Nonlinear Differ. Equ. Appl. (2017) 24:65

1021-9722/17/060001-22
published online November 11, 2017
DOI 10.1007/s00030-017-0489-z

Nonlinear Differential Equations and Applications NoDEA

Multi-bump bound states for a Schrödinger system via Lyapunov–Schmidt Reduction

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Abstract. Motivated by problems arising in nonlinear optics and Bose–Einstein condensates, we consider in $\mathbb{R}^N$ $(N \leq 3)$ the following $n \times n$ system of coupled Schrödinger equations

$$\begin{cases} -\varepsilon^2 \Delta u_i + V_i(x) u_i = u_i \sum_{\ell=1}^n \beta_{i\ell} u_\ell^2, \\ u_i > 0, \quad \lim_{|x| \to \infty} u_i(x) = 0, \end{cases} \qquad i = 1, \ldots, n,$$

where $\varepsilon > 0$ is a parameter, β_{ij} are constants satisfying $\beta_{ii} > 0$, and V_i are positive potentials that admit some common critical points $a_1, \ldots, a_k$ satisfying certain non-degenerate assumption. Then for any subsets $J \subset \{1, 2, \ldots, k\}$, using a Lyapunov–Schmidt reduction method, we prove the existence of multi-bump bound solutions which as $\varepsilon \to 0$ concentrate on $\cup_{j \in J} a_j$.

Mathematics Subject Classification. 35J60, 35B33.

Keywords. Nonlinear Schrödinger systems, Non-degenerate critical points, Multi-bump bound states, Variational methods.

1. Introduction

In this paper, we consider the following $n \times n$ system of coupled Schrödinger equations in $\mathbb{R}^N$

$$\begin{cases} -\varepsilon^2 \Delta u_i + V_i(x) u_i = u_i \sum_{\ell=1}^n \beta_{i\ell} u_\ell^2, \\ u_i > 0, \quad \lim_{|x| \to \infty} u_i(x) = 0, \end{cases} \qquad i = 1, \ldots, n, \qquad (S_\varepsilon)$$

where $\varepsilon > 0$ is a parameter, $N \leq 3$, β_{ij} are physical constants satisfying

$$\beta_{ii} > 0, \quad \beta_{ij} = \beta_{ji}.$$

This work was partially supported by a grant from the Simons Foundation (No. 210368 to Marcello Lucia). The second author was supported by National Science Foundation of China (No. 11571040).

Problem (S_ε) arises in the Hartree–Fock theory for a double condensate i.e. a binary mixture of Bose–Einstein condensate in two different hyperfine states $|1\rangle$ and $|2\rangle$(see [11]). This system can be described by considering the condensate amplitudes (u_1, u_2), and the β_{ii} and β_{12} are the intraspecies and interspecies scattering lengths. The sign of the scattering length β_{12} determines whether the interactions of states $|1\rangle$ and $|2\rangle$ are repulsive ($\beta_{12} > 0$) or attractive ($\beta_{12} < 0$).

For $n = 1$, the system (S_ε) reduces to a scalar semilinear problem with a subcritical nonlinearity

$$-\varepsilon^2\Delta u + V(x)u = u^p, \quad x \in \mathbb{R}^N, \tag{1.1}$$

which has been extensively investigated under various assumptions. In [12], Floer and Weinstein considered Problem (1.1) in $\mathbb{R}$ with a bounded function V having a non-degenerate critical point. Using a Lyapunov–Schmidt reduction, they established for small $\varepsilon > 0$ the existence of solutions u_ε to (1.1) which concentrate near the given non-degenerate critical point of V as ε tends to 0. Their method and results were later generalized by Oh [19,20] to the higher-dimensional case who also obtained the existence of multi-bump solutions concentrating near several non-degenerate critical points of V as ε tends to 0.

This strategy of finding solutions using a Lyapunov–Schmidt reduction has been applied successfully for bounded potentials by Ambrosetti et al. [1], Del Pino and Felmer [9,10], Cao et al. [6], Ambrosetti et al. [2], Cao and Heinz [7]. In [13] Li and in [14] considered Problem (1.1) on $\mathbb{R}^N$ with a potential allowed to be unbounded. By pushing further the Lyapunov–Schmidt reduction used previously, they proved existence of multi-bound solution for the scalar case.

The system (S_ε) with $n = 2$ with trap potentials (possibly unbounded) satisfying

$$0 < \inf_{x\in\mathbb{R}^N} V_i(x) < \liminf_{|x|\to\infty} V_i(x), \quad i = 1, 2$$

has been considered by Lin and Wei [17]. In a range of the parameter β_{12}

$$-\infty < \beta_{12} < \beta_0,$$

where $\beta_0 \in (0, \sqrt{\beta_{11}\beta_{22}})$ is a constant depending only on N, they proved existence of a least energy solution and study also its asymptotic behavior as $\varepsilon \to 0$. They proved how trap potentials and the interspecies scattering length affect the locations of spikes. They used the Nehari's manifold to construct least energy solutions and derive their asymptotic behaviors by some techniques of singular perturbation problems.

Several results have also been obtained for the system (S_ε) with constant potentials $V_i = \lambda_i$. Firstly, almost the same condition on β, Lin and Wei [15] obtained the least energy solution by minimizing the certain Nehari manifold. For $\lambda_1 = 1$, Sirakov [24] discussed the problem for all $\beta_{12} \in \mathbb{R}$ and analyzed for which β_{12} Problem (S_ε) admits a least energy solution. We also refer to Lin and Wei [16], Bartsch et al. [4] for the existence of bound states of Schrödinger systems.

For more recent work, we first want to refer the readers to work by Long and Peng [18], where segregated vector solutions were obtained for a class of Bose–Einstein systems related to (S_ε). One also can refer to the work by Peng et al. [22], where the authors considered the existence of multiple solutions for linearly coupled nonlinear elliptic systems with critical exponent. For more results related to elliptic systems, one can refer the work by Peng et al. [23].

In the present paper, we aim to prove existence of multi-bump bound states to system (S_ε) for potentials $V_1, \dots, V_n$ that admit some common critical points which satisfy certain non-degenerate assumption, but without assuming any behavior at infinity. More precisely, we will work under the following assumptions:

($\mathbf{A_1}$) $V_i \in C^2(\mathbb{R}^N, \mathbb{R})$ and $\inf_{x\in\mathbb{R}^N} V_i(x) > 0$;

($\mathbf{A_2}$) There exists a finite set $\mathcal{A} \subset \mathbb{R}^N$ for which

$$\nabla V_1(a) = \cdots = \nabla V_n(a) = 0, \quad \forall a \in \mathcal{A},$$

and for each $a \in \mathcal{A}$, we can find a non-degenerate solution $\mathbf{U}_a(x) = (U_{a,1}(x), \dots, U_{a,n}(x))$ in $[H^1(\mathbb{R}^N)]^n$ to the following $n \times n$ system in $\mathbb{R}^N$

$$\begin{cases} -\Delta u_i + V_i(a)u_i = u_i \sum_{\ell=1}^n \beta_{i\ell} u_\ell^2, \\ u_i > 0, \quad u_i(0) = \max_{x\in\mathbb{R}^N} u_i(x), \\ \lim_{|x|\to\infty} u_i(x) = 0; \end{cases} \tag{1.2}$$

($\mathbf{A_3}$) For each $a \in \mathcal{A}$, the matrix $\sum_{\ell=1}^n \left(\|U_{a,\ell}\|_{L^2}^2 D^2 V_\ell(a)\right)$ is invertible.

Remark 1.1. We say that a solution $\mathbf{U} := (U_1, \dots, U_n)$ of (1.2) is non-degenerate, if the set of solutions $\mathbf{f} \in [H^2(\mathbb{R}^N)]^n$ to the following $n \times n$ linear system:

$$\Delta f_i - V_i(a) f_i + \left(3\beta_{ii} U_i^2 + \sum_{\ell\neq i} \beta_{i\ell} u_\ell^2\right) f_i + 2U_i \sum_{\ell\neq i} \beta_{i\ell} U_\ell f_\ell = 0, \tag{1.3}$$

are given by $\mathbf{f} \in span\left\{\frac{\partial \mathbf{U}_a}{\partial y_1}, \dots, \frac{\partial \mathbf{U}_a}{\partial y_N}\right\}$.

Remark 1.2. **(i)** When (1.2) is a 2×2 system, it is known by a result of Dancer and Wei [8] that Problem (1.2) has a non-degenerate solution for all $\beta \in (0, +\infty)\backslash(I^a \cup E^a)$ for some interval I^a and countable set E^a that depend on the values of $(V_1(a), V_2(a))$.
More recently, for constant potentials $V_i \equiv 1$ ($i = 1, 2$), Peng and Wang [21] have proved that Problem (1.2) has a non-degenerate solution for all $\beta_{12} \in (0, +\infty)\backslash I$, where $I = [a, b]$ is an interval with $a = \min\{\beta_{11}, \beta_{22}\}$ and $b = \max\{\beta_{11}, \beta_{22}\}$. Furthermore, one can check that system (1.2) admits no positive solution when $\beta_{12} \in (a, b)$.

(ii) Assume $D^2V_i(a)$ is positive definite (resp. negative definite) for some $i \in \{1, \dots, n\}$, and $D^2V_j(a)$ is semi-positive definite (resp. semi-negative definite) for $j \neq i$. Then in this case the condition $(\mathbf{A_3})$ is satisfied since we easily verify

$$\sum_{i=1}^{n} t_i D^2 V_i(a) \text{ is positive definite}, \quad \forall t_i > 0 \text{ (resp. negative definite)}.$$

Under these assumption, we will prove that for ε small enough, the system (S_ε) admits solutions $\mathbf{u}_\varepsilon$ that are small perturbation of $\sum_{a\in\mathcal{A}} \mathbf{U}_a\left(\frac{x-a}{\varepsilon}\right)$. More specifically, given $\eta > 0$, we introduce a cut-off function $\mathcal{X}_\eta : \mathbb{R}^n \to [0,1]$ that satisfies

$$\mathcal{X}_\eta \in C^\infty(\mathbb{R}), \quad \mathcal{X}_\eta(x) = \begin{cases} 1 & \text{for } x \in B_\eta(0), \\ 0 & \text{for } x \in \mathbb{R}^N \backslash B_{2\eta}(0), \end{cases} \quad |\nabla \mathcal{X}_\eta(x)| \leq \frac{C}{\eta}, \tag{1.4}$$

with $\eta := \eta(\varepsilon) > 0$ chosen in such a way to ensure the supports of the functions $\mathcal{X}_\eta(x)\mathbf{U}_a\left(\frac{x-a}{\varepsilon}\right)$ to be disjoint.

Using a Lyapunov–Schmidt reduction method, we will be able to use the family of compactly supported function $(\chi_\eta \mathbf{U}_a)_{a\in\mathcal{A}}$ to construct solutions to Problem (S_ε) that concentrate at the points $a \in \mathcal{A}$ as $\varepsilon \to 0$. Our main result reads more precisely as follows.

Theorem 1.3. *Suppose that assumptions* $(\mathbf{A_1})$ *to* $(\mathbf{A_3})$ *hold. Then, there exists* $\varepsilon_0 > 0$ *such that for each* $0 < \varepsilon < \varepsilon_0$ *the system* (S_ε) *admits a solution of the form*

$$\mathbf{u}_\varepsilon(x) = \sum_{a\in\mathcal{A}} \alpha_a(\chi_\eta \mathbf{U}_a)\left(\frac{x-a}{\varepsilon} + P_a\right) + \mathbf{w}_\varepsilon\left(\frac{x}{\varepsilon}\right)$$

with

$$|\alpha - 1|^2 + \sum_{\ell=1}^{n} \int_{\mathbb{R}^N} \{|\nabla w_\ell|^2 + V_\ell(\varepsilon x) w_\ell^2\} = O(\varepsilon^4) \quad \textit{and} \quad |P_a| = O(\varepsilon). \tag{1.5}$$

Since the assumptions (A_1)–(A_3) also hold with each non-empty subset $\mathcal{A}_0 \subset \mathcal{A}$, Theorem 1.3 can be applied with $\mathcal{A}_0$ and provides the following multiplicity result.

Corollary 1.4. *Suppose that* (A_1) *to* (A_3) *hold. Then* (S_ε) *has at least* $2^{|\mathcal{A}|} - 1$ *solutions.*

Henceforth, the same C will stand for a various positive constant, and we will use the asymptotic notation $f = O(t)$ to denote a quantity such that $|\frac{f}{t}| \leq C$.

2. Functional framework and decomposition lemma

By considering the rescaled function $\mathbf{u}(\varepsilon x)$, one easily check that Problem (S_ε) is equivalent to the following $n \times n$ system

$$\begin{cases} -\Delta u_i + V_i(\varepsilon x) u_i = u_i \sum_{\ell=1}^{n} \beta_{i\ell} u_\ell^2, \\ u_i > 0, \quad \lim_{|x|\to\infty} u_i(x) = 0, \end{cases} \quad i = 1, \ldots, n. \tag{2.1}$$

This nonlinear problem will be handled by working in the functional space

$$E_\varepsilon := \left\{ \mathbf{u} \in [H^1(\mathbb{R}^N)]^n \,:\, \int_{\mathbb{R}^N} V_i(\varepsilon x) u_i^2 < \infty \quad \forall i \in \{1, \ldots, n\} \right\},$$

endowed with the inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle_\varepsilon = \sum_{\ell=1}^{n} \int_{\mathbb{R}^N} \{ \nabla u_\ell \nabla v_\ell + V_\ell(\varepsilon x) u_\ell v_\ell \}.$$

The associated norm will be denoted $\| \cdot \|_\varepsilon$, and we easily verify that $(E_\varepsilon, \langle \cdot, \cdot \rangle_\varepsilon)$ is a Hilbert space that embeds continuously in $H^1(\mathbb{R}^n)$. In this space, by setting $G(\mathbf{u}) := \frac{1}{4} \int_{\mathbb{R}^N} \sum_{i,j} \beta_{ij} u_i^2 u_j^2$, Problem (2.1) is the Euler-Lagrange equation of the following action functional:

$$J_\varepsilon(\mathbf{u}) := \frac{1}{2} \sum_{\ell=1}^{n} \int_{\mathbb{R}^N} \{ |\nabla u_\ell|^2 + V_\ell(\varepsilon x) u_\ell^2 \} - \int_{\mathbb{R}^N} G(\mathbf{u}), \quad \mathbf{u} \in E_\varepsilon. \tag{2.2}$$

For any $f : \mathbb{R}^N \to \mathbb{R}$ and $\tau \in \mathbb{R}^N$, the translated function $x \mapsto f(x + \tau)$ will be denoted as $f^{[\tau]}$. Using this notation, we introduce the following family of functions depending on $\varepsilon > 0$, $a \in \mathcal{A}$, $P \in \mathbb{R}^N$

$$\boldsymbol{\sigma}_{\varepsilon,a,P} := (\mathcal{X}_\eta \mathbf{U}_a)^{[-\tau]}, \quad \tau := \frac{a}{\varepsilon} + P, \tag{2.3}$$

where $\mathbf{U}_a$ is defined in (A_2), and χ_η has been given in (1.4). The reason of using a cut-off function χ_η is due to the fact that $\mathbf{U}_a$ may fail to be in E_ε (since we do not make any assumptions on the potential V_i at infinity).

Setting $\alpha := \min \left\{ \frac{\|a - a'\|}{4} \,:\, a, a' \in \mathcal{A},\, a \neq a' \right\}$, for each $\varepsilon > 0$ we consider $\eta := \eta(\varepsilon)$ and fix $\delta > 0$ satisfying

$$\varepsilon \eta < \alpha \quad \text{and} \quad \delta < \alpha. \tag{2.4}$$

Under condition (2.4), the supports of the functions $\boldsymbol{\sigma}_{\varepsilon,a,P}$ are pairwise disjoint, and we consider the collection of linear combination of such functions:

$$\Sigma_{\varepsilon,\delta} := \left\{ \sum_{a \in \mathcal{A}} \alpha_a \, \boldsymbol{\sigma}_{\varepsilon,a,P_a} \,:\, |\alpha_a - 1| < \delta,\, P_a \in B_\delta(0) \right\}.$$

This set is parametrized with the variables$(\alpha_a, P_a)_{a \in \mathcal{A}}$, and defines a $|\mathcal{A}|\,(N+1)$-dimensional manifold in E_ε. We consider a δ-tubular neighborhood of $\Sigma_{\varepsilon,\delta}$

$$W_{\delta,\varepsilon} := \{ \mathbf{u} \in E_\varepsilon \,:\, \|\mathbf{u} - \boldsymbol{\sigma}\|_\varepsilon < \delta \text{ for some } \boldsymbol{\sigma} \in \Sigma_{\varepsilon,\delta} \}.$$

In order to give (for $\delta > 0$ small enough) a good parametrization of $W_{\delta,\varepsilon}$, for each $\varepsilon > 0$, $a \in \mathcal{A}$ and $P \in \mathbb{R}^N$ we define

$$F_{\varepsilon,a,P} = \left\{ \mathbf{f} \in E_\varepsilon : \begin{array}{c} \langle \mathbf{f}, \boldsymbol{\sigma}_{\varepsilon,a,P} \rangle_\varepsilon = 0, \\ \left\langle \mathbf{f}, \dfrac{\partial \boldsymbol{\sigma}_{\varepsilon,a,P}}{\partial P_j} \right\rangle_\varepsilon = 0, \; j = 1, \ldots, N \end{array} \right\}. \tag{2.5}$$

Following the arguments by Bahri and Coron [3] (see also Cao et al. [6]), we have the following decomposition lemma.

Lemma 2.1. *There exist* $\delta_0, \varepsilon_0 > 0$ *such that for* $0 < \delta < \delta_0$, $0 < \varepsilon < \varepsilon_0$ *and* $\mathbf{u} \in W_{\delta,\varepsilon}$, *the following minimization problem*

$$\inf \{ \|\mathbf{u} - \boldsymbol{\sigma}\|_\varepsilon \,:\, \boldsymbol{\sigma} \in \Sigma_{4\delta} \}$$

has a unique solution which must be in $\Sigma_{2\delta}$. *Hence, for each* $\mathbf{u} \in W_{\delta,\varepsilon}$, *there are unique*

$$(P_a)_{a\in\mathcal{A}} \in \prod_{a\in\mathcal{A}} B_\delta(0), \quad (\alpha_a)_{a\in\mathcal{A}} \in \prod_{a\in\mathcal{A}} (1-\delta, 1+\delta) \quad \mathbf{w} \in \bigcap_{a\in\mathcal{A}} F_{\varepsilon,a,P_a}$$

such that

$$\mathbf{u} = \sum_{a\in\mathcal{A}} \alpha_a \boldsymbol{\sigma}_{\varepsilon,a,P_a} + \mathbf{w}. \tag{2.6}$$

3. Invertibility of the linearized problem

For any $\mathbf{u}$ belonging to the δ-tubular neighborhood $W_{\varepsilon,\delta}$, formula (2.6) allows to write $\mathbf{u}$ uniquely with the parameters $(P_a, \alpha_a)_{a\in\mathcal{A}}$. By setting $\mathbf{P} = (P_a)_{a\in\mathcal{A}}$, and $\boldsymbol{\alpha} := (\alpha_a)_{a\in\mathcal{A}}$, we can rewrite the functional (2.2) with these new parameters

$$I_\varepsilon(\mathbf{P}, \boldsymbol{\alpha}, \mathbf{w}) := J_\varepsilon\left(\sum_{a\in\mathcal{A}} \alpha_a \boldsymbol{\sigma}_{\varepsilon,a,P_a} + \mathbf{w}\right),$$

with $(\mathbf{P}, \boldsymbol{\alpha}, \mathbf{w}) \in \prod_{a\in\mathcal{A}} B_\delta(0) \times \prod_{a\in\mathcal{A}} (1-\delta, 1+\delta) \times \bigcap_{a\in\mathcal{A}} F_{\varepsilon,a,P_a}$.

Taking into account the orthogonality conditions (2.5) and the fact that the supports of the functions $\boldsymbol{\sigma}_{\varepsilon,a,P_a}$ are disjoint (by (2.4)), we have

$$I_\varepsilon(\mathbf{P}, \boldsymbol{\alpha}, \mathbf{w}) = \frac{1}{2}\left\{\sum_{a\in\mathcal{A}} \alpha_a^2 \left\|\boldsymbol{\sigma}_{\varepsilon,a,P_a}\right\|_\varepsilon^2 + \|\mathbf{w}\|_\varepsilon^2\right\} - \int_{\mathbb{R}^N} G\left(\sum_{a\in\mathcal{A}} \alpha_a \boldsymbol{\sigma}_{\varepsilon,a,P_a} + \mathbf{w}\right). \tag{3.1}$$

It is then easy to check that $(\mathbf{P}, \boldsymbol{\alpha}, \mathbf{w})$ is a critical point of I_ε if and only if $\sum_{a\in\mathcal{A}} \alpha_a \boldsymbol{\sigma}_{\varepsilon,a,P_a} + \mathbf{w}$ is a critical point of J_ε (for the details about this argument, one can see the paper by Cao et al. [6]). Hence, we are reduced to solve the following three equations

$$\begin{cases} \frac{\partial I_\varepsilon}{\partial \boldsymbol{\alpha}}(\mathbf{P}, \boldsymbol{\alpha}, \mathbf{w}) = 0 & \text{in } \prod_{a\in\mathcal{A}} \mathbb{R}, \\ \frac{\partial I_\varepsilon}{\partial \mathbf{w}}(\mathbf{P}, \boldsymbol{\alpha}, \mathbf{w}) = 0 & \text{in } \mathcal{F}_{\varepsilon,\mathbf{P}}^*, \\ \frac{\partial I_\varepsilon}{\partial \mathbf{P}}(\mathbf{P}, \boldsymbol{\alpha}, \mathbf{w}) = 0 & \text{in } \prod_{a\in\mathcal{A}} \mathbb{R}^N \end{cases} \tag{3.2}$$

where $\mathcal{F}_{\varepsilon,\mathbf{P}} := \bigcap_{a\in\mathcal{A}} F_{\varepsilon,a,P_a}$ and $\mathcal{F}_{\varepsilon,\mathbf{P}}^*$ stands for the dual space of $\mathcal{F}_{\varepsilon,\mathbf{P}}$.

To find a solution, we will first apply an implicit function Theorem (used previously by Li [13], and Li–Nirenberg [14]) to show that for each $\mathbf{P}$ there exists $(\boldsymbol{\alpha}(\mathbf{P}), \mathbf{w}(\mathbf{P}))$ that solves the two first equation in (3.2):

$$\frac{\partial I_\varepsilon}{\partial \boldsymbol{\alpha}}(\mathbf{P}, \boldsymbol{\alpha}(\mathbf{P}), \mathbf{w}(\mathbf{P})) = 0 \qquad \frac{\partial I_\varepsilon}{\partial \mathbf{w}}(\mathbf{P}, \boldsymbol{\alpha}(\mathbf{P}), \mathbf{w}(\mathbf{P})) = 0. \tag{3.3}$$

Hence, we will be reduced to solve the finite dimensional problem

$$\frac{\partial}{\partial \mathbf{P}} \left\{ I_\varepsilon \left(\mathbf{P}, \boldsymbol{\alpha}(\mathbf{P}), \mathbf{w}(\mathbf{P}) \right) \right\} = 0, \tag{3.4}$$

which will be done by applying a topological degree argument. To reach this goal, we first fix $\mathbf{P}$ and look at the map S defined by

$$S : \prod_{a\in\mathcal{A}} \mathbb{R} \times \mathcal{F}_{\varepsilon,\mathbf{P}} \to \prod_{a\in\mathcal{A}} \mathbb{R} \times \mathcal{F}^*_{\varepsilon,\mathbf{P}} \quad (\boldsymbol{\alpha}, \mathbf{w}) \mapsto \left(\frac{\partial I_\varepsilon}{\partial \boldsymbol{\alpha}}, \frac{\partial I_\varepsilon}{\partial \mathbf{w}} \right), \tag{3.5}$$

and therefore solving (3.3) is equivalent to solve $S(\boldsymbol{\alpha}, \mathbf{w}) = 0$. The two components of the map S can be computed explicitly. The partial derivatives of I_ε with respect to each α_a at a point $\mathbf{q} = (\mathbf{P}, \boldsymbol{\alpha}, \mathbf{w})$:

$$\begin{aligned}\frac{\partial I_\varepsilon}{\partial \alpha_a}(\mathbf{q}) &= \alpha_a \|\boldsymbol{\sigma}_{\varepsilon,a,P_a}\|_\varepsilon^2 - \int_{\mathbb{R}^N} \left\langle \nabla G \left(\sum_{a\in\mathcal{A}} \alpha_a \boldsymbol{\sigma}_{\varepsilon,a,P_a} + \mathbf{w} \right), \boldsymbol{\sigma}_{\varepsilon,a,P_a} \right\rangle \\ &= \alpha_a \|\boldsymbol{\sigma}_{\varepsilon,a,P_a}\|_\varepsilon^2 - \int_{\mathbb{R}^N} \left\langle \nabla G \left(\alpha_a \boldsymbol{\sigma}_{\varepsilon,a,P_a} + \mathbf{w} \right), \boldsymbol{\sigma}_{\varepsilon,a,P_a} \right\rangle \end{aligned} \tag{3.6}$$

and the derivative with respect to the variable $\mathbf{w}$ is given by the linear form

$$\frac{\partial I_\varepsilon}{\partial \mathbf{w}}(\mathbf{q}) : F_{\varepsilon,a,P} \to \mathbb{R}, \quad \boldsymbol{\varphi} \mapsto \langle \mathbf{w}, \boldsymbol{\varphi} \rangle_\varepsilon - \int_{\mathbb{R}^N} \left\langle \nabla G \left(\sum_{a\in\mathcal{A}} \alpha_a \boldsymbol{\sigma}_{\varepsilon,a,P_a} + \mathbf{w} \right), \boldsymbol{\varphi} \right\rangle. \tag{3.7}$$

In the sequel, we will also need to compute the derivative of S. At each point $\mathbf{q} = (\mathbf{P}, \boldsymbol{\alpha}, \mathbf{w})$, the derivative $DS_{(\mathbf{q})}$ is a linear map

$$DS_{(\mathbf{q})} : \prod_{a\in\mathcal{A}} \mathbb{R} \times \mathcal{F}_{\varepsilon,\mathbf{P}} \to \prod_{a\in\mathcal{A}} \mathbb{R} \times (\mathcal{F}_{\varepsilon,\mathbf{P}})^\star$$

which can be represented as

$$DS_{(\mathbf{q})} := \begin{pmatrix} \dfrac{\partial^2 I_\varepsilon}{\partial \boldsymbol{\alpha}^2}(\mathbf{q}) & \dfrac{\partial^2 I_\varepsilon}{\partial \boldsymbol{\alpha}\, \partial \mathbf{w}}(\mathbf{q}) \\ \dfrac{\partial^2 I_\varepsilon}{\partial \boldsymbol{\alpha}\, \partial \mathbf{w}}(\mathbf{q}) & \dfrac{\partial^2 I_\varepsilon}{\partial^2 \mathbf{w}}(\mathbf{q}) \end{pmatrix}. \tag{3.8}$$

Each entry of (3.8) can easily be computed, and the final expression can be simplified since the family of functions $\boldsymbol{\sigma}_{\varepsilon,a,P}$ have compact support, with supports that are pairwise disjoint. Firstly,

$$\frac{\partial^2 I_\varepsilon}{\partial \alpha_a^2}(\mathbf{q}) = \|\boldsymbol{\sigma}_{\varepsilon,a,P_a}\|_\varepsilon^2 - \int_{\mathbb{R}^N} \boldsymbol{\sigma}^t_{\varepsilon,a,P_a} D^2 G \left(\alpha_a \boldsymbol{\sigma}_{\varepsilon,a,P_a} + \mathbf{w} \right) \boldsymbol{\sigma}_{\varepsilon,a,P_a}, \tag{3.9}$$

and for $a, a' \in \mathcal{A}$ with $a \neq a'$

$$\frac{\partial^2 I_\varepsilon}{\partial \alpha_a\, \partial \alpha_{a'}}(\mathbf{q}) = - \int_{\mathbb{R}^N} \boldsymbol{\sigma}^t_{\varepsilon,a,P_a} D^2 G \left(\alpha_a \boldsymbol{\sigma}_{\varepsilon,a,P_a} + \mathbf{w} \right) \boldsymbol{\sigma}_{\varepsilon,a',P_{a'}} = 0. \tag{3.10}$$

We easily see that the mixed derivative can be identified with the linear map

$$\frac{\partial^2 I_\varepsilon}{\partial \alpha_a \partial \mathbf{w}}(\mathbf{q}) : F_{\varepsilon,a,P} \to \mathbb{R}, \qquad \boldsymbol{\varphi} \mapsto - \int_{\mathbb{R}^N} \boldsymbol{\sigma}^t_{\varepsilon,a,P_a} D^2 G \left(\alpha_a \boldsymbol{\sigma}_{\varepsilon,a,P_a} + \mathbf{w} \right) \boldsymbol{\varphi}, \tag{3.11}$$

and the derivative with respect to the variable $\mathbf{w}$ is given by the bilinear form

$$\frac{\partial^2 I_\varepsilon}{\partial \mathbf{w}^2}(\mathbf{q}) : F_{\varepsilon,a,P} \times F_{\varepsilon,a,P} \longrightarrow \mathbb{R},$$
$$(\boldsymbol{\varphi}, \boldsymbol{\eta}) \longmapsto \langle \boldsymbol{\varphi}, \boldsymbol{\eta} \rangle_\varepsilon - \int_{\mathbb{R}^N} \boldsymbol{\varphi}^t D^2 G\left(\sum_{a\in\mathcal{A}} \alpha_a \boldsymbol{\sigma}_{\varepsilon,a,P_a} + \mathbf{w}\right) \boldsymbol{\eta}. \tag{3.12}$$

We will now study the coercivity of this bilinear form and the invertibility of the linear operator $L_{\varepsilon,P} : F_{\varepsilon,a,P} \to F_{\varepsilon,a,P}$ associated to this bilinear form defined by

$$\langle L_{\varepsilon,P}(\mathbf{u}), \boldsymbol{\eta} \rangle_\varepsilon = \frac{\partial^2 I_\varepsilon}{\partial \mathbf{w}^2}(\mathbf{q}, \mathbf{u}, \boldsymbol{\eta}).$$

Proposition 3.1. *There exist constants* $0 < \delta_1 \leq \delta_0, 0 < \varepsilon_1 \leq \varepsilon_0$ *and* $C_0 > 0$ *(independent of* ε, δ, $\mathbf{P}$*) such that when* $0 < \varepsilon < \varepsilon_1, 0 < \delta < \delta_1$ *and* $\mathbf{P} \in \mathcal{B}_\delta$*, we have*

$$\|L_{\varepsilon,\mathbf{P}}(\mathbf{w})\|_\varepsilon \geq \frac{1}{C_0}\|\mathbf{w}\|_\varepsilon, \quad \forall \mathbf{w} \in \bigcap_{a\in\mathcal{A}} F_{\varepsilon,a,P_a}. \tag{3.13}$$

As a consequence, $L_{\varepsilon,\mathbf{P}} : \mathcal{F}_{\varepsilon,\mathbf{P}} \to \mathcal{F}_{\varepsilon,\mathbf{P}}$ *is invertible and* $\|L^{-1}_{\varepsilon,\mathbf{P}}\|_\varepsilon \leq C_0$.

Before presenting the proof of Proposition 3.1, we firstly give some preliminary notations. Given $a \in \mathcal{A}$, we define on $[H^1(\mathbb{R}^N)]^n$ an equivalent inner product $\langle \cdot, \cdot \rangle_a$ as follows

$$\langle \mathbf{f}, \mathbf{g} \rangle_a := \sum_{\ell=1}^n \int_{\mathbb{R}^N} \left\{ \nabla f_\ell \nabla g_\ell + V(a) f_\ell g_\ell \right\}. \tag{3.14}$$

The space $[H^1(\mathbb{R}^N)]^n$ endowed with this inner product will be denoted $\mathbf{H}_a$, and the corresponding norm will be denoted $\|\cdot\|_a$. Given a non-degenerate solution $\mathbf{U}_a$ of (1.2), we are naturally led to consider the analogue of F_{ε,a,P_a} in E_ε:

$$F_a := \left\{ \boldsymbol{\Phi} \in \mathbf{H}_a : \begin{array}{c} \langle \boldsymbol{\Phi}, \mathbf{U}_a \rangle_a = 0, \\ \left\langle \boldsymbol{\Phi}, \dfrac{\partial \mathbf{U}_a}{\partial x_j} \right\rangle_a = 0, \quad j = 1, \ldots, N \end{array} \right\}, \tag{3.15}$$

In the Hilbert space $\mathbf{H}_a$, the linearization at $\mathbf{U}_a$ of the Problem (1.2) is described by the following bilinear form

$$(\boldsymbol{\psi}, \boldsymbol{\phi}) \mapsto \langle \boldsymbol{\psi}, \boldsymbol{\phi} \rangle_a + \int_{\mathbb{R}^N} \boldsymbol{\psi}^t D^2 G(\mathbf{U}_a) \boldsymbol{\phi},$$

to which we can associate the linear operator $L_a : \mathbf{H}_a \to \mathbf{H}_a$ defined as

$$\langle L_a(\mathbf{u}), \boldsymbol{\phi} \rangle_a = \langle \mathbf{u}, \boldsymbol{\phi} \rangle_a + \int_{\mathbb{R}^N} \mathbf{u}^t D^2 G(\mathbf{U}_a) \boldsymbol{\phi}.$$

Remark 3.2. By the definition of L_a and the fact that $\mathbf{U}_a$ is a solution of (1.2) and $\frac{\partial \mathbf{U}_a}{\partial x_j}$ is such that $L_a(\frac{\partial \mathbf{U}_a}{\partial x_j}) = 0$, we have that for any $\boldsymbol{\phi} \in F_a$, $L_a(\boldsymbol{\phi}) \in F_a$. Hence, the restriction L_a to $F_\mathbf{a}$ defines an operator $F_a \to F_a$.

Lemma 3.3. *Let $a \in \mathcal{A}$ and U_a be a non-degenerate solution to (1.2). There exists a constant $C > 0$ such that*

$$\|L_a(\mathbf{w})\|_a \geq C\|\mathbf{w}\|_a \quad \forall \mathbf{w} \in F_a.$$

Proof. Arguing by contradiction, assume there exists a sequence $(\mathbf{w}_n)$ in F_a such that

$$\|\mathbf{w}_n\|_a = 1 \quad \text{and} \quad \|L_a(\mathbf{w}_n)\|_a \to 0.$$

Hence, $\mathbf{w}_n$ weakly to $\mathbf{w}$ in $\mathbf{H}_a$ and also $o(1) = \langle L_a(\mathbf{w}_n), \boldsymbol{\psi}\rangle_a \to \langle L_a(\mathbf{w}), \boldsymbol{\psi}\rangle_a$ for all $\boldsymbol{\psi} \in \mathbf{H}_a$. We conclude that $L_a(\mathbf{w}) = 0$. Since $\mathbf{w} \in F_a$, the non-degeneracy condition implies $\mathbf{w} = 0$. Therefore,

$$o(1) = \langle L_a(\mathbf{w}_n), \mathbf{w}_n\rangle_a = \langle \mathbf{w}_n, \mathbf{w}_n\rangle_a - \int_{\mathbb{R}^N} \mathbf{w}_n^t D^2F(\mathbf{U_a})\mathbf{w}_n. \tag{3.16}$$

Since $\mathbf{U}_a \in L^2(\mathbb{R}^N)$ and the sequence $w_{i,n}w_{j,n}$ converges weakly in L^2 (for $i, j \in \{1, \dots, n\}$), the last term in (3.16) converges to zero. Thus, we deduce the strong convergence $\|\mathbf{w}_n\|_a \to 0$ in contradiction with the assumption $\|\mathbf{w}_n\|_a = 1$. □

We now collect several estimates for elements in $F_{\varepsilon,a,P}$.

Lemma 3.4. *Assume $(\mathbf{A_1})$, $(\mathbf{A_2})$ hold, and let $\mathbf{U}_a$ be a non-degenerate solution of (1.2) with $a \in \mathcal{A}$. Then, by setting $\tau := \frac{a}{\varepsilon} + P$ there exists a constant $C_0 > 0$ such that*

$$\|L(\chi_\eta \mathbf{w}^{[\tau]})\|_a \geq C_0\|\chi_\eta \mathbf{w}^{[\tau]}\|_a + O\left(\varepsilon^2 + \|\mathbf{U}_a\|_{H^1(B_\eta^c)}\right)\|\mathbf{w}\|_a, \tag{3.17}$$

(remember that $\mathbf{w}^{[\tau]}$ stands for the function $\mathbf{w}(\cdot + \tau)$), which can be equivalently written as

$$(1 - C_0)\|\chi_\eta \mathbf{w}^{[\tau]}\|_a^2 \geq \int_{\mathbb{R}^N} \left(\chi_\eta \mathbf{w}^{[\tau]}\right)^t D^2G(\mathbf{U}_a)\left(\chi_\eta \mathbf{w}^{[\tau]}\right) + O\left(\varepsilon^4 + \|\mathbf{U}_a\|_{H^1(B_\eta^c)}^2\right)\|\mathbf{w}\|_a^2. \tag{3.18}$$

Proof. Consider then in $\mathbf{H}_a$ the projection $\boldsymbol{\varphi}_a$ of $\chi_\eta \mathbf{w}^{[\tau]}$ on the space spanned by $\mathbf{U}_a$, $\frac{\partial \mathbf{U}_a}{\partial x_j}$ $(j = 1, \dots, N)$. Using Lemma A.2 of the "Appendix" we deduce that

$$\left\|\chi_\eta \mathbf{w}^{[\tau]} - \boldsymbol{\varphi}_a\right\|_a^2 = \left\|\chi_\eta \mathbf{w}^{[\tau]}\right\|_a^2 + O\left(\varepsilon^4 + \|\mathbf{U}_a\|_{H^1(B_\eta^c)}^2\right)\|\mathbf{w}\|_a^2 \tag{3.19}$$

and also

$$\int_{\mathbb{R}^N} \left(\chi_\eta \mathbf{w}^{[\tau]} - \boldsymbol{\varphi}_a\right)^t D^2G(\mathbf{U}_a)\left(\chi_\eta \mathbf{w}^{[\tau]} - \boldsymbol{\varphi}_a\right) = \int_{\mathbb{R}^N} \left(\chi_\eta \mathbf{w}^{[\tau]}\right)^t D^2G(\mathbf{U}_a)\left(\chi_\eta \mathbf{w}^{[\tau]}\right) + O\left(\varepsilon^4 + \|\mathbf{U}_a\|_{H^1(B_\eta^c)}^2\right)\|\mathbf{w}\|_a^2. \tag{3.20}$$

The inequality (3.17) follows by applying Lemma 3.3 with the function $\mathbf{w} = \chi_\eta \mathbf{w}^{[\tau]} - \boldsymbol{\varphi}_a$ combined with the estimates (3.19) and (3.20). □

Now we give the proof of Proposition 3.1.

Proof. Let us set $\tau_a := \frac{a}{\varepsilon} + P$. Using that the supports of the functions $\boldsymbol{\sigma}_{\varepsilon,a,P_a}$ are disjoint with $G(0) = 0$, together with the homogeneity of G (of degree four), and applying (3.18) we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} \mathbf{w}^t G\left(\sum_{a\in\mathcal{A}} \boldsymbol{\sigma}_{\varepsilon,a,P_a}\right)\mathbf{w} &= \sum_{a\in\mathcal{A}} \int_{\mathbb{R}^N} \mathbf{w}^t D^2 G\left((\chi_\eta \mathbf{U}_a)^{[-\tau_a]}\right)\mathbf{w} \\ &= \sum_{a\in\mathcal{A}} \int_{\mathbb{R}^N} \left(\chi_\eta \mathbf{w}^{[\tau_a]}\right)^t D^2 G(\mathbf{U}_a)\left(\chi_\eta \mathbf{w}^{[\tau_a]}\right) \\ &\le (1 - C_0) \sum_{a\in\mathcal{A}} \|\chi_\eta \mathbf{w}^{[\tau_a]}\|_a^2. \qquad (3.21)\end{aligned}$$

We estimate the right hand-side of (3.21) as follows:

$$\int_{B_{2\eta}(\tau_a)} \left|\nabla(\chi_\eta^{[\tau_a]} w_\ell)\right|^2 \le \left(1 + \frac{1}{\eta}\right)\int_{B_{2\eta}(\tau_a)} \left|\nabla w_\ell\right|^2 + \frac{1}{\eta}\left(1 + \frac{1}{\eta}\right)\int_{B_{2\eta}(\tau_a)} w_\ell^2 \qquad (3.22)$$

and

$$\begin{aligned}\int_{B_{2\eta}(\tau_a)} V_\ell(a)(\chi_\eta^{[\tau_a]} w_\ell)^2 &= O\left(\varepsilon^2 + \|\mathbf{U}_a\|^2_{H^1(B^c_\eta)}\right)\int_{B_{2\eta}(\tau_a)} w_\ell^2 \\ &\quad + \int_{B_{2\eta}(0)} V_\ell\left(a + \varepsilon[x + P_a]\right)\left(\chi_\eta w_\ell^{[-\tau_a]}\right)^2 \\ &\le O\left(\varepsilon^2 + \|\mathbf{U}_a\|^2_{H^1(B^c_\eta)}\right)\int_{B_{2\eta}(\tau_a)} w_\ell^2 + \int_{B_{2\eta}(\tau_a)} V_\ell(\varepsilon x) w_\ell^2 \qquad (3.23)\end{aligned}$$

Adding inequalities (3.22) and (3.23) on the collection of balls $B_{2\eta}(\tau)$ (pairwise disjoint by (2.4)), we deduce

$$\sum_{a\in\mathcal{A}} \|\chi_\eta \mathbf{w}^{[\tau_a]}\|_a^2 \le \left(1 + \frac{1}{\eta}\right)\|\mathbf{w}\|_\varepsilon^2 + O\left(\varepsilon^2 + \|\mathbf{U}_a\|^2_{H^1(B^c_\eta)} + \frac{1}{\eta}\right)\|\mathbf{w}\|_2^2. \qquad (3.24)$$

By combining (3.21) and (3.24), with the embedding $E_\varepsilon \hookrightarrow H^1$, we obtain

$$\frac{1}{1 - C_0}\int_{\mathbb{R}^N} \mathbf{w}^t G\left(\sum_{a\in\mathcal{A}} \boldsymbol{\sigma}_{\varepsilon,a,P_a}\right)\mathbf{w} \le \|\mathbf{w}\|_\varepsilon^2 \left\{1 + O\left(\varepsilon^2 + \|\mathbf{U}_a\|^2_{H^1(B^c_\eta)} + \frac{1}{\eta}\right)\right\}.$$

Hence, by choosing ε, δ small enough, we find a constant $C_1 \in (0, 1)$ such that

$$\int_{\mathbb{R}^N} \mathbf{w}^t G\left(\sum_{a\in\mathcal{A}} \boldsymbol{\sigma}_{\varepsilon,a,P_a}\right)\mathbf{w} \le C_1 \|\mathbf{w}\|_\varepsilon^2.$$

This concludes the proof. □

Proposition 3.1 is a key step in proving the following property:

Proposition 3.5. *Let* $0 < \varepsilon < \varepsilon_1, 0 < \delta < \delta_1$ *and* $\mathbf{P} \in \mathcal{B}_\delta$. *Then, at the point* $\mathbf{q}_0 = (\mathbf{P}, \mathbf{1}, \mathbf{0})$, *the operator* $DS_{(\mathbf{q}_0)}$ *is invertible and there is a constant* C *which is independent of* ε, δ *and* $\mathbf{P}$ *such that*

$$\left\| \left[DS_{(\mathbf{q}_0)}\right]^{-1} \right\| \leq C. \tag{3.25}$$

Proof. Since G is homogeneous of degree 4, the map $x \mapsto \nabla G(x)$ is homogenous of degree 3 and the Euler formula for homogenous map gives $\mathbf{r}^t\, D^2G(\mathbf{r})\,\mathbf{s} = 3\langle \nabla G(\mathbf{r}), \mathbf{s}\rangle$. Hence, for any $\mathbf{f} \in H^1$ we have

$$\int_{\mathbb{R}^N} \boldsymbol{\sigma}^t_{\varepsilon,a,P} D^2G\left(\boldsymbol{\sigma}_{\varepsilon,a,P}\right) \mathbf{f} \;=\; 3\int_{\mathbb{R}^N} \langle \nabla G(\boldsymbol{\sigma}_{\varepsilon,a,P}), \mathbf{f}\rangle. \tag{3.26}$$

Using successively the definition of $\boldsymbol{\sigma}_{\varepsilon,a,P}$, the homogeneity of the function G together with the equation satisfied by $\mathbf{U}_a$, the equality (3.26) becomes

$$\int_{\mathbb{R}^N} \boldsymbol{\sigma}^t_{\varepsilon,a,P} D^2G\left(\boldsymbol{\sigma}_{\varepsilon,a,P}\right) \mathbf{f} \;=\; 3\int_{\mathbb{R}^N} \langle \nabla G(\mathbf{U}_a), \chi^3_\eta \mathbf{f}^{[-\tau]}\rangle \;=\; 3\left\langle \mathbf{U}_a, \chi^3_\eta \mathbf{f}^{[-\tau]}\right\rangle_a. \tag{3.27}$$

We now estimate the second derivative $\frac{\partial^2 I_\varepsilon}{\partial \alpha_a^2}$ at the point $\mathbf{q}_0 = (\mathbf{P}, \mathbf{1}, \mathbf{0})$:

$$\begin{aligned}\frac{\partial^2 I_\varepsilon}{\partial \alpha_a^2}(\mathbf{q}_0) &= \|\boldsymbol{\sigma}_{\varepsilon,a,P_a}\|^2_\varepsilon - \int_{\mathbb{R}^N} \boldsymbol{\sigma}^t_{\varepsilon,a,P_a} D^2G\left(\boldsymbol{\sigma}_{\varepsilon,a,P_a}\right)\boldsymbol{\sigma}_{\varepsilon,a,P_a} \qquad \text{(by (3.9))}\\ &= \|\boldsymbol{\sigma}_{\varepsilon,a,P_a}\|^2_\varepsilon - 3\left\langle \mathbf{U}_a, \chi^3_\eta \mathbf{U}_a\right\rangle_a. \qquad \text{(by (3.27))}\end{aligned}$$

Hence, using (A.5) and (A.3), we obtain

$$\left|\frac{\partial^2 I_\varepsilon}{\partial \alpha_a^2}(\mathbf{q}_0)\right| \;=\; -2\|U_a\|^2_a + O\left(\varepsilon^2 + \|U_a\|^2_{H^1(B^c_\eta)}\right).$$

To estimate $\frac{\partial^2 I_\varepsilon}{\partial \alpha_a \partial \mathbf{w}}$ at the point $\mathbf{q}_0 := (\mathbf{P}, \mathbf{1}, \mathbf{0})$, we first use (3.11):

$$\frac{\partial^2 I_\varepsilon}{\partial \alpha_a \partial \mathbf{w}}(\mathbf{q}_0) : F_{\varepsilon,a,P} \to \mathbb{R}, \quad \boldsymbol{\varphi} \mapsto -\int_{\mathbb{R}^N} \boldsymbol{\sigma}^t_{\varepsilon,a,P} D^2G\left(\boldsymbol{\sigma}_{\varepsilon,a,P}\right)\boldsymbol{\varphi}. \tag{3.28}$$

Hence, applying (3.27) with Lemma A.2, we obtain

$$\left\|\frac{\partial^2 I_\varepsilon}{\partial \alpha_a \partial \mathbf{w}}(\mathbf{q}_0)\right\| = O\left(\varepsilon^2 + \|\mathbf{U}\|_{H^1(B^c_\eta)}\right).$$

□

4. Reduction to finite dimension via implicit function lemma

In this section, we aim to solve (3.3). This will be done by applying the following implicit function lemma (already used by Li [13])

Lemma 4.1. *Let* X, Y *be Banach spaces,* $B_d(x_0) = \{x \in X : \|x - x_0\|_X \leq d\}$ $(d > 0)$ *and* $S : B_d(x_0) \to Y$ *be a* C^1 *map. Assume* $DS_{(x_0)}$ *is invertible and satisfies, for some* $\theta \in (0, 1)$,

$$\left\|\left[DS_{(x_0)}\right]^{-1}(S(x_0))\right\|_X \leq (1-\theta)d, \tag{4.1}$$

$$\left\|\left[DS_{(x_0)}\right]^{-1}\right\| \left\|DS_{(x)} - DS_{(x_0)}\right\| \;\leq\; \theta \quad \forall x \in B_d(x_0). \tag{4.2}$$

Then there is a unique solution in $B_d(z_0)$ *of* $S(x) = 0$.

Proof. The proof relies on a fix point argument, and for the sake of completeness we provide a short proof. Setting $f(x) := [DS_{(x_0)}]^{-1}(S(x))$ we want to solve the equivalent problem $x - f(x) = x$ in $B_d(x_0)$. On the one hand, using (4.2), the mean value theorem and $Df_{(x_0)} = \mathrm{id}$ we easily check that $x \mapsto x - f(x)$ defines a map from X into itself that satisfies

$$\big\|[y - f(y)] - [x - f(x)]\big\| \le \theta\|y - x\| \quad \forall x, y \in B_d(x_0). \tag{4.3}$$

On the other hand, using (4.3) and assumption (4.1) for any $x \in B_d(x_0)$ we get

$$\|x - f(x) - x_0\| \le \theta\|x - x_0\| + \|f(x_0)\| \le \theta d + (1-\theta)d = d. \tag{4.4}$$

Hence, (4.3) and (4.4) show that the map $x \mapsto x - f(x)$ is contraction map in the complete metric space $B_d(x_0)$. The conclusion follows from the Banach fixed point Theorem. □

Lemma 4.1 will be applied with the spaces

$$X = \left(\prod_{a\in\mathcal{A}} \mathbb{R}\right) \times \mathcal{F}_{\varepsilon,\mathbf{P}} \quad Y = \left(\prod_{a\in\mathcal{A}} \mathbb{R}\right) \times \mathcal{F}^{\star}_{\varepsilon,\mathbf{P}}$$

endowed with the product norm, and S defined by (3.5) in a neighborhood of $x_0 = (\mathbf{1}, \mathbf{0})$. Using this lemma we can reduce the system of equations (3.3) to a finite dimension problem:

Proposition 4.2. *For* $0 < \varepsilon < \varepsilon_0, 0 < \delta < \delta_0$ *and for any* $\mathbf{P} \in \prod_{a\in\mathcal{A}} B_\delta(0)$, *the problem* (3.3) *admits a unique solution* $(\boldsymbol{\alpha}(\mathbf{P}), \mathbf{w}(\mathbf{P}))$ *in* $\left(\prod_{a\in\mathcal{A}}\mathbb{R}\right) \times \mathcal{F}_{\varepsilon,\mathbf{P}}$ *with*

$$|\mathbf{1} - \alpha(\mathbf{P})| + \|\mathbf{w}(\mathbf{P})\|_\varepsilon \le C\left(\varepsilon^2 + \sum_{a\in\mathcal{A}} \|\mathbf{U}_a\|_{H^1(B^c_\eta)}\right). \tag{4.5}$$

Proof. Consider the map $S(\boldsymbol{\alpha}, \mathbf{w})$ defined by (3.5). We divide the proof in three steps.

Step 1: We show that

$$\|S(\mathbf{1}, \mathbf{0})\| \le C\left(\varepsilon^2 + \sum_{a\in\mathcal{A}} \|\mathbf{U}_a\|_{H^1(B^c_\eta)}\right). \tag{4.6}$$

We first estimate the derivative $\frac{\partial I_\varepsilon}{\partial \boldsymbol{\alpha}}$ which has been computed in (3.6). Using the definition of $\boldsymbol{\sigma}_{a,\varepsilon,P}$, the fact that ∇G is homogeneous of degree 3 and that $\mathbf{U}_a$ is a solution of (1.2), we get

$$\begin{aligned}\frac{\partial I_\varepsilon}{\partial \alpha_a}(\mathbf{P}, \mathbf{1}, \mathbf{0}) &= \|\boldsymbol{\sigma}_{\varepsilon,a,P_a}\|^2_\varepsilon - \int_{\mathbb{R}^N} \langle \nabla G\left(\boldsymbol{\sigma}_{\varepsilon,a,P_a}\right), \boldsymbol{\sigma}_{\varepsilon,a,P_a}\rangle \\ &= \|\boldsymbol{\sigma}_{\varepsilon,a,P_a}\|^2_\varepsilon - \left\langle \mathbf{U}_a, \chi^4_\eta \mathbf{U}_a\right\rangle_a \\ &= O\left(\varepsilon^2 + \|\mathbf{U}_a\|^2_{H^1(B^c_\eta)}\right)\end{aligned} \tag{4.7}$$

where the last estimate follows from (A.3) and (A.5) proved in “Appendix”.

The derivative $\frac{\partial I_\varepsilon}{\partial \mathbf{w}} \in \mathcal{F}^*_{\varepsilon,\mathbf{P}}$ has been computed in (3.7), and at $(\mathbf{1},\mathbf{0})$ it is given by the linear form:

$$\frac{\partial I_\varepsilon}{\partial \mathbf{w}} = \sum_{a\in\mathcal{A}} \int_{\mathbb{R}^N} \langle \nabla G(\boldsymbol{\sigma}_{\varepsilon,a,P_a}), \cdot\rangle . \tag{4.8}$$

Hence, given $\mathbf{w} \in \mathcal{F}_{\varepsilon,\mathbf{P}}$, by using the homogeneity of the function G and the fact that $\mathbf{U}_a$ solves (1.2), we obtain

$$\int_{\mathbb{R}^N} \langle \nabla G(\boldsymbol{\sigma}_{\varepsilon,a,P_a}), \mathbf{w}\rangle = \int_{\mathbb{R}^N} \left\langle \nabla G(\mathbf{U}_a), \chi_\eta^3 \mathbf{w}^{\tau_a}\right\rangle = \left\langle \mathbf{U}_a, \chi_\eta^3 \mathbf{w}^{\tau_a}\right\rangle_a . \tag{4.9}$$

By applying now Lemma A.2 we conclude that at the point $\mathbf{q} = (\mathbf{P},\mathbf{1},\mathbf{0})$:

$$\left\|\frac{\partial I_\varepsilon}{\partial \mathbf{w}}(\mathbf{q})\right\| = O\left(\varepsilon^2 + \|\mathbf{U}_a\|_{H^1(B^c_\eta)}\right).$$

Step 2: For $|\boldsymbol{\alpha}-\mathbf{1}| < \frac{1}{2}$ and $\|\mathbf{w}\|_\varepsilon < \frac{1}{2}$, we have the following estimates

$$\|DS_{(\boldsymbol{\alpha},\mathbf{w})} - DS_{(\mathbf{1},\mathbf{0})}\| \le C\left(|\boldsymbol{\alpha}-\mathbf{1}| + \|\mathbf{w}\|_{H^1}\right). \tag{4.10}$$

Indeed, setting $\mathbf{q} := (\mathbf{P},\boldsymbol{\alpha},\mathbf{w})$ and $\mathbf{q}_0 := (\mathbf{P},\mathbf{1},\mathbf{0})$

$$DS(\mathbf{q}) - DS(\mathbf{q}_0) = \begin{pmatrix} \frac{\partial^2 I_\varepsilon}{\partial \boldsymbol{\alpha}^2}(\mathbf{q}) - \frac{\partial^2 I_\varepsilon}{\partial \boldsymbol{\alpha}^2}(\mathbf{q}_0) & \frac{\partial^2 I_\varepsilon}{\partial \boldsymbol{\alpha}\partial \mathbf{w}}(\mathbf{q}) - \frac{\partial^2 I_\varepsilon}{\partial \boldsymbol{\alpha}\partial \mathbf{w}}(\mathbf{q}_0) \\ \frac{\partial^2 I_\varepsilon}{\partial \boldsymbol{\alpha}\partial \mathbf{w}}(\mathbf{q}) - \frac{\partial^2 I_\varepsilon}{\partial \boldsymbol{\alpha}\partial \mathbf{w}}(\mathbf{q}_0) & \frac{\partial^2 I_\varepsilon}{\partial \mathbf{w}^2}(\mathbf{q}) - \frac{\partial^2 I_\varepsilon}{\partial \mathbf{w}^2}(\mathbf{q}_0) \end{pmatrix}$$

First we note that, from the definition of G, for any $\mathbf{r},\mathbf{s} \in \mathbb{R}^n$, we have

$$\begin{aligned} \Big|\partial_{ij}G(\mathbf{s}) - \partial_{ij}G(\mathbf{r})\Big| &= \Big|\int_0^1 \frac{d}{dt}\partial_{ij}G(\mathbf{r}+t[\mathbf{s}-\mathbf{r}])\,dt\Big| \\ &\le C\Big|\mathbf{s}-\mathbf{r}\Big| \int_0^1 \Big|\mathbf{r}+t[\mathbf{s}-\mathbf{r}]\Big| dt \\ &\le C\Big|\mathbf{s}-\mathbf{r}\Big|\left(|\mathbf{s}|+|\mathbf{r}|\right) \end{aligned} \tag{4.11}$$

Applying (4.11) with $\mathbf{s} = \alpha_a\boldsymbol{\sigma}_{\varepsilon,a,P_a} + \mathbf{w}$ and $\mathbf{r} = \boldsymbol{\sigma}_{\varepsilon,a,P_a}$, together with the estimate (A.4) for any $\mathbf{f},\mathbf{g} \in H^1(\mathbb{R}^N)$ and $\mathbf{w} \in F_{\varepsilon,a,P}$ (with $\|\mathbf{w}\|_{H^1} \le 1/2$) we have

$$\begin{aligned} &\left|\int_{\mathbb{R}^N} \mathbf{f}^t\left[D^2G(\alpha_a\boldsymbol{\sigma}_{\varepsilon,a,P_a}+\mathbf{w}) - D^2G(\boldsymbol{\sigma}_{\varepsilon,a,P_a})\right]\mathbf{g}\right| \\ &\qquad \le C\left(|\alpha_a - 1| + \|\mathbf{w}\|_{H^1}\right)\|\mathbf{f}\|_{H^1}\|\mathbf{g}\|_{H^1} \end{aligned} \tag{4.12}$$

The second derivative with respect to α_a given in (3.9), can be estimated using (4.12) and (A.4):

$$\begin{aligned} &\left|\frac{\partial^2 I_\varepsilon}{\partial \alpha_a^2}(\mathbf{P},\boldsymbol{\alpha},\mathbf{w}) - \frac{\partial^2 I_\varepsilon}{\partial \alpha_a^2}(\mathbf{q}_0)\right| \\ &\quad \le \int_{\mathbb{R}^N} \left|\boldsymbol{\sigma}^t_{\varepsilon,a,P_a}\left\{D^2G(\alpha_a\boldsymbol{\sigma}_{\varepsilon,a,P_a}+\mathbf{w}) - D^2G(\boldsymbol{\sigma}_{\varepsilon,a,P_a})\right\}\boldsymbol{\sigma}_{\varepsilon,a,P_a}\right| \\ &\quad \le C\left(|\alpha_a - 1| + \|\mathbf{w}\|_{H^1}\right). \end{aligned} \tag{4.13}$$

Similarly, using the expression of the mixed partial derivatives $\frac{\partial^2 I_\varepsilon}{\partial \mathbf{w}\partial \alpha_a}$ given by the linear map (3.11), the estimate (4.12) and (A.4) show

$$\left|\int_{\mathbb{R}^N} \boldsymbol{\varphi}^t \left(D^2G\left(\alpha_a \boldsymbol{\sigma}_{\varepsilon,a,P_a} + \mathbf{w}\right) - D^2G\left(\boldsymbol{\sigma}_{\varepsilon,a,P_a}\right)\right)\boldsymbol{\sigma}_{\varepsilon,a,P_a}\right| \\ \leq C\left(|\alpha_a - 1| + \|\mathbf{w}\|_{H^1}\right)\|\boldsymbol{\varphi}\|_{H^1}$$

for all $\boldsymbol{\varphi} \in H^1(\mathbb{R}^N)$, namely

$$\left\|\frac{\partial^2 I_\varepsilon}{\partial \mathbf{w}\partial \alpha_a}(\boldsymbol{\alpha}, \mathbf{w}) - \frac{\partial^2 I_\varepsilon}{\partial \mathbf{w}\partial \alpha_a}(\mathbf{1}, \mathbf{0})\right\| \leq C\left(|\alpha_a - \mathbf{1}| + \|\mathbf{w}\|_{H^1}\right) \tag{4.14}$$

The estimate of the bilinear form $\frac{\partial^2 I_\varepsilon}{\partial \mathbf{w}^2}(\boldsymbol{\alpha}, \mathbf{w}) - \frac{\partial^2 I_\varepsilon}{\partial \mathbf{w}^2}(\mathbf{1}, \mathbf{0})$ (defined on $\mathcal{F}_{\varepsilon,\mathbf{P}} \times \mathcal{F}_{\varepsilon,\mathbf{P}}$) follows also immediately from (4.12), and we get

$$\left\|\frac{\partial^2 I_\varepsilon}{\partial \mathbf{w}^2}(\boldsymbol{\alpha}, \mathbf{w}) - \frac{\partial^2 I_\varepsilon}{\partial \mathbf{w}^2}(\mathbf{1}, \mathbf{0})\right\| \leq C\left\{|\boldsymbol{\alpha} - \mathbf{1}| + \|\mathbf{w}\|_{H^1}\right\}. \tag{4.15}$$

The estimate (4.10) follows by combining (4.13), (4.14), and (4.15).

Step 3: From Step 1 and Step 2, together with Proposition 3.5 we get a constant $C_0 > 0$ such that

$$\left\|\left[DS_{(\mathbf{1},\mathbf{0})}\right]^{-1}\left(S(\mathbf{1}, \mathbf{0})\right)\right\| \leq C_0\left(\varepsilon^2 + \sum_{a\in\mathcal{A}} \|\mathbf{U}_a\|_{H^1(B^c_\eta)}\right), \tag{4.16}$$

$$\left\|\left[DS_{(\mathbf{1},\mathbf{0})}\right]^{-1}\right\|\left\|DS_{(\boldsymbol{\alpha},\mathbf{w})} - DS_{(\mathbf{1},\mathbf{0})}\right\| \leq C_0\left(|\boldsymbol{\alpha} - \mathbf{1}|^2 + \|\mathbf{w}\|^2_{H^1}\right)^{1/2}. \tag{4.17}$$

In order to apply Lemma 4.1, we choose

$$\theta = \frac{1}{2}, \qquad d := 2C_0\left(\varepsilon^2 + \sum_{a\in\mathcal{A}} \|\mathbf{U}_a\|_{H^1(B^c_\eta)}\right).$$

With this choice we get

$$\left\|\left[DS_{(\mathbf{1},\mathbf{0})}\right]^{-1}\left(S(\mathbf{1}, \mathbf{0})\right)\right\| \leq (1 - \theta)d, \tag{4.18}$$

$$\left\|\left[DS_{(\mathbf{1},\mathbf{0})}\right]^{-1}\right\|\left\|DS_{(\boldsymbol{\alpha},\mathbf{w})} - DS_{(\mathbf{1},\mathbf{0})}\right\| \leq \theta \quad \forall(\boldsymbol{\alpha}, \mathbf{w}) \in B_d(\mathbf{1}, \mathbf{0}). \tag{4.19}$$

Hence we can apply Lemma 4.1, which concludes the proof of the proposition. □

5. Solving finite dimension problem via degree arguments

By Proposition 4.2, we can define for each $\mathbf{P} \in \mathcal{B}_\delta$

$$R_\varepsilon(\mathbf{P}) = I_\varepsilon(\mathbf{P}, \boldsymbol{\alpha}(\mathbf{P}), \mathbf{w}(\mathbf{P})) = J_\varepsilon\left(\sum_{a\in\mathcal{A}} \alpha_a(\mathbf{P})\boldsymbol{\sigma}_{\varepsilon,a,P_a} + \mathbf{w}(\mathbf{P})\right),$$

where $(\boldsymbol{\alpha}(\mathbf{P}), \mathbf{w}(\mathbf{P}))$ is a solution to (3.3). It is easy to verify that $\mathbf{P} \in \prod_{a\in\mathcal{A}} B_\delta(0)$ is a critical point of R_ε if and only if $(\mathbf{P}, \alpha(\mathbf{P}), \mathbf{w}(\mathbf{P}))$ is critical point of $I_\varepsilon(\alpha, \mathbf{P}, \omega)$ (see for instance Cao and Heinz [7]). Thus solving (3.4) is equivalent to solve

$$\frac{\partial R_\varepsilon(\mathbf{P})}{\partial \mathbf{P}} = 0. \tag{5.1}$$

Existence of a solution to (5.1) can be obtained by computing the Brouwer degree of the map ∇R_ε.

Proposition 5.1. *Assume* $(A_1)-(A_2)$ *are satisfied. Then,*

$$\frac{\partial R_\varepsilon}{\partial P_a}(\mathbf{P}) = \varepsilon^2\left(\sum_{\ell=1}^{n}\|\mathbf{U}_{a,\ell}\|_2^2\, D^2 V_{\ell\,(a)}\right)(P_a) + O\left(\varepsilon^3\right). \tag{5.2}$$

If furthermore (A_3) *holds, then there exist* $\varepsilon_0, \delta_0 > 0$ *such that for* $\varepsilon \in (0, \varepsilon_0)$ *and* $\delta \in (0, \delta_0)$

$$\deg\left(0, \nabla R_\varepsilon, \mathcal{B}_\delta\right) = (-1)^{\sum_{a\in\mathcal{A}} n_a}. \tag{5.3}$$

where n_a *stands for the number of negative eigenvalues of* $\sum_{\ell=1}^{n}\|U_{a,\ell}\|^2 D^2 V_\ell(a)$*.*

Proof. By setting $\mathbf{q} := (\mathbf{P}, \boldsymbol{\alpha}(\mathbf{P}), \mathbf{w}(\mathbf{P}))$, the derivative of R_ε with respect to $\mathbf{P}$ is given by

$$\frac{\partial R_\varepsilon(\mathbf{P})}{\partial \mathbf{P}} = \frac{\partial I_\varepsilon}{\partial \mathbf{P}}(\mathbf{q}) + \underbrace{\frac{\partial I_\varepsilon}{\partial \boldsymbol{\alpha}}\left(\frac{\partial \boldsymbol{\alpha}}{\partial \mathbf{P}}(\mathbf{q})\right)}_{=0} + \frac{\partial I_\varepsilon}{\partial \mathbf{w}}\left(\frac{\partial \mathbf{w}}{\partial \mathbf{P}}(\mathbf{q})\right), \tag{5.4}$$

where the second term on the right hand-side of (5.4) is zero since $(\boldsymbol{\alpha}(\mathbf{P}), \mathbf{w}(\mathbf{P}))$ solves the first equation in (3.3). Concerning the third term of (5.4), we emphasize that here $\dfrac{\partial I_\varepsilon}{\partial \mathbf{w}}$ is a linear form on the entire Hilbert space E_ε and that $\frac{\partial \mathbf{w}}{\partial \mathbf{P}}$ may not belong to $\mathcal{F}_{\varepsilon,\mathbf{P}}$. Hence, we decompose

$$\frac{\partial \mathbf{w}}{\partial P_a} = \mathbf{f}_\perp + \mathbf{f}_0, \tag{5.5}$$

with $\mathbf{f}_\perp \in \mathcal{F}_{\varepsilon,\mathbf{P}}$ and $\mathbf{f}_0 \in \operatorname{span}\left\{\boldsymbol{\sigma}_{\varepsilon,a,P_a}, \frac{\partial \boldsymbol{\sigma}_{\varepsilon,a,P_a}}{\partial y_1}, \ldots, \frac{\partial \boldsymbol{\sigma}_{\varepsilon,a,P_a}}{\partial y_N}\right\}$. Since, $\mathbf{w} \in \mathcal{F}_{\varepsilon,a}$, by differentiating with respect to the point P_a the identities

$$\langle \mathbf{w}, \boldsymbol{\sigma}_{a,P_a,\varepsilon}\rangle_\varepsilon = 0, \qquad \left\langle \mathbf{w}, \frac{\partial \boldsymbol{\sigma}_{a,P_a,\varepsilon}}{\partial y_i}\right\rangle_\varepsilon = 0,$$

we derive (as in Cao and Heinz [7, Appendix D]):

$$\left\langle \frac{\partial \mathbf{w}}{\partial P_{a,i}}, \boldsymbol{\sigma}_{a,P_a,\varepsilon}\right\rangle_\varepsilon = 0, \quad \left\langle \frac{\partial \mathbf{w}}{\partial P_{a,i}}, \frac{\partial \boldsymbol{\sigma}_{a,P_a,\varepsilon}}{\partial y_i}\right\rangle_\varepsilon = O\left(\|\mathbf{w}\|_\varepsilon\right). \tag{5.6}$$

Therefore, from $\dfrac{\partial I_\varepsilon}{\partial \mathbf{w}}(\mathbf{f}_\perp) = 0$ (by (3.3)) and (5.6) we get

$$\left|\frac{\partial I_\varepsilon}{\partial \mathbf{w}}\left(\frac{\partial \mathbf{w}}{\partial \mathbf{P}}(\mathbf{q})\right)\right| \le \left\|\frac{\partial I_\varepsilon}{\partial \mathbf{w}}\right\| O\left(\|\mathbf{w}\|_\varepsilon\right) = O\left(\varepsilon^4 + \|\mathbf{U}_a\|^2_{H^1(B_\eta^c)} + \|\mathbf{w}\|_\varepsilon^2\right).$$

We can then rewrite (5.4) as follows:

$$\frac{\partial R_\varepsilon(\mathbf{P})}{\partial \mathbf{P}} = \frac{\partial I_\varepsilon}{\partial \mathbf{P}}(\mathbf{q}) + \frac{\partial I_\varepsilon}{\partial \mathbf{w}}\left(\frac{\partial \mathbf{w}}{\partial \mathbf{P}}(\mathbf{q})\right) = \frac{\partial I_\varepsilon}{\partial \mathbf{P}}(\mathbf{q}) + O\left(\varepsilon^4 + \|\mathbf{U}_a\|^2_{H^1(B_\eta^c)} + \|\mathbf{w}\|_\varepsilon^2\right). \tag{5.7}$$

So, we need to estimate $\dfrac{\partial I_\varepsilon}{\partial \mathbf{P}}$. From the definition of I_ε and using the facts that the functions $\boldsymbol{\sigma}_{\varepsilon,a,P_a}$ have disjoint supports together with $\mathbf{w} \in \mathcal{F}_{\varepsilon,\mathbf{P}}$, we obtain for $i = 1, \ldots, N$:

$$\begin{aligned}\frac{\partial I_\varepsilon}{\partial P_{a,i}}(\mathbf{q}) &= DJ_{\left(\sum_{a\in\mathcal{A}} \alpha_a \boldsymbol{\sigma}_{\varepsilon,a,P_a}+\mathbf{w}\right)}\left(\alpha_a \frac{\partial \boldsymbol{\sigma}_{\varepsilon,a,P_a}}{\partial P_{a,i}}\right)\\ &= \alpha_a^2 \underbrace{\left\langle \boldsymbol{\sigma}_{\varepsilon,a,P_a}, \frac{\partial \boldsymbol{\sigma}_{\varepsilon,a,P_a}}{\partial P_{a,i}}\right\rangle_\varepsilon}_{I} - \underbrace{\int_{\mathbb{R}^N}\left\langle \nabla G\left(\alpha_a \boldsymbol{\sigma}_{\varepsilon,a,P_a} + \mathbf{w}\right), \alpha_a \frac{\partial \boldsymbol{\sigma}_{\varepsilon,a,P_a}}{\partial P_{a,i}}\right\rangle}_{II}.\end{aligned} \tag{5.8}$$

To estimate I and II observe that the functions $\mathbf{U}_a$, χ_η and the nonlinearity G are even, which implies

$$\int_{\mathbb{R}^N}\left\langle \nabla \boldsymbol{\sigma}_{\varepsilon,a,P_a}, \nabla \frac{\partial \boldsymbol{\sigma}_{\varepsilon,a,P_a}}{\partial P_{a,i}}\right\rangle = \int_{\mathbb{R}^N}\left\langle \nabla(\chi_\eta \mathbf{U}_a), \nabla\left(\frac{\partial(\chi_\eta \mathbf{U}_a)}{\partial y_i}\right)\right\rangle = 0, \tag{5.9}$$

$$\int_{\mathbb{R}^N}\left\langle \nabla G\left(\alpha_a \boldsymbol{\sigma}_{\varepsilon,a,P_a}\right), \frac{\partial \boldsymbol{\sigma}_{\varepsilon,a,P_a}}{\partial P_{a,i}}\right\rangle = \int_{\mathbb{R}^N}\left\langle \nabla G\left(\alpha_a \chi_\eta \mathbf{U}_a\right), \left(\frac{\partial(\chi_\eta \mathbf{U}_a)}{\partial y_i}\right)\right\rangle = 0. \tag{5.10}$$

Estimate of (I):

Using (5.9) and the definition of $\boldsymbol{\sigma}_{\varepsilon,a,P_a}$ we get

$$\begin{aligned}\left\langle \boldsymbol{\sigma}_{\varepsilon,a,P_a}, \frac{\partial \boldsymbol{\sigma}_{\varepsilon,a,P_a}}{\partial P_{a,i}}\right\rangle_\varepsilon &= -\sum_{\ell=1}^n \int_{\mathbb{R}^N} V_\ell(\varepsilon x)\frac{\partial(\chi_\eta U_{a,\ell})^2}{\partial x_i}\left(x - \frac{a}{\varepsilon} - P_a\right) dx\\ &= \varepsilon \sum_{\ell=1}^n \int_{\mathbb{R}^N} \frac{\partial V_\ell}{\partial x_i}\left(a + \varepsilon[y + P_a]\right)(\chi_\eta U_{a,\ell})^2\, dy\\ &= \varepsilon^2 \sum_{\ell=1}^n \left\langle \nabla\left(\frac{\partial V_\ell}{\partial y_i}\right)(a), P_a\right\rangle \|U_{a,\ell}\|_{L^2}^2\\ &\quad + \varepsilon\, O(\varepsilon^2 + \|U_{a,\ell}\|^2_{L^2(B_\eta^c)}),\end{aligned} \tag{5.11}$$

where the last equality follows by applying Lemma A.3 with $W := \frac{\partial V_\ell}{\partial y_i}$ at $a \in \mathcal{A}$ and the even function $f(x) = (\chi_\eta U_{a,\ell})^2$.

Estimate of (II):

Setting $\boldsymbol{\sigma} := \boldsymbol{\sigma}_{\varepsilon,a,P_a}$, a Taylor expansion of G gives

$$\begin{aligned}\left\langle \nabla G_{(\alpha\boldsymbol{\sigma}+\mathbf{w})} - \nabla G_{(\alpha\boldsymbol{\sigma})}, \frac{\partial \boldsymbol{\sigma}}{\partial P_{a,i}}\right\rangle &= \int_0^1 D^2 G_{(\alpha\boldsymbol{\sigma}+s\mathbf{w})}\left[\frac{\partial \boldsymbol{\sigma}}{\partial P_{a,i}}, \mathbf{w}\right] ds\\ &= D^2 G_{(\mathbf{U}_a^{[-\tau_a]})}\left[\frac{\partial \boldsymbol{\sigma}}{\partial x_i}, \mathbf{w}\right] + R,\end{aligned} \tag{5.12}$$

where the remainder r is given by

$$r = \int_0^1\int_0^1 D^3 G_{(\mathbf{U}_a^{[-\tau_a]} + t[\alpha\boldsymbol{\sigma} - \mathbf{U}_a^{[-\tau_a]} + s\mathbf{w}])}\left[\frac{\partial \boldsymbol{\sigma}}{\partial x_i}, \mathbf{w}, \alpha\boldsymbol{\sigma} - \mathbf{U}_a^{[-\tau_a]} + s\mathbf{w}\right] ds dt,$$

whose L^1-norm can easily be estimated as

$$\|r\|_{L^1} = O\left(\|\mathbf{w}\|^2_{H^1} + |\alpha - 1|^2 + \|\mathbf{U}_a\|^2_{H^1(B^c_\eta)}\right). \tag{5.13}$$

From the equation satisfied by $\mathbf{U}_a$ we derive that

$$-\Delta\frac{\partial \mathbf{U}_a}{\partial x_i} + \mathbf{V}(a)\frac{\partial \mathbf{U}_a}{\partial x_i} = D^2G_{(\mathbf{U_a})}\frac{\partial \mathbf{U}_a}{\partial x_i}, \tag{5.14}$$

which implies with the estimate (A.7)

$$\int_{\mathbb{R}^N} D^2G_{(\mathbf{U_a})}\left[\frac{\partial \mathbf{U}_a}{\partial x_i}, \chi_\eta \mathbf{w}^{\tau_a}\right] = \left\langle \frac{\partial \mathbf{U}_a}{\partial x_i}, \chi_\eta \mathbf{w}^{\tau_a}\right\rangle_a = O\left(\varepsilon^4 + \|\mathbf{w}\|^2_\varepsilon\right). \tag{5.15}$$

Therefore, using (5.10), (5.12), (5.13) and (5.15) we deduce that

$$\begin{aligned}\int_{\mathbb{R}^N}\langle \nabla G(\alpha\boldsymbol{\sigma} + \mathbf{w}), \frac{\partial \boldsymbol{\sigma}}{\partial P_{a,i}}\rangle &= \int_{\mathbb{R}^N} D^2G_{(\mathbf{U}_a)}\left[\frac{\partial(\chi_\eta \mathbf{U}_a)}{\partial x_i}, \mathbf{w}^{[\tau_a]}\right] + \int_{\mathbb{R}^N} R \\ &= \left\langle \frac{\partial \mathbf{U}_a}{\partial x_i}, \chi_\eta \mathbf{w}^{\tau_a}\right\rangle_a \\ &\quad + O\left(|\alpha - 1|^2 + \|\mathbf{w}\|^2_{H^1} + \|\mathbf{U}_a\|^2_{H^1(B^c_\eta)}\right) \\ &= O\left(\varepsilon^4 + |\alpha - 1|^2 + \|\mathbf{w}\|^2_{H^1} + \|\mathbf{U}_a\|^2_{H^1(B^c_\eta)}\right).\end{aligned} \tag{5.16}$$

By applying the estimates (5.11), (5.16) in (5.8), and using (4.5) with (A.12), we can complete the estimate of (5.7) as follows

$$\frac{\partial R_\varepsilon(\mathbf{P})}{\partial P_a} = \varepsilon^2 \underbrace{\left(\sum_{\ell=1}^n \|\mathbf{U}_{a,\ell}\|^2_2\, D^2V_{\ell\,(a)}\right)}_{M_a}(P_a) + O\left(\varepsilon^3\right). \tag{5.17}$$

This proves (5.2).

Let us now prove the degree formula (5.3). Under the assumption (A3), each of the matrices M_a is invertible. So, there exists a constant C_0 such that

$$\left|M_a(P_a)\right| \geq C_0, \quad \forall P_a \in \partial B_\delta(a),\ \forall a \in \mathcal{A}.$$

Hence, setting $M(\mathbf{P}) := \left(M_a(P_a)\right)_{a\in\mathcal{A}}$, we can choose $\varepsilon > 0$ small enough such that

$$\left|\frac{\nabla R_\varepsilon(\mathbf{P})}{\varepsilon^2}\right| = \left|M(\mathbf{P})\right| + O(\varepsilon) > 0, \tag{5.18}$$

for all $\mathbf{P} \in \partial\left(\prod_{a\in\mathcal{A}} B_\delta(a)\right)$. Therefore, (5.18) implies that the homotopy given by

$$H(t, \mathbf{P}) = (1 - t)\nabla R_\varepsilon(\mathbf{P}) + tM(\mathbf{P}), \quad t \in [0, 1],\ \mathbf{P} \in \mathcal{B}_\delta$$

satisfies $|H(t, \mathbf{P})| \neq 0$ for all $t \in [0, 1]$. By the classical property of the Brouwer Degree, we have

$$\deg(0, \nabla R_\varepsilon, \mathcal{B}_\delta) = \deg(0, M, \mathcal{B}_\delta) = (-1)^{\sum_{a\in\mathcal{A}} n_a},$$

where n_a stands for the number of negative eigenvalues of the matrix M_a. This completes the proof of the proposition. □

Since the degree computed in Proposition 5.1 is non-zero, the existence property for the Brouwer degree, we deduce the following

Corollary 5.2. *Suppose the assumptions* $(A1)$*–*(A_2) *are satisfied. Then, there exists* $\varepsilon_0 > 0$ *such that for each* $\varepsilon \in (0, \varepsilon_0)$ *the system* (5.1) *admits a solution* $\mathbf{P}^\varepsilon \in \mathcal{B}_\delta$*.*
Moreover, setting $\tau_a := \frac{a}{\varepsilon} + P_a^\varepsilon$*,* $\sum_{a\in\mathcal{A}} \alpha_a(\mathbf{P}^\varepsilon)\mathbf{U}^{[\tau_a]} + \mathbf{w}(\mathbf{P}^\varepsilon)$ *is a solution of* (2.1)*.*

6. Proof of main results

In this section we give the proof of main result.

Proof of Theorem 1.3. By Corollary 5.2, we know that $\mathbf{u}^\varepsilon := \sum_{a\in\mathcal{A}} \alpha_a(\mathbf{P}^\varepsilon)$ $\boldsymbol{\sigma}_{\varepsilon,a,P_a^\varepsilon} + \mathbf{w}(\mathbf{P}^\varepsilon)$ is a solution of (2.1). Thus, a solution to the original problem (S_ε) is given by

$$\mathbf{u}_\varepsilon(x) := \mathbf{u}^\varepsilon(\frac{x}{\varepsilon}) = \sum_{a\in\mathcal{A}} \alpha_a(\mathbf{P}^\varepsilon)(\chi_\eta \mathbf{U}_a)\left(\frac{x-a}{\varepsilon} - P_a^\varepsilon\right) + \mathbf{w}(\mathbf{P}^\varepsilon)\left(\frac{x}{\varepsilon}\right).$$

The estimate in (1.5) on $(\alpha, \mathbf{w})$ follows from (4.5) and (A.12). Also, by using (5.1) and (5.2) we have

$$0 = \varepsilon^2 M_a(P_a^\varepsilon) + O(\varepsilon^3).$$

Since M_a is invertible, we deduce that $|\mathbf{P}^\varepsilon| = O(\varepsilon)$. Hence, (1.5) holds. This completes the proof of main result. □

Acknowledgements

This paper was partially done while the second author was visiting the Department of Mathematics at Vanderbilt University, and the Department of Mathematics at College of Staten Island (CUNY) supported by the AMS Fan Fund China Exchange program (2015). The author is grateful to both departments for their hospitality during his stay. The first author thanks the support of the Center of PDE at the East China Normal University (Shanghai) where this work was finalized.

A. Appendix: Some estimates

In this appendix, we collect some technical estimates that have been used in our paper. We start with two preliminary observations:

(i) Given $W \in C^2(\mathbb{R}^N)$ and $a \in \mathbb{R}^N$, the definition of χ_η (see (1.4)) and a first order Taylor expansion imply for $\varepsilon \in (0, \varepsilon_0)$, $P \in B_1(0)$:

$$\begin{aligned}&\chi_\eta(x)\left\{W\left(a+\varepsilon[x+P]\right) - W(a) - \varepsilon DW_{(a)}(x+P)\right\}\\ &\quad= \varepsilon^2\chi_\eta(x)\int_0^1\int_0^1 (x+P)^t D^2W_{(a+st\varepsilon[x+P])}(x+P)ds\,dt\\ &\quad= O\left(\varepsilon^2\left[|x|^2+|P|^2\right]\right),\end{aligned} \tag{A.1}$$

where we used $\varepsilon|x+P| = O(1)$ on the ball $B_\eta(0)$ since $P \in B_1(0)$ and $\varepsilon\eta = O(1)$ (see (2.4)).

(ii) Given $s \geq 1$, the definition of χ_η implies

$$\left\|\chi_\eta^s \mathbf{f}\right\|_{L^q} = \|\mathbf{f}\|_{L^q} + O\left(\|\mathbf{f}\|_{L^q(B_\eta^c)}\right) \quad \forall \mathbf{f} \in L^q(\mathbb{R}^N)\ (q \geq 1), \tag{A.2}$$

and for all $\mathbf{f}, \mathbf{g} \in H^1(\mathbb{R}^N)$ it holds

$$\langle \mathbf{f}, \chi_\eta^s \mathbf{g}\rangle_a = \langle \mathbf{f}, \mathbf{g}\rangle_a + O\left(\|\mathbf{f}\|_{H^1(B_\eta^c)}\|\mathbf{g}\|_{H^1(B_\eta^c)}\right). \tag{A.3}$$

In the rest of this appendix, we will always assume that $\varepsilon \in (0, \varepsilon_0)$, $P \in B_1(0)$.

Lemma A.1. *Given $a \in \mathcal{A}$, $P \in B_1(0)$. Then, for $q \leq 5$ we have*

$$\|\boldsymbol{\sigma}_{\varepsilon,a,P}\|_q = \|\mathbf{U}_a\|_q + O\left(\left\|\mathbf{U}_a\right\|_{H^1(B_\eta^c)}\right), \tag{A.4}$$

$$\|\boldsymbol{\sigma}_{\varepsilon,a,P}\|_\varepsilon^2 = \|\mathbf{U}_a\|_a^2 + O\left(\varepsilon^2 + \|\mathbf{U}_a\|_{H^1(B_\eta^c)}^2\right). \tag{A.5}$$

Proof. Estimate (A.4) is a special case of (A.2). For (A.5) we note that (setting $\tau := \frac{a}{\varepsilon} + P$)

$$\begin{aligned}\left\|\boldsymbol{\sigma}_{\varepsilon,a,P}\right\|_\varepsilon^2 &= \left\|(\chi_\eta \mathbf{U}_a)^{[-\tau]}\right\|_\varepsilon^2\\ &= \left\|\chi_\eta \mathbf{U}_a\right\|_a^2 + \int_{\mathbb{R}^N}\left\{V\left(a+\varepsilon[x+P]\right) - V(a)\right\}(\chi_\eta \mathbf{U}_a)^2.\end{aligned}$$

We conclude by applying (A.1), (A.3) with the fact that $(1+|x|^2)\mathbf{U}_a^2 \in L^1(\mathbb{R}^N)$. □

Lemma A.2. *Let $s \geq 1$. Then, for any $\mathbf{w} \in F_{\varepsilon,a,P}$ and setting $\tau := \frac{a}{\varepsilon} + P$ we have*

$$\left\langle \chi_\eta^s \mathbf{w}^{[\tau]}, \mathbf{U}_a\right\rangle_a = O\left(\varepsilon^2 + \|\mathbf{U}_a\|_{H^1(B_\eta^c)}\right)\|\mathbf{w}\|_\varepsilon, \tag{A.6}$$

$$\left\langle \chi_\eta^s \mathbf{w}^{[\tau]}, \frac{\partial \mathbf{U}_a}{\partial x_j}\right\rangle_a = O\left(\varepsilon^2 + \|\mathbf{U}_a\|_{H^1(B_\eta^c)}\right)\|\mathbf{w}\|_\varepsilon \quad j = 1, \ldots, N. \tag{A.7}$$

Proof. Let us first prove (A.6). We have

$$\begin{aligned}\nabla(\chi_\eta^s w_\ell^{[\tau]})\nabla U_{a,\ell} &= \nabla w_\ell^{[\tau]}\nabla(\chi_\eta U_{a,\ell}) + \nabla(\chi_\eta^s w_\ell^{[\tau]})\nabla U_{a,\ell} - \nabla w_\ell^{[\tau]}\nabla(\chi_\eta U_{a,\ell})\\ &= \nabla w_\ell^{[\tau]}\nabla(\chi_\eta U_{a,\ell}) + \nabla\chi_\eta\left\{s\chi_\eta^{s-1}w_\ell^{[\tau]}\nabla U_{a,\ell} - U_{a,\ell}\nabla w_\ell^{[\tau]}\right\}\\ &\quad + \chi_\eta(\chi_\eta^{s-1}-1)\nabla w_\ell^{[\tau]}\nabla U_{a,\ell}.\end{aligned}$$

Using then the property of the function χ_η (see (1.4)) we easily deduce that

$$\int_{\mathbb{R}^N} \nabla(\chi_\eta^s w_\ell^{[\tau]})\nabla U_{a,\ell} = \int_{\mathbb{R}^N} \nabla w_\ell^{[\tau]}\nabla(\chi_\eta U_{a,\ell}) + O\left(\|U_{a,\ell}\|_{H^1(B_\eta^c)}\right)\|w_\ell\|_{H^1}. \tag{A.8}$$

Furthermore,

$$\begin{aligned} V_\ell(a)\chi_\eta^s w_\ell^{[\tau]} U_{a,\ell} = {} & V_\ell\left(a+\varepsilon[x+P]\right) w_\ell^{[\tau]}\chi_\eta U_{a,\ell} \\ & + \left(V_\ell(a) - V_\ell\left(a+\varepsilon[x+P]\right)\right)\chi_\eta w_\ell^{[\tau]} U_{a,\ell} \\ & + V_\ell(a)\left(\chi_\eta^{s-1}-1\right)\chi_\eta w_\ell^{[\tau]} U_{a,\ell}. \end{aligned}$$

Hence by using (A.1) and the fact that $(1+|x|^4)\mathbf{U}_a^2 \in L^1(\mathbb{R}^N)$ we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} V_\ell(a)\chi_\eta^s w_\ell^{[\tau]} U_{a,\ell} = {} & \int_{\mathbb{R}^N} V(\varepsilon x) w_\ell (\chi_\eta U_{a,\ell})^{[-\tau]} \\ & + O\left(\varepsilon^2 + \|U_{a,\ell}\|_{L^2(B_\eta^c)}\right)\|w_\ell\|_{L^2}. \end{aligned} \tag{A.9}$$

Using $\langle \mathbf{w}, (\chi_\eta \mathbf{U}_a)^{[-\tau]}\rangle_\varepsilon = 0$ (Since $\mathbf{w} \in F_{\varepsilon,a,P}$) together with (A.8) and (A.9), we obtain

$$\left\langle \chi_\eta^s \mathbf{w}^{[\tau]}, \mathbf{U}_a \right\rangle_a = O\left(\varepsilon^2 + \|\mathbf{U}_a\|_{H^1(B_\eta^c)}\right)\|\mathbf{w}\|_\varepsilon. \tag{A.10}$$

The proof of (A.7) is similar. □

Lemma A.3. *Let $W \in C^1(\mathbb{R}^N)$ be such that $W(a) = 0$ for some $a \in \mathbb{R}^N$, $f \in L^1(\mathbb{R}^N, (1+|x|^2)dx)$ be an even function and consider χ_η given by (1.4). Then, for $i = 1,\ldots,N$ we have*

$$\int_{\mathbb{R}^N} W(a+\varepsilon[x+P])(\chi_\eta f)(x)dx = \varepsilon\langle\nabla W(a), P\rangle\int_{\mathbb{R}^N} f + O\left(\varepsilon^2 + \|f\|_{L^1(B_\eta^c)}\right). \tag{A.11}$$

Proof. Using (A.2) we have, for $x \in B_\eta(0)$ and $P \in B_1(0)$,

$$\begin{aligned} \int_{\mathbb{R}^N} W\left(a+\varepsilon[x+P]\right)(\chi_\eta f)dx &= \varepsilon\int_{B_\eta(0)}\langle\nabla W(a), x+P\rangle(\chi_\eta f)dx + O(\varepsilon^2), \\ &= \varepsilon\langle\nabla W(a), P\rangle\int_{\mathbb{R}^N} f(x)dx \\ &\quad + O\left(\varepsilon^2 + \|f\|_{L^1(B_\eta^c(0))}\right), \end{aligned}$$

where we have used that the map $x \mapsto \langle\nabla W(a), x\rangle f(x)$ is an odd function. This proves (A.11). □

Finally, given function $f \in L^1(\mathbb{R}^N)$ satisfying $f(x) = O\left(|x|^{-\alpha}\right)$ for some $\alpha > N$, we have

$$\int_{B_\eta^c} |f| = O(\varepsilon^{\alpha-N})$$

whenever $\varepsilon\eta = O(1)$. Since for the functions $\mathbf{U}_a$ solution of (1.2) there exists a constant $\delta > 0$ such that (same arguments as in Busca and Sirakov [5])

$$|\mathbf{U}(x)|, |\partial_i \mathbf{U}(x)| \leq Ce^{-\delta|x|},$$

we conclude

$$\|\mathbf{U}_a\|_{H^1(B_\eta^c)} = O(\varepsilon^k), \quad \forall k \geq 1. \tag{A.12}$$

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Received: 27 September 2016.
Accepted: 30 October 2017.