^1(\mathbb{R}^d)$ with $\mathcal{P}_\omega(u) \le 0$. Hence, Lemma 4.3 together with (4.60) gives us that

$$\lim_{\omega\to\omega_*} d(\omega) = \lim_{\omega\to\omega_*} \widetilde{d}(\omega) \gtrsim \lim_{\omega\to\omega_*} c(\omega)^{-\frac{d-2}{2}} = \infty. \tag{4.61}$$

Thus, we obtain the desired result. □

Lemma 4.7. *Assume* $d \ge 3$, (N1) *through* (N4) *and* (K2)*. Then, there exists a constant* $C(d) > 0$ *with the following property: let* $0 < \omega_1 < \omega_2 < \omega_*$*, and let* Q_{ω_1} *and* Q_{ω_2} *be minimizers of the variational problems for* $d(\omega_1)$ *and* $d(\omega_2)$*, respectively. Moreover, assume that*

$$2(\omega_2 - \omega_1) < \frac{\|\nabla Q_{\omega_1}\|_{L^2}^2}{2^*\mathcal{M}(Q_{\omega_1})}. \tag{4.62}$$

Then, we have

$$d(\omega_1) \le d(\omega_2) - \mathcal{M}(Q_{\omega_2})(\omega_2 - \omega_1) + C(d)\frac{\mathcal{M}(Q_{\omega_2})^2}{d(\omega_2)}|\omega_2 - \omega_1|^2, \tag{4.63}$$

$$d(\omega_2) \le d(\omega_1) + \mathcal{M}(Q_{\omega_1})(\omega_2 - \omega_1) + C(d)\frac{\mathcal{M}(Q_{\omega_1})^2}{d(\omega_1)}|\omega_2 - \omega_1|^2. \tag{4.64}$$

In particular, $d(\omega)$ *is continuous and strictly increasing on* $(0, \omega_*)$*.*

Proof of Lemma 4.7. Let us begin with a proof of (4.63). Put $Q_{\omega_2,\lambda} := Q_{\omega_2}(\sqrt{\lambda}\cdot)$ for $\lambda > 0$. Since $P_{\omega_2}(Q_{\omega_2}) = 0$, we see that

$$\begin{aligned}\mathcal{P}_{\omega_1}(Q_{\omega_2,\lambda}) &= \frac{\lambda^{1-\frac{d}{2}}}{2^*}\|\nabla Q_{\omega_2}\|_{L^2}^2 + \lambda^{-\frac{d}{2}}\omega_1\mathcal{M}(Q_{\omega_2}) - \lambda^{-\frac{d}{2}}\int_{\mathbb{R}^d} F(|Q_{\omega_2}|)\,dx \\ &= \frac{\lambda^{-\frac{d}{2}}}{2^*}\left\{ \lambda\|\nabla Q_{\omega_2}\|_{L^2}^2 - 2^*(\omega_2 - \omega_1)\mathcal{M}(Q_{\omega_2}) - \|\nabla Q_{\omega_2}\|_{L^2}^2 \right\}.\end{aligned} \tag{4.65}$$

We define $\lambda_* > 1$ by

$$\lambda_* := 1 + \frac{2^*(\omega_2 - \omega_1)\mathcal{M}(Q_{\omega_2})}{\|\nabla Q_{\omega_2}\|_{L^2}^2}. \tag{4.66}$$

Then, we see from (4.65) that $\mathcal{P}_{\omega_1}(Q_{\omega_2,\lambda_*}) = 0$. Thus,

$$d(\omega_1) = \widetilde{d}(\omega_1) \le \frac{1}{d}\|\nabla Q_{\omega_2,\lambda_*}\|_{L^2}^2 = \lambda_*^{1-\frac{d}{2}}\frac{1}{d}\|\nabla Q_{\omega_2}\|_{L^2}^2 = \lambda_*^{-\frac{d-2}{2}} d(\omega_2). \tag{4.67}$$

Moreover, it follows from Taylor's expansion that there exists $\theta_* \in (0, 1)$ depending on ω_1 and ω_2,

$$\begin{aligned}\lambda_*^{-\frac{d-2}{2}} &= 1 - \frac{(\omega_2 - \omega_1)\mathcal{M}(Q_{\omega_2})}{\frac{1}{d}\|\nabla Q_{\omega_2}\|_{L^2}^2} \\ &\quad + \frac{2^*}{4}\left\{1 + \theta_*\frac{2^*(\omega_2 - \omega_1)\mathcal{M}(Q_{\omega_2})}{\|\nabla Q_{\omega_2}\|_{L^2}^2}\right\}^{-\frac{d+2}{2}}\left(\frac{(\omega_2 - \omega_1)\mathcal{M}(Q_{\omega_2})}{\frac{1}{d}\|\nabla Q_{\omega_2}\|_{L^2}^2}\right)^2 \\ &\le 1 - \frac{\mathcal{M}(Q_{\omega_2})}{d(\omega_2)}(\omega_2 - \omega_1) + C(d)\frac{\mathcal{M}(Q_{\omega_2})^2}{d(\omega_2)^2}|\omega_2 - \omega_1|^2,\end{aligned} \tag{4.68}$$

where $C(d) > 0$ is some constant depending only on d: note here that we do not need any assumption like (4.62). Putting (4.67) and (4.68) together, we obtain the desired inequality

$$d(\omega_1) \le d(\omega_2) - \mathcal{M}(Q_{\omega_1})(\omega_2 - \omega_1) + C(d)\frac{\mathcal{M}(Q_{\omega_2})^2}{d(\omega_2)}|\omega_2 - \omega_1|^2. \tag{4.69}$$

Next, we prove (4.64). Put $Q_{\omega_1,\lambda} := Q_{\omega_1}(\sqrt{\lambda}\cdot)$ for $\lambda > 0$. Since $P_{\omega_1}(Q_{\omega_1}) = 0$, we see that

$$\begin{aligned}\mathcal{P}_{\omega_2}(Q_{\omega_1,\lambda}) &= \frac{\lambda^{1-\frac{d}{2}}}{2^*}\|\nabla Q_{\omega_1}\|_{L^2}^2 + \lambda^{-\frac{d}{2}}\omega_2\mathcal{M}(Q_{\omega_1}) - \lambda^{-\frac{d}{2}}\int_{\mathbb{R}^d} F(|Q_{\omega_1}|)\,dx \\ &= \frac{\lambda^{-\frac{d}{2}}}{2^*}\left\{ \lambda\|\nabla Q_{\omega_1}\|_{L^2}^2 + 2^*(\omega_2 - \omega_1)\mathcal{M}(Q_{\omega_1}) - \|\nabla Q_{\omega_1}\|_{L^2}^2 \right\}.\end{aligned} \tag{4.70}$$

Here, by the assumption (4.62), we can define $\lambda_* \in (\frac{1}{2}, 1)$ by

$$\lambda_* := 1 - \frac{2^*(\omega_2 - \omega_1)\mathcal{M}(Q_{\omega_1})}{\|\nabla Q_{\omega_1}\|_{L^2}^2}. \tag{4.71}$$

Then, we see from (4.70) that $\mathcal{P}_{\omega_2}(Q_{\omega_1,\lambda_*}) = 0$. Thus,

$$d(\omega_2) = \widetilde{d}(\omega_2) \le \frac{1}{d}\|\nabla Q_{\omega_1,\lambda_*}\|_{L^2}^2 = \lambda_*^{1-\frac{d}{2}}\frac{1}{d}\|\nabla Q_{\omega_1}\|_{L^2}^2 = \lambda_*^{-\frac{d-2}{2}} d(\omega_1). \tag{4.72}$$

Moreover, it follows from Taylor's expansion that there exists $\theta_* \in (0,1)$ depending on ω_1 and ω_2,

$$\begin{aligned}\lambda_*^{-\frac{d-2}{2}} &= 1 + \frac{(\omega_2 - \omega_1)\mathcal{M}(Q_{\omega_1})}{\frac{1}{d}\|\nabla Q_{\omega_1}\|_{L^2}^2} \\ &\quad + \frac{2^*}{4}\left\{1 - \theta_*\frac{2^*(\omega_2 - \omega_1)\mathcal{M}(Q_{\omega_1})}{\|\nabla Q_{\omega_1}\|_{L^2}^2}\right\}^{-\frac{d+2}{2}}\left(\frac{(\omega_2 - \omega_1)\mathcal{M}(Q_{\omega_1})}{\frac{1}{d}\|\nabla Q_{\omega_1}\|_{L^2}^2}\right)^2,\end{aligned} \tag{4.73}$$

which together with the assumption (4.62) gives us that

$$\lambda_*^{-\frac{d-2}{2}} \le 1 + \frac{\mathcal{M}(Q_{\omega_1})}{d(\omega_1)}(\omega_2 - \omega_1) + C(d)\frac{\mathcal{M}(Q_{\omega_1})^2}{d(\omega_1)^2}|\omega_2 - \omega_1|^2, \tag{4.74}$$

where $C(d) > 0$ is some constant depending only on d. Putting (4.72) and (4.74) together, we obtain the inequality

$$d(\omega_2) \le d(\omega_1) + \mathcal{M}(Q_{\omega_1})(\omega_2 - \omega_1) + C(d)\frac{\mathcal{M}(Q_{\omega_1})^2}{d(\omega_1)}|\omega_2 - \omega_1|^2. \tag{4.75}$$

Hence, the claim (4.64) holds. □

Lemma 4.8. *Assume* $d \ge 3$, (N1) *through* (N4) *and* (K2)*. Then, there exist a sequence* $\{\omega_n\}$ *in* $(0, \omega_*)$ *and a sequence* $\{M_n\}$ *in* $(0, \infty)$ *such that* $\lim_{n\to\infty} \omega_n = \omega_*$*, and*

$$d(\omega) > d(\omega_n) + M_n(\omega - \omega_n) \tag{4.76}$$

for all n *and all* $\omega \in (0, \omega_*) \setminus \{\omega_n\}$*. Furthermore,*

$$\lim_{n\to\infty} M_n = \infty, \tag{4.77}$$

and

$$\mathcal{M}(Q_{\omega_n}) = M_n \tag{4.78}$$

for any n *and any* $Q_{\omega_n} \in \mathcal{G}_{\omega_n}$.

Proof of Lemma 4.8. Define a sequence $\{a_n\}$ by

$$a_n := \frac{2^{n+1} d((1-2^{-n-1})\omega_*)}{\omega_*}. \tag{4.79}$$

Note here that Lemma 4.6 and the continuity of $d(\omega)$ (see Lemma 4.7) imply

$$\lim_{n\to\infty} a_n = \infty. \tag{4.80}$$

We shall show that for each number n, there exists a minimizer of $d(\omega) - a_n\omega - \omega^2$ on $(0, \omega_*)$. Put $y_n(\omega) := d(\omega) - a_n\omega - \omega^2$. It follows from $d(\omega) > 0$ on $(0, \omega_*)$ (see Lemma 4.3) and the definition of a_n (see 4.79) that for any $\omega \in (0, (1-2^{-n})\omega_*)$,

$$\begin{aligned} y_n(\omega) &> -a_n\omega - \omega^2 \\ &> -a_n(1-2^{-n})\omega_* - \left\{(1-2^{-n})\omega_*\right\}^2 \\ &> a_n(2^{-n} - 2^{-n-1})\omega_* - a_n(1-2^{-n-1})\omega_* - \left\{(1-2^{-n-1})\omega_*\right\}^2 \\ &= y_n((1-2^{-n-1})\omega_*). \end{aligned} \tag{4.81}$$

On the other hand, it follows from Lemma 4.6 and the continuity of $d(\omega)$ (see Lemma 4.7) that

$$\lim_{\omega\to\omega_*} y_n(\omega) = \infty, \tag{4.82}$$

so that for any n, we can take $\gamma_n \in [(1-2^{-n})\omega_*, \omega_*)$ such that $y_n(\omega) > y_n((1-2^{-n-1})\omega_*)$ for all $\omega \in (\gamma_n, \omega_*)$. Moreover, we can take a minimizer of $y_n(\omega)$ on the compact interval $[(1-2^{-n})\omega_*, \gamma_n]$. Hence, we are able to take a minimizer ω_n of $y_n(\omega)$ on $(0, \omega_*)$. In particular, we see from (4.81) that $\omega_n \geq (1-2^{-n})\omega_*$ and therefore

$$\lim_{n\to\infty} \omega_n = \omega_*. \tag{4.83}$$

Next, we shall show that

$$d(\omega) > d(\omega_n) + (a_n + 2\omega_n)(\omega - \omega_n) \tag{4.84}$$

for all n and all $\omega \in (0, \omega_*) \setminus \{\omega_n\}$. Since $a_n\omega + \omega^2$ is convex as a function of ω, we see that

$$a_n\omega + \omega^2 > a_n\omega_n + \omega_n^2 + (a_n + 2\omega_n)(\omega - \omega_n) \tag{4.85}$$

for all $\omega \in (0, \omega_*) \setminus \{\omega_n\}$. This together with the fact that ω_n is a minimizer of $y_n(\omega) = d(\omega) - a_n\omega - \omega^2$ shows

$$\begin{aligned} d(\omega) &= d(\omega) - a_n\omega - \omega^2 + a_n\omega + \omega^2 \\ &> d(\omega_n) - a_n\omega_n - \omega_n^2 + a_n\omega_n + \omega_n^2 + (a_n + 2\omega_n)(\omega - \omega_n) \\ &= d(\omega_n) + (a_n + 2\omega_n)(\omega - \omega_n) \end{aligned} \tag{4.86}$$

for all $\omega \in (0, \omega_*) \setminus \{\omega_n\}$. Thus, we have proved (4.84). Furthermore, putting $M_n := 2\omega_n + a_n$, we find from (4.84) and (4.80) that (4.76) and (4.77) holds.

Finally, we shall prove

$$M_n = \mathcal{M}(Q_{\omega_n}) \tag{4.87}$$

for all $Q_{\omega_n} \in \mathcal{G}_{\omega_n}$. Take an arbitrary ground state $Q_n \in \mathcal{G}_{\omega_n}$. Then, it is also a minimizer of the variational problem for $d(\omega_n)$ (see Corollary 4.3). Hence, we see from (4.63) in Lemma 4.7 and (4.76) that Q_n satisfies that for any $\omega < \omega_n$ sufficiently close to ω_n,

$$\begin{aligned}\mathcal{M}(Q_n)(\omega_n - \omega) &\le d(\omega_n) - d(\omega) + o(\omega_n - \omega)\\ &\le M_n(\omega_n - \omega) + o(\omega_n - \omega).\end{aligned} \tag{4.88}$$

Dividing the both sides above by $\omega_n - \omega$, and then taking $\omega \to \omega_n$, we find that

$$\mathcal{M}(Q_n) \le M_n. \tag{4.89}$$

Similarly, the opposite inequality follows from (4.64) in Lemma 4.7. Hence, we have completed the proof. □

Lemma 4.9. *Assume $d \ge 3$,* (N1) *through* (N4) *and* (K2)*. Let $\omega_0 \in (0, \omega_*)$, and let Q_{ω_0} be a ground state to the Eq.* (1.10) *with $\omega = \omega_0$. Moreover, assume that*

$$d(\omega) > d(\omega_0) + \mathcal{M}(Q_{\omega_0})(\omega - \omega_0) \tag{4.90}$$

for all $\omega \in (0, \omega_*) \setminus \{\omega_0\}$. Then, for any $\varepsilon \in (0, \omega_0)$, there exists $\delta > 0$ such that if a function $\psi_0 \in H^1(\mathbb{R}^d)$ satisfies

$$\|\psi_0 - Q_{\omega_0}\|_{H^1} \le \delta, \tag{4.91}$$

then the solution ψ to (NLS) *with $\psi(0) = \psi_0$ obeys that*

$$d(\omega_0 - \varepsilon) < \frac{1}{d}\|\nabla \psi(t)\|_{L^2}^2 < d(\omega_0 + \varepsilon) \tag{4.92}$$

for all $t \in I_{\max}$, where $I_{\max}$ denotes the maximal existence interval of ψ.

Proof of Lemma 4.9. Let $\varepsilon \in (0, \omega_0)$, and let $\delta > 0$ be a constant to be chosen later. Furthermore, let ψ_0 be a function in $H^1(\mathbb{R}^d)$ satisfying

$$\|\psi_0 - Q_{\omega_0}\|_{H^1} \le \delta. \tag{4.93}$$

Since $\mathcal{P}_{\omega_0}(Q_{\omega_0}) = 0$, we see from (4.13) and (4.93) that

$$d(\omega_0) = \mathcal{S}_{\omega_0}(Q_{\omega_0}) = \frac{1}{d}\|\nabla Q_{\omega_0}\|_{L^2}^2 = \frac{1}{d}\|\nabla \psi_0\|_{L^2}^2 + O(\delta). \tag{4.94}$$

Furthermore, it follows from Corollary 4.3, (4.63) in Lemma 4.7 and the assumption (4.90) that if δ is sufficiently small dependently on ε, then

$$d(\omega_0 - \varepsilon) < \frac{1}{d}\|\nabla \psi_0\|_{L^2}^2 < d(\omega_0 + \varepsilon). \tag{4.95}$$

Thus, from the point of view of the invariance of $\mathcal{A}_{\omega_0,+}$ and $\mathcal{A}_{\omega_0,-}$ (see Lemma 4.5), it suffices for the desired result (4.92) to show that

$$\mathcal{S}_{\omega_0+\varepsilon}(\psi_0) < d(\omega_0 + \varepsilon), \tag{4.96}$$

$$\mathcal{S}_{\omega_0-\varepsilon}(\psi_0) < d(\omega_0 - \varepsilon). \tag{4.97}$$

We shall prove (4.96). To this end, we put

$$\eta(\varepsilon) := d(\omega_0 + \varepsilon) - d(\omega_0) - \varepsilon \mathcal{M}(Q_{\omega_0}). \tag{4.98}$$

Then, it follows from the assumption (4.90) that $\eta(\varepsilon) > 0$. Furthermore, we see from Taylor's expansion around Q_{ω_0} and the assumption (4.90) that

$$\begin{aligned} \mathcal{S}_{\omega_0+\varepsilon}(\psi_0) &= \mathcal{S}_{\omega_0+\varepsilon}(Q_{\omega_0}) + \mathcal{S}'_{\omega_0+\varepsilon}(Q_{\omega_0})(\psi_0 - Q_{\omega_0}) + o(\|\psi_0 - Q_{\omega_0}\|_{H^1}) \\ &= \mathcal{S}_{\omega_0}(Q_{\omega_0}) + \varepsilon \mathcal{M}(Q_{\omega_0}) \\ &\quad + \langle (\omega_0+\varepsilon) Q_{\omega_0} - \Delta Q_{\omega_0} + f(Q_{\omega_0}), \psi_0 - Q_{\omega_0} \rangle_{H^{-1},H^1} + o(\delta) \\ &= d(\omega_0) + \varepsilon \mathcal{M}(Q_{\omega_0}) + \varepsilon (Q_{\omega_0}, \psi_0 - Q_{\omega_0})_{L^2_{real}} + o(\delta) \\ &\le d(\omega_0) + \varepsilon \mathcal{M}(Q_{\omega_0}) + \varepsilon\delta \|Q_{\omega_0}\|_{L^2} + o(\delta) \\ &= d(\omega_0+\varepsilon) - \eta(\varepsilon) + \varepsilon\delta \|Q_{\omega_0}\|_{L^2} + o(\delta). \end{aligned} \tag{4.99}$$

Thus, taking δ sufficiently small dependently on ε and ω_0, we find that (4.96) holds. Similarly, we can verify that if δ is sufficiently small dependently on ε and ω_0, then (4.97) holds. Hence, we have completed the proof. □

Finally, we give a proof of Theorem 4.2:

Proof of Theorem 4.2. Let $\{\omega_n\}$ be a sequence found in Lemma 4.8, so that $\lim_{n\to\infty} \omega_n = \omega_*$ and

$$d(\omega) > d(\omega_n) + \mathcal{M}(Q)(\omega - \omega_n) \tag{4.100}$$

for any n, any $\omega \in (0, \omega_*) \setminus \{\omega_n\}$ and any $Q \in \mathcal{G}_{\omega_n}$. We shall prove that for each $n \ge 1$, the set $\mathcal{G}_{\omega_n}$ is stable in the sense described in Sect. 4.1. Suppose for contradiction that there exists a number m such that $\mathcal{G}_{\omega_m}$ is unstable. Then, we could take $\varepsilon_0 > 0$ with the following property: for any positive integer k, there exists $\psi_{k,0} \in H^1(\mathbb{R}^d)$ and $t_k \in I_{\max,k}$ such that

$$\inf_{Q \in \mathcal{G}_m} \|\psi_{k,0} - Q\|_{H^1} \le \frac{1}{k}, \tag{4.101}$$

and

$$\inf_{Q \in \mathcal{G}_m} \|\psi_k(t_k) - Q\|_{H^1} \ge \varepsilon_0, \tag{4.102}$$

where ψ_k is the solution to (NLS) with $\psi_k(0) = \psi_{k,0}$, and $I_{\max,k}$ is the maximal existence interval of ψ_k. Since

$$\mathcal{S}_{\omega_m}(Q) = d(\omega_m) \tag{4.103}$$

for all $Q \in \mathcal{G}_{\omega_m}$ (see Corollary 4.3), we see from the conservation law of action and (4.101) that

$$\mathcal{S}_{\omega_m}(\psi_k(t_k)) = \mathcal{S}_{\omega_m}(\psi_{k,0}) = d(\omega_m) + o_k(1). \tag{4.104}$$

Moreover, we see from (4.101), Lemma 4.8, 4.9 and the continuity of $d(\omega)$ (see Lemma 4.7) that

$$\frac{1}{d}\|\nabla \psi_k(t_k)\|_{L^2}^2 = d(\omega_m) + o_k(1). \tag{4.105}$$

Thus, we find from (4.104), (4.105) and (4.13) that

$$\mathcal{P}_{\omega_m}(\psi_k(t_k)) = \mathcal{S}_{\omega_m}(\psi_k(t_k)) - \frac{1}{d}\|\nabla\psi_k(t_k)\|_{L^2}^2 = o_k(1). \tag{4.106}$$

Hence, it follows from Lemma 4.4 and Proposition 4.2 that there exists a subsequence of $\{\psi_k(t_k)\}$ (still denoted by the same symbol), a ground state Q_∞ and a sequence $\{y_k\}$ in $\mathbb{R}^d$ such that

$$\lim_{k\to\infty} \psi_k(\cdot + y_k, t_k) = Q_\infty \quad \text{strongly in } H^1(\mathbb{R}^d). \tag{4.107}$$

Since $Q_\infty(\cdot - y_k)$ is still a ground state for any number k, we see that

$$\lim_{k\to\infty} \|\psi_k(t_k) - Q_\infty(\cdot - y_k)\|_{H^1} = 0. \tag{4.108}$$

However, this contradicts (4.102). Hence, the claim of Theorem 4.2 is true. □

4.4. Limiting profile of ground state

In this subsection, we discuss the limiting profile of ground states. To this end, we introduce a function H_ω on $[0,\infty)$ as

$$H_\omega(s) := \frac{1}{2}\omega s^2 - F(s) = \frac{1}{2}s^2\Big\{\omega - 2\frac{F(s)}{s^2}\Big\}, \tag{4.109}$$

so that

$$\mathcal{P}_\omega(u) = \frac{1}{2^*}\|\nabla u\|_{L^2}^2 + \int_{\mathbb{R}^d} H_\omega(|u|)\,dx. \tag{4.110}$$

Recall here that the maximum of the function $2F(s)/s^2$ is ω_* (cf. Lemma 4.2), so that for any $\omega \in (0,\omega_*)$,

$$\inf_{s\ge 0} H_\omega(s) < 0. \tag{4.111}$$

Moreover, we see from Lemma 4.1 and Lemma 4.2 that

$$\omega_1 := \lim_{s\to\infty} \frac{2F(s)}{s^2} < \omega_* \tag{4.112}$$

and therefore for any $\omega \in (\omega_1,\omega_*)$, we can take $r(\omega) > 0$ such that

$$\inf_{s\ge r(\omega)} H_\omega(s) > 0. \tag{4.113}$$

This together with (4.111) implies that for any $\omega \in (\omega_1,\omega_*)$, H_ω has its minimum.

Lemma 4.10. *Assume $d \ge 3$,* (N1) *through* (N4) *and* (K2)*. Furthermore, let $\omega \in (\omega_1,\omega_*)$, Q_ω be a ground state to the Eq.* (1.10)*, and $s(\omega)$ be a point for which*

$$H_\omega(s(\omega)) = \min_{s\ge 0} H_\omega(s). \tag{4.114}$$

Then, we have

$$\|Q_\omega\|_{L^\infty} \le s(\omega). \tag{4.115}$$

Proof of Lemma 4.10. Since Q_ω becomes a minimizer of the variational problem (4.12) (see Corollary 4.3), we see from Lemma 4.3 that

$$\widetilde{d}(\omega) = \frac{1}{d}\|\nabla Q_\omega\|_{L^2}^2. \tag{4.116}$$

We define

$$\Phi_\omega(x) := \min\{|Q_\omega|, s(\omega)\}. \tag{4.117}$$

Then, it is easy to verify that $\Phi \in H^1(\mathbb{R}^d)$. In particular, we have

$$\nabla\Phi_\omega(x) = \begin{cases} \nabla|Q_\omega|(x) & \text{when } |Q_\omega(x)| \le s(\omega), \\ 0 & \text{when } s(\omega) < |Q_\omega(x)|. \end{cases} \tag{4.118}$$

Furthermore, we see that

$$\begin{aligned} &\mathcal{P}_\omega(Q_\omega) \\ &\quad \ge \frac{1}{2^*}\big\|\nabla|Q_\omega|\big\|_{L^2}^2 + \int_{\{|Q_\omega|\le s(\omega)\}} H_\omega(|Q_\omega|)\,dx + \int_{\{s(\omega)\le|Q_\omega|\}} H_\omega(|Q_\omega|)\,dx \\ &\quad \ge \frac{1}{2^*}\int_{\{|Q_\omega|\le s(\omega)\}} \big|\nabla|Q_\omega|\big|^2\,dx + \int_{\{|Q_\omega|\le s(\omega)\}} H_\omega(|Q_\omega|)\,dx \\ &\qquad + \int_{\{s(\omega)\le|Q_\omega|\}} H_\omega(s(\omega))\,dx \\ &\quad = \frac{1}{2^*}\big\|\nabla\Phi_\omega\big\|_{L^2}^2 + \int_{\mathbb{R}^d} H_\omega(\Phi_\omega)\,dx = \mathcal{P}_\omega(\Phi_\omega). \end{aligned} \tag{4.119}$$

This together with $\mathcal{P}_\omega(Q_\omega) = 0$ shows that $\mathcal{P}_\omega(\Phi_\omega) \le 0$. Hence, it follows from the definition of $\widetilde{d}(\omega)$ (see (4.14)) that

$$\widetilde{d}(\omega) \le \frac{1}{d}\|\nabla\Phi_\omega\|_{L^2}^2. \tag{4.120}$$

Moreover, we see from (4.116) and (4.118) that

$$\frac{1}{d}\|\nabla\Phi_\omega\|_{L^2}^2 \le \frac{1}{d}\|\nabla Q_\omega\|_{L^2}^2 = \widetilde{d}(\omega). \tag{4.121}$$

Thus, we find from (4.120) and (4.121) that

$$\int_{\{s(\omega)\le|Q_\omega|\}} \big|\nabla|Q_\omega|\big|^2\,dx = 0. \tag{4.122}$$

This implies that the measure of $\{s(\omega) \le |Q_\omega|\}$ is zero, or $\nabla|Q_\omega|(x) = 0$ when $s(\omega) \le |Q_\omega(x)|$. In the former case, we have the desired result (4.115). We consider the latter case. To this end, we define

$$\Psi_\omega(x) := \max\{|Q_\omega| - s(\omega), 0\}. \tag{4.123}$$

Then, we can verify that

$$\nabla\Psi_\omega(x) = \begin{cases} 0 & \text{when } |Q_\omega(x)| \le s(\omega), \\ \nabla|Q_\omega|(x) & \text{when } s(\omega) \le |Q_\omega(x)|. \end{cases} \tag{4.124}$$

Since (4.122) implies that $\nabla|Q_\omega|(x) = 0$ a.e. when $s(\omega) \le |Q_\omega(x)|$, we find from (4.124) that $\nabla\Psi_\omega = 0$ a.e. in $\mathbb{R}^d$. Furthermore, since $\Psi_\omega \in L^2(\mathbb{R}^d)$, we conclude that $\Psi_\omega(x) \equiv 0$. This means that $|Q_\omega(x)| \equiv s(\omega)$ when $\{s(\omega) \le |Q_\omega|\}$ and therefore we obtain the desired result (4.115). □

Now, we give the main result in this subsection.

Theorem 4.4. *Assume* $d \geq 3$, (N1) *through* (N4) *and* (K2)*. Furthermore, assume that* $2F(s)/s^2$ *takes its maximum at only one point, say* s_**, that is,*

$$\left\{s > 0\colon \frac{2F(s)}{s^2} = \omega_*\right\} = \{s_*\}. \tag{4.125}$$

Then, we have the followings:

(i) *Let* $s_{\max}(\omega) := \sup\{s \geq 0\colon H_\omega(s) = \min_{r\geq 0} H_\omega(r)\}$ *and* $s_{\min}(\omega) := \inf\{s \geq 0\colon H_\omega(s) = \min_{r\geq 0} H_\omega(r)\}$.

$$\lim_{\omega\to\omega_*} s_{\max}(\omega) = \lim_{\omega\to\omega_*} s_{\min}(\omega) = s_*. \tag{4.126}$$

(ii) *Let* $\omega \in (0, \omega_*)$*, and let* Φ_ω *be a ground state to the equation* (1.10) *with the following properties:* Φ_ω *is non-negative; radially symmetric about the origin; and strictly decreasing in the radial direction, that is,* $x\cdot\nabla\Phi_\omega(x) < 0$ *for* $x \in \mathbb{R}^d \setminus \{0\}$*. Then, for any compact subset* Ω *of* $\mathbb{R}^d$,

$$\lim_{\omega\to\omega_*} \Phi_\omega = s_* \tag{4.127}$$

uniformly in Ω.

Remark 4.4. (i) The nonlinearity (1.20) satisfies the conditions of Theorem 4.4. (ii) We can prove in a way similar to the proof of Proposition 2.1 in [3] that for any ground state Q_ω to (1.10) there exist $\theta \in \mathbb{R}$, $y \in \mathbb{R}^d$ and a ground state Φ_ω to (1.10) which is positive, radially symmetric about the origin and strictly decreasing in the radial direction, such that $Q_\omega(x) = e^{i\theta}\Phi_\omega(x-y)$ for all $x \in \mathbb{R}^d$.

Proof of Theorem 4.4. Let $\varepsilon > 0$. If $\omega \in (\omega_* - \frac{2\varepsilon}{(s_*+\varepsilon)^2}, \omega_*)$ and $s \leq s_* + \varepsilon$, then

$$H_\omega(s) \geq \frac{1}{2}s^2(\omega - \omega_*) \geq \frac{1}{2}(s_* + \varepsilon)^2(\omega - \omega_*) > -\varepsilon. \tag{4.128}$$

Moreover, it follows from Lemma 4.1 and the assumption (4.125) that there exists $\alpha(\varepsilon) \in (0, \omega_*)$ such that when $|s - s_*| \geq \varepsilon$,

$$2F(s)/s^2 \leq \alpha(\varepsilon). \tag{4.129}$$

Hence, if $\omega \in (\alpha(\varepsilon), \omega_*)$ and $|s - s_*| \geq \varepsilon$, then

$$H_\omega(s) \geq \frac{1}{2}s^2(\omega - \alpha(\varepsilon)) > 0. \tag{4.130}$$

Put $\delta(\varepsilon) := \max\{2\varepsilon/(s_*+\varepsilon)^2, \omega_*-\alpha(\varepsilon), \omega_*-\omega_1\}$, where $\omega_1 := \lim_{s\to\infty} 2F(s)/s^2$ (see (4.112)). Then, it follows from (4.130) and $H_\omega(s_*) < 0$ that if $\omega \in (\omega_* - \delta(\varepsilon), \omega_*)$, then

$$s_* - \varepsilon \leq s_{\min}(\omega) \leq s_{\max}(\omega) \leq s_* + \varepsilon. \tag{4.131}$$

Since we can take an arbitrarily small ε and $\lim_{\varepsilon\to 0} \delta(\varepsilon) = 0$, the first claim follows.

It remains to prove the second claim. We find from (4.128) and (4.130) that for any $\omega \in (\omega_* - \delta(\varepsilon), \omega_*)$,

$$\begin{aligned} 0 &= \mathcal{P}_\omega(\Phi_\omega) \\ &\geq \frac{1}{2^*}\|\nabla \Phi_\omega\|_{L^2}^2 + \int_{\{s_*-\varepsilon \leq \Phi_\omega \leq s_*+\varepsilon\}} H_\omega(\Phi_\omega)\, dx \\ &\geq \frac{1}{2^*}\|\nabla \Phi_\omega\|_{L^2}^2 - \int_{\{s_*-\varepsilon \leq \Phi_\omega \leq s_*+\varepsilon\}} \varepsilon\, dx \\ &\geq \frac{d-2}{2}\widetilde{d}(\omega) - \int_{\{s_*-\varepsilon \leq \Phi_\omega\}} \varepsilon\, dx \\ &= \frac{d-2}{2} d(\omega) - \varepsilon\big|\{s_* - \varepsilon \leq \Phi_\omega\}\big|. \end{aligned} \tag{4.132}$$

Since Φ_ω is strictly decreasing in the radial direction, there exists $R_\omega(\varepsilon) > 0$ such that

$$\{x \in \mathbb{R}^d \colon s_* - \varepsilon \leq \Phi_\omega(x)\} = \{x \in \mathbb{R}^d \colon |x| \leq R_\omega(\varepsilon)\}. \tag{4.133}$$

In particular, $|x| \leq R_\omega(\varepsilon)$ implies

$$-\varepsilon < \Phi_\omega(x) - s_*. \tag{4.134}$$

Furthermore, it follows from (4.132) and (4.133) that

$$R_\omega(\varepsilon) \gtrsim \{\varepsilon^{-1} d(\omega)\}^{\frac{1}{d}}. \tag{4.135}$$

On the other hand, Lemma 4.10 together with (4.131) shows that for any $\omega \in (\omega_* - \delta(\varepsilon), \omega_*)$,

$$\|\Phi_\omega\|_{L^\infty} \leq s_{\max}(\omega) \leq s_* + \varepsilon. \tag{4.136}$$

Putting (4.134) and (4.136) together, we find that when $|x| \leq R_\omega(\varepsilon)$,

$$|\Phi_\omega - s_*| \leq \varepsilon. \tag{4.137}$$

Since ε is arbitrary, the claim (4.127) follows from (4.135) and (4.137). □

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A. Elementary properties of nonlinear terms

We collect well-known facts derived from assumptions (N1) through (N4).

We see from the conditions (N1) and (N2) that

$$|f(z)| \leq C_1(|z|^p + |z|^q) \tag{A.1}$$

for any $z \in \mathbb{C}$. Furthermore, we see from (N3) and (A.1) that

$$|F(z)| \leq 2C_1(|z|^{p+1} + |z|^{q+1}) \tag{A.2}$$

for any $z \in \mathbb{C}$.

Lemma A.1. *Let $d \geq 1$. Assuming the conditions* (N3) *and* (N4)*, we have that for any $z \in \mathbb{C}$,*

$$F(z) = F(|z|). \tag{A.3}$$

Moreover, under the condition (N3)*, the condition* (N4) *is equivalent to that for any $z \in \mathbb{C}$,*

$$\bar{z}f(z) = |z|f(|z|). \tag{A.4}$$

In particular, under the conditions (N3) and (N4), we have that for any $\theta \in \mathbb{R}$ and any non-zero $z \in \mathbb{C}$,

$$f(e^{i\theta}z) = e^{i\theta}f(z). \tag{A.5}$$

Lemma A.2. *Let $d \geq 1$, and assume the conditions* (N3) *and* (N4)*. Furthermore, assume $f(0) = 0$. Then, we can regard f as a real-valued function on $\mathbb{R}$. Moreover, for any $s \in \mathbb{R}$,*

$$\frac{dF}{ds}(s) = f(s), \tag{A.6}$$

$$\frac{d}{ds}\frac{F(s)}{s^2} = \frac{G(s)}{s^3}. \tag{A.7}$$

Assuming (N1) *further, we have that*

$$F(z) = o(|z|^2) \quad \text{as } |z| \to 0. \tag{A.8}$$

B. Tools for compactness

Lemma B.1. (Fröhlich, Lieb and Loss [11]) *Let $d \geq 1$, $1 < p_1 < p_2 < p_3$, and let C_1, C_2, C_3 be constants. Then, there exists constants $C > 0$ and $\eta > 0$ such that for any measurable function u on $\mathbb{R}^d$ satisfying*

$$\|u\|_{L^{p_1}} \leq C_1, \tag{B.1}$$

$$C_2 \leq \|u\|_{L^{p_2}}, \tag{B.2}$$

$$\|u\|_{L^{p_3}} \leq C_3, \tag{B.3}$$

we have

$$|\{x \in \mathbb{R}^d \colon |u(x)| > \eta\}| \geq C. \tag{B.4}$$

Lemma B.2. (Lieb [18]) *Let $\{u_n\}$ be a bounded sequence in $H^1(\mathbb{R}^d)$. Assume that there exists $C > 0$ and $\eta > 0$ such that*

$$|\{x \in \mathbb{R}^d \colon |u_n(x)| > \eta\}| \geq C \tag{B.5}$$

for all n. Then, there exists a subsequence of $\{u_n\}$ (still denoted by the same symbol), a non-trivial function $u_\infty \in H^1(\mathbb{R}^d)$ and a sequence $\{y_n\}$ in $\mathbb{R}^d$ such that

$$\lim_{n\to\infty} u_n(\cdot + y_n) = u_\infty \qquad \text{weakly in } H^1(\mathbb{R}^d). \tag{B.6}$$

Lemma B.3. (Brezis and Lieb [5], Coti Zelati-Rabinowitz [9]) *Let $\{u_n\}$ be a bounded sequence in $H^1(\mathbb{R}^d)$ such that*

$$\lim_{n\to\infty} u_n(x) = u_\infty(x) \qquad \text{almost all } x \in \mathbb{R}^d \tag{B.7}$$

for some function $u_\infty \in H^1(\mathbb{R}^d)$*. Then, for any* $2 \le r \le 2^*$,

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \big||u_n|^r - |u_n - u_\infty|^r - |u_\infty|^r\big|\, dx = 0 \tag{B.8}$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \big||\nabla u_n|^2 - |\nabla\{u_n - u_\infty\}|^2 - |\nabla u_\infty|^2\big|\, dx = 0. \tag{B.9}$$

Furthermore, if we assume (N2) and (N3), then

$$\lim_{n\to\infty} \{\mathcal{P}_\omega(u_n) - \mathcal{P}_\omega(u_n - u_\infty) - \mathcal{P}_\omega(u_\infty)\} = 0 \tag{B.10}$$

for all $\omega > 0$*.*

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