Nonlinear Differ. Equ. Appl. (2017) 24:47

1021-9722/17/040001-31
published online July 6, 2017
DOI 10.1007/s00030-017-0469-3

Nonlinear Differential Equations and Applications NoDEA

Estimate for the number of limit cycles of Abel equation via a geometric criterion on three curves

Jianfeng Huang and Haihua Liang

Abstract. This paper is devoted to the investigation of Abel equation $\dot{x} = S(t,x) = \sum_{i=0}^{3} a_i(t)x^i$, where $a_i \in \mathrm{C}^\infty([0,1])$. A solution $x(t)$ with $x(0) = x(1)$ is called a *periodic solution*. And an orbit $x = x(t)$ is called a *limit cycle* if $x(t)$ is a isolated periodic solution. By means of Lagrange interpolation formula, we give a criterion to estimate the number of limit cycles of the equation. This criterion is only concerned with $S(t,x)$ on three non-intersecting curves. Applying our main result, we prove that the maximum number of limit cycles of the equation is 4 if $a_2(t)a_0(t) < 0$. To the best of our knowledge, this is a nontrivial supplement for a classical result which says that the equation has at most 3 limit cycles when $a_2(t) \neq 0$ and $a_0(t) \equiv 0$. We also study a planar polynomial system with homogeneous nonlinearities:

$$\dot{x} = ax - y + P_n(x,y), \quad \dot{y} = x + ay + Q_n(x,y),$$

where $a \in \mathbb{R}$ and P_n, Q_n are homogeneous polynomials of degree $n \geq 2$. Denote by $\psi(\theta) = \cos(\theta) \cdot Q_n\big(\cos(\theta), \sin(\theta)\big) - \sin(\theta) \cdot P_n\big(\cos(\theta), \sin(\theta)\big)$. We prove that if $(n-1)a\psi(\theta) + \dot{\psi}(\theta) \neq 0$, then the polynomial system has at most 1 limit cycle surrounding the origin, and the multiplicity is no more than 2.

Mathematics Subject Classification. Primary 34C07, Secondary 37C05, Tertiary 35F60.

Keywords. Abel equation, Coefficients without fixed signs, Limit cycles, Upper bound, Planar polynomial system with homogeneous nonlinearities.

The first author is supported by the NSF of China (No. 11401255) and the China Scholarship Council (No. 201606785007), the Fundamental Research Funds for the Central Universities (No. 21614325) and the China Scholarship Council (No. 201606785007). The second author is supported by the NSF of Guangdong Province (No. 2015A030313669).

Contents

1. Introduction and statements of main results

Consider a non-autonomous differential equation

$$\dot{x} = \frac{dx}{dt} = S(t, x), \tag{1}$$

where $S \in \mathrm{C}^{\infty}([0, 1] \times \mathbb{R})$.

A solution $x(t)$ of (1) is called a *periodic solution*, if it is defined in $[0, 1]$ with $x(0) = x(1)$. Moreover, an orbit $x = x(t)$ in the strip $[0, 1] \times \mathbb{R}$ is called a *periodic orbit* (resp. *limit cycle*) of (1), if $x(t)$ is a periodic solution (resp. isolated periodic solution) of the equation.

One of the simplest type of (1) is that

$$S(t, x) = \sum_{i=0}^{m} a_i(t) x^i,$$

where $a_i \in \mathrm{C}^{\infty}([0, 1])$, $i = 0, 1, \ldots, m$. This type of equation is usually named generalized Abel equation. It is not only a powerful tool in solving the problems for limit cycles of planar differential systems (see Cherkas [14], Devlin et al. [17], Lins-Neto [26] and Lloyd [31]), but also extensively applied in the biological studies (for instance, harvesting model, see [5,18,37]). Under these backgrounds, an important problem for the generalized Abel equation is to estimate its number of limit cycles.

Over the past three decades, a large number of investigations have been carried out for generalized Abel equation. Up till now, the problem mentioned above is not completely solved. The works on generalized Abel equation are firstly motivated by Lins-Neto [26] and Lloyd [29–31]. Both authors prove that the equation has at most one (resp. two) limit cycle(s) if $m = 1$ (resp. $m = 2$). However, the difficulty starts from $m = 3$. Also in [26], it is proved that the number of limit cycles of the equation is not bounded when $m = 3$ (see also Panov [34]). Such result is easily extended for the equation with $m > 3$ (see Gasull and Guillamon [19]). For these reasons, a more specific problem arises: Give a suitable classification (additional hypotheses) for generalized Abel equation, such that the number of limit cycles of the equation can be estimate.

The most classical hypothesis on which many results depend is the fixed sign hypothesis for one of the first two non-zero coefficients. For instance, when $m = 3$, the equation is called Abel equation. It is well-known that if $a_3(t)$ keeps the sign, then Abel equation has at most three limit cycles (see Gasull and Llibre [21], Lins-Neto [26], Lloyd [31] and Pliss [35]). Also for the case that $a_2(t)$ does not change sign and $a_0(t) \equiv 0$, the number of limit cycles of Abel equation is still no more than three (see Gasull and Llibre [21]). For generalized Abel equation with $m > 3$ and $a_m(t) \equiv 1$, Ilyashenko [24] gives an upper bound for the number of limit cycles associated with the bounds of $|a_i(t)|$, $i = 0, \ldots, m-1$. Some generalized Abel equations with coefficients of definite signs are also studied by several authors (see Alkoumi and Torres [1], Gasull and Guillamon [19] and Panov [33]).

The hypotheses presented above are easily invalid in some cases (especially when $a_m(t), \ldots, a_0(t)$ are trigonometrical polynomials). In recent years, two families of investigations which admit the coefficients without fixed signs are proposed. The first one requires the symmetric conditions for some coefficients of the equation (see [3,9,10]). The second one depends on the fixed sign hypotheses for some linear combinations of the coefficients. It is started and motivated by Álvarez et al. [4]. They prove that if $S(t,x) = a_3(t)x^3 + a_2(t)x^2$ with

$$a \cdot a_3(t) + b \cdot a_2(t) \neq 0, \qquad a, b \in \mathbb{R}, \tag{2}$$

then the equation has at most one non-zero limit cycle. Later, Huang and Zhao [23] consider a generalized case $S(t,x) = a_m(t)x^m + a_n(t)x^n + a_l(t)x^l$. Under the hypotheses of the parity of m, n, l and the inequality

$$a_l(t) \cdot \left(a_m(t)\lambda^{m-n} + a_n(t)\right) \cdot S\left(t, (\pm 1)^{m-n+1}\lambda\right) \neq 0, \qquad \lambda \neq 0,$$

they also obtain the upper bound for the number of limit cycles. In paper [2], Álvarez, Bravo and Fernández study the generalized Abel equation and give a criterion to estimate the number of limit cycles, which can be computationally checked by the algebraic methods.

For more relevant works, see [7,8,11,15,20,25], etc.

The aim of this paper is to estimate the upper bound for the number of limit cycles of the Abel equation

$$\dot{x} = \frac{dx}{dt} = S(t,x) = a_3(t)x^3 + a_2(t)x^2 + a_1(t)x + a_0(t), \tag{3}$$

where $x \in \mathbb{R}$ and $a_i \in \mathrm{C}^\infty([0,1])$, $i = 0, \ldots, 3$.

Instead of the restriction on the signs of $a_3(t), \ldots, a_0(t)$, our main result is only based on the hypothesis for $S(t,x)$ on three curves $x = \lambda_1(t)$, $x = \lambda_2(t)$, $x = \lambda_3(t)$ which possess the following property

(H) $\lambda_i \in \mathrm{C}^\infty([0,1])$, $\lambda_i(1) = \lambda_i(0)$ *and* $\lambda_i(t) - \lambda_j(t) \neq 0$, *where* $i, j = 1, 2, 3$ *and* $i \neq j$.

Throughout this paper we will use the notation "λ_i" to represent not only the function λ_i itself but also the function value "$\lambda_i(t)$" for the sake of brevity. Moreover, unless specially stated, "λ_i" is supposed to satisfy **(H)** whenever the notation is used. Define

$$\omega(t) = \frac{1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)} \begin{vmatrix} S\big(t,\lambda_1\big) & \lambda_1 & 1 \\ S\big(t,\lambda_2\big) & \lambda_2 & 1 \\ S\big(t,\lambda_3\big) & \lambda_3 & 1 \end{vmatrix}, \tag{4}$$

where $S(t,x)$ is defined as in (3). A straightforward calculation leads to

$$\omega(t) = a_3(t)\cdot\big(\lambda_1+\lambda_2+\lambda_3\big) + a_2(t). \tag{5}$$

We are now ready to state the following fundamental result.

Theorem 1.1. *Let b_1, b_2, b_3 be three real constants satisfying*

$$\begin{cases} b_1\lambda_1 + b_2\lambda_2 + b_3\lambda_3 \equiv 0, \\ b_1 + b_2 + b_3 = 0. \end{cases} \tag{6}$$

If either $\omega(t) \geq 0$ and $(b_i-1)\big(S(t,\lambda_i) - \dot{\lambda}_i\big) \geq 0$ for $i = 1,2,3$, or $\omega(t) \leq 0$ and $(b_i-1)\big(S(t,\lambda_i) - \dot{\lambda}_i\big) \leq 0$ for $i = 1,2,3$, then (3) has at most

(i) 2 limit cycles in each connected component of $\big([0,1]\times\mathbb{R}\big)\backslash\big\{(t,\lambda_i)\big|t \in [0,1], i \in \{1,2,3\}, b_i \neq 1\big\}$, counted with multiplicities.

(ii) $2(n+1)$ limit cycles in $[0,1]\times\mathbb{R}$, where $n = \#\big\{b_i\big|i \in \{1,2,3\}, b_i \neq 1\big\}$ and $\#$ represents the cardinality of a set.

Clearly, the equalities in (6) always hold for $b_1 = b_2 = b_3 = 0$. As an important consequence of Theorem 1.1 we have

Theorem 1.2. *If either $\omega(t) \geq 0$ and $S(t,\lambda_i) - \dot{\lambda}_i \leq 0$ for $i = 1,2,3$, or $\omega(t) \leq 0$ and $S(t,\lambda_i) - \dot{\lambda}_i \geq 0$ for $i = 1,2,3$, then (3) has at most*

(i) 2 limit cycles in each connected component of $\big([0,1]\times\mathbb{R}\big)\backslash\big\{(t,\lambda_i)\big|t \in [0,1],\ i = 1,2,3\big\}$, counted with multiplicities.

(ii) 8 limit cycles in $[0,1]\times\mathbb{R}$.

Remark 1.3. The hypotheses on ω and S, which play a crucial role in our above result, are natural. Firstly, by (5) one can see that the assumption on ω essentially extend condition (2). Secondly, the assumption on S is a "transversal" condition from the geometric point of view. Furthermore, if (3) has a limit cycle $x = \lambda(t)$, then there are many choices of the curves $x = \lambda_1(t)$, $x = \lambda_2(t)$ and $x = \lambda_3(t)$, satisfying either $S(t,\lambda_i) \geq \dot{\lambda}_i$ for $i = 1,2,3$, or $S(t,\lambda_i) \leq \dot{\lambda}_i$ for $i = 1,2,3$. For instance, observe that the equation $\dot{x} = S(t,x) + \varepsilon$ has a limit cycle $x = \eta_\varepsilon(t)$ located near $x = \lambda(t)$ as $\varepsilon > 0$ (or < 0) sufficiently small. We can choose three small positive (or negative) numbers $\varepsilon_1 > \varepsilon_2 > \varepsilon_3$ and then take $\lambda_i(t) = \eta_{\varepsilon_i}(t)$. So $S(t,\lambda_i) = \dot{\lambda}_i - \varepsilon_i$ for $i = 1,2,3$, which implies the above inequalities. Compare with the results in [4] and [23], our restrictions for the curves are less (essentially, the works in [4] and [23] require some straight lines hypotheses). This is an improvement for the previous results.

An example with all the coefficients changing signs will be given after the proof of Theorem 1.2. Also together with the bifurcation method, we will show in a second example that there exist an equation which has at least 4 limit cycles. See Sect. 4 for details.

As we mentioned above, Gasull and Llibre [21] proved that the number of limit cycles of (3) is no more than 3 when $a_2(t) \neq 0$ and $a_0(t) \equiv 0$. It is natural to ask that whether if this upper bound can be enlarged once $a_0(t)$ does not vanish? By virtue of Theorem 1.2, we can provide a positive answer to this question, see the following corollary.

Corollary 1.4. *If $a_2(t) \cdot a_0(t) < 0$, then (3) has at most 4 limit cycles, counted with multiplicities. This upper bound is sharp.*

Remark 1.5. the conclusion of Corollary 1.4 shows that the upper bound appearing in Theorem 1.2 seems not so rough.

Next we consider the case where b_1, b_2, b_3 in Theorem 1.1 are not all zeros. This case occurs when $(\lambda_1 - \lambda_3)/(\lambda_1 - \lambda_2)$ is a constant, which leads to the second important conclusion of Theorem 1.1 below.

Theorem 1.6. *Assume that $b \triangleq (\lambda_1 - \lambda_3)/(\lambda_1 - \lambda_2)$ is a constant. If either $\omega(t) \geq 0$, $(b-2)\big(S(t,\lambda_1) - \dot{\lambda_1}\big) \geq 0$ and $(b+1)\big(S(t,\lambda_2) - \dot{\lambda_2}\big) \leq 0$, or $\omega(t) \leq 0$, $(b-2)\big(S(t,\lambda_1) - \dot{\lambda_1}\big) \leq 0$ and $(b+1)\big(S(t,\lambda_2) - \dot{\lambda_2}\big) \geq 0$, then (3) has at most*

- *(i) 2 limit cycles in each connected component of $\big([0,1] \times \mathbb{R}\big) \backslash \big\{(t,\lambda_i) \big| t \in [0,1],\ i = 1,2\big\}$, counted with multiplicities.*
- *(ii) 6 limit cycles in $[0,1] \times \mathbb{R}$.*

Corollary 1.7. *Consider equation*

$$\dot{x} = a_3(t)x^3 + a_2(t)x^2 + a_1(t)x. \tag{7}$$

If there exist two different non-zero real numbers κ_1 and κ_2 such that

$$\big(a_3(t)\kappa_1^2 + a_2(t)\kappa_1\big) \cdot \big(a_3(t)\kappa_2^2 + a_2(t)\kappa_2 + a_1(t)\big) < 0,$$

then (7) has at most 6 limit cycles.

A third example will be given after the proof of Corollary 1.7, see Sect. 5 for details.

It is notable that Hilbert's 16th problem on many planar polynomial differential systems can be reduced to the problem of determining the maximal number of limit cycles of (3), where the coefficients $a_0(t) \equiv 0$ and $a_1(t), a_2(t), a_3(t)$ are homogeneous trigonometrical polynomials (see Gasull and Llibre [21] and Lins-Neto [26], etc). Take, for instance, the planar polynomial system with homogeneous nonlinearities:

$$\begin{cases} \dfrac{dx}{dt} = ax - y + P_n(x,y), \\ \dfrac{dy}{dt} = x + ay + Q_n(x,y), \end{cases} \tag{8}$$

where P_n, Q_n are homogeneous polynomials of degree $n \geq 2$.

In fact, (8) is a system which has been extensively studied and gained wide attention. So far, plenty of works have been carried out for the bifurcation of (8) with small perturbations, see for instance [6,22,27,28,32,36] and the references therein. In contrast, only a few results for the non-bifurcation case are obtained. Here we summarize the representative ones as below: Let

$$\begin{aligned}\varphi(\theta) &= \cos(\theta)\cdot P_n\big(\cos(\theta),\sin(\theta)\big)+\sin(\theta)\cdot Q_n\big(\cos(\theta),\sin(\theta)\big),\\ \psi(\theta) &= \cos(\theta)\cdot Q_n\big(\cos(\theta),\sin(\theta)\big)-\sin(\theta)\cdot P_n\big(\cos(\theta),\sin(\theta)\big).\end{aligned} \tag{9}$$

(I) If $\varphi(\theta)-a\psi(\theta)\not\equiv 0$ does not change sign, then (8) has at most 1 limit cycle surrounding the origin (see Coll, Gasull and Prohens [16]).

(II) If $\psi(\theta)\big(\varphi(\theta)-a\psi(\theta)\big)\not\equiv 0$ does not change sign, then (8) has at most 1 (resp. 2) limit cycle(s) surrounding the origin when n is even (resp. odd) (see Carbonell and Llibre [12]).

(III) If $(n-1)\big(\varphi(\theta)-2a\psi(\theta)\big)-\dot{\psi}(\theta)\not\equiv 0$ does not change sign, then (8) has at most 2 limit cycles surrounding the origin (see Gasull and Llibre [21]).

(IV) If either $(n-1)\big(\varphi(\theta)-2a\psi(\theta)\big)-\dot{\psi}(\theta)\equiv 0$, or $\psi(\theta)\big(\varphi(\theta)-a\psi(\theta)\big)\equiv 0$, then (8) has at most 1 limit cycle surrounding the origin (see Gasull and Llibre [21]).

Now as an application of our theorems, we give a new criterion on the upper bound for the number of limit cycles of (8).

Proposition 1.8. *Consider planar polynomial system* (8)*. Let* $\psi(\theta)$ *be defined as in* (9)*. Suppose that*

$$(n-1)a\psi(\theta)+\dot{\psi}(\theta)\neq 0.$$

Then system (8) *has at most 1 limit cycle surrounding the origin, and the multiplicity is no more than 2.*

We remark that when n is even, $\psi(\theta+\pi)=-\psi(\theta)$ and therefore $(n-1)a\psi(\theta)+\dot{\psi}(\theta)$ always has zeros. So the hypothesis in Proposition 1.8 implies that n is odd.

Two examples will be provided to illustrate the application of Proposition 1.8. One of them has exactly 1 limit cycle and the other one violates all the previous criterions (I)–(IV). See Sect. 6 for details.

The rest of this paper is organized as follows: in Sect. 2 we give several preliminary results. In Sect. 3 we prove Theorem 1.1. Theorem 1.2 and Corollary 1.4 are obtained in Sect. 4. Theorem 1.6 and Corollary 1.7 are proved in Sect. 5. Finally, in Sect. 6 we prove Proposition 1.8.

2. Preliminaries

In this section we mainly give four lemmas and one proposition that are useful for the proofs of the theorems.

Let $x(t, x_0)$ be the solution of (3) with $x(0, x_0) = x_0$. It is well-known that (see Lloyd [31] for instance)

$$\frac{\partial x}{\partial x_0}(t, x_0) = \exp \int_0^t \frac{\partial S}{\partial x}\big(s, x(s, x_0)\big) ds. \tag{10}$$

Moreover, for the return map

$$H(x_0) = x(1, x_0),$$

we have

$$\begin{aligned} \dot{H}(x_0) &= \exp \int_0^1 \frac{\partial S}{\partial x}\big(t, x(t, x_0)\big) dt, \\ \ddot{H}(x_0) &= \dot{H}(x_0) \cdot \int_0^1 \frac{\partial^2 S}{\partial x^2}\big(t, x(t, x_0)\big) \cdot \frac{\partial x}{\partial x_0}(t, x_0) dt, \end{aligned} \tag{11}$$

where $\dot{}$ and $\ddot{}$ represent the first-order and second-order derivatives, respectively.

When $x(t, x_0)$ is periodic with $H'(x_0) \neq 1$ (resp. $\big(H - id\big)'(x_0) = \cdots = \big(H - id\big)^{(n-1)}(x_0) = 0$ and $\big(H - id\big)^{(n)}(x_0) \neq 0$), we say that $x = x(t, x_0)$ is a *hyperbolic* limit cycle (resp. a limit cycle with multiplicity n).

Firstly we have the following lemma.

Lemma 2.1. *Let* $U = \big\{(t, x)\big| t \in [0, 1], x \in \big(c_1(t), c_2(t)\big)\big\}$, *where* $c_i \in \mathrm{C}^\infty([0, 1]) \bigcup \{+\infty, -\infty\}$, $c_i(0) = c_i(1)$ *and* $i = 1, 2$.

Let $F \in \mathrm{C}^2(U)$ *with* $F(1, x) = F(0, x)$. *Assume that*

$$G(t, x) \triangleq \frac{\partial S}{\partial x}(t, x) + \frac{\partial F}{\partial x}(t, x) \cdot S(t, x) + \frac{\partial F}{\partial t}(t, x),$$

where $(t, x) \in U$.

(i) If $x = x(t, x_0)$ *is a periodic orbit of* (3) *in* U, *then*

$$\int_0^1 \frac{\partial S}{\partial x}\big(t, x(t, x_0)\big) dt = \int_0^1 G\big(t, x(t, x_0)\big) dt. \tag{12}$$

(ii) Assume that $E \subseteq [0, 1]$ *is a non-empty open set. If*

$$\left.\frac{\partial G}{\partial x}\right|_U \geq 0 (\leq 0), \quad \left.\frac{\partial G}{\partial x}\right|_{U \cap (E \times \mathbb{R})} \neq 0,$$

then (3) *has at most 2 limit cycles in* U, *counted with multiplicities. In addition:*

(ii.a) If there exist two different limit cycles of (3) *in* U, *then the one above is unstable and the one below is stable (resp. the one above is stable and the one below is unstable) when* $(\partial G/\partial x)|_U \geq 0$ *(resp.* $(\partial G/\partial x)|_U \leq 0$*).*

(ii.b) If there exists a limit cycle with multiplicity 2 of (3) *in* U, *then it is unstable from above and stable from below (resp. stable from above and unstable from below) when* $(\partial G/\partial x)|_U \geq 0$ *(resp.* $(\partial G/\partial x)|_U \leq 0$*).*

Proof. Let $x(t,x_0)$ be the solution of (3) with $x(0,x_0)=x_0$. Let $I\subset\mathbb{R}$ be the maximal interval such that $\big(t,x(t,x_0)\big)\in U$ for $(t,x_0)\in[0,1]\times I$. By assumption, we get

$$\begin{aligned}&\int_0^1\left(\frac{\partial F}{\partial x}\big(t,x(t,x_0)\big)\cdot S\big(t,x(t,x_0)\big)+\frac{\partial F}{\partial t}\big(t,x(t,x_0)\big)\right)dt\\&\quad=\int_0^1\left(\frac{\partial F}{\partial x}\big(t,x(t,x_0)\big)\cdot\frac{dx}{dt}(t,x_0)+\frac{\partial F}{\partial t}\big(t,x(t,x_0)\big)\right)dt\\&\quad=F\big(1,x(1,x_0)\big)-F\big(0,x(0,x_0)\big)\\&\quad=F\big(0,H(x_0)\big)-F(0,x_0),\end{aligned}$$

where $x_0\in I$ and $H(x_0)$ represents the return map of (3). Therefore,

$$\begin{aligned}&\int_0^1\frac{\partial S}{\partial x}\big(t,x(t,x_0)\big)dt\\&\quad=\int_0^1G\big(t,x(t,x_0)\big)dt+F(0,x_0)-F\big(0,H(x_0)\big),\qquad x_0\in I.\end{aligned}\tag{13}$$

(i) Clearly, Eq. (12) is implied by (13) if $x=x(t,x_0)$ is a periodic orbit.
(ii) Define a function as

$$g(x_0)=\int_0^1G\big(t,x(t,x_0)\big)dt,\qquad x_0\in I.$$

Equation (13) becomes

$$\int_0^1\frac{\partial S}{\partial x}\big(t,x(t,x_0)\big)dt=g(x_0)+F(0,x_0)-F\big(0,H(x_0)\big),\qquad x_0\in I.$$

When $x=x(t,x_0)$ is a limit cycle in U, it follows from (11) that

$$\begin{aligned}&\dot H(x_0)=\exp g(x_0),\\&\ddot H(x_0)=\exp g(x_0)\cdot\left(\dot g(x_0)+\frac{\partial F}{\partial x_0}(0,x_0)\cdot\big(1-\exp g(x_0)\big)\right).\end{aligned}\tag{14}$$

In what follows we consider the case $(\partial G/\partial x)|_U\ge 0$, $\big(\partial G/\partial x\big)\big|_{U\cap(E\times\mathbb{R})}\neq 0$ (it is a similar argument for the other case). Note that $\partial x/\partial x_0>0$ by (10). We obtain

$$\begin{aligned}\dot g(x_0)=&\int_{[0,1]\setminus E}\frac{\partial G}{\partial x}\big(t,x(t,x_0)\big)\cdot\frac{\partial x}{\partial x_0}(t,x_0)dt\\&+\int_E\frac{\partial G}{\partial x}\big(t,x(t,x_0)\big)\cdot\frac{\partial x}{\partial x_0}(t,x_0)dt\\&>0.\end{aligned}\tag{15}$$

Hence, $g(x_0)$ is a strictly increasing function. In view of (14), we have:

(a) If $x_1>x_2$ are initial values of two consecutive limit cycles in U, then $x=x(t,x_1)$ (resp. $x=x(t,x_2)$) is unstable (resp. stable) when it is hyperbolic.

(b) At most 1 limit cycle of (3) in U is non-hyperbolic. In addition, if such limit cycle exists, then it is of multiplicity 2, unstable from above and stable from below.

Clearly, statement (a) shows that U contains at most 2 limit cycles of (3) if it only contains hyperbolic limit cycles.

Now we claim that U contains no other limit cycles when it contains a non-hyperbolic one (a limit cycle of multiplicity 2). In fact, assume for a contradiction that $x = x(t, x_1)$ and $x = x(t, x_2)$ are two consecutive limit cycles of (3) in U, where one of them is non-hyperbolic and $x_1 > x_2$. From statement (b), $x = x(t, x_1)$ is hyperbolic stable (resp. $x = x(t, x_2)$ is hyperbolic unstable) if $x = x(t, x_2)$ (resp. $x = x(t, x_1)$) is non-hyperbolic. However, statement (a) tells us that $x = x(t, x_1)$ (resp. $x = x(t, x_2)$) is unstable (resp. stable) when it is hyperbolic. This shows a contradiction. As a result, our claim is valid.

Based on the above discussion, (3) has at most 2 limit cycles in U, counted with multiplicties. In addition, statements (ii.a) and (ii.b) follows from statements (a) and (b), respectively.

The proof of Lemma 2.1 is finished. □

Next we give a lemma which is essentially proved in [4, Proposition 6].

Lemma 2.2. *Let $L \in \mathrm{C}^1(\mathbb{R})$ and $p \in \mathrm{C}^1([0, 1])$. Consider differential equation*

$$\frac{dx}{dt} = p(t)L(x). \tag{16}$$

The following statements hold.

(i) Suppose that $\int_0^1 p(s)ds = 0$. Then (16) has no limit cycles.

(ii) Suppose that $\int_0^1 p(s)ds \neq 0$. Then an orbit $x = x(t)$ of (16) is periodic if and only if $L\big(x(0)\big) = 0$. In addition, $x = x(t)$ is a hyperbolic limit cycle when $L\big(x(0)\big) = 0$ and $\big(dL/dx\big)(x(0)) \neq 0$.

Proof. Let $x(t)$ be an arbitrary solution of (16). We firstly claim that for a point t in the domain,

$$\operatorname{sgn}\big(x(t) - x(0)\big) = \operatorname{sgn}\left(\int_0^t p(s)ds\right) \cdot \operatorname{sgn}\Big(L\big(x(0)\big)\Big). \tag{17}$$

In fact, if $L\big(x(0)\big) = 0$, then $x(t) \equiv x(0)$. Equation (17) holds. If $L\big(x(0)\big) \neq 0$, then $L\big(x(t)\big) \neq 0$, i.e. $L\big(x(t)\big)$ does not change sign. We have

$$\int_{x(0)}^{x(t)} \frac{dx}{L(x)} = \int_0^t p(s)ds,$$

which also implies Eq. (17).

(i) According to (17), Eq. (16) has no limit cycles when $\int_0^1 p(s)ds = 0$.

(ii) Suppose that $\int_0^1 p(s)ds \neq 0$. Then (17) tells us that $x = x(t)$ is a periodic orbit if and only if $L\big(x(0)\big) = 0$. Furthermore, when $L\big(x(0)\big) = 0$ and $\big(dL/dx\big)(x(0)) \neq 0$, the first derivative for the return map of (16) on $x(0)$ is

$$\exp\int_0^1 p(t)\frac{dL}{dx}\big(x(0)\big)dt = \exp\left(\frac{dL}{dx}\big(x(0)\big)\cdot\int_0^1 p(t)dt\right)\neq 1.$$

Hence, $x = x(t)$ is a hyperbolic limit cycle. The conclusion follows. □

Recall that without further remark, in this paper $x = \lambda_1(t)$, $x = \lambda_2(t)$, $x = \lambda_3(t)$ always represent three curves satisfying hypothesis **(H)**, and $\omega(t)$ represents the function defined in (4) (i.e. (5)), respectively. We give the rest two lemmas and one proposition below.

Lemma 2.3. *Let b_1, b_2, b_3 be three real constants satisfying (6). If $\omega(t) \equiv 0$ and $(b_i - 1)\big(S(t,\lambda_i) - \dot{\lambda}_i\big) \equiv 0$ for $i = 1, 2, 3$, then the following statements hold for (3).*

- *(i) Each connected component of $\big([0,1]\times\mathbb{R}\big)\backslash\big\{(t,\lambda_i)\big|t\in[0,1], i\in\{1,2,3\}, b_i \neq 1\big\}$ contains at most 1 limit cycle of (3), counted with multiplicity.*
- *(ii) $x = \lambda_i(t)$ is not semi-stable when it is a limit cycle of (3), $i = 1, 2, 3$.*

Proof. Firstly, since $\omega(t) \equiv 0$, we get that

$$\begin{aligned}\frac{d}{dt}\left(\frac{\lambda_1-\lambda_3}{\lambda_2-\lambda_3}\right) &= \frac{1}{(\lambda_2-\lambda_3)^2}\begin{vmatrix}\dot{\lambda}_1 & \lambda_1 & 1\\ \dot{\lambda}_2 & \lambda_2 & 1\\ \dot{\lambda}_3 & \lambda_3 & 1\end{vmatrix}\\ &= \frac{1}{(\lambda_2-\lambda_3)^2}\begin{vmatrix}S(t,\lambda_1) & \lambda_1 & 1\\ S(t,\lambda_2) & \lambda_2 & 1\\ S(t,\lambda_3) & \lambda_3 & 1\end{vmatrix}\\ &= \frac{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}{\lambda_2-\lambda_3}\omega(t)\\ &\equiv 0,\end{aligned}$$

which means that $(\lambda_1-\lambda_3)/(\lambda_2-\lambda_3)$ is a constant.

Secondly, taking transformation

$$y = \frac{x-\lambda_3}{\lambda_2-\lambda_3},\ \text{i.e. } x = \lambda_3 + (\lambda_2-\lambda_3)y, \tag{18}$$

Equation (3) becomes

$$\dot{y} = \tilde{S}(t,y) = \frac{S\big(t,\lambda_3+(\lambda_2-\lambda_3)y\big) - \big(\dot{\lambda}_2-\dot{\lambda}_3\big)y - \dot{\lambda}_3}{\lambda_2-\lambda_3}. \tag{19}$$

According to hypothesis **(H)**, the limit cycles of (3) one-to-one correspond to the limit cycles of (19). Furthermore, we know that $\tilde{S}(t,y)$ is a polynomial in y of degree no more than 3, and

$$\begin{aligned}&\tilde{S}(t,0) = \frac{S(t,\lambda_3)-\dot{\lambda}_3}{\lambda_2-\lambda_3},\\ &\tilde{S}(t,1) = \frac{S(t,\lambda_2)-\dot{\lambda}_2}{\lambda_2-\lambda_3},\\ &\tilde{S}\left(t,\frac{\lambda_1-\lambda_3}{\lambda_2-\lambda_3}\right) = \frac{S(t,\lambda_1)-\dot{\lambda}_1}{\lambda_2-\lambda_3} + \frac{d}{dt}\left(\frac{\lambda_1-\lambda_3}{\lambda_2-\lambda_3}\right) = \frac{S(t,\lambda_1)-\dot{\lambda}_1}{\lambda_2-\lambda_3}.\end{aligned} \tag{20}$$

Since $b_1+b_2+b_3=0$ from (6), at least one of b_1,b_2,b_3 is not equal to 1. In what follows we prove the lemma in three cases, respectively.

Case 1. $b_i\neq 1$ for $i=1,2,3$.

By assumption, $S(t,\lambda_i)-\dot{\lambda}_i\equiv 0$ for $i=1,2,3$. Hence (20) tells us that

$$\tilde{S}(t,0)=\tilde{S}(t,1)=\tilde{S}\left(t,\frac{\lambda_1-\lambda_3}{\lambda_2-\lambda_3}\right)\equiv 0. \tag{21}$$

Observe that $(\lambda_1-\lambda_3)/(\lambda_2-\lambda_3)\neq 0,1$ from hypothesis **(H)**. $\tilde{S}(t,y)$ can be written as

$$\tilde{S}(t,y)=\tilde{a}_3(t)\cdot\left(y-\frac{\lambda_1-\lambda_3}{\lambda_2-\lambda_3}\right)(y-1)y,$$

where $\tilde{a}_3(t)$ is the first coefficient of $\tilde{S}(t,y)$ in y. As a result, using Lemma 2.2, Eq. (19) has either no limit cycles, or three limit cycles $y=0$, $y=1$ and $y=(\lambda_1-\lambda_3)/(\lambda_2-\lambda_3)$, which are all hyperbolic. That is to say, (3) has either no limit cycles, or three limit cycles $x=\lambda_1(t)$, $x=\lambda_2(t)$ and $x=\lambda_3(t)$, which are all hyperbolic. Our assertion for Case 1 holds.

Case 2. Two of b_1,b_2,b_3 are not equal to 1.

Without loss of generality, suppose that $b_1=1$ and $b_2\neq 1, b_3\neq 1$. Then $S(t,\lambda_2)-\dot{\lambda}_2=S(t,\lambda_3)-\dot{\lambda}_3\equiv 0$. Together with (4) and (6), we have

$$\begin{aligned}\begin{vmatrix} S(t,\lambda_1)-\dot{\lambda}_1 & 0 & 0\\ \dot{\lambda}_2 & \lambda_2 & 1\\ \dot{\lambda}_3 & \lambda_3 & 1\end{vmatrix} &= \begin{vmatrix} S(t,\lambda_1)-\dot{\lambda}_1-b_2\dot{\lambda}_2-b_3\dot{\lambda}_3 & -b_2\lambda_2-b_3\lambda_3 & -b_2-b_3\\ \dot{\lambda}_2 & \lambda_2 & 1\\ \dot{\lambda}_3 & \lambda_3 & 1\end{vmatrix}\\ &= \begin{vmatrix} S(t,\lambda_1) & \lambda_1 & 1\\ S(t,\lambda_2) & \lambda_2 & 1\\ S(t,\lambda_3) & \lambda_3 & 1\end{vmatrix}\\ &= (\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)\cdot\omega(t)\\ &\equiv 0,\end{aligned}$$

i.e. $S(t,\lambda_1)-\dot{\lambda}_1\equiv 0$. Thus, (21) is also valid. Following a similar argument in Case 1, (3) has either no limit cycles, or three limit cycles $x=\lambda_1(t)$, $x=\lambda_2(t)$ and $x=\lambda_3(t)$, which are all hyperbolic. As a consequence, there exists at most 1 limit cycle of (3) in $\big([0,1]\times\mathbb{R}\big)\backslash\big\{(t,\lambda_i)|t\in[0,1], i\in\{2,3\}\big\}$, counted with multiplicity. And $x=\lambda_i(t)$ is not semi-stable when it is a limit cycle of (3), $i=1,2,3$. Our assertion for Case 2 is valid.

Case 3. One of b_1,b_2,b_3 is not equal to 1.

Without loss of generality, suppose that $b_1=b_2=1$ and $b_3\neq 1$. Then we obtain $b_3=-2$ and $2\lambda_3=\lambda_1+\lambda_2$ by (6), and $S(t,\lambda_3)-\dot{\lambda}_3\equiv 0$.

Recalling that $\tilde{S}(t,y)$ is a polynomial in y of degree no more than 3, it can be written as

$$\tilde{S}(t,y)=\tilde{a}_3(t)y^3+\tilde{a}_2(t)y^2+\tilde{a}_1(t)y+\tilde{a}_0(t).$$

From (3), (5), (19) and a direct calculation, we get

$$\begin{aligned}\tilde{a}_2(t) &= (\lambda_2 - \lambda_3)(3a_3(t)\lambda_3 + a_2(t)) \\ &= (\lambda_2 - \lambda_3)\big(a_3(t)(\lambda_1 + \lambda_2 + \lambda_3) + a_2(t)\big) \\ &= (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) \cdot \omega(t) \\ &\equiv 0, \\ \tilde{a}_0(t) &= \tilde{S}(t, 0) \\ &= \frac{S(t, \lambda_3) - \dot{\lambda_3}}{\lambda_2 - \lambda_3} \\ &\equiv 0.\end{aligned}$$

Thus, (19) is reduced to $\dot{y} = \tilde{S}(t, y) = \tilde{a}_3(t)y^3 + \tilde{a}_1(t)y$, which is a Bernoulli equation. The result follows from the expression of the general solution and a straightforward calculation.

Based on the above, (19) has at most 1 limit cycle in each connected component of $[0, 1]\times(\mathbb{R}\backslash\{0\})$, counted with multiplicity. And $y = 0$ is not semi-stable when it is a limit cycle of (19). Applying transformation (18), Eq. (3) has at most 1 limit cycle in each connected component of $\big([0, 1] \times \mathbb{R}\big)\backslash\big\{(t, \lambda_3)\big|t \in [0, 1]\big\}$, counted with multiplicity. And $x = \lambda_3(t)$ is not semi-stable when it is a limit cycle of (3). Our assertion for Case 3 follows.

The proof of Lemma 2.3 is finished.

Lemma 2.4. *Let b_1, b_2, b_3 be three real constants satisfying (6). Assume that*

$$q(t, x) = \prod_{\lambda_i \in I} (x - \lambda_i), \qquad I = \big\{\lambda_i \big| i \in \{1, 2, 3\}, b_i \neq 1\big\}.$$

Then $(b_i - 1) \cdot q(t, \lambda_i) \equiv 0$ for $i = 1, 2, 3$, and

$$(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3) \begin{vmatrix} q(t, \lambda_1) & \lambda_1 & 1 \\ q(t, \lambda_2) & \lambda_2 & 1 \\ q(t, \lambda_3) & \lambda_3 & 1 \end{vmatrix} \geq 0. \tag{22}$$

Proof. Firstly, we have by assumption that $(b_i - 1) \cdot q(t, \lambda_i) \equiv 0$ for $i = 1, 2, 3$.

Secondly, since $b_1 + b_2 + b_3 = 0$ from (6), at least one of b_1, b_2, b_3 is not equal to 1. Thus the proof of (22) can be split into three cases: (i) $b_i \neq 1$ for $i = 1, 2, 3$. (ii) Two of b_1, b_2, b_3 are not equal to 1. (iii) One of b_1, b_2, b_3 are not equal to 1. In each case (22) follows from the definition of $q(t, x)$ and a direct calculation. □

Before giving the final proposition of this section, we need a little bit more preparation.

Let $f(t, x)$ be a function defined as follows:

$$f(t, x) = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3). \tag{23}$$

We know by (3) that $S(t,x) - a_3(t)f(t,x)$ is a polynomial in x of degree no more than 2. According to Lagrange interpolation formula,

$$S(t,x) = a_3(t)f(t,x) + \sum_{i=1}^{3}\left(\frac{S(t,\lambda_i)}{\prod_{j\neq i}(\lambda_i-\lambda_j)}\cdot\prod_{j\neq i}(x-\lambda_j)\right). \tag{24}$$

Hence, we get

$$\begin{aligned}
\frac{\partial S}{\partial x}(t,x) &= a_3(t)\cdot\frac{\partial f}{\partial x}(t,x)\\
&\quad+\left(\frac{S(t,\lambda_1)(\lambda_2-\lambda_3)-S(t,\lambda_2)(\lambda_1-\lambda_3)+S(t,\lambda_3)(\lambda_1-\lambda_2)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}\right)\cdot 2x\\
&\quad-\sum_{i=1}^{3}\frac{\sum_{j\neq i}\lambda_j}{\prod_{j\neq i}(\lambda_i-\lambda_j)}\cdot S(t,\lambda_i)\\
&= a_3(t)\cdot\frac{\partial f}{\partial x}(t,x)+\frac{1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}\begin{vmatrix} S(t,\lambda_1) & \lambda_1 & 1\\ S(t,\lambda_2) & \lambda_2 & 1\\ S(t,\lambda_3) & \lambda_3 & 1\end{vmatrix}\cdot 2x\\
&\quad-\sum_{i=1}^{3}\frac{\sum_{j\neq i}\lambda_j}{\prod_{j\neq i}(\lambda_i-\lambda_j)}\cdot S(t,\lambda_i)\\
&= a_3(t)\cdot\frac{\partial f}{\partial x}(t,x)+2\omega(t)\cdot x-\sum_{i=1}^{3}\frac{\sum_{j\neq i}\lambda_j}{\prod_{j\neq i}(\lambda_i-\lambda_j)}\cdot S(t,\lambda_i).
\end{aligned}\tag{25}$$

Define $\lambda_0 = \lambda_1+\lambda_2+\lambda_3$ and

$$F_i(t,x) = \ln|x-\lambda_i|, \qquad (t,x)\in\big([0,1]\times\mathbb{R}\big)\backslash\big\{(t,\lambda_i)\big|t\in[0,1]\big\},\ i=1,2,3. \tag{26}$$

We have by (5) that $\omega(t) = a_3(t)\lambda_0 + a_2(t)$ and

$$\begin{aligned}
&\frac{\partial F_i}{\partial x}(t,x)\cdot S(t,x)+\frac{\partial F_i}{\partial t}(t,x)\\
&\quad=\frac{S(t,x)-\dot{\lambda}_i}{x-\lambda_i}\\
&\quad=a_3(t)x^2+\big(a_3(t)\lambda_i+a_2(t)\big)x\\
&\qquad+\big(a_3(t)\lambda_i^2+a_2(t)\lambda_i+a_1(t)\big)+\frac{S(t,\lambda_i)-\dot{\lambda}_i}{x-\lambda_i}\\
&\quad=a_3(t)\left(x-\lambda_0+\lambda_i\right)x+\omega(t)x\\
&\qquad+\big(a_3(t)\lambda_i^2+a_2(t)\lambda_i+a_1(t)\big)+\frac{S(t,\lambda_i)-\dot{\lambda}_i}{x-\lambda_i}.
\end{aligned}\tag{27}$$

Now let b_1, b_2, b_3 be three real constants satisfying (6). Taking

$$\begin{aligned}
&F(t,x) = (b_1-1)\cdot F_1(t,x)+(b_2-1)\cdot F_2(t,x)+(b_3-1)\cdot F_3(t,x),\\
&(t,x)\in\big([0,1]\times\mathbb{R}\big)\backslash\big\{(t,\lambda_i)\big|t\in[0,1],i\in\{1,2,3\},b_i\neq 1\big\}
\end{aligned}\tag{28}$$

in Lemma 2.1, we obtain the following proposition.

Proposition 2.5. *Let b_1, b_2, b_3 be three real constants satisfying (6). If either*

(I.1) $\omega(t) \geq 0$ *and* $(b_i - 1)(S(t, \lambda_i) - \dot{\lambda}_i) \geq 0$ *for* $i = 1, 2, 3,$

or

(I.2) $\omega(t) \leq 0$ *and* $(b_i - 1)(S(t, \lambda_i) - \dot{\lambda}_i) \leq 0$ *for* $i = 1, 2, 3,$

then the following statements hold.

- (i) *Suppose that U is a connected component of $\big([0,1] \times \mathbb{R}\big) \backslash \big\{(t, \lambda_i) \big| t \in [0,1], i \in \{1,2,3\}, b_i \neq 1\big\}$. Then U contains at most 2 limit cycles of (3), counted with multiplicities. In addition:*
  - (i.a) *If there exist two different limit cycles of (3) in U, then the one above is stable and the one below is unstable (resp. the one above is unstable and the one below is stable) when (I.1) (resp. (I.2)) holds.*
  - (i.b) *If there exists a limit cycle with multiplicity 2 of (3) in U, then it is stable from above and unstable from below (resp. unstable from above and stable from below) when (I.1) (resp. (I.2)) holds.*
- (ii) *Suppose that U_1 and U_2 are two consecutive connected components of $\big([0,1] \times \mathbb{R}\big) \backslash \big\{(t, \lambda_i) \big| t \in [0,1], i \in \{1,2,3\}, b_i \neq 1\big\}$, with a limit cycle $x = \lambda(t)$ of (3) being common boundary.*
  - (ii.a) *Either U_1 or U_2 contains at most 1 limit cycle of (3), counted with multiplicity.*
  - (ii.b) *If $U_1 \cup U_2$ contains 3 limit cycles (counted with multiplicities) of (3), then $x = \lambda(t)$ is not semi-stable.*
  - (ii.c) *If each of U_1 and U_2 contains 1 limit cycle (counted with multiplicity) of (3) and $x = \lambda(t)$ is semi-stable, then $x = \lambda(t)$ is unstable from above and stable from below (resp. stable from above and unstable from below) when (I.1) (resp. (I.2)) holds.*

Proof. Denote by

$$E = [0,1] \backslash \big\{t \big| t \in [0,1], \omega(t) = 0 \text{ and } (b_i - 1)(S(t,\lambda_i) - \dot{\lambda}_i) = 0 \text{ for all } i = 1,2,3\big\},$$
$$V = \big([0,1] \times \mathbb{R}\big) \backslash \big\{(t, \lambda_i) \big| t \in [0,1], i \in \{1,2,3\}, b_i \neq 1\big\}.$$

Firstly, according to Lemma 2.3, we know that statements (i) and (ii) hold if $E = \phi$.

In what follows we consider the case $E \neq \phi$. We will only give the proof for case **(I.1)** because case **(I.2)** can be dealt with in an analogous way.

Take $\lambda_0 = \lambda_1 + \lambda_2 + \lambda_3$. Let $f(t, x)$ be defined as in (23). Let $F_1(t, x)$, $F_2(t, x)$, $F_3(t, x)$ be defined as in (26) and $F(t, x)$ be defined as in (28), respectively. Substituting $F(t, x)$ in Lemma 2.1, it follows from (25) and (27) that

$$\begin{aligned}
\frac{\partial G}{\partial x}(t,x) &= \frac{\partial^2 S}{\partial x^2}(t,x) + \frac{\partial^2 F}{\partial x^2}(t,x) \cdot S(t,x) + \frac{\partial F}{\partial x}(t,x) \cdot \frac{\partial S}{\partial x}(t,x) + \frac{\partial^2 F}{\partial t \partial x}(t,x) \\
&= a_3(t) \cdot \frac{\partial^2 f}{\partial x^2}(t,x) + 2\omega(t) \\
&\quad + \sum_{i=1}^{3} (b_i - 1) \left(a_3(t)\,(2x - \lambda_0 + \lambda_i) + \omega(t) - \frac{S(t,\lambda_i) - \dot{\lambda}_i}{(x - \lambda_i)^2} \right)
\end{aligned}$$

$$
\begin{aligned}
&= a_3(t)\cdot\big(6x-2\lambda_0\big)+2\omega(t)\\
&\qquad+\sum_{i=1}^{3}(b_i-1)\left(a_3(t)\,(2x-\lambda_0+\lambda_i)+\omega(t)-\frac{S(t,\lambda_i)-\dot{\lambda}_i}{(x-\lambda_i)^2}\right)\\
&= 2\left(\sum_{i=1}^{3}b_i\right)a_3(t)\cdot x+\left(\sum_{i=1}^{3}b_i\lambda_i-\sum_{i=1}^{3}b_i\lambda_0\right)a_3(t)\\
&\qquad-\left(1-\sum_{i=1}^{3}b_i\right)\omega(t)-\sum_{i=1}^{3}(b_i-1)\cdot\frac{S(t,\lambda_i)-\dot{\lambda}_i}{(x-\lambda_i)^2}.
\end{aligned}
$$

Since (6) holds from assumption, we get

$$
\frac{\partial G}{\partial x}(t,x)=-\omega(t)-\sum_{i=1}^{3}\frac{(b_i-1)\big(S(t,\lambda_i)-\dot{\lambda}_i\big)}{(x-\lambda_i)^2}, \tag{29}
$$

where $(t,x)\in V$. As a result,

$$
\left.\frac{\partial G}{\partial x}\right|_V\le 0,\qquad \left.\frac{\partial G}{\partial x}\right|_{V\cap(E\times\mathbb{R})}\ne 0. \tag{30}
$$

(i) Note that U is a connected component of V. By means of (30) and statement (ii) of Lemma 2.1, U contains at most 2 limit cycles of (3), counted with multiplicities. Furthermore, if this upper bound is achieved, then statements (i.a) and (i.b) are valid.

(ii) By assumption, we suppose $U_1\subset\{(t,x)|t\in[0,1],x>\lambda(t)\}$ and $U_2\subset\{(t,x)|t\in[0,1],x<\lambda(t)\}$ for convenience.

Now consider a perturbation of (3)

$$
\dot{x}=S_\varepsilon(t,x)=S(t,x)+\varepsilon\cdot q(t,x), \tag{31}
$$

where

$$
q(t,x)=\prod_{\lambda_i\in I}(x-\lambda_i),\qquad I=\big\{\lambda_i\big|i\in\{1,2,3\},b_i\ne 1\big\}.
$$

We know that $S_\varepsilon(t,x)$ is a polynomial in x of degree no more than 3, i.e. (31) is of the form (3). Recall that we suppose **(I.1)** holds for (3). For $\varepsilon\ge 0$, it follows from (4) and Lemma 2.4 that

$$
(b_i-1)\big(S_\varepsilon(t,\lambda_i)-\dot{\lambda}_i\big)=(b_i-1)\big(S(t,\lambda_i)-\dot{\lambda}_i\big)\ge 0,\qquad i=1,2,3,
$$

and

$$
\frac{\begin{vmatrix}S_\varepsilon(t,\lambda_1)&\lambda_1&1\\S_\varepsilon(t,\lambda_2)&\lambda_2&1\\S_\varepsilon(t,\lambda_3)&\lambda_3&1\end{vmatrix}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}=\frac{\begin{vmatrix}S(t,\lambda_1)&\lambda_1&1\\S(t,\lambda_2)&\lambda_2&1\\S(t,\lambda_3)&\lambda_3&1\end{vmatrix}+\varepsilon\begin{vmatrix}q(t,\lambda_1)&\lambda_1&1\\q(t,\lambda_2)&\lambda_2&1\\q(t,\lambda_3)&\lambda_3&1\end{vmatrix}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}\ge\omega(t)\ge 0.
$$

Hence, $(31)|_{\varepsilon\ge 0}$ also satisfies condition **(I.1)**. Using statement (i), each of U_1 and U_2 contains at most 2 limit cycles of $(31)|_{\varepsilon\ge 0}$, counted with multiplicities. In addition:

(a) If U_1 (U_2) contains two different limit cycles of $(31)|_{\varepsilon\ge 0}$, then the one above is stable and the one below is unstable.

(b) If U_1 (U_2) contains one limit cycle with multiplicity 2 of $(31)|_{\varepsilon\geq 0}$, then it is stable from above and unstable from below.

Since $x=\lambda(t)$ is a common boundary of U_1 and U_2 as well as a limit cycle of (3), we have $\lambda \in I$ and $S(t,\lambda)-\dot{\lambda}=0$. Therefore, λ is a simple zero of q in x, and $S_\varepsilon(t,\lambda)-\dot{\lambda}=0$ (i.e. $x=\lambda(t)$ is a limit cycle of (31)). This implies that

$$\operatorname{sgn}\left(\left.\frac{\partial q}{\partial x}\right|_{x=\lambda(t)}\right)=\operatorname{sgn}\left(q|_{U_1}\right)=-\operatorname{sgn}\left(q|_{U_2}\right)\neq 0, \tag{32}$$

and

$$\begin{aligned}\dot{H}_\varepsilon\big(\lambda(0)\big)&=\exp\int_0^1\left(\frac{\partial S}{\partial x}\big(t,\lambda(t)\big)+\varepsilon\frac{\partial q}{\partial x}\big(t,\lambda(t)\big)\right)dt\\&=\dot{H}_0\big(\lambda(0)\big)\cdot\exp\int_0^1\varepsilon\frac{\partial q}{\partial x}\big(t,\lambda(t)\big)dt,\end{aligned}\tag{33}$$

where H_ε is the return map of (31). As a consequence, for $\varepsilon>0$,

$$\begin{cases}\dot{H}_\varepsilon\big(\lambda(0)\big)>1,\ S_\varepsilon|_{U_1}>S|_{U_1}, & \text{if } \dot{H}_0\big(\lambda(0)\big)=1,\ \left.\dfrac{\partial q}{\partial x}\right|_{x=\lambda(t)}>0,\\ \dot{H}_\varepsilon\big(\lambda(0)\big)<1,\ S_\varepsilon|_{U_2}>S|_{U_2}, & \text{if } \dot{H}_0\big(\lambda(0)\big)=1,\ \left.\dfrac{\partial q}{\partial x}\right|_{x=\lambda(t)}<0.\end{cases}\tag{34}$$

In order to prove statement (ii.a), we assume to the contrary that each of U_1 and U_2 contains 2 limit cycles of (3) (i.e. $(31)|_{\varepsilon=0}$), counted with multiplicities. By statements (a) and (b), $x=\lambda(t)$ is a semi-stable limit cycle of $(31)|_{\varepsilon=0}$, stable from above and unstable from below. Furthermore, together with (34), there exist at least 3 limit cycles of (31) in U_1 (resp. U_2) as $\varepsilon>0$ sufficiently small and $(\partial q/\partial x)|_{x=\lambda(t)}>0$ (resp. $(\partial q/\partial x)|_{x=\lambda(t)}<0$), where one of them is near $x=\lambda(t)$. This shows a contradiction. As a result, statement (ii.a) holds.

Suppose that $U_1\cup U_2$ contains 3 limit cycles (counted with multiplicities) of (3) (i.e. $(31)|_{\varepsilon=0}$). Then one of them contains two limit cycles and the other one contains one. Assume for a contradiction that $x=\lambda(t)$ is semi-stable for (3). By statements (a) and (b), it is stable from above and unstable from below. In view of (34) again, for $\varepsilon>0$ sufficiently small we obtain:

(c) If U_1 contains 1 limit cycle (counted with multiplicity) of (3) and $(\partial q/\partial x)|_{x=\lambda(t)}>0$, then U_1 contains at least 2 limit cycles of (31), where the one below is stable from below and near $x=\lambda(t)$.

(d) If U_1 contains 2 limit cycles (counted with multiplicities) of (3) and $(\partial q/\partial x)|_{x=\lambda(t)}>0$, then U_1 contains at least 3 limit cycles of (31).

(e) If U_1 contains 1 limit cycle (counted with multiplicity) of (3) and $(\partial q/\partial x)|_{x=\lambda(t)}<0$, then U_2 contains at least 3 limit cycles of (31).

(f) If U_1 contains 2 limit cycles (counted with multiplicities) of (3) and $(\partial q/\partial x)|_{x=\lambda(t)}<0$, then U_2 contains at least 2 limit cycles of (31), where the one above is unstable from above and near $x=\lambda(t)$.

However, recall that each of U_1 and U_2 contains at most 2 limit cycles of $(31)|_{\varepsilon>0}$, counted with multiplicities. This contradicts to statements (d) and (e). And the stabilities of limit cycles of $(31)|_{\varepsilon>0}$ shown in statement (a) contradict to those shown in statements (c) and (f). Based on the above, $x = \lambda(t)$ is not semi-stable. Statement (ii.b) follows.

Finally, suppose that each of U_1 and U_2 contains 1 limit cycles (counted with multiplicity) of (3) and $x = \lambda(t)$ is semi-stable. Assume to the contrary that $x = \lambda(t)$ is stable from above and unstable from below. Then according to (34), the following statements are valid as $\varepsilon > 0$ sufficiently small.

(g) If $(\partial q/\partial x)|_{x=\lambda(t)} > 0$, then U_1 contains 2 limit cycles of (31), where the one below is stable and near $x = \lambda(t)$.

(h) If $(\partial q/\partial x)|_{x=\lambda(t)} < 0$, then U_2 contains 2 limit cycles of (31), where the one above is unstable and near $x = \lambda(t)$.

However, these contradict to the stabilities of limit cycles of $(31)|_{\varepsilon>0}$ shown in statement (a). As a consequence, $x = \lambda(t)$ is unstable from above and stable from below. Statement (ii.c) holds.

The proof of Proposition 2.5 is finished. □

3. Proof of Theorem 1.1

By virtue of the results given in the previous sections, we now begin to prove Theorem 1.1.

Proof of Theorem 1.1. Firstly, denote by

$$V = \big([0,1] \times \mathbb{R}\big) \backslash \big\{(t, \lambda_i) \big| t \in [0,1], i \in \{1,2,3\}, b_i \neq 1\big\}.$$

From assumption and statement (i) of Proposition 2.5, (3) has at most 2 limit cycles in each connected component of V, counted with multiplicities. Statement (i) holds.

Secondly, since $b_1 + b_2 + b_3 = 0$ from (6), at least one of b_1, b_2, b_3 is not equal to 1. Hence $n \in \{1,2,3\}$ by assumption. In what follows we divide the proof of statement (ii) into three cases.

Case 1. $n = 1$.

Without loss of generality, suppose that $b_1 \neq 1$ and $b_2 = b_3 = 1$. Then V has two connected components, with $x = \lambda_1(t)$ being common boundary.

When $x = \lambda_1(t)$ is not a limit cycle, all the limit cycles of (3) are located in V. Hence their number is at most $2 \times 2 = 2(n+1)$.

When $x = \lambda_1(t)$ is a limit cycle, statement (ii.a) of Proposition 2.5 tells us that one of the connected components of V contains at most 1 limit cycle of (3), counted with multiplicity. This implies that the number of limit cycles of (3) is no more than $1 + 1 + 2 = 2(n+1)$.

Case 2. $n = 2$.

Suppose that $b_1 \neq 1$, $b_2 \neq 1$ and $b_3 = 1$, without loss of generality. Then V has three consecutive connected components U_1, U_2 and U_3. For convenience, we suppose that $x = \lambda_1(t)$ (resp. $x = \lambda_2(t)$) is the common boundary of U_1 and U_2 (resp. U_2 and U_3).

When neither $x = \lambda_1(t)$ nor $x = \lambda_2(t)$ is a limit cycle, all the limit cycles of (3) are located in V. Thus their number is at most $2 \times 3 = 2(n+1)$.

When one of $x = \lambda_1(t)$ and $x = \lambda_2(t)$ is a limit cycle and the other one is not, it follows from statement (ii.a) of Proposition 2.5 that at least one of U_1, U_2 and U_3 contains at most 1 limit cycle of (3), counted with multiplicity. Therefore (3) has no more than $1+1+2+2 = 2(n+1)$ limit cycles.

When $x = \lambda_1(t)$ and $x = \lambda_2(t)$ are both limit cycles, the estimate for the number of limit cycles of (3) is obtained by the following subcases.

(2.a) Both U_1 and U_3 contain at most 1 limit cycle (counted with multiplicity), respectively. Then (3) has at most $1+1+1+1+2 = 2(n+1)$ limit cycles.

(2.b) One of U_1 and U_3 contains at most 1 limit cycle (counted with multiplicity) and the other one contains 2. Then statement (ii.a) of Proposition 2.5 tells us that U_2 contains at most 1 limit cycle, counted with multiplicity. Hence, the number of limit cycles is still no more than $1+1+1+1+2 = 2(n+1)$.

(2.c) Both U_1 and U_3 contain 2 limit cycles (counted with multiplicities), respectively. Then from statements (i.a) and (i.b) of Proposition 2.5, the number of limit cycles (counted with multiplicities) in $U_2 \cup \{x = \lambda_1(t)\} \cup \{x = \lambda_2(t)\}$ is even. Moreover, using statement (ii.a) of Proposition 2.5, U_2 contains at most 1 limit cycle, counted with multiplicity. Together with statement (ii.b) of Proposition 2.5, the multiplicities of $x = \lambda_1(t)$ and $x = \lambda_2(t)$ are both odd when this upper bound is exactly achieved. Thus, (3) has no limit cycles in U_2, which means that the total number of limit cycles is no more than $1+1+2+2 = 2(n+1)$.

Case 3. $n = 3$.

By assumption, V has four consecutive connected components U_1, U_2, U_3 and U_4. For convenience, we suppose $x = \lambda_1(t)$, $x = \lambda_2(t)$ and $x = \lambda_3(t)$ are common boundaries of U_1 and U_2, U_2 and U_3, and U_3 and U_4, respectively.

When none of $x = \lambda_1(t)$, $x = \lambda_2(t)$ and $x = \lambda_3(t)$ are limit cycles, the limit cycles of (3) only appear in V. Hence their number is no more than $2 \times 4 = 2(n+1)$.

When one of $x = \lambda_1(t)$, $x = \lambda_2(t)$ and $x = \lambda_3(t)$ is a limit cycle and the other two are not, it follows from statement (ii.a) of Proposition 2.5 that at least one of U_1, U_2, U_3 and U_4 contains at most 1 limit cycle of (3), counted with multiplicity. Thus (3) has at most $1+1+2+2+2 = 2(n+1)$ limit cycles.

When two of $x = \lambda_1(t)$, $x = \lambda_2(t)$ and $x = \lambda_3(t)$ are limit cycles and the rest one is not, the estimate for the number of limit cycles of (3) can be known by the following subcases.

(3.a) The two limit cycles in $\{x = \lambda_i(t) | i = 1, 2, 3\}$ are consecutive. Without loss of generality, suppose that they are $x = \lambda_1(t)$ and $x = \lambda_2(t)$. Then following a similar argument in statements (2.a), (2.b) and (2.c), there exist at most 6 limit cycles in $U_1 \cup U_2 \cup U_3 \cup \{x = \lambda_i(t) | i = 1, 2\}$. Recall that U_4 has at most 2 limit cycles. The number of limit cycles is no more than $6 + 2 = 2(n+1)$.

(3.b) The two limit cycles in $\{x = \lambda_i(t)|i = 1, 2, 3\}$ are not consecutive, i.e. they are $x = \lambda_1(t)$ and $x = \lambda_3(t)$. Then according to statement (ii.a) of Proposition 2.5, at least two of U_1, U_2, U_3 and U_4 contain at most 1 limit cycle (counted with multiplicity), respectively. Thus, the number of limit cycles is no more than $1 + 1 + 1 + 1 + 2 + 2 = 2(n + 1)$.

When $x = \lambda_1(t)$, $x = \lambda_2(t)$ and $x = \lambda_3(t)$ are all limit cycles, we get the estimate for the number of limit cycles of (3) by the following subcases.

(3.c) Either U_1 or U_4 contains at most 1 limit cycle, counted with multiplicity. Suppose that U_4 does, without loss of generality. Then again following a similar argument in cases (2.a), (2.b) and (2.c), there exist at most 6 limit cycles in $U_1 \cup U_2 \cup U_3 \cup \{x = \lambda_i(t)|i = 1, 2\}$. The number of limit cycles of (3) is no more than $6 + 1 + 1 = 2(n + 1)$.

(3.d) Each of U_1 and U_4 contains 2 limit cycles, and $U_2 \cup U_3$ contain at most 1 limit cycle, counted with multiplicities. Then the number of limit cycles is no more than $1 + 1 + 1 + 2 + 1 + 2 = 2(n + 1)$.

(3.e) Each of U_1 and U_4 contains 2 limit cycles, and $U_2 \cup U_3$ contains at least 2 limit cycles, counted with multiplicities. Then according to case (ii.a) of Proposition 2.5, each of U_2 and U_3 contains exactly 1 limit cycle, counted with multiplicity. Together with case (ii.b) of Proposition 2.5, both numbers of limit cycles (counted with multiplicities) in $U_1 \cup U_2 \cup \{x = \lambda_1(t)\}$ and $U_3 \cup U_4 \cup \{x = \lambda_3(t)\}$ are even. Hence, using cases (i.a) and (i.b) of Proposition 2.5 for U_1 and U_4, the stability of $x = \lambda_2(t)$ can be known. More precisely:
   - When $\omega(t) \geq 0$ and $(b_i - 1)\big(S(t, \lambda_i) - \dot{\lambda}_i\big) \geq 0$ for $i = 1, 2, 3$, $x = \lambda_2(t)$ is semi-stable, stable from above and unstable from below.
   - When $\omega(t) \leq 0$ and $(b_i - 1)\big(S(t, \lambda_i) - \dot{\lambda}_i\big) \leq 0$ for $i = 1, 2, 3$, $x = \lambda_2(t)$ is semi-stable, unstable from above and stable from below.

   However, both of these two cases contradict to case (ii.c) of Proposition 2.5. As a result, this subcase (3.e) does not exist.

Based on the argument for the above three cases, statement (ii) holds. The proof of Theorem 1.1 is finished. □

4. Proofs of Theorem 1.2 and Corollary 1.4

In this section we mainly prove Theorem 1.2 and Corollary 1.4. We also give an example, in which all the coefficients with respect to x change signs, to show the application of our result.

Proof of Theorem 1.2. Take $b_1 = b_2 = b_3 = 0$. Then b_1, b_2 and b_3 satisfy (6). In addition, $(b_i - 1)\big(S(t, \lambda_i) - \dot{\lambda}_i\big) = -\big(S(t, \lambda_i) - \dot{\lambda}_i\big)$, $i = 1, 2, 3$. Thus, our assertion immediately follows from Theorem 1.1.

Example 1. Consider equation

$$\begin{aligned}\frac{dx}{dt} &= S(t,x)\\ &= -2\pi\Bigg(\cos(2\pi t)x^3 + \big(1-3\cos(2\pi t)-2\cos^2(2\pi t)\big)x^2\\ &\qquad\qquad - \big(2+4\cos(2\pi t)-4\cos^2(2\pi t)-\cos^3(2\pi t)\big)x\\ &\qquad\qquad - \big(6-6\cos(2\pi t)+\cos^3(2\pi t)\big)\Bigg). \end{aligned} \tag{35}$$

We know that (35) is of the form (3) with

$$a_3(t) = -2\pi\cos(2\pi t), \qquad a_2(t) = -2\pi\big(1-3\cos(2\pi t)-2\cos^2(2\pi t)\big).$$

If we take

$$\lambda_1(t)=1, \quad \lambda_2(t)=3+\cos(2\pi t), \quad \lambda_3(t)=-1+\cos(2\pi t),$$

then it follows from (5) that

$$\omega(t) = a_3(t)\cdot\big(\lambda_1+\lambda_2+\lambda_3\big)+a_2(t) = -2\pi < 0.$$

Furthermore, a direct calculation shows that

$$\begin{aligned} S(t,\lambda_1) &= 2\pi\big(7-2\cos^2(2\pi t)\big),\\ S(t,\lambda_2) &= 2\pi\big(3+2\cos(2\pi t)\big),\\ S(t,\lambda_3) &= 2\pi\big(3-2\cos(2\pi t)\big), \end{aligned}$$

which implies that

$$\begin{aligned} S(t,\lambda_1)-\dot{\lambda_1} &= 2\pi\big(7-2\cos^2(2\pi t)\big) > 0,\\ S(t,\lambda_2)-\dot{\lambda_2} &= 2\pi\big(3+2\cos(2\pi t)+\sin(2\pi t)\big) > 0,\\ S(t,\lambda_3)-\dot{\lambda_3} &= 2\pi\big(3-2\cos(2\pi t)+\sin(2\pi t)\big) > 0. \end{aligned}$$

According to Theorem 1.2, the number of limit cycles of (35) is no more than 8.

Remark 4.1. We emphasize that Eq. (35) is an Abel equation which has no coefficients with respect to x keeping signs.

Example 2. Consider equation

$$\frac{dx}{dt} = S(t,x) = a_3(t)\left(x^2-1\right)x + \varepsilon\left(-\alpha x^2+\alpha+\beta e^{2A_3(t)}\right), \tag{36}$$

where $\alpha,\beta>0$, $a_3\in \mathrm{C}^\infty([0,1])$ and $A_3(t)=\int_0^t a_3(s)ds$ with $A_3(1)=0$. If we take $\lambda_1(t)=1$, $\lambda_2(t)=0$ and $\lambda_3(t)=-1$, then

$$\omega(t) = -\varepsilon\alpha, \qquad S(t,\lambda_1)=S(t,\lambda_3)=\varepsilon\beta e^{2A_3}, \qquad S(t,\lambda_2)=\varepsilon\left(\alpha+\beta e^{2A_3}\right).$$

From Theorem 1.2, Eq. (36) has at most 8 limit cycles when $\varepsilon\neq 0$.

In the following we show that there exist some equations which are of the form (36) with at least 4 limit cycles as $\varepsilon \neq 0$ small enough.

Let us regard (36) as a perturbation of $(36)_{\varepsilon=0}$, and ε is the small perturbation parameter. It is not hard to know that the first integral and the solution of $(36)_{\varepsilon=0}$ can be written in

$$\mathcal{H}(t,x) = e^{-2A_3}\left(1-\frac{1}{x^2}\right) \text{ and } x = \pm\left(1-he^{2A_3}\right)^{-1/2}, \tag{37}$$

respectively, where $h \le \min_{t\in[0,1]}\left\{e^{-2A_3}\right\} := h_r$. Moreover, the solutions well defined in [0,1] are all periodic due to $A_3(1)=0$. Thus, when $\varepsilon \neq 0$ is small enough, using the first order analysis (about ε), the number of limit cycles bifurcated from the family of periodic solutions, is exactly the twice of number of simple zeros of the following function:

$$\begin{aligned}\Phi(h) &= \int_0^1 \left(\frac{\partial \mathcal{H}}{\partial x}\cdot\left(-\alpha x^2+\alpha+\beta e^{2A_3}\right)\right)\Big|_{H(t,x)=h} dt \\ &= \mp 2\left(\alpha h\int_0^1 \left(1-he^{2A_3}\right)^{1/2} dt - \beta\int_0^1 \left(1-he^{2A_3}\right)^{3/2} dt\right),\ h\in(0,h_r).\end{aligned} \tag{38}$$

In fact, by (37) each simple zero of Φ implies two symmetric solutions of $(36)_{\varepsilon=0}$, which give rise to two limit cycles of $(36)_{\varepsilon\neq 0}$ that tend to these two solutions respectively as ε goes to 0. For a similar argument see [20].

Now denote by

$$J_{k,\eta}(h) = \int_0^1 e^{kA_3}\left(1-he^{2A_3}\right)^{\eta} dt, \qquad k\in\mathbb{Z},\ \eta\in\mathbb{R}.$$

Then $\Phi(h) = \mp 2\left(\alpha\cdot hJ_{0,1/2} - \beta\cdot J_{0,3/2}\right)$. One can check that

$$\begin{aligned}\dot{J}_{k,\eta} &= -\eta J_{k+2,\eta-1}, \\ J_{k,\eta} &= J_{k,\eta-1} - hJ_{k+2,\eta-1} = J_{k,\eta-2} - 2hJ_{k+2,\eta-2} + h^2 J_{k+4,\eta-2}.\end{aligned}$$

Hence, the Wronskian determinant for $hJ_{0,1/2}$ and $J_{0,3/2}$ is

$$\begin{aligned} & W(hJ_{0,1/2}, J_{0,3/2}) \\ &\quad = \begin{vmatrix} hJ_{0,1/2} & J_{0,3/2} \\ J_{0,1/2}-\frac{1}{2}hJ_{2,-1/2} & -\frac{3}{2}J_{2,1/2}\end{vmatrix} \\ &\quad = \begin{vmatrix} hJ_{0,-1/2}-h^2J_{2,-1/2} & J_{0,-1/2}-2hJ_{2,-1/2}+h^2J_{4,-1/2} \\ J_{0,-1/2}-\frac{3}{2}hJ_{2,-1/2} & -\frac{3}{2}J_{2,-1/2}+\frac{3}{2}hJ_{4,-1/2}\end{vmatrix} \\ &\quad = -\left(J_{0,-1/2}-hJ_{2,-1/2}\right)^2 + \frac{1}{2}h^2\left(J_{0,-1/2}\cdot J_{4,-1/2} - J_{2,-1/2}^2\right) \\ &\quad = -J_{0,1/2}^2 + \frac{1}{2}h^2\left(J_{0,-1/2}\cdot J_{4,-1/2} - J_{2,-1/2}^2\right). \end{aligned} \tag{39}$$

We have the following proposition.

Proposition 4.2. *Suppose there exists $h_0 \in (0, h_r)$ such that $W(hJ_{0,1/2}, J_{0,3/2})|_{h=h_0} > 0$. Then there exist $\alpha, \beta > 0$ such that Φ has at least 2 positive simple zeros. Therefore Eq.* (36) *has at least 4 limit cycles as $\varepsilon \neq 0$ sufficiently small.*

Proof. Firstly, since $J_{0,3/2} > 0$, we get

$$\Phi(h) = \mp 2\alpha \cdot J_{0,3/2}\left(\frac{hJ_{0,1/2}}{J_{0,3/2}} - \frac{\beta}{\alpha}\right),$$
$$\frac{d}{dh}\left(\frac{hJ_{0,1/2}}{J_{0,3/2}}\right) = -\frac{W(hJ_{0,1/2}, J_{0,3/2})}{J_{0,3/2}^2}.$$

Secondly, due to (39), $W(hJ_{0,1/2}, J_{0,3/2})|_{h=0} < 0$. Together with assumption, $hJ_{0,1/2}/J_{0,3/2}$ has a local maximum in the interval $(0, h_0)$. Observe that $hJ_{0,1/2}/J_{0,3/2}$ is always positive. Consequently, there exist fixed $\alpha, \beta > 0$ such that $hJ_{0,1/2}/J_{0,3/2} - \beta/\alpha$ (i.e. Φ) has at least 2 positive simple zeros in $(0, h_0)$. From the previous mention, the conclusion follows. □

We would like to show a concrete example. Take

$$h_0 = \frac{3}{4}, \quad A_3(t) = \frac{1}{2}\ln\left(\frac{4}{3} - \frac{4}{3}\left(2(1-2\mu)\left(t - \frac{1}{2}\right)^2 + \mu\right)^2\right), \quad 0 < \mu < \frac{1}{2}.$$

Then $A_3(0) = A_3(1) = 0$, and one can check by (39) that

$$W(hJ_{0,1/2}, J_{0,3/2})|_{h=3/4} = \frac{128\mu^3 + 32\mu^2 + 12\mu + 5}{280\big((2-4\mu)\mu\big)^{1/2}} \cdot \arctan\left(\frac{1}{2\mu} - 1\right)^{1/2} - \frac{(1+4\mu)^2}{24}.$$

So the assumption of Proposition 4.2 holds as $\mu > 0$ is sufficiently small. For instance, $W(hJ_{0,1/2}, J_{0,3/2})|_{h=3/4} = 1.9244 > 0$ when $\mu = 10^{-4}$ (in this situation we have $h_0 = 3/4 \in (0, h_r) = (0, 25\ 000\ 000/33\ 333\ 333)$). That is to say, there exactly exists an equation of the form (36) with at least 4 limit cycles.

To our knowledge, for Theorem 1.2, the example with 3 limit cycles is easy to find (such as an equation with constant coefficients). However, up till now, even if there is no restriction for Eq. (3), only some special cases with at least 4 limit cycles are known, because they all come from the bifurcation studies and the calculations are very difficult. This is the reason that we can only find the example which has 4 limit cycles.

Nevertheless, such example is still very significant, because in the following we can prove Corollary 1.4 with its help. This shows that the upper bound appearing in Theorem 1.2 seems not so rough.

Proof of Corollary 1.4. By assumption, $\mathrm{sgn}(a_0(t)) = -\mathrm{sgn}(a_2(t)) \neq 0$. Hence there exists a small constant $\lambda_0 > 0$, such that

$$\begin{aligned}\mathrm{sgn}\big(a_3(t)\cdot\lambda_0 + a_2(t)\big) &= \mathrm{sgn}(a_2(t)),\\ \mathrm{sgn}\big(S(t,x)\big) &= \mathrm{sgn}\left(a_3(t)x^3 + a_2(t)x^2 + a_1(t)x + a_0(t)\right)\\ &= \mathrm{sgn}(a_0(t))\\ &= -\mathrm{sgn}(a_2(t)),\end{aligned} \tag{40}$$

where $x \in (0,\lambda_0)$.

Now define three constant functions $\lambda_1(t) = \lambda_0/6$, $\lambda_2(t) = \lambda_0/3$ and $\lambda_3(t) = \lambda_0/2$. It follows from (40) that

$$\begin{aligned}&\mathrm{sgn}\big(a_3(t)\cdot(\lambda_1+\lambda_2+\lambda_3) + a_2(t)\big) = \mathrm{sgn}(a_2(t)),\\ &\mathrm{sgn}\big(S(t,\lambda_i) - \dot{\lambda}_i\big) = \mathrm{sgn}\big(S(t,\lambda_i)\big) = -\mathrm{sgn}(a_2(t)), \qquad i = 1,2,3.\end{aligned}$$

Together with (5), we know that Eq. (3) satisfies the assumption of Theorem 1.2. As a consequence, each region of $[0,1]\times(-\infty,\lambda_1)$, $[0,1]\times(\lambda_1,\lambda_2)$, $[0,1]\times(\lambda_2,\lambda_3)$ and $[0,1]\times(\lambda_3,+\infty)$ contains at most 2 limit cycles (counted with multiplicities) of (3).

Observe that from the second equality in (40), neither $[0,1]\times(\lambda_1,\lambda_2)$ nor $[0,1]\times(\lambda_2,\lambda_3)$ contains limit cycles of (3). For the same reason, all of $x = \lambda_1$, $x = \lambda_2$ and $x = \lambda_3$ are not limit cycles of (3).

Based on the above, (3) has at most 4 limit cycles, counted with multiplicities. We can use Eq. (36) and Proposition 4.2 to show that such upper bound is sharp. In fact, when $\varepsilon \neq 0$, $a_2(t)a_0(t) = -\varepsilon^2\alpha\big(\alpha + \beta e^{2A_3}\big) < 0$, which satisfies the assumption of the corollary. On the other hand, the previous argument tells us that there exists an example in which the equation has exactly 4 limit cycles. The conclusion follows. $\square$

5. Proofs of Theorem 1.6 and Corollary 1.7

At the beginning of this section, we give the proof of Theorem 1.6.

Proof of Theorem 1.6. By assumption, let $b_1 = b - 1$, $b_2 = -b$ and $b_3 = 1$. It is easy to check that $b_1 + b_2 + b_3 = 0$ and

$$b_1\lambda_1 + b_2\lambda_2 + b_3\lambda_3 = \lambda_1\cdot\left(\frac{\lambda_1-\lambda_3}{\lambda_1-\lambda_2} - 1\right) - \lambda_2\cdot\frac{\lambda_1-\lambda_3}{\lambda_1-\lambda_2} + \lambda_3 = 0.$$

Hence, b_1, b_2 and b_3 are three constants satisfying (6). Moreover,

$$\begin{aligned}(b_1-1)\big(S(t,\lambda_1)-\dot{\lambda}_1\big) &= (b-2)\big(S(t,\lambda_1)-\dot{\lambda}_1\big),\\ (b_2-1)\big(S(t,\lambda_2)-\dot{\lambda}_2\big) &= -(b+1)\big(S(t,\lambda_2)-\dot{\lambda}_2\big),\\ (b_3-1)\big(S(t,\lambda_1)-\dot{\lambda}_1\big) &= 0.\end{aligned}$$

Thus, according to assumption and Theorem 1.1, (3) has at most

(a) 2 limit cycles in each connected component of $\big([0,1]\times\mathbb{R}\big)\backslash\big\{(t,\lambda_i)\big|t \in [0,1], i\in\{1,2\}, b_i \neq 1\big\}$, counted with multiplicities.

(b) $2(n+1)$ limit cycles on $[0,1]\times\mathbb{R}$, where $n=\#\{b_i|i\in\{1,2\},b_i\neq 1\}$ and $\#$ represents the cardinality of a set.

Clearly, $\big([0,1]\times\mathbb{R}\big)\backslash\big\{(t,\lambda_i)\big|t\in[0,1],i=1,2\big\}\subseteq\big([0,1]\times\mathbb{R}\big)\backslash\big\{(t,\lambda_i)\big|t\in[0,1],i\in\{1,2\},b_i\neq 1\big\}$, and $n\le 2$. We therefore obtain statements (i) and (ii) of the theorem. □

Now we are in position to prove Corollary 1.7.

Proof of Corollary 1.7. Firstly, define three constant functions

$$\lambda_1(t)=0,\quad \lambda_2(t)=\kappa_2,\quad \lambda_3(t)=\kappa_1-\kappa_2. \tag{41}$$

It is clear that λ_1, λ_2 and λ_3 satisfy hypothesis **(H)**, and

$$b\triangleq\frac{\lambda_1-\lambda_3}{\lambda_1-\lambda_2}=\frac{\kappa_1}{\kappa_2}-1\in\mathbb{R}. \tag{42}$$

Secondly, Eq. (7) is of the form (3). Denote by $S(t,x)=a_3(t)x^3+a_2(t)x^2+a_1(t)x$. It follows from (41) and (42) that

$$\begin{aligned}&(b-2)\left(S(t,\lambda_1)-\dot{\lambda_1}\right)=(b-2)\cdot S(t,0)=0,\\&(b+1)\left(S(t,\lambda_2)-\dot{\lambda_2}\right)=(b+1)\cdot S(t,\kappa_2)=\kappa_1\left(a_3(t)\kappa_2^2+a_2(t)\kappa_2+a_1(t)\right).\end{aligned}$$

In addition, let $\omega(t)$ be defined as in (4) (i.e. (5)). Then

$$\omega(t)=a_3(t)\cdot\big(\lambda_1+\lambda_2+\lambda_3\big)+a_2(t)=a_3(t)\kappa_1+a_2(t).$$

From assumption,

$$\operatorname{sgn}\left((b+1)\left(S(t,\lambda_2)-\dot{\lambda_2}\right)\right)=-\operatorname{sgn}\big(\omega(t)\big)\neq 0.$$

As a result, we know by Theorem 1.6 that Eq. (7) has at most 6 limit cycles. □

Example 3. Consider equation

$$\frac{dx}{dt}=2\cos(2\pi t)x^3+\big(1-2\cos(2\pi t)\big)x^2-\big(1+4\cos(2\pi t)\big)x. \tag{43}$$

Then (43) is of the form (7) with

$$a_3(t){=}2\cos(2\pi t),\quad a_2(t){=}1-2\cos(2\pi t),\quad a_1(t){=}-\big(1+4\cos(2\pi t)\big).$$

Taking $\kappa_1=1$ and $\kappa_2=-1$, we have

$$\begin{aligned}&a_3(t)\kappa_1^2+a_2(t)\kappa_1=1>0,\\&a_3(t)\kappa_2^2+a_2(t)\kappa_2+a_1(t)=-2<0.\end{aligned}$$

According to Corollary 1.7, (43) has at most 6 limit cycles.

6. Application on planar polynomial system with homogeneous nonlinearities

In this final section, we apply our theorems to study the limit cycles for the planar polynomial system (8) which has been extensively studied by a large number of authors. Now the proof of Proposition 1.8 is given below.

Proof of Proposition 1.8. Firstly, system (8) in polar coordinates can be written in the form

$$\begin{cases} \dfrac{dr}{dt} = ar + r^n\varphi(\theta), \\ \dfrac{d\theta}{dt} = 1 + r^{n-1}\psi(\theta), \end{cases} \tag{44}$$

where $\varphi(\theta)$ and $\psi(\theta)$ are defined as in (9).

It is known that the limit cycles surrounding the origin of system (8) do not intersect the curve $1 + r^{n-1}\psi(\theta) = 0$ (see Carbonell and Llibre [13]). Therefore, these limit cycles can be investigated by equation

$$\frac{dr}{d\theta} = \frac{ar + r^n\varphi(\theta)}{1 + r^{n-1}\psi(\theta)}, \qquad \theta \in [0, 2\pi]. \tag{45}$$

Furthermore, using the transformation introduced by Cherkas [14]

$$\rho = \frac{r^{n-1}}{1 + r^{n-1}\psi(\theta)}, \quad \theta = 2\pi\tau, \tag{46}$$

Equation (45) becomes

$$\frac{d\rho}{d\tau} = \mathcal{S}(\tau, \rho) = a_3\big(\theta(\tau)\big)\rho^3 + a_2\big(\theta(\tau)\big)\rho^2 + a_1\big(\theta(\tau)\big)\rho, \qquad \tau \in [0, 1], \tag{47}$$

where

$$\begin{aligned} a_3(\theta) &= 2(n-1)\pi \cdot \psi(\theta)\big(a\psi(\theta) - \varphi(\theta)\big), \\ a_2(\theta) &= 2(n-1)\pi \cdot \big(\varphi(\theta) - 2a\psi(\theta)\big) - 2\pi\dot{\psi}(\theta), \\ a_1(\theta) &= 2a(n-1)\pi. \end{aligned} \tag{48}$$

We claim that $\psi(\theta) \neq 0$. In fact, since ψ is differentiable and periodic, there exist $\theta_{\max}, \theta_{\min} \in [0, 2\pi]$ such that

$$\psi(\theta_{\max}) = \max_{\theta\in[0,2\pi]}\{\psi(\theta)\}, \quad \psi(\theta_{\min}) = \min_{\theta\in[0,2\pi]}\{\psi(\theta)\}, \quad \dot{\psi}(\theta_{\max}) = \dot{\psi}(\theta_{\min}) = 0.$$

From assumption,

$$\begin{aligned} (n-1)a\psi(\theta_{\max}) &= (n-1)a\psi(\theta_{\max}) + \dot{\psi}(\theta_{\max}) \neq 0, \\ (n-1)a\psi(\theta_{\min}) &= (n-1)a\psi(\theta_{\min}) + \dot{\psi}(\theta_{\min}) \neq 0, \end{aligned}$$

which implies that $\psi(\theta_{\max}) \neq 0$, $\psi(\theta_{\min}) \neq 0$ and $a \neq 0$. Thus, $\mathrm{sgn}\big(\psi(\theta_{\max})\big) = \mathrm{sgn}\big(\psi(\theta_{\min})\big) \neq 0$ (otherwise there exists $\theta_0 \in [0, 2\pi]$ satisfying $\dot{\psi}(\theta_0) + (n-1)a\psi(\theta_0) = 0$). We know $\psi(\theta) \neq 0$.

We also emphasize that the limit cycles of system (8) in polar coordinates are all located in region $[0, 2\pi] \times \mathbb{R}^+$. And transformation (46) sends the region

$[0, 2\pi] \times \mathbb{R}^+$ to the region U_1 (resp. $U_2 \cup U_3$) when $\psi(\theta) > 0$ (resp. $\psi(\theta) < 0$), where

$$\begin{aligned} U_1 &= \big\{(\tau, \rho) \big| 0 < \rho < 1/\psi(\theta(\tau)),\ \tau \in [0, 1]\big\}, \\ U_2 &= \big\{(\tau, \rho) \big| \rho < 1/\psi(\theta(\tau)),\ \tau \in [0, 1]\big\}, \\ U_3 &= \big\{(\tau, \rho) \big| \rho > 0,\ \tau \in [0, 1]\big\}. \end{aligned}$$

Now by a direct calculation (47) can be rewritten into

$$\frac{1}{(\psi\rho - 1)(\psi\rho)^2} \cdot \frac{d(\psi\rho)}{d\theta} = (n-1)\frac{a\psi(\theta) - \varphi(\theta)}{\psi(\theta)} - \frac{(n-1)a\psi(\theta) + \dot{\psi}(\theta)}{\psi^2(\theta)\rho}.$$

Hence, if $\rho_1(\theta) \neq \rho_2(\theta)$ are two limit cycles of (47) located in U_1 (resp. $U_2 \cup U_3$) when $\psi(\theta) > 0$ (resp. $\psi(\theta) < 0$), then

$$\begin{aligned} &\int_0^{2\pi} \frac{(n-1)a\psi(\theta) + \dot{\psi}(\theta)}{\psi^2(\theta)} \cdot \frac{1}{\rho_i(\theta)} d\theta \\ &\quad = \int_0^{2\pi} (n-1)\frac{a\psi(\theta) - \varphi(\theta)}{\psi(\theta)} d\theta, \quad i = 1, 2. \end{aligned} \tag{49}$$

However, since $\big((n-1)a\psi + \dot{\psi}\big)/\psi^2 \neq 0$, we get

$$\int_0^{2\pi} \frac{(n-1)a\psi(\theta) + \dot{\psi}(\theta)}{\psi^2(\theta)} \cdot \frac{1}{\rho_1(\theta)} d\theta \neq \int_0^{2\pi} \frac{(n-1)a\psi(\theta) + \dot{\psi}(\theta)}{\psi^2(\theta)} \cdot \frac{1}{\rho_2(\theta)} d\theta,$$

which contradicts to (49). As a result, when $\psi(\theta) > 0$ (resp. $\psi(\theta) < 0$), Eq. (47) has at most 1 limit cycle (not counted with multiplicity) in U_1 (resp. $U_2 \cup U_3$).

The rest of the proof is to obtain the multiplicity of the limit cycle of Eq. (47). To this end we apply Theorem 1.6. Define three functions

$$\lambda_1(\tau) = 0, \quad \lambda_2(\tau) = \frac{\varepsilon}{\psi\big(\theta(\tau)\big)}, \quad \lambda_3(\tau) = \frac{1 - \varepsilon}{\psi\big(\theta(\tau)\big)}, \qquad \varepsilon \in (-1/2, 1/2)\backslash\{0\}.$$

By the previous claim, they are well-defined in $[0, 1]$ and satisfy hypothesis **(H)**.

According to (48), we have

$$\begin{aligned} \omega(\tau) &\triangleq a_3\big(\theta(\tau)\big) \cdot \big(\lambda_1(\tau) + \lambda_2(\tau) + \lambda_3(\tau)\big) + a_2\big(\theta(\tau)\big) \\ &= \frac{a_3(\theta)}{\psi(\theta)} + a_2(\theta) \\ &= -2\pi\left((n-1)a\psi(\theta) + \dot{\psi}(\theta)\right). \end{aligned}$$

It follows from assumption that

$$\mathrm{sgn}\big(\omega(\tau)\big) = -\mathrm{sgn}\left((n-1)a\psi(\theta) + \dot{\psi}(\theta)\right) \neq 0. \tag{50}$$

Moreover, a direct calculation shows that $\mathcal{S}(\tau,\lambda_1)=0$ and

$$\begin{aligned}\mathcal{S}(\tau,\lambda_2)-\frac{d\lambda_2}{d\tau}&=\mathcal{S}(\tau,\lambda_2)-\frac{d\theta}{d\tau}\cdot\frac{d\lambda_2}{d\theta}\\&=\Big(a_3(\theta)\lambda_2^2+a_2(\theta)\lambda_2+a_1(\theta)\Big)\frac{\varepsilon}{\psi(\theta)}+2\pi\varepsilon\cdot\frac{\dot\psi(\theta)}{\psi^2(\theta)}\\&=\Big(a_3(\theta)\varepsilon^2+a_2(\theta)\psi(\theta)\varepsilon+2\pi\psi(\theta)\Big((n-1)a\psi(\theta)+\dot\psi(\theta)\Big)\Big)\\&\quad\frac{\varepsilon}{\psi^3(\theta)}.\end{aligned}$$

Again following assumption, for $\varepsilon\neq0$ sufficiently small, we obtain

$$\operatorname{sgn}\left(\mathcal{S}(\tau,\lambda_2)-\frac{d\lambda_2}{d\tau}\right)=\operatorname{sgn}\Big(\varepsilon\big((n-1)a\psi(\theta)+\dot\psi(\theta)\big)\Big)\neq0.\qquad(51)$$

Observe that

$$b\triangleq\frac{\lambda_1-\lambda_3}{\lambda_1-\lambda_2}=\frac{1}{\varepsilon}-1\in\mathbb{R}.$$

For $\varepsilon\neq0$ sufficiently small, (50) and (51) tell us that

$$\operatorname{sgn}\left((b+1)\left(\mathcal{S}(\tau,\lambda_2)-\frac{d\lambda_2}{d\tau}\right)\right)=\operatorname{sgn}\Big((n-1)a\psi(\theta)+\dot\psi(\theta)\Big)=-\operatorname{sgn}\big(\omega(\tau)\big).$$

Meanwhile, we have

$$(b-2)\left(\mathcal{S}(\tau,\lambda_1)-\frac{d\lambda_1}{d\tau}\right)=0.$$

As a consequence, when $\varepsilon\neq0$ is fixed and small enough, we know by Theorem 1.6 that any arbitrary limit cycle of (47) in the connected component of $\big([0,1]\times\mathbb{R}\big)\backslash\big\{(\tau,\lambda_i)\big|\tau\in[0,1],\ i=1,2\big\}$, has multiplicity at most 2. Furthermore, if we take ε sufficiently small, then all the non-zero limit cycles of (47) do not intersect the curves $\rho=\lambda_1(\tau)$ and $\rho=\lambda_2(\tau)$. Thus the multiplicity of any arbitrary limit cycle of (47) is no more than 2.

Based on the above, system (8) has at most 1 limit cycle surrounding the origin, and the multiplicity is no more than 2.

The proof of Proposition 1.8 is finished.

Example 4. Consider system

$$\begin{cases}\dfrac{dx}{dt}=x-y-(x+y)\big(x^2+y^2\big)^k,\\[2mm]\dfrac{dy}{dt}=x+y+(x-y)\big(x^2+y^2\big)^k,\end{cases}\qquad k\in\mathbb{Z}^+.\qquad(52)$$

We know that (52) is of the form (8) with $a=1$, $n=2k+1$ and

$$P_n(x,y)=-(x+y)\big(x^2+y^2\big)^k,\qquad Q_n(x,y)=(x-y)\big(x^2+y^2\big)^k.$$

A straightforward calculation shows that $\psi(\theta)=1$ and $(n-1)a\psi(\theta)+\dot\psi(\theta)=2k>0$. Hence by Theorem 1.2, the number of limit cycles of system (52) surrounding the origin is at most 1.

Actually, one can check that system (52) has a limit cycle $x^2 + y^2 = 1$ surrounding the origin.

Example 5. Consider planar system

$$\begin{cases} \dfrac{dx}{dt} = x - y - x^3 + \dfrac{3}{2}x^2y - xy^2 + \dfrac{1}{2}y^3, \\ \dfrac{dy}{dt} = x + y - \dfrac{3}{2}x^3 - x^2y - \dfrac{1}{2}xy^2 - y^3. \end{cases} \tag{53}$$

Then (53) is of the form (8) with $a = 1$, $n = 3$ and

$$P_3(x, y) = -x^3 + \frac{3}{2}x^2y - xy^2 + \frac{1}{2}y^3, \quad Q_3(x, y) = -\frac{3}{2}x^3 - x^2y - \frac{1}{2}xy^2 - y^3.$$

It is easy to check that

$$\psi(\theta) = -\frac{1}{2} - \cos^2(\theta). \tag{54}$$

Consequently,

$$(n - 1)a\psi(\theta) + \dot{\psi}(\theta) = -2 - \cos(2\theta) + \sin(2\theta) < 0.$$

Applying Proposition 1.8 again, the number of limit cycles of system (53) surrounding the origin is at most 1.

For system (53), we can also obtain by (9) that $\varphi(\theta) = -1$. Together with (54),

$$\begin{aligned} &\varphi(\theta) - a\psi(\theta) = \frac{1}{2}\cos(2\theta), \\ &\psi(\theta)\big(\varphi(\theta) - a\psi(\theta)\big) = -\frac{1}{2}\cos(2\theta)\left(\frac{1}{2} + \cos^2(\theta)\right), \\ &(n - 1)\big(\varphi(\theta) - 2a\psi(\theta)\big) - \dot{\psi}(\theta) = 2 + 2\cos(2\theta) - \sin(2\theta). \end{aligned}$$

Clearly, all of these three equalities have indefinite signs, which violate the conditions of the representative results (I)–(IV) shown in our introduction (Sect. 1). That is to say, our Proposition 1.8 is indeed an improvement of the previous works.

References

[1] Alkoumi, N.M.H., Torres, P.J.: Estimates on the number of limit cycles of a generalized Abel equation. Discrete Cont. Dyn. A **31**, 25–34 (2011)

[2] Álvarez, A., Bravo, J.L., Fernández, M.: Limit cycles of Abel equations of the first kind. J. Math. Anal. Appl. **423**, 734–745 (2015)

[3] Álvarez, M.J., Bravo, J.L., Fernández, M.: Existence of non-trivial limit cycles in Abel equations with symmetries. Nonlinear Anal. Theor. Meth. Appl. **84**, 18–28 (2013)

[4] Álvarez, M.J., Gasull, A., Giacomini, H.: A new uniqueness criterion for the number of periodic orbits of Abel equations. J. Differ. Equ. **234**, 161–176 (2007)

[5] Benardete, D.M., Noonburg, V.W., Pollina, B.: Qualitative tools for studying periodic solutions and bifurcations as applied to the periodically harvested logistic equation. Am. Math. Mon. **115**, 202–219 (2008)

[6] Benterki, R., Llibre, J.: Limit cycles of polynomial differential equations with quintic homogeneous nonlinearities. J. Math. Anal. Appl. **407**, 16–22 (2013)

[7] Blinov, M., Yomdin, Y.: Generalized centre conditions and multiplicities for polynomial Abel equations of small degrees. Nonlinearity **12**, 3869–3876 (1999)

[8] Bravo, J.L., Fernández, M.: Limit cycles of non-autonomous scalar odes with two summands. Commun. Pure Appl. Anal. **12–2**, 1091–1102 (2013)

[9] Bravo, J.L., Fernández, M., Gasull, A.: Limit cycles for some Abel equations having coefficients without fixed signs. Int. J. Bifurc. Chaos **19**, 3869–3876 (2009)

[10] Bravo, J.L., Torregrosa, J.: Abel-like differential equations with no periodic solutions. J. Math. Anal. Appl. **342**, 931–942 (2008)

[11] Briskin, M., Françoise, J.-P., Yomdin, Y.: The Bautin ideal of the Abel equation. Nonlinearity **11**, 41–53 (1998)

[12] Carbonell, M., Llibre, J.: Limit cycles of a class of polynomial systems. Proc. R. Soc. Edinb. **109A**, 187–199 (1988)

[13] Carbonell, M., Llibre, J.: Hopf bifurcation, averaging methods and Liapunov quantities for polynomial systems with homogeneous nonlinearities. In: Proceedings of European Conference on Iteration Theory-ECIT 87, pp. 145–160. World Scientific, Singapore (1989)

[14] Cherkas, L.A.: Number of limit cycles of an autonomous second-order system. Differ. Uravn. **12**, 944–946 (1975)

[15] Cima, A., Gasull, A., Mañosas, F.: Periodic orbits in complex Abel equations. J. Differ. Equ. **232**, 314–328 (2007)

[16] Coll, T., Gasull, A., Prohens, R.: Differential equations defined by the sum of two quasi-homogeneous vector fields. Can. J. Math. **49**, 212–231 (1997)

[17] Devlin, J., Lloyd, N.G., Pearson, J.M.: Cubic systems and Abel equations. J. Differ. Equ. **147**, 435–454 (1998)

[18] Fan, M., Wang, K.: Optimal harvesting policy for single population with periodic coefficients. Math. Biosci. **152**, 165–177 (1998)

[19] Gasull, A., Guillamon, A.: Limit cycles for generalized Abel equations. Int. J. Bifurc. Chaos **16**, 3737–3745 (2006)

[20] Gasull, A., Li, C., Torregrosa, J.: A new Chebyshev family with applications to Abel equations. J. Differ. Equ. **252**, 1635–1641 (2012)

[21] Gasull, A., Llibre, J.: Limit cycles for a class of Abel equation. SIAM J. Math. Anal. **21**, 1235–1244 (1990)

[22] Gasull, A., Yu, J., Zhang, X.: Vector fields with homogeneous nonlinearities and many limit cycles. J. Differ. Equ. **258**, 3286–3303 (2015)

[23] Huang, J., Zhao, Y.: Periodic solutions for equation $\dot{x} = A(t)x^m + B(t)x^n + C(t)x^l$ with $A(t)$ and $B(t)$ changing signs. J. Differ. Equ. **253**, 73–99 (2012)

[24] Ilyashenko, Y.: Hilbert-type numbers for Abel equations, growth and zeros of holomorphic functions. Nonlinearity **13**, 1337–1342 (2000)

[25] Ilyashenko, Y.: Selected topics in differential equations with real and complex time. In: Ilyashenko, Y., Rousseau, C (eds.) Normal Forms, Bifurcations and Finiteness Problems in Differential Equations. NATO Science Series II: Mathematics, Physics and Chemistry, vol. 137, pp. 317–354. Kluwer Academic, Dordrecht (2004)

[26] Lins-Neto, A.: On the number of solutions of the equation $dx/dt = \sum_{j=0}^{n} a_j(t)x^j$, $0 \leq t \leq 1$, for which $x(0) = x(1)$. Invent. Math. **59**, 67–76 (1980)

[27] Llibre, J., Świrszcz, G.: On the limit cycles of polynomial vector fields. Dyn. Contin. Discrete Impuls. Syst. **18**, 203–214 (2011)

[28] Llibre, J., Valls, C.: Classification of the centers, their cyclicity and isochronicity for a class of polynomial differential systems generalizing the linear systems with cubic homogeneous nonlinearities. J. Differ. Equ. **46**, 2192–2204 (2009)

[29] Lloyd, N.G.: The number of periodic solutions of the equation $\dot{z} = z^N + p_1(t)z^{N-1} + \cdots + p_N(t)$. Proc. Lond. Math. Soc. **27**, 667–700 (1973)

[30] Lloyd, N.G.: On a class of differential equations of Riccati type. J. Lond. Math. Soc. **10**, 1–10 (1979)

[31] Lloyd, N.G.: A note on the number of limit cycles in certain two-dimensional systems. J. Lond. Math. Soc. **20**, 277–286 (1979)

[32] Lloyd, N.G., Pearson, J.M.: Bifurcation of limit cycles and integrability of planar dynamical systems in complex form. J. Phys. A Math. Gen. **32**, 1973–1984 (1999)

[33] Panov, A.A.: The number of periodic solutions of polynomial differential equations. Math. Notes **64**, 622–628 (1998)

[34] Panov, A.A.: On the diversity of Poincaré mappings for cubic equations with variable coefficients. Funct. Anal. Appl. **33**, 310–312 (1999)

[35] Pliss, V.A.: Non-Local Problems of the Theory of Oscillations. Academic Press, New York (1966)

[36] Sibirskii, K.S.: On the number of limit cycles in the neighborhood of a singular point. Differ. Equ. **1**, 36–47 (1965)

[37] Xu, C., Boyce, M.S., Daley, D.J.: Harvesting in seasonal environments. J. Math. Biol **50–6**, 663–682 (2005)

Jianfeng Huang
Department of Mathematics
Jinan University
Guangzhou 510632
People's Republic of China
e-mail: thuangjf@jnu.edu.cn

Haihua Liang
School of Mathematics and Systems Science
Guangdong Polytechnic Normal University
Guangzhou 510665
People's Republic of China
e-mail: lianghhgdin@126.com

Received: 24 February 2017.
Accepted: 23 June 2017.