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Some remarks on the structure of finite Morse index solutions to the Allen–Cahn equation in $\mathbb{R}^2$

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Abstract. For a solution of the Allen–Cahn equation in $\mathbb{R}^2$, under the natural linear growth energy bound, we show that the blowing down limit is unique. Furthermore, if the solution has finite Morse index, the blowing down limit satisfies the multiplicity one property.

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1. Introduction

Let $u \in C^2(\mathbb{R}^2)$, $|u| < 1$ be a solution to the problem

$$\Delta u = W'(u) \tag{1.1}$$

where W is a standard double-well potential, i.e. $W \in C^2([-1,1])$ satisfying $W > 0$ in $(-1,1)$, $W(\pm 1) = 0$ and $W''(\pm 1) > 0$.

Assume the energy of u grows linearly, i.e. there exists a constant $C > 0$ such that

$$\int_{B_R(0)} \left[\frac{1}{2}|\nabla u|^2 + W(u)\right] \leq CR, \quad \forall R > 0. \tag{1.2}$$

For $\varepsilon \to 0$, let the blowing down sequence be

$$u_\varepsilon(x,y) := u(\varepsilon^{-1}x, \varepsilon^{-1}y).$$

By (1.2), we can assume that, up to a subsequence of $\varepsilon \to 0$,

$$\varepsilon|\nabla u_\varepsilon|^2 dxdy \rightharpoonup \mu_1,$$
$$\frac{1}{\varepsilon}W(u_\varepsilon)dxdy \rightharpoonup \mu_2,$$

weakly as Radon measures on any compact set of $\mathbb{R}^2$. Denote $\mu = \mu_1/2 + \mu_2$ and $\Sigma = \mathrm{spt}(\mu)$ the support of μ.

We can also assume the matrix valued measures

$$\varepsilon\nabla u_\varepsilon \otimes \nabla u_\varepsilon dxdy \rightharpoonup [\tau_{\alpha\beta}]\mu_1,$$

where $[\tau_{\alpha\beta}]$, $1 \leq \alpha, \beta \leq 2$, is measurable with respect to μ_1. Moreover, τ is nonnegative definite μ_1-almost everywhere and it satisfies

$$\sum_{\alpha=1}^{2} \tau_{\alpha\alpha} = 1, \quad \mu_1 - a.e.$$

By [4,5] and [6], we have the following characterization about the convergence of u_ε:

Theorem 1.1. (i) *$u_\varepsilon \to \pm 1$ uniformly on any compact set of $\mathbb{R}^2 \setminus \Sigma$;*
(ii) *there exists $N \in \mathbb{N}$ and N unit vectors $e_\alpha, 1 \leq \alpha \leq N$, such that $\Sigma = \cup_{\alpha=1}^{N} L_\alpha$, where*

$$L_\alpha := \{te_\alpha : t \geq 0\};$$

(iii) *$\mu_1 = 2\mu_2 = \sigma_0 \sum_{\alpha=1}^{N} n_\alpha \mathcal{H}^1 \lfloor_{L_\alpha}$, where σ_0 is a constant and $n_\alpha \in \mathbb{N}$;*
(iv) *$I - \tau = e_\alpha \otimes e_\alpha$ on $L_\alpha \setminus \{0\}$;*
(v) *$\sum_{\alpha=1}^{N} n_\alpha e_\alpha = 0$.*

In the above, the constant σ_0 is defined as follows. There exists a function $g \in C^2(\mathbb{R})$ satisfying

$$\begin{cases} g'' = W'(g), \quad \text{on } \mathbb{R}, \\ g(0) = 0, \\ \lim_{t \to \pm\infty} g(t) = \pm 1. \end{cases} \tag{1.3}$$

Moreover, the following identity holds for g:

$$g'(t) = \sqrt{2W(g(t))} > 0, \quad \text{on } \mathbb{R}. \tag{1.4}$$

As $t \to \pm\infty$, $g(t)$ converges to ± 1 exponentially. Hence the following quantity is finite:

$$\sigma_0 := \int_{-\infty}^{+\infty} \left[\frac{1}{2} \left| g'(t) \right|^2 + W(g(t)) \right] dt = \int_{-\infty}^{+\infty} \left| g'(t) \right|^2 dt.$$

In Theorem 1.1, we do not claim the uniqueness of Σ as well as (n_α), because they are obtained by a compactness argument. They may depend on the choice of subsequences of $\varepsilon \to 0$. This is similar to the problem on uniqueness of tangent cones to generalized minimal submanifolds (stationary varifolds, minimizing currents). Our first main result is

Theorem 1.2. *Σ and $(n_1, \ldots, n_N)$ are uniquely determined by u.*

This uniqueness comes from a convexity structure, first discovered in Smyrnelis [11]. This convexity is basically a consequence of the Modica inequality [10]. Note that in higher dimensions it is expected that the Modica inequality implies a kind of mean convexity.

Next we further assume that u has finite Morse index, i.e. the maximal dimension of linear subspaces of

$$\left\{\varphi \in C_0^\infty(\mathbb{R}^2) : \int_{\mathbb{R}^2} \left[|\nabla\varphi|^2 + W''(u)\varphi^2\right] \leq 0\right\}$$

is finite. This is equivalent to the fact that u is stable outside a compact set (see [1]), i.e. there exists a compact set K such that for any $\varphi \in C_0^\infty(\mathbb{R}^2 \setminus K)$,

$$\int_{\mathbb{R}^2} \left[|\nabla\varphi|^2 + W''(u)\varphi^2\right] \geq 0.$$

Our second result is a multiplicity one property on the blowing down limit.

Theorem 1.3. *Let u be a solution of (1.1) with finite Morse index. Then in the blowing down limit, $n_\alpha = 1$ for every $\alpha = 1, \ldots, N$.*

As in [2], we introduce the following notations. Assume e_α are in clockwise order. For each $\alpha = 1, \ldots, N$, let $L_\alpha^\pm$ be the rays generated by the vector $(e_\alpha + e_{\alpha+1})/2$ and $(e_\alpha + e_{\alpha-1})/2$ respectively (with obvious modification at the end points $\alpha = 1, N$). Denote Ω_α to be the sector bounded by $L_\alpha^\pm$. Our final result says

Theorem 1.4. *Let u be a solution of (1.1) in $\mathbb{R}^2$ with finite Morse index, and Ω_α be defined as above. In each Ω_α, which we assume to be the sector $\{(x, y) : -\lambda_- x < y < \lambda_+ x, x \geq 0\}$ for two positive constants $\lambda_\pm$, there exist three constants C, R_0 and t_α such that*

$$\sup_{-\lambda_- x < y < \lambda_+ x} \left|u(x, y) - g(y - t_\alpha)\right| \leq Ce^{-\frac{x}{C}}, \quad \forall x > R_0.$$

If we have known Theorem 1.3, this theorem will follow from the refined asymptotic result in [2] or [4, Lemma 4.3]. Here the point is, we can prove Theorem 1.3 and Theorem 1.4 at the same time. This will be achieved by generalizing Gui's method in [4] to the multiple interface setting.

In [4] the multiplicity one case was treated. The method amounts to viewing the equation as an evolution problem in the form

$$\frac{d^2u}{dt^2} = \nabla\mathcal{J}(u).$$

Let $\mathcal{M}$ be the manifold of one dimensional solutions. (This mainfold is the real line $\mathbb{R}$, formed by translations of the one dimensional solution g.) Take the nearest point $P(u)$ on $\mathcal{M}$ to u. Then $u - P(u)$ almost lies in the subspace orthogonal to the first eigenfunction of $P(u)$. By some more computations we get

$$\frac{d^2}{dt^2}\|u - P(u)\|^2 \geq \mu\|u - P(u)\|^2 + \mathcal{R}, \tag{1.5}$$

where μ is a positive constant. The norm $\|\cdot\|$ is usually taken to be a L^2 one. The remainder term $\mathcal{R}$ is of the order $O(e^{-ct})$ for some constant $c > 0$. This then implies the exponential convergence of $\|u - P(u)\|^2$, and the exponential convergence of $u(t)$ with some more work.

The existence of the positive constant μ is related to the positivity of the second eigenvalue of the linearized problem at g (the first eigenvalue is 0 due to the translation invariance of (1.1)), which is a nondegeneracy condition for g.

Our main contribution is to show that a similar phenomena holds in the multiple interface case. Here we have to choose several pieces of the one dimensional solution g, patched together suitably, to approximate u. Then some boundary terms appear in the process of deducing (1.5). But fortunately all of these boundary terms give a positive term, hence (1.5) still holds and the exponential convergence follows.

The above three results show that the main results in [2,7–9] hold for finite Morse index solutions with linear energy growth.

It is conjectured that finite Morse index solutions of (1.1) satisfy the energy growth bound (1.2). This conjecture has recently been proved by the author and Wei [13], where results of the current paper is also used. On the other hand, if a solution satisfies the conclusion of Theorem 1.4, it has finite Morse index (see [7]).

Finally, concerning the multiplicity one property established in Theorem 1.4, we refer the reader to one interesting conjecture in [3], which says that in some cases there is no interfaces clustering (that is, there is only one transition layer) for solutions of the singularly perturbed Allen–Cahn equation

$$\varepsilon\Delta u_\varepsilon = \frac{1}{\varepsilon}W'(u_\varepsilon),$$

provided u_ε satisfies some stability condition. Whether this conjecture is true still remains open.

In this paper, a point in $\mathbb{R}^2$ is denoted by $X = (x, y)$.

The organization of this paper is as follows. In Section 2 we prove Theorem 1.2. Theorem 1.3 and Theorem 1.4 is proved in Section 3 at the same time.

2. Uniqueness of the blowing down limit

By direct integration by parts, we get the stationary condition

$$\int_{\mathbb{R}^2}\left[\left(\frac{1}{2}|\nabla u|^2 + W(u)\right)\mathrm{div}Y - DY(\nabla u, \nabla u)\right] = 0, \quad \forall Y \in C_0^\infty(\mathbb{R}^2, \mathbb{R}^2).$$

Following [11], this condition implies the existence of a function $U \in C^3(\mathbb{R}^2)$ satisfying

$$\nabla^2 U = \begin{bmatrix} u_x^2 - u_y^2 + 2W(u) & 2u_x u_y \\ 2u_x u_y & u_y^2 - u_x^2 + 2W(u) \end{bmatrix}.$$

Moreover, by the Modica inequality (see [10])

$$\frac{1}{2}|\nabla u|^2 \le W(u), \quad \text{in } \mathbb{R}^2,$$

U is convex. After subtracting an affine function, we can assume $U(0) = 0$ and $\nabla U(0) = 0$. Hence by the convexity of U, $U \ge 0$ in $\mathbb{R}^2$.

Lemma 2.1. *There exists a constant C such that,*

$$U(x,y) \le C\left(|x|+|y|\right), \quad in \ \ \mathbb{R}^2.$$

Proof. By definition,

$$\Delta U = 4W(u). \tag{2.1}$$

Then for any $R>0$,

$$\fint_{\partial B_R(0)} U = \int_0^R \frac{d}{dr}\left(\fint_{\partial B_r(0)} U\right) = \int_0^R \left(\frac{1}{2\pi r}\int_{B_r(0)} 4W(u)\right) \le CR,$$

where we have used (1.2).

The conclusion follows from this integral bound and the convexity of U. □

By this linear growth bound and the convexity of U, as $\varepsilon \to 0$,

$$U_\varepsilon(x,y) := \varepsilon U(\varepsilon^{-1}x, \varepsilon^{-1}y) \to U_\infty(x,y)$$

uniformly on compact sets of $\mathbb{R}^2$. Here U_∞ is a 1-homogeneous, nonnegative convex function. By convexity, this limit is independent of subsequences of $\varepsilon \to 0$.

Take a sequence $\varepsilon_i \to 0$ such that the blowing down limit of u_{ε_i} is $\Sigma = \cup_{\alpha=1}^N \{te_\alpha : t \ge 0\}$ and the density on $\{te_\alpha : t \ge 0\}$ is n_α. Then outside Σ, by the strict convexity of W near ± 1,

$$|\nabla u_{\varepsilon_i}(X)|^2 + W(u_{\varepsilon_i}(X)) \le Ce^{-c\varepsilon_i^{-1} dist(X,\Sigma)}.$$

Because

$$\nabla^2 U_{\varepsilon_i} = \begin{bmatrix} \varepsilon_i u^2_{\varepsilon_i,x} - \varepsilon_i u^2_{\varepsilon_i,y} + \frac{2}{\varepsilon_i}W(u_{\varepsilon_i,}) & 2\varepsilon_i u_{\varepsilon_i,x}u_{\varepsilon_i,y} \\ 2\varepsilon_i u_{\varepsilon_i,x}u_{\varepsilon_i,y} & \varepsilon_i u^2_{\varepsilon_i,y} - \varepsilon_i u^2_{\varepsilon_i,x} + \frac{2}{\varepsilon_i}W(u) \end{bmatrix},$$

we also have

$$|\nabla^2 U_{\varepsilon_i}(X)|^2 \le Ce^{-c\varepsilon_i^{-1} dist(X,\Sigma)}.$$

Hence $\nabla^2 U_\infty \equiv 0$ in $\mathbb{R}^2 \setminus \Sigma$, that is, U_∞ is linear in every connected component of $\mathbb{R}^2 \setminus \Sigma$. Thus the set $\{U_\infty < 1\}$ is a convex polygon with its vertex points lying on Σ. Now it is clear that Σ is uniquely determined by U_∞. Since U_∞ is independent of the choice of subsequences of $\varepsilon \to 0$, Σ also does not depend on the choice of subsequences of $\varepsilon \to 0$.

In a neighborhood of $\{te_\alpha : t > 0\}$, using the $(e_\alpha, e_\alpha^\perp)$ coordinates, the matrix valued measure $\nabla^2 U_{\varepsilon_i} dxdy$ can be written as

$$\nabla^2 U_{\varepsilon_i} dxdy = \begin{bmatrix} \varepsilon_i u^2_{\varepsilon_i,e_\alpha} - \varepsilon_i u^2_{\varepsilon_i,e_\alpha^\perp} + \frac{2}{\varepsilon_i}W(u_{\varepsilon_i,}) & 2\varepsilon_i u_{\varepsilon_i,e_\alpha}u_{\varepsilon_i,e_\alpha^\perp} \\ 2\varepsilon_i u_{\varepsilon_i,e_\alpha}u_{\varepsilon_i,e_\alpha^\perp} & \varepsilon_i u^2_{\varepsilon_i,e_\alpha^\perp} - \varepsilon_i u^2_{\varepsilon_i,e_\alpha} + \frac{2}{\varepsilon_i}W(u) \end{bmatrix} dxdy,$$

By Theorem 1.1, after passing to the limit, we obtain that in a neighborhood of $\{te_\alpha : t > 0\}$, the limit of $\nabla^2 U_{\varepsilon_i} dxdy$ equals

$$\begin{bmatrix} 0 & 0 \\ 0 & 2n_\alpha \sigma_0 \mathcal{H}^1 \lfloor_{\{te_\alpha : t \ge 0\}} \end{bmatrix}.$$

Hence across the ray $\{te_\alpha : t \geq 0\}$, ∇U_∞ has a jump $2n_\alpha \sigma_0 e_\alpha^\perp$. In other words, let $e^\pm = \nabla U_\infty$ on each side of $\{te_\alpha : t \geq 0\}$, then

$$e^+ - e^- = 2n_\alpha \sigma_0 e_\alpha^\perp.$$

Thus n_α is uniquely determined by U_∞. This finishes the proof of Theorem 1.2.

3. The multiplicity one property

Since u is assumed to have finite Morse index, it is stable outside a compact set. Then standard argument using the stable De Giorgi theorem gives the following

Lemma 3.1. *For any* $X_i = (x_i, y_i) \in u^{-1}(0) \to \infty$,

$$u_i(x, y) := u(x_i + x, y_i + y)$$

converges to a one dimensional solution $g(e \cdot X)$ *in* $C^2_{loc}(\mathbb{R}^2)$, *where* e *is a unit vector.*

Recall the sector Ω_α introduced in Section 1. The nodal set of u in Ω_α can be described in the following way.

Lemma 3.2. *There exists an* $R_1 > 0$ *large such that, for each* α, *in* $\Omega_\alpha \setminus B_{R_1}(0)$, $\{u = 0\}$ *consists of* n_α *curves, which can be represented by the graph of functions defined on* L_α, *with their* C^1 *norm converging to* 0 *at infinity.*

Proof. Take an Ω_α, which we assume to be the sector $\{(x, y) : -\lambda_- x < y < \lambda_+ x, x > 0\}$ for two constants $\lambda_\pm > 0$. L_α is assumed to be the ray $\{x > 0, y = 0\}$. By [12, Theorem 5], for all ε small, there exists a constant $t_\varepsilon \in (-1/2, 1/2)$, such that

$$\{u_\varepsilon = t_\varepsilon\} \cap (B_2(0) \setminus B_{1/2}(0)) \cap \Omega_\alpha$$

consists of n_α curves in the form

$$y = h^i_\varepsilon(x), \quad \text{for } 1/2 \leq x \leq 2, \quad 1 \leq i \leq n_\alpha,$$

where $\|h^i_\varepsilon\|_{C^{1,1/2}([1/2,2])}$ is uniformly bounded. By [6], for each i, h^i_ε converges to 0 in $C^1([1/2, 2])$ as $\varepsilon \to 0$.

By Lemma 3.1, for each $t \in [-3/4, 3/4]$, $\{u_\varepsilon = t\}$ consists of n_α curves, in the form

$$y = h^i_\varepsilon(x, t), \quad \text{for } 1/2 \leq x \leq 2, \quad 1 \leq i \leq n_\alpha,$$

which lies in an $O(\varepsilon)$ neighborhood of $\{u_\varepsilon = t_\varepsilon\}$. Moreover, after a scaling and using Lemma 3.1, we get

$$\lim_{\varepsilon \to 0} \sup_{1/2 \leq x \leq 2} \left| \frac{d}{dx} h^i_\varepsilon(x, t) \right| = 0.$$

Rescaling back to u we conclude the proof. □

Now we are in the following situation:

(H1) There are two positive constants $R > 0$ large and $\lambda > 0$.

(H2) The domain $\mathcal{D} := \{(x, y) : |y| < \lambda x, x > R\}$.

(H3) $u \in C^2(\overline{\mathcal{D}})$ satisfies (1.1) in $\mathcal{D}$.

(H4) $\{u = 0\}$ consists of N curves $\{y = f_i(x)\}$, $1 \le i \le N$, where $f_i \in C^\infty([R, +\infty))$ satisfying

$$f_1 < f_2 < \cdots < f_N,$$
$$\lim_{x \to +\infty} f_i'(x) = 0, \quad \forall 1 \le i \le N.$$

The last condition implies that

$$\lim_{x \to +\infty} \frac{|f_i(x)|}{|x|} = 0, \quad \forall 1 \le i \le N.$$

The main goal in this section is to prove

Theorem 3.3. *We must have $N = 1$. Moreover, there exists a constant t such that for all $x > R$,*

$$\left|f(x) - t\right| \le Ce^{-\frac{x}{C}},$$

and

$$\sup_{-\lambda x < y < \lambda x} \left|u(x, y) - g(y - t)\right| \le Ce^{-\frac{x}{C}},$$

where the constant C depends only on W.

Theorem 1.3 and 1.4 follow by combining this theorem with Theorem 1.1, Theorem 1.2 and Lemma 3.2.

Possibly by a change of sign, assume $u < 0$ in $\{y < f_1(x)\}$.

Lemma 3.4. *For any $1 \le i \le N$ and $t \to +\infty$,*

$$u^t(x, y) := u(t + x, f_i(t) + y)$$

converges to $g(y)$ in $C^2_{loc}(\mathbb{R}^2)$.

Proof. This is a consequence of Lemma 3.1 and Lemma 3.2. Note that $\{u^t = 0\} = \{y = f^t(x)\}$ where $f^t(x) := f_i(x + t) - f_i(t)$. As $t \to +\infty$, $\frac{df^t}{dx}$ converges to 0 uniformly on any compact set of $\mathbb{R}$. Hence by noting that $f^t(0) = 0$, f^t also converges to 0 uniformly on any compact set of $\mathbb{R}$. This implies that the limit $u^\infty = 0$ on $\{y = 0\}$. From this we see $u_\infty(x, y) \equiv g(y)$. Since this limit is independent of subsequences of $t \to +\infty$, we finish the proof. $\square$

Lemma 3.5. *In $\overline{\mathcal{D}}$,*

$$1 - u(x, y)^2 \le Ce^{-c \min_i |y - f_i(x)|}.$$

Proof. By the previous lemma, for any $M > 0$, if we have chosen R large enough, $u^2 > 1 - \sigma(M)$ in $\{(x, y) : |y - f_i(x)| > M, \forall i\}$, where $\sigma(M)$ is a constant depending on M satisfying $\lim_{M \to +\infty} \sigma(M) = 0$. By choosing M large (then $\sigma(M)$ can be made small so that W is strictly convex in $(-1, -1 + \sigma(M)) \cup (1 - \sigma(M), 1))$, in $\{(x, y) : |y - f_i(x)| > M, \forall i\}$,

$$\Delta W(u) \ge cW(u).$$

From this we deduce the exponential decay

$$W(u) \le Ce^{-c\,dist(X, \cup_i \{(x,y): |y-f_i(x)|<M\})}.$$

Finally, because $|f_i'(x)| < 1$, the distance to $\{y = f_i(x)\}$ is comparable to $|y - f_i(x)|$. This finishes the proof. □

As a consequence,

$$1 - u(x,y)^2 \sim O(e^{-cx}) \quad \text{on } \{y = \pm\lambda x, x > R\}. \tag{3.1}$$

Another consequence of this exponential decay is:

Corollary 3.6. *In $\mathcal{D}$,*

$$|u_x(x,y)| + |u_{xx}(x,y)| \le Ce^{-\frac{\min_i |y-f_i(x)|}{C}}.$$

This follows from standard gradient estimates.

This exponential decay implies that

$$\int_{-\lambda x}^{\lambda x} \left[u_x(x,y)^2 + u_{xx}(x,y)^2\right] dy \le C, \quad \forall x > R. \tag{3.2}$$

Next we show that different components of $\{u = 0\}$ are $O(1)$ separated.

Lemma 3.7. *For any $1 \le i \le N-1$,*

$$\lim_{x\to+\infty} \left(f_{i+1}(x) - f_i(x)\right) = +\infty.$$

Proof. By Lemma 3.4, for any $t \to +\infty$,

$$u^t(x,y) := u(x+t, y+f_i(t))$$

converges uniformly to $g(y)$ on any compact set of $\mathbb{R}^2$.

From this we see, for any $L > 0$, if t is large enough, $u^t > 0$ on $\{x = 0, 0 < y < L\}$ and $u^t < 0$ on $\{x = 0, -L < y < 0\}$. The conclusion follows from this claim directly. □

The following identity is the Hamiltonian identity of [4]. For completeness we include a proof here.

Proposition 3.8. *For any $x > R$,*

$$\int_{-\lambda x}^{\lambda x} \left[\frac{u_y^2 - u_x^2}{2} + W(u)\right] dy = N\sigma_0 + O(e^{-cx}).$$

Proof. First, differentiating in x, integrating by parts and using (3.1) leads to

$$\frac{d}{dx}\int_{-\lambda x}^{\lambda x} \left[\frac{u_y^2 - u_x^2}{2} + W(u)\right] dy = O(e^{-cx}). \tag{3.3}$$

Next, by Lemma 3.5, for any $\delta > 0$, there exists an $L > 0$ such that for all $x > R$,

$$\int_{\{y\in(-\lambda x,\lambda x): |y-f_i(x)|>L, \forall i\}} \left[\frac{u_y^2 - u_x^2}{2} + W(u)\right] dy \le \delta. \tag{3.4}$$

While for each $i = 1, \ldots, N$, by Lemma 3.4, we have

$$\lim_{x\to+\infty}\int_{f_i(x)-L}^{f_i(x)+L}\left[\frac{u_y^2-u_x^2}{2}+W(u)\right]dy$$
$$=\int_{-L}^{L}\left[\frac{1}{2}g'(y)^2+W(g(y))\right]dy=\sigma_0+O(\delta), \tag{3.5}$$

where in the last step we have used the exponential convergence of g at infinity.

Combining (3.4) and (3.5), by noting that δ can be arbitrarily small, we get

$$\lim_{x\to+\infty}\int_{-\lambda x}^{\lambda x}\left[\frac{u_y^2-u_x^2}{2}+W(u)\right]dy=N\sigma_0.$$

The conclusion of this lemma follows by combining this identity and (3.3). □

We need the following form of nondegeneracy of the one dimensional solution g.

Proposition 3.9. *There exist two constants $L_0 > 0$ and $\mu > 0$ so that the following holds. For any constants $L^+ > L_0$ and $L^- > L_0$ and $v \in H^1((-L^-, L^+))$ satisfying*

$$\int_{-L^-}^{L^+} v(t)g'(t)dt = 0, \tag{3.6}$$

we have

$$\int_{-L^-}^{L^+}\left[\left|\frac{dv}{dt}(t)\right|^2+W''(g(t))v(t)^2\right]dt\geq\mu\int_{-L^-}^{L^+}v(t)^2dt. \tag{3.7}$$

Proof. Assume by the contrary, there exist $L_j^\pm \to +\infty$ and $v_j \in H^1((-L_j^-, L_j^+))$ satisfying

$$\int_{-L_j^-}^{L_j^+} v_j(t)g'(t)dt = 0, \tag{3.8}$$

and

$$\int_{-L_j^-}^{L_j^+} v_j(t)^2dt = 1, \tag{3.9}$$

but

$$\int_{-L_j^-}^{L_j^+}\left[\left|\frac{dv_j}{dt}(t)\right|^2+W''(g(t))v_j(t)^2\right]dt\leq\frac{1}{j}. \tag{3.10}$$

From the last two assumptions we deduce that

$$\int_{-L_j^-}^{L_j^+}\left|\frac{dv_j}{dt}(t)\right|^2dt\leq C, \tag{3.11}$$

for some constant C depending only on $\sup|W''|$. Hence the 1/2-Hölder seminorm of v_j is uniformly bounded. Then by (3.9), $\sup|v_j|$ is also uniformly bounded. Assume v_j converges to v_∞ in $C_{loc}(\mathbb{R})$.

By the exponential decay of g' at infinity, (3.8) can be passed to the limit, which gives

$$\int_{-\infty}^{+\infty} v_\infty(t)g'(t)dt = 0. \tag{3.12}$$

(3.9) and (3.11) can also be passed to the limit, leading to

$$\int_{-\infty}^{+\infty} \left[v_\infty(t)^2 + \left| \frac{dv_\infty}{dt}(t) \right|^2 \right] dt \le C + 1. \tag{3.13}$$

Because g converges to ± 1 at $\pm\infty$ respectively, there exists an R_2 such that

$$W''(g(t)) \ge c_0 := \frac{1}{2} \min \{W''(-1), W''(1)\} > 0, \quad \text{in } \{|t| \ge R_2\}. \tag{3.14}$$

Thus for any $R \ge R_2$,

$$\begin{aligned} \int_{-R}^{R} \left[\left| \frac{dv_\infty}{dt}(t) \right|^2 + W''(g(t)) v_\infty(t)^2 \right] dt &\le \liminf_{j\to+\infty} \int_{-R}^{R} \left[\left| \frac{dv_j}{dt}(t) \right|^2 + W''(g(t)) v_j(t)^2 \right] dt \\ &\le \liminf_{j\to+\infty} \int_{-L_j^-}^{L_j^+} \left[\left| \frac{dv_j}{dt}(t) \right|^2 + W''(g(t)) v_j(t)^2 \right] dt \\ &\le 0. \end{aligned}$$

By (3.13), we can let $R \to +\infty$, which leads to

$$\int_{-\infty}^{+\infty} \left[\left| \frac{dv_\infty}{dt}(t) \right|^2 + W''(g(t)) v_\infty(t)^2 \right] dt \le 0.$$

Then by the spectrum theory for $-\frac{d^2}{dt^2} + W''(g(t))$ (see for example [2, Lemma 1.1]) and (3.12), $v_\infty \equiv 0$.

By the convergence of v_j in $C_{loc}(\mathbb{R})$,

$$\lim_{j\to+\infty} \int_{-R_2}^{R_2} v_j(t)^2 dt = 0. \tag{3.15}$$

Substituting this into (3.10), by noting (3.14), we get

$$\int_{(-L_j^-, -R_2)\cup(R_2, L_j^+)} v_j(t)^2 dt \le C \left(\frac{1}{j} + \int_{-R_2}^{R_2} v_j(t)^2 dt \right) \to 0.$$

Combining this with (3.15) we get a contradiction with (3.9). Thus under the assumptions (3.8) and (3.9), (3.10) cannot hold. □

With these preliminaries, we come to the proof of Theorem 3.3.

Proof of Theorem 3.3. Given a tuple $(t_1, \dots, t_N)$ with $t_1 < \dots < t_N$, define

$$G(y; t_1, \dots, t_N) = \begin{cases} g(y - t_1), & y < t_1^+, \\ \min\{g(y - t_1), -g(y - t_2)\}, & t_1^+ = t_2^- < x < t_2^+, \\ \min\{-g(y - t_2), g(y - t_3)\}, & t_2^+ = t_3^- < x < t_3^+, \\ \cdots . \end{cases}$$

In the above,

$$t_i^+ := \frac{t_i + t_{i+1}}{2}, \quad t_i^- := \frac{t_{i-1} + t_i}{2},$$

and we adopt the convention that $t_1^- = -\lambda x$ and $t_N^+ = \lambda x$.

Note that $G(y; t_i)$ is continuous, while its derivative in y has a jump at t_i^+. (In fact, the left and right derivatives at each t_i^+ only differ by a sign.)

Next we define

$$F(x; t_1, \ldots, t_N) := \int_{-\lambda x}^{\lambda x} \left|u(x, y) - G(y; t_1, \ldots, t_N)\right|^2 dy.$$

We divide the proof into three steps.

Step 1 As $x \to +\infty$, $\int_{-\lambda x}^{\lambda x} \left|u(x, y) - G(y; f_i(x))\right|^2 dy \to 0$.

This follows from Lemma 3.4 and Lemma 3.5.

Step 2 By Step 1,

$$\lim_{x \to +\infty} F(x; f_1(x), \ldots, f_N(x)) = 0.$$

Moreover, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that, if $|t_i - f_i(x)| > \delta$ for some i, then

$$\liminf_{x \to +\infty} F(x; t_1, \ldots, t_N) \geq \varepsilon. \tag{3.16}$$

Direct calculations give

$$\frac{\partial F}{\partial t_i}(x; t_1, \ldots, t_N) = 2(-1)^i \int_{t_i^-}^{t_i^+} \left[u(x, y) - (-1)^{i-1} g(y - t_i)\right] g'(y - t_i) dy. \tag{3.17}$$

$$\begin{aligned} \frac{\partial^2 F}{\partial t_i^2}(x; t_1, \ldots, t_N) = {} & 2 \int_{t_i^-}^{t_i^+} g'(y - t_i)^2 dy \\ & + 2(-1)^{i+1} \int_{t_i^-}^{t_i^+} \left[u(x, y) - (-1)^{i-1} g(y - t_i)\right] g''(y - t_i) dy \\ & + O(e^{-c \min\{t_i - t_{i-1}, t_{i+1} - t_i\}}). \end{aligned} \tag{3.18}$$

By Step 1, Lemma 3.5 and the exponential decay of g'' at infinity, there exists a $\sigma > 0$ such that, for any $(t_1, \ldots, t_N)$ satisfying $|t_i - f_i(x)| < \sigma$, $\frac{\partial^2 F}{\partial t_i^2}(x; t_i) > \sigma$.

Finally, $\frac{\partial^2 F}{\partial t_i \partial t_j}(x; t_i) = 0$ if $|i - j| > 1$ and

$$\left|\frac{\partial^2 F}{\partial t_i \partial t_{i+1}}(x; t_i)\right| \leq C e^{-c(t_{i+1} - t_i)}.$$

Combining this with (3.18) we see the matrix $[\frac{\partial^2 F}{\partial t_i \partial t_j}(x; t_i)]$ is positively definite for those $(t_1, \ldots, t_N)$ satisfying the condition that $|t_i - f_i(x)|$ is small enough for all i.

Combining the above analysis, we see for all x large, there exists a unique tuple $(t_i(x))$ such that

$$F(x; t_i(x)) = \min_{(t_i) \in \mathbb{R}^N} F(x; t_i).$$

Moreover,

$$\lim_{x\to+\infty} |t_i(x) - f_i(x)| = 0, \quad \forall 1 \le i \le N. \tag{3.19}$$

By the implicit function theorem, for each i, $t_i(x)$ is twice differentiable in x.

Lemma 3.7 and (3.19) imply that for any $1 \le i \le N-1$,

$$t_{i+1}(x) - t_i(x) \to +\infty, \quad \text{as } x \to +\infty. \tag{3.20}$$

Let

$$v(x,y) := u(x,y) - G(y; t_i(x)).$$

Clearly

$$\lim_{x\to+\infty} \|v\|_{L^2(-\lambda x,\lambda x)} = \lim_{x\to+\infty} F(x; t_i(x)) = 0. \tag{3.21}$$

In the following we denote $g^* := G(y; t_i(x))$ and

$$g_i(y) := (-1)^{i-1} g(y - t_i(x)), \quad \text{for } y \in (t_i^-, t_i^+).$$

By definition,

$$0 = \frac{\partial F}{\partial t_i}(x; t_i(x)) = 2\int_{t_i^-(x)}^{t_i^+(x)} (u - g_i)\, g_i'. \tag{3.22}$$

Differentiating (3.22) with respect to x leads to

$$\begin{aligned}
&\left[\int_{t_i^-(x)}^{t_i^+(x)} |g_i'|^2 - (u-g_i)\, g_i''\right] t_i'(x) + \int_{t_i^-(x)}^{t_i^+(x)} u_x g_i' \\
&= -\left[u(x,t_i^+(x)) - g_i(t_i^+(x))\right] g_i'(t_i^+(x)) \frac{t_i'(x) + t_{i+1}'(x)}{2} \\
&\quad + \left[u(x,t_i^-(x)) - g_i(t_i^-(x))\right] g_i'(t_i^-(x)) \frac{t_i'(x) + t_{i-1}'(x)}{2}.
\end{aligned} \tag{3.23}$$

Note that by the result in Step 1 and the exponential decay of g'' at infinity,

$$\begin{aligned}
\lim_{x\to+\infty} \int_{t_i^-(x)}^{t_i^+(x)} (u-g_i)\, g'' &\le \lim_{x\to+\infty} \left[\int_{t_i^-(x)}^{t_i^+(x)} (u-g_i)^2\right]^{\frac{1}{2}} \left[\int_{t_i^-(x)}^{t_i^+(x)} \left|g''\right|^2\right]^{1/2} \\
&= 0,
\end{aligned}$$

while by (3.20), there exists a constant $c > 0$ such that

$$\int_{t_i^-(x)}^{t_i^+(x)} |g_i'|^2 \ge c, \quad \text{for all } x \text{ large}.$$

By Lemma 3.5 and (3.20), $u(x, t_i^\pm(x))$ and $g_i(t_i^\pm(x))$ all converge to 0 as $x \to +\infty$. Thus by (3.23) we obtain

$$t_i'(x) = -\frac{\int_{t_i^-(x)}^{t_i^+(x)} u_x g_i'}{\left[\int_{t_i^-(x)}^{t_i^+(x)} |g_i'|^2\right] + o(1)} + o(1)\sum_{j\neq i} \frac{\left|\int_{t_j^-(x)}^{t_j^+(x)} u_x g_j'\right|}{\left[\int_{t_j^-(x)}^{t_j^+(x)} |g_j'|^2\right] + o(1)}$$
$$+O(e^{-cx}) \to 0, \quad \text{as } x \to +\infty. \tag{3.24}$$

Differentiating this once again we see $t_i''(x)$ also converges to 0 as $x \to +\infty$.

Similar to the calculation in [4, page 927], we have

$$\begin{aligned}
&\int_{t_i^-(x)}^{t_i^+(x)} \left[\left(\frac{u_y^2 - u_x^2}{2} + W(u)\right) - \frac{|g_i'|^2}{2} - W(g_i)\right] \\
&= \int_{t_i^-(x)}^{t_i^+(x)} \left(\frac{u_y^2 - |g_i'|^2}{2} + W(u) - W(g_i) - \frac{u_x^2}{2}\right) \\
&= \int_{t_i^-(x)}^{t_i^+(x)} \left[W(u) - W(g_i) - \frac{W'(u) + W'(g_i)}{2}(u - g_i)\right] \\
&\quad + \frac{1}{2}\int_{t_i^-(x)}^{t_i^+(x)} \left[(u - g_i)\, u_{xx} - u_x^2\right] + \mathcal{B},
\end{aligned} \tag{3.25}$$

where $\mathcal{B}$ is the boundary terms coming from integrating by parts. In the above we have used

$$\begin{aligned}
\int_{t_i^-(x)}^{t_i^+(x)} u_y^2 - |g_i'|^2 &= \int_{t_i^-(x)}^{t_i^+(x)} (u_y - g_i')(u_y + g_i') \\
&= -\int_{t_i^-(x)}^{t_i^+(x)} (u - g_i)(u_{yy} + g_i'') \\
&\quad + \left[u(x, t_i^+(x)) - g_i(t_i^+(x))\right]\left[u_y(x, t_i^+(x)) + g_i'(t_i^+(x))\right] \\
&\quad - \left[u(x, t_i^-(x)) - g_i(t_i^-(x))\right]\left[u_y(x, t_i^-(x)) + g_i'(t_i^-(x))\right] \\
&= -\int_{t_i^-(x)}^{t_i^+(x)} (u - g_i)\left[W'(u) + W'(g_i)\right] + \int_{t_i^-(x)}^{t_i^+(x)} u_{xx}(u - g_i) \\
&\quad + \left[u(x, t_i^+(x)) - g_i(t_i^+(x))\right]\left[u_y(x, t_i^+(x)) + g_i'(t_i^+(x))\right] \\
&\quad - \left[u(x, t_i^-(x)) - g_i(t_i^-(x))\right]\left[u_y(x, t_i^-(x)) + g_i'(t_i^-(x))\right].
\end{aligned}$$

Summing (3.25) in i and using the Hamiltonian identity (Proposition 3.8), we obtain

$$\begin{aligned}
\int_{-\lambda x}^{\lambda x} \left[u_{xx}(u - g^*) - u_x^2\right] &= \sum_i \int_{t_i^-(x)}^{t_i^+(x)} \left[(u - g_i)\, u_{xx} - u_x^2\right] \\
&= -2\sum_i \left[u(x, t_i^+(x)) - g_i(t_i^+(x))\right] g_i'(t_i^+(x)) + o(\|v\|^2) \\
&\quad + 2\sum_i \left[\int_{t_i^+(x)}^{+\infty} |g_i'|^2 + \int_{-\infty}^{t_i^-(x)} |g_i'|^2\right] + O(e^{-cx}).
\end{aligned} \tag{3.26}$$

On the other hand, similar to [4, Eq. (4.35)], we have

$$\begin{aligned}
\int_{t_i^-(x)}^{t_i^+(x)} u_{xx}\,(u-g_i) &= \int_{t_i^-(x)}^{t_i^+(x)} \left(W'(u)-u_{yy}\right)(u-g_i)\\
&= \int_{t_i^-(x)}^{t_i^+(x)} \left[W'(u)-W'(g_i)-W''(g_i)\,(u-g_i)\right](u-g_i)\\
&\quad + \int_{t_i^-(x)}^{t_i^+(x)} \left(g_i''-u_{yy}\right)(u-g_i)+W''(g_i)\,(u-g_i)^2\\
&= o(\|v\|^2)+\int_{t_i^-(x)}^{t_i^+(x)} \left|\,(u-g_i)_y\,\right|^2+W''(g_i)\,(u-g_i)^2\\
&\quad - \left[u(x,t_i^+(x))-g_i(t_i^+(x))\right]\left[u_y(x,t_i^+(x))-g_i'(t_i^+(x))\right]\\
&\quad + \left[u(x,t_i^-(x))-g_i(t_i^-(x))\right]\left[u_y(x,t_i^-(x))-g_i'(t_i^-(x))\right].
\end{aligned} \tag{3.27}$$

Summing (3.27) in i we get

$$\begin{aligned}
\int_{-\lambda x}^{\lambda x} u_{xx}\,(u-g^*) &= o(\|v\|^2)+\sum_i \int_{t_i^-(x)}^{t_i^+(x)} \left|\,(u-g_i)_y\,\right|^2+W''(g_i)\,(u-g_i)^2\\
&\quad + 2\sum_i \left[u(x,t_i^+(x))-g_i(t_i^+(x))\right] g_i'(t_i^+(x))+O(e^{-cx}).
\end{aligned} \tag{3.28}$$

By (3.22) and (3.20), Proposition 3.9 applies to $u-g_i$ in $(t_i^-(x),t_i^+(x))$, which gives

$$\int_{t_i^-(x)}^{t_i^+(x)} \left|\,(u-g_i)_y\,\right|^2+W''(g_i)\,(u-g_i)^2 \geq \mu \int_{t_i^-(x)}^{t_i^+(x)} (u-g_i)^2\,. \tag{3.29}$$

Hence

$$\begin{aligned}
&\int_{-\lambda x}^{\lambda x} u_{xx}\,(u-g^*) \geq (\mu+o(1))\,\|v\|^2\\
&+2\sum_i \left[u(x,t_i^+(x))-g_i(t_i^+(x))\right] g_i'(t_i^+(x))+O(e^{-cx}).
\end{aligned} \tag{3.30}$$

Combining this with (3.26), we deduce that

$$\begin{aligned}
\int_{-\lambda x}^{\lambda x} u_x^2 &\geq (\mu+o(1))\,\|v\|^2+4\sum_i \left[u(x,t_i^+(x))-g_i(t_i^+(x))\right] g_i'(t_i^+(x))\\
&\quad - 2\sum_i \left[\int_{t_i^+(x)}^{+\infty} |g_i'|^2+\int_{-\infty}^{t_i^-(x)} |g_i'|^2\right]+O(e^{-cx}).
\end{aligned} \tag{3.31}$$

Differentiating $\|v\|^2$ twice in x leads to

$$\begin{aligned}\frac{1}{2}\frac{d}{dx}\|v\|^2 &= \sum_i \int_{t_i^-(x)}^{t_i^+(x)} (u-g_i)\left[u_x + g_i' t_i'(x)\right] \\ &= \sum_i \int_{t_i^-(x)}^{t_i^+(x)} (u-g_i)\, u_x, \quad \text{(by (3.22))}\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2}\frac{d^2}{dx^2}\|v\|^2 &= \sum_i \int_{t_i^-(x)}^{t_i^+(x)} \left[u_x^2 + u_x g_i' t_i'(x) + u_{xx}(u-g_i)\right] \\ &\geq 2\sum_i \int_{t_i^-(x)}^{t_i^+(x)} u_x^2 - \frac{3}{2}\sum_i \frac{\left(\int_{t_i^-(x)}^{t_i^+(x)} u_x g_i'\right)^2}{\int_{t_i^-(x)}^{t_i^+(x)} |g_i'|^2} \quad \text{(by (3.26) and (3.24))} \\ &- 2\sum_i \left[u(x,t_i^+(x)) - g_i(t_i^+(x))\right] g_i'(t_i^+(x)) \\ &\quad +2\sum_i \left[\int_{t_i^+(x)}^{+\infty} |g_i'|^2 + \int_{-\infty}^{t_i^-(x)} |g_i'|^2\right] \\ &\geq \sum_i \frac{1}{2}\int_{t_i^-(x)}^{t_i^+(x)} u_x^2 - 2\sum_i \left[u(x,t_i^+(x)) - g_i(t_i^+(x))\right] g_i'(t_i^+(x)) \\ &\qquad \text{(by Cauchy-Schwarz)} \\ &+ \sum_i \left[\int_{t_i^+(x)}^{+\infty} |g_i'|^2 + \int_{-\infty}^{t_i^-(x)} |g_i'|^2\right] \\ &\geq \frac{1}{2}\left[\mu + o(1)\right]\|v\|^2. \quad \text{(by (3.31))}\end{aligned}$$

By noting (3.21), from this inequality we deduce that

$$\|v\|^2 \leq Ce^{-cx}, \quad \text{for all } x \text{ large.} \tag{3.32}$$

Step 3 Note that

$$g_i(t_i^+(x))g_i'(t_i^+(x)) = \int_{t_i^+(x)}^{+\infty} |g_i'|^2 + g_i g_i'' \leq \int_{t_i^+(x)}^{+\infty} |g_i'|^2,$$

because g_i is close to 1 in $(t_i^+(x), +\infty)$ (see (3.20)) and hence $g_i'' = W'(g_i) < 0$ in this interval. We also have $g_i(t_i^+(x))g_i'(t_i^+(x)) > 0$, because $g_i(t_i^+(x)) > 0$ and $g_i'(t_i^+(x)) > 0$.

Then for all x large, by noting that $g_i(t_i^+(x))$ is close to 1 and $u(x, t_i^+(x)) - g_i(t_i^+(x))$ is close to 0, we obtain

$$\begin{aligned}\left|\left[u(x,t_i^+(x)) - g_i(t_i^+(x))\right] g_i'(t_i^+(x))\right| &\leq \frac{1}{2} g_i(t_i^+(x))g_i'(t_i^+(x)) \\ &\leq \frac{1}{2}\int_{t_i^+(x)}^{+\infty} |g_i'|^2.\end{aligned}$$

Substituting this into (3.26), we get

$$\begin{aligned}\int_{-\lambda x}^{\lambda x} u_x^2 &\le \int_{-\lambda x}^{\lambda x} u_{xx}\,(u-g_*) + o(\|v\|^2) + O(e^{-cx}) \\ &\le \left[\int_{-\lambda x}^{\lambda x} u_{xx}^2\right]^{\frac{1}{2}} \|v\| + o(\|v\|^2) + O(e^{-cx}) \\ &\le Ce^{-cx}. \quad \text{(by (3.2) and (3.32))}\end{aligned} \tag{3.33}$$

Then by (3.24) and the Cauchy-Schwarz inequality, we get

$$|t_i'(x)| \le Ce^{-cx}, \quad \forall i.$$

Thus for all $1 \le i \le N$, $\lim_{x\to+\infty} t_i(x)$ exists and it is finite. By noting (3.19), for each i, the limit $\lim_{x\to+\infty} f_i(x)$ also exists. In particular, this limit is finite. Then for all $1 \le i \le N-1$,

$$\lim_{x\to+\infty} \left(f_{i+1}(x) - f_i(x)\right)$$

also exists and it is finite. However, this is a contradiction with Lemma 3.7 if $N \ge 2$. Hence we must have $N = 1$.

Finally, the exponential convergence of $u(x,\cdot)$ follows from (3.33), and the exponential convergence of $f_i(x)$ follows from this exponential convergence and the (uniform) positive lower bound on g' and $u_y(x,\cdot)$ in the part where $|g| < 1/2$ and $|u| < 1/2$. □

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