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Sharp Trudinger-Moser inequalities with monomial weights

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Abstract. In this paper, we will study the Trudinger-Moser inequalities with the monomial weight $|x_1|^{A_1} \dots |x_N|^{A_N}$ in $\mathbb{R}^N$ with $A_1 \geq 0, \dots, A_N \geq 0$. Moreover, we investigate the Trudinger-Moser inequalities on both domains of finite and infinite volume. More importantly, we will exhibit the best constants for our results. In the particular case $A_1 = \cdots = A_N = 0$, we recover many results about the Trudinger-Moser inequalities without weight established in the literature.

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1. Introduction

Functional and geometric inequalities with monomial weights have been studied extensively recently. For example, motivated by an open question raised by Haim Brezis [5,6], Cabré and Ros-Oton studied in [7] the problem of the regularity of stable solutions to reaction-diffusion problems of double revolution and then established in [8] the Sobolev, Morrey, Trudinger and isoperimetric inequalities with weight x^A. Here

$$x^A = |x_1|^{A_1} \dots |x_N|^{A_N}$$
$$A_1 \geq 0, \dots, A_N \geq 0$$
$$A = (A_1, \dots, A_N).$$

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In [4], Bakry, Gentil and Ledoux used the stereographic projection combined with the Curvature-Dimension condition to prove the following Sobolev inequality with monomial weight: for $a \geq 0, N + a > 2$, there exists $S(N, a) > 0$ such that for all smooth, compactly supported function f on $\mathbb{R}^{N-1} \times \mathbb{R}_+$:

$$\left[\int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}_+} |u(x)|^{\frac{2(N+a)}{N+a-2}} x_N^a dx\right]^{\frac{N+a-2}{2(N+a)}} \leq S(N,a) \left[\int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}_+} |\nabla u(x)|^2 x_N^a dx\right]^{\frac{1}{2}}.$$

The best constant $S(N, a)$ was also calculated explicitly in [4]. In [20], V.H. Nguyen employed the mass transport approach to prove again and extend the above result. Moreover, he also studied the best constants and extremal functions for the Gagliardo–Nirenberg inequalities and logarithmic Sobolev inequalities with the weight x_N^a and with arbitrary norm.

The main purpose of this article is to study the sharp Trudinger-Moser inequalities in $\mathbb{R}^N$ with monomial weight x^A. Denote

$$\mathbb{R}_*^N = \left\{(x_1, ..., x_N) \in \mathbb{R}^N \text{ such that } x_i > 0 \text{ whenever } A_i > 0\right\}$$

and $\Omega^* = \Omega \cap \mathbb{R}_*^N$. Let

$$D = N + A_1 + \cdots + A_N$$

and denote

$$m_A(E) = \int_E x^A dx.$$

In [8], the authors set up the following weighted Trudinger inequality.

Theorem A. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Then for each $u \in C_c^1(\Omega)$ with $\int_\Omega |\nabla u|^D x^A dx \leq 1$, we have*

$$\int_\Omega \exp\left\{c_1 |u|^{\frac{D}{D-1}}\right\} x^A dx \leq c_2 m_A(\Omega)$$

where c_1 and c_2 are constants depending only on D.

Our first main result in this paper is to exhibit the best constant in the above result. More precisely, we will prove the following sharp Trudinger-Moser inequality on finite-volume domains:

Theorem 1.1. *There exists $C_0(D) > 0$ such that for all u such that u is a Lipschitz continuous function in $\mathbb{R}_*^N$, $m_A\left\{x \in \mathbb{R}_*^N : |u(x)| > t\right\}$ is finite for every positive t, $m_A(supp(u)) < \infty$ and $\int_{\mathbb{R}_*^N} |\nabla u|^D x^A dx \leq 1$, we have*

$$\int_{\mathbb{R}_*^N} \left[\exp\left(\alpha_{D,A} |u|^{\frac{D}{D-1}}\right) - 1\right] x^A dx \leq C_0(D) m_A(supp(u))$$

where

$$\alpha_{D,A} = D\left(\int_{\partial B_1^*} x^A d\sigma\right)^{\frac{1}{D-1}}$$

is the best constant.

The best constant $\alpha_{D,A}$ can actually be computed as follows:

$$\begin{aligned}\int_{B_1^*} x^A dx &= \int_0^1 \int_{\partial B_r^*} x^A d\sigma dr \\ &= \int_0^1 r^{D-1}\left(\int_{\partial B_1^*} x^A d\sigma\right) dr \\ &= \frac{1}{D}\left(\int_{\partial B_1^*} x^A d\sigma\right).\end{aligned}$$

Hence by [8]:

$$\begin{aligned}\int_{\partial B_1^*} x^A d\sigma &= D\int_{B_1^*} x^A dx \\ &= D\frac{\Gamma\left(\frac{A_1+1}{2}\right)\Gamma\left(\frac{A_2+1}{2}\right)...\Gamma\left(\frac{A_N+1}{2}\right)}{2^k\Gamma\left(1+\frac{D}{2}\right)}\end{aligned}$$

where k is the number of strictly positive entries of A. So

$$\alpha_{D,A} = D\left(D\frac{\Gamma\left(\frac{A_1+1}{2}\right)\Gamma\left(\frac{A_2+1}{2}\right)...\Gamma\left(\frac{A_N+1}{2}\right)}{2^k\Gamma\left(1+\frac{D}{2}\right)}\right)^{\frac{1}{D-1}}.$$

When $A = \overrightarrow{0}$, we recover the well-known Trudinger-Moser inequality on bounded domains proved by J. Moser in [19]. It is worth mentioning that the theorems of J. Moser in [19] are the sharp versions with best constants of the earlier results of Pohozaev [21], Trudinger [24] and Yudovich [25] about the embedding $W_0^{1,N}(\Omega) \subset L_{\varphi_N}(\Omega)$. Here $\Omega \subset \mathbb{R}^N$ is a bounded domain and $L_{\varphi_N}(\Omega)$ is the Orlicz space associated with the Young function $\varphi_N(t) = \exp\left(\alpha\left|t\right|^{N/(N-1)}\right) - 1$ for some $\alpha > 0$. We also mention that when the volume of Ω is infinite, the Trudinger-Moser inequality in [19] becomes meaningless. Thus, it is interesting and nontrivial to extend such inequalities to domains with infinite measure. In this direction, we state here the following three such results in the Euclidean spaces that could be found in [1,3,9,10,13,14,18,22]:

Theorem B. *Let $0 \le \beta < N$ and $0 \le \alpha < \alpha_N = N\omega_{N-1}^{\frac{1}{N-1}}$, where ω_{N-1} is the area of the surface of the unit $N-$ ball. There hold*

$$\sup_{u\in W^{1,N}(\mathbb{R}^N):\ \|\nabla u\|_N\le 1} \frac{1}{\|u\|_N^{N-\beta}}\int_{\mathbb{R}^N}\phi_N\left(\alpha\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right)\frac{dx}{|x|^\beta} < \infty. \tag{1.1}$$

$$\sup_{u\in W^{1,N}(\mathbb{R}^N):\ \|\nabla u\|_N^N+\|u\|_N^N\le 1} \int_{\mathbb{R}^N}\phi_N\left(\alpha_N\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right)\frac{dx}{|x|^\beta} < \infty. \tag{1.2}$$

$$\sup_{u\in W^{1,N}(\mathbb{R}^N):\ \|\nabla u\|_N\le 1} \frac{1}{\|u\|_N^{N-\beta}} \int_{\mathbb{R}^N} \frac{\phi_N\left(\alpha_N\left(1-\frac{\beta}{N}\right)|u|^{\frac{N}{N-1}}\right)}{\left(1+|u|^{\frac{N}{N-1}\left(1-\frac{\beta}{N}\right)}\right)} \frac{dx}{|x|^{\beta}} < \infty. \tag{1.3}$$

Here

$$\phi_N(t) = e^t - \sum_{j=0}^{N-2} \frac{t^j}{j!}.$$

Moreover, the constant α_N is sharp.

Our next purpose of this article is to establish the sharp Trudinger-Moser inequalities on the whole domain in the sense of [10,17,18]:

Theorem 1.2. *There exists a constant $C(D,A) > 0$ such that for all $u \in C_c^\infty\left(\overline{\mathbb{R}_*^N}\right) : \int_{\mathbb{R}_*^N} |\nabla u|^D x^A dx \le 1$, there holds*

$$\int_{\mathbb{R}_*^N} \frac{\Phi_D\left(\alpha_{D,A}|u|^{\frac{D}{D-1}}\right)}{\left(1+|u|^{\frac{D}{D-1}}\right)} x^A dx \le C(D,A) \int_{\mathbb{R}_*^N} |u|^D x^A dx. \tag{1.4}$$

Here

$$\Phi_D(t) = \sum_{k\in\mathbb{N}:k\ge D-1} \frac{t^k}{k!}.$$

The constant $\alpha_{D,A}$ is sharp. Moreover, the inequality does not hold when $1+|u|^{\frac{D}{D-1}}$ is replaced by $1+|u|^p$ with $p < \frac{D}{D-1}$.

As consequences, we get the following versions of the Trudinger-Moser inequalities on the whole domain in the spirit of [1,12,14,22]:

Theorem 1.3. *1/For all $\alpha < \alpha_{D,A}$, there exists a constant $C(D,A) > 0$ such that for all $u \in C_c^\infty\left(\overline{\mathbb{R}_*^N}\right) : \int_{\mathbb{R}_*^N} |\nabla u|^D x^A dx \le 1$, there holds*

$$\int_{\mathbb{R}_*^N} \Phi_D\left(\alpha|u|^{\frac{D}{D-1}}\right) x^A dx \le \frac{C(D,A)}{\alpha_{D,A}-\alpha} \int_{\mathbb{R}_*^N} |u|^D x^A dx. \tag{1.5}$$

2/ For any $M > 1$, there exists $C(D,A,M) > 0$ such that for all $u \in C_c^\infty\left(\overline{\mathbb{R}_*^N}\right) : \int_{\mathbb{R}_*^N} |\nabla u|^D x^A dx < 1$, there holds

$$\int_{\mathbb{R}_*^N} \Phi_D\left(M_{D,A}(u)\,\alpha_{D,A}|u|^{\frac{D}{D-1}}\right) x^A dx \le C(D,A,M) \frac{\int_{\mathbb{R}_*^N} |u|^D x^A dx}{1-\int_{\mathbb{R}_*^N} |\nabla u|^D x^A dx} \tag{1.6}$$

where

$$M_{D,A}(u) = \frac{M^{\frac{1}{D-1}}}{\left(M - 1 + \int_{\mathbb{R}_*^N} |\nabla u|^D x^A dx\right)^{\frac{1}{D-1}}}.$$

3/ There exists a constant $C(D,A) > 0$ *such that for all* $u \in C_c^\infty\left(\overline{\mathbb{R}_*^N}\right)$: $\int_{\mathbb{R}_*^N} |\nabla u|^D x^A dx + \int_{\mathbb{R}_*^N} |u|^D x^A dx \leq 1$, *there holds*

$$\int_{\mathbb{R}_*^N} \Phi_D\left(\alpha_{D,A} |u|^{\frac{D}{D-1}}\right) x^A dx \leq C(D,A). \tag{1.7}$$

The constant $\alpha_{D,A}$ *is sharp.*

It is interesting to mention that if we just consider the restriction under the seminorm $\int_{\mathbb{R}_*^N} |\nabla u|^D x^A dx \leq 1$, the inequality (1.5) fails at the critical case $\alpha = \alpha_{D,A}$. Hence, (1.5) can be considered as the sharp subcritical Trudinger-Moser inequality with monomial weight. Also, Statement 3 claims that if we want to achieve the sharp constant $\alpha_{D,A}$, we have to use the constraint of full norm $\int_{\mathbb{R}_*^N} |\nabla u|^D x^A dx + \int_{\mathbb{R}_*^N} |u|^D x^A dx \leq 1$. Thus, (1.7) is the sharp critical Trudinger-Moser inequality with monomial weight. Finally, Statement 2 is the extension of these two results in the spirit of Lions [16]. It is easy to see that (1.5) without the asymptotic behavior and (1.7) are just easy consequences of (1.6). Surprisingly, we will show next that these three inequalities are actually equivalent. More specifically, let us denote

$$STM(\alpha) = \sup_{u \in C_c^\infty\left(\overline{\mathbb{R}_*^N}\right) : \int_{\mathbb{R}_*^N} |\nabla u|^D x^A dx \leq 1} \frac{1}{\int_{\mathbb{R}_*^N} |u|^D x^A dx} \int_{\mathbb{R}_*^N} \Phi_D\left(\alpha |u|^{\frac{D}{D-1}}\right) x^A dx$$

$$TM = \sup_{u \in C_c^\infty\left(\overline{\mathbb{R}_*^N}\right) : \int_{\mathbb{R}_*^N} |\nabla u|^D x^A dx + \int_{\mathbb{R}_*^N} |u|^D x^A dx \leq 1} \int_{\mathbb{R}_*^N} \Phi_D\left(\alpha_{D,A} |u|^{\frac{D}{D-1}}\right) x^A dx$$

and

$$ITM_M = \sup_{u \in C_c^\infty\left(\overline{\mathbb{R}_*^N}\right) : \int_{\mathbb{R}_*^N} |\nabla u|^D x^A dx \leq 1} \frac{1 - \int_{\mathbb{R}_*^N} |\nabla u|^D x^A dx}{\int_{\mathbb{R}_*^N} |u|^D x^A dx} \times \int_{\mathbb{R}_*^N} \Phi_D\left(M_{D,A}(u) \alpha_{D,A} |u|^{\frac{D}{D-1}}\right) x^A dx.$$

Then our next result is that

Theorem 1.4. *For $M > 1$, we have*

$$ITM_M = \frac{M}{M-1}TM = \frac{M}{M-1} \sup_{\alpha \in (0,\alpha_{D,A})} \left[\frac{1-\left(\frac{\alpha}{\alpha_{D,A}}\right)^{D-1}}{\left(\frac{\alpha}{\alpha_{D,A}}\right)^{D-1}} \right] STM\left(\alpha\right).$$

See [11] for a similar result for the nonweighted case.

Our paper is organized as follows: Preliminaries and some useful lemmata will be provided in Sect. 2. Our first main result about the Trudinger-Moser inequality on bounded domains will be proved in Sect. 3. Finally, in Sect. 4, we will investigate several versions of the Trudinger-Moser inequalities on the whole domain and will also establish the equivalency of some of them.

2. Preliminaries

A result by Talenti in [23] states that whenever balls minimize the isoperimetric quotient with a weight w, there exists a radial rearrangement which preserves $\int f\left(u\right) wdx$ and decreases $\int \left|\nabla u\right|^p wdx$. As pointed out in [8], by combining this fact with the results in [8] about the isoperimetric inequalities with monomial weights and the layer cake representation (see [15]), one could deduce the following rearrangement results:

Lemma 2.1. *Let u be a Lipschitz continuous function in $\mathbb{R}_*^N$ such that $m_A \left\{x \in \mathbb{R}_*^N : \left|u\left(x\right)\right| > t\right\}$ is finite for every positive t. Then there exists a radial rearrangement u^* of u such that*

(i) $m_A\left(\left\{|u| > t\right\}\right) = m_A\left(\left\{u^* > t\right\}\right)$ *for all* t,

(ii) u^* *is radially decreasing*

(iii) *for every Young function Φ (that is, Φ maps $[0,\infty)$ into $[0,\infty)$, vanishes at 0, and is convex and increasing):*

$$\int_{\mathbb{R}_*^N} \Phi\left(\left|\nabla u^*\right|\right) x^A dx \le \int_{\mathbb{R}_*^N} \Phi\left(\left|\nabla u\right|\right) x^A dx.$$

(iv) *If $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing, then* $\int_{\mathbb{R}_*^N} \Psi\left(u\right) x^A dx = \int_{\mathbb{R}_*^N} \Psi\left(u^*\right) x^A dx.$

Now, we consider the following Moser type sequence:

$$M_n(x) = \left(\frac{1}{\int_{\partial B_1^*} x^A d\sigma} \right)^{\frac{1}{D}} \begin{cases} \left(\frac{n}{D}\right)^{\frac{D-1}{D}}, & 0 \le |x| \le e^{-\frac{n}{D}}, \\ \left(\frac{D}{n}\right)^{\frac{1}{D}} \log\left(\frac{1}{|x|}\right), & e^{-\frac{n}{D}} < |x| < 1 \\ 0, & |x| \ge 1 \end{cases} . \tag{2.1}$$

Then, we have that

$$\begin{aligned}\int_{\mathbb{R}_*^N} |\nabla M_n|^D x^A dx &= \left(\int_{\partial B_1^*} x^A d\sigma\right) \int_0^1 |M_n'(r)|^D r^{D-1} dr \\ &= \left(\int_{\partial B_1^*} x^A d\sigma\right) \int_{e^{-\frac{n}{D}}}^1 \frac{D}{n\int_{\partial B_1^*} x^A d\sigma} \frac{1}{r} dr \\ &= 1.\end{aligned}$$

Next, we state the following result by Adams [2]:

Lemma 2.2. *Let* $1 < p < \infty$ *and* $a(s,t)$ *be a non-negative measurable function on* $[0,\infty) \times [0,\infty)$ *such that (a.e.)*

$$a(s,t) \leq 1, \text{ when } 0 < s < t, \tag{2.2}$$

$$\sup_{t>0} \left(\int_t^\infty a(s,t)^{p'} ds\right)^{1/p'} = b < \infty. \tag{2.3}$$

Then there is a constant $c_0 = c_0(p,b)$ *such that if for* $\phi \geq 0$,

$$\int_0^\infty \phi(s)^p ds \leq 1, \tag{2.4}$$

then

$$\int_0^\infty e^{-F(t)} dt \leq c_0 \tag{2.5}$$

where

$$F(t) = t - \left(\int_0^\infty a(s,t)\phi(s) ds\right)^{p'}. \tag{2.6}$$

We now state the following result that the proof could be found in [17,18]:

Lemma 2.3. *Given any sequence* $s = \{s_k\}_{k\geq 0}$, *let*

$$\|s\|_1 = \sum_{k=0}^\infty |s_k|,$$

$$\|s\|_D = \left(\sum_{k=0}^\infty |s_k|^D\right)^{1/D},$$

$$\|s\|_{q,(e)} = \left(\sum_{k=0}^\infty |s_k|^q e^k\right)^{1/q}$$

and

$$\mu(h) = \inf\left\{\|s\|_{q,(e)} : \|s\|_1 = h,\ \|s\|_D \leq 1\right\}.$$

Then for $h > 1$, *we have*

$$\mu(h) \sim \frac{\exp\left(\frac{h^{\frac{D}{D-1}}}{q}\right)}{h^{\frac{1}{D-1}}}.$$

Using the above lemma, we can obtain the following Radial Sobolev inequality in the spirit of Ibrahim-Masmoudi-Nakanishi [10]:

Theorem 2.1. (Radial Sobolev). *There exists a constant $C > 0$ such that for any radially nonnegative nonincreasing function φ satisfying $\varphi(R) > 1$ and*

$$\left(\int_{\partial B_1^*} x^A d\sigma\right) \int_R^\infty |\varphi'(t)|^D t^{D-1} dt \leq K$$

for some $R,\ K > 0$, then we have

$$\frac{\exp\left[\frac{\alpha_{D,A}}{K^{\frac{1}{D-1}}}\varphi^{\frac{D}{D-1}}(R)\right]}{\varphi^{\frac{D}{D-1}}(R)} R^D \leq C \frac{\int_R^\infty |\varphi(t)|^D t^{D-1} dt}{K^{\frac{D}{D-1}}}.$$

We also have the following observation:

Lemma 2.4. *We have*

$$STM(\alpha) = \sup_{u \in C_c^\infty\left(\overline{\mathbb{R}_*^N}\right) : \int_{\mathbb{R}_*^N} |\nabla u|^D x^A dx = 1 = \int_{\mathbb{R}_*^N} |u|^D x^A dx} \int_{\mathbb{R}_*^N} \Phi_D\left(\alpha |u|^{\frac{D}{D-1}}\right) x^A dx.$$

Proof. First, it's easy to see that

$$STM(\alpha) = \sup_{u \in C_c^\infty\left(\overline{\mathbb{R}_*^N}\right) : \int_{\mathbb{R}_*^N} |\nabla u|^D x^A dx = 1} \frac{1}{\int_{\mathbb{R}_*^N} |u|^D x^A dx} \int_{\mathbb{R}_*^N} \Phi_D\left(\alpha |u|^{\frac{D}{D-1}}\right) x^A dx.$$

Next, for any $u \in C_c^\infty\left(\overline{\mathbb{R}_*^N}\right) \backslash \{0\} : \int_{\mathbb{R}_*^N} |\nabla u|^D x^A dx = 1$, we set

$$v(x) = u(\lambda x)$$

with

$$\lambda^D = \int_{\mathbb{R}_*^N} |u|^D x^A dx.$$

Then it is easy to verify that

$$\int_{\mathbb{R}_*^N} |\nabla v|^D x^A dx = \int_{\mathbb{R}_*^N} |\nabla u|^D x^A dx = 1$$

and

$$\int_{\mathbb{R}_*^N} |v|^D x^A dx = \frac{1}{\lambda^D} \int_{\mathbb{R}_*^N} |u|^D x^A dx = 1.$$

Also,

$$\frac{1}{\int_{\mathbb{R}_*^N} |u|^D x^A dx} \int_{\mathbb{R}_*^N} \Phi_D\left(\alpha |u|^{\frac{D}{D-1}}\right) x^A dx$$
$$= \frac{1}{\lambda^D} \frac{1}{\int_{\mathbb{R}_*^N} |v|^D x^A dx} \lambda^D \int_{\mathbb{R}_*^N} \Phi_D\left(\alpha |v|^{\frac{D}{D-1}}\right) x^A dx$$
$$= \int_{\mathbb{R}_*^N} \Phi_D\left(\alpha |v|^{\frac{D}{D-1}}\right) x^A dx.$$

Hence

$$STM(\alpha) = \sup_{u \in C_c^\infty\left(\overline{\mathbb{R}_*^N}\right) : \int_{\mathbb{R}_*^N} |\nabla u|^D x^A dx = 1 = \int_{\mathbb{R}_*^N} |u|^D x^A dx} \int_{\mathbb{R}_*^N} \Phi_D\left(\alpha |u|^{\frac{D}{D-1}}\right) x^A dx.$$

□

3. Sharp Trudinger-Moser inequality on bounded domains

Proof of Theorem 1.1. Using Lemma 2.1, we can now assume that u is radially decreasing with $\overline{supp\,(u)} = \overline{B_R^*}$. Then

$$\int_{\mathbb{R}_*^N} \left[\exp\left(\alpha_{D,A} |u|^{\frac{D}{D-1}}\right) - 1\right] x^A dx$$
$$= \int_0^R \left(\int_{\partial B_r^*} \left[\exp\left(\alpha_{D,A} |u|^{\frac{D}{D-1}}\right) - 1\right] x^A d\sigma \right) dr$$
$$= \int_0^R \left[\exp\left(\alpha_{D,A} |u|^{\frac{D}{D-1}}\right) - 1\right] r^{D-1} \left(\int_{\partial B_1^*} x^A d\sigma \right) dr$$
$$= \left(\int_{\partial B_1^*} x^A d\sigma \right) \int_0^R \left[\exp\left(\alpha_{D,A} |u|^{\frac{D}{D-1}}\right) - 1\right] r^{D-1} dr.$$

Also

$$\int_{\mathbb{R}_*^N} |\nabla u|^D x^A dx = \left(\int_{\partial B_1^*} x^A d\sigma \right) \int_0^R |u'|^D r^{D-1} dr.$$

Set

$$v(t) = Bu\left(Re^{-\frac{t}{D}}\right)$$

then

$$
\begin{aligned}
&\int_0^R \left[\exp\left(\alpha_{D,A}\left|u\right|^{\frac{D}{D-1}}\right)-1\right]r^{D-1}dr \\
&\quad=\int_0^\infty \left[\exp\left(\alpha_{D,A}\left|u\left(Re^{-\frac{t}{D}}\right)\right|^{\frac{D}{D-1}}\right)-1\right]\left(Re^{-\frac{t}{D}}\right)^{D-1}R\frac{1}{D}e^{-\frac{t}{D}}dt \\
&\quad=R^D\frac{1}{D}\int_0^\infty\left[\exp\left(\frac{\alpha_{D,A}}{B^{\frac{D}{D-1}}}\left|v\left(t\right)\right|^{\frac{D}{D-1}}\right)-1\right]e^{-t}dt
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^R \left|u'\left(r\right)\right|^D r^{D-1}dr &= \int_0^R \left|u'\left(Re^{-\frac{t}{D}}\right)\right|^D\left(Re^{-\frac{t}{D}}\right)^{D-1}R\frac{1}{D}e^{-\frac{t}{D}}dt \\
&=\int_0^R \left|v\left(t\right)\right|^D\left(\frac{D}{RB}e^{\frac{t}{D}}\right)^D\left(Re^{-\frac{t}{D}}\right)^{D-1}R\frac{1}{D}e^{-\frac{t}{D}}dt \\
&=\left(\frac{D}{B}\right)^D\frac{1}{D}\int_0^R\left|v'\left(t\right)\right|^D dt.
\end{aligned}
$$

So if we choose B such that

$$
\left(\frac{D}{B}\right)^D\frac{1}{D}\left(\int_{\partial B_1^*}x^A d\sigma\right)=1
$$

i.e.

$$
B=D\left(\frac{1}{D}\int_{\partial B_1^*}x^A d\sigma\right)^{\frac{1}{D}},
$$

we get that

$$
\int_0^R\left|v'\left(t\right)\right|^D dt\le 1.
$$

Using Lemma 2.2 with

$$
\begin{aligned}
a\left(s,t\right)&=\begin{cases}1 & 0\le s\le t\\ 0 & t<s\end{cases}, \\
\phi&=v',
\end{aligned}
$$

we get that there exists a constant $C_0=C_0\left(D\right)$ such that

$$
\begin{aligned}
&\frac{1}{D}\int_0^\infty\left[\exp\left(\frac{\alpha_{D,A}}{B^{\frac{D}{D-1}}}\left|v\left(t\right)\right|^{\frac{D}{D-1}}\right)-1\right]e^{-t}dt \\
&\quad=\frac{1}{D}\int_0^\infty\left[\exp\left(\left|v\left(t\right)\right|^{\frac{D}{D-1}}\right)-1\right]e^{-t}dt\le C_0.
\end{aligned}
$$

Combining these estimates, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N_*}\left[\exp\left(\alpha_{D,A}\left|u\right|^{\frac{D}{D-1}}\right)-1\right]x^A dx &\leq C_0\left(D\right)R^D\left(\int_{\partial B_1^*}x^A d\sigma\right)\\ &= C_0\left(D\right)\frac{\displaystyle\int_{\partial B_1^*}x^A d\sigma}{\displaystyle\int_{B_1^*}x^A dx}m_A\left(B_R^*\right).\end{aligned}$$

Finally, we note that

$$\begin{aligned}\int_{B_1^*}x^A dx &= \int_0^1\int_{\partial B_r^*}x^A d\sigma dr\\ &= \int_0^1 r^{D-1}\left(\int_{\partial B_1^*}x^A d\sigma\right)dr\\ &= \frac{1}{D}\left(\int_{\partial B_1^*}x^A d\sigma\right).\end{aligned}$$

Hence

$$\int_{\mathbb{R}^N_*}\left[\exp\left(\alpha_{D,A}\left|u\right|^{\frac{D}{D-1}}\right)-1\right]x^A dx \leq C_1\left(D\right)m_A\left(\operatorname{supp}\left(u\right)\right).$$

Now, to show that the constant $\alpha_{D,A}$ is sharp, we will take into consideration the Moser sequence M_n. Indeed, for all $\alpha > \alpha_{D,A}$:

$$\begin{aligned}&\int_{\mathbb{R}^N_*}\left[\exp\left(\alpha\left|M_n\right|^{\frac{D}{D-1}}\right)-1\right]x^A dx\\ &= \left(\int_{\partial B_1^*}x^A d\sigma\right)\int_0^1\left[\exp\left(\alpha\left|M_n\right|^{\frac{D}{D-1}}\right)-1\right]r^{D-1}dr\\ &\gtrsim \int_0^{e^{-\frac{n}{D}}}\exp\left(\alpha\left|\left(\frac{1}{\displaystyle\int_{\partial B_1^*}x^A d\sigma}\right)^{\frac{1}{D}}\left(\frac{n}{D}\right)^{\frac{D-1}{D}}\right|^{\frac{D}{D-1}}\right)r^{D-1}dr\\ &\gtrsim \exp\left[\frac{\alpha}{\alpha_{D,A}}n\right]e^{-n}\to\infty \text{ as } n\to\infty.\end{aligned}$$

Actually, from the above argument, we can deduce that for any positive function f such that $f\left(n\right)\to\infty$ as $n\to\infty$, we get

$$\sup_{\int_{\mathbb{R}^N_*}\left|\nabla u\right|^D x^A dx\leq 1}\frac{1}{m_A\left(\operatorname{supp}\left(u\right)\right)}\int_{\mathbb{R}^N_*}\left[\exp\left(\alpha_{D,A}\left|u\right|^{\frac{D}{D-1}}\right)-1\right]f\left(\left|u\right|\right)dx=\infty.$$

$\square$

4. Sharp Trudinger-Moser inequalities on $\mathbb{R}_*^N$

Proof of Theorem 1.2. By Lemma 2.1, we may assume that u is a smooth, nonnegative and radially nonincreasing function. Let $R_1 = R_1(u)$ be such that

$$\int_{B_{R_1}^*} |\nabla u|^D x^A dx = \left(\int_{\partial B_1^*} x^A d\sigma \right) \int_0^{R_1} |u_r|^D r^{D-1} dr \leq 1 - \varepsilon_0,$$

$$\int_{\mathbb{R}_*^N \setminus B_{R_1}^*} |\nabla u|^D x^A dx = \left(\int_{\partial B_1^*} x^A d\sigma \right) \int_{R_1}^{\infty} |u_r|^D r^{D-1} dr \leq \varepsilon_0.$$

Here $\varepsilon_0 \in (0,1)$ is fixed and does not depend on u.

Denote

$$\omega_{N-1,A} = \int_{\partial B_1^*} x^A d\sigma,$$

by the Holder's inequality, we have

$$\begin{aligned} u(r_1) - u(r_2) &\leq \int_{r_1}^{r_2} -u_r dr \\ &\leq \left(\frac{1-\varepsilon_0}{\omega_{N-1,A}} \right)^{1/D} \left(\ln \frac{r_2}{r_1} \right)^{\frac{D-1}{D}} \quad \text{for } 0 < r_1 \leq r_2 \leq R_1, \end{aligned} \tag{4.1}$$

and

$$u(r_1) - u(r_2) \leq \left(\frac{\varepsilon_0}{\omega_{N-1,A}} \right)^{1/D} \left(\ln \frac{r_2}{r_1} \right)^{\frac{D-1}{D}} \quad \text{for } R_1 \leq r_1 \leq r_2. \tag{4.2}$$

We define $R_0 := \inf \{ r > 0 : u(r) \leq 1 \} \in [0, \infty)$. Hence $u(s) \leq 1$ when $s \geq R_0$. WLOG, we assume $R_0 > 0$.

Now, we split the integral as follows:

$$\begin{aligned} \int_{\mathbb{R}_*^N} \frac{\Phi_D \left(\alpha_{D,A} |u|^{\frac{D}{D-1}} \right)}{1 + u^{\frac{D}{D-1}}} x^A dx &= \int_{B_{R_0}^*} + \int_{\mathbb{R}_*^N \setminus B_{R_0}^*} \frac{\Phi_D \left(\alpha_{D,A} |u|^{\frac{D}{D-1}} \right)}{1 + u^{\frac{D}{D-1}}} x^A dx \\ &= I + J. \end{aligned}$$

First, we will estimate J. Since $u \leq 1$ on $\mathbb{R}_*^N \setminus B_{R_0}^*$, we have

$$\begin{aligned} J &= \int_{\mathbb{R}_*^N \setminus B_{R_0}^*} \frac{\Phi_D \left(\alpha_{D,A} |u|^{\frac{D}{D-1}} \right)}{1 + u^{\frac{D}{D-1}}} x^A dx \\ &\leq C \int_{\mathbb{R}_*^N} |u|^D x^A dx. \end{aligned} \tag{4.3}$$

Hence, now, we just need to deal with the integral I.

Case 1: $0 < R_0 \leq R_1$.

In this case, using (4.1), we have for $0 < r \leq R_0$:

$$u(r) \leq 1 + \left(\frac{1-\varepsilon_0}{\omega_{N-1,A}} \right)^{1/D} \left(\ln \frac{R_0}{r} \right)^{\frac{D-1}{D}}.$$

By using

$$(a+b)^{\frac{D}{D-1}} \leq (1+\varepsilon)a^{\frac{D}{D-1}} + C(\varepsilon)\, b^{\frac{D}{D-1}},$$

where

$$A(\varepsilon) = \left(1 - \frac{1}{(1+\varepsilon)^{D-1}}\right)^{\frac{1}{1-D}},$$

we get

$$u^{\frac{D}{D-1}}(r) \leq (1+\varepsilon_1)\left(\frac{1-\varepsilon_0}{\omega_{N-1,A}}\right)^{1/(D-1)} \ln\frac{R_0}{r} + A(\varepsilon_1).$$

Thus, we can estimate the integral I as follows:

$$\begin{aligned}
I &= \int_{B^*_{R_0}} \frac{\Phi_D\left(\alpha_{D,A}|u|^{\frac{D}{D-1}}\right)}{\left(1+u^{\frac{D}{D-1}}\right)} x^A dx \\
&\leq \int_{B^*_{R_0}} \exp\left(\alpha_{D,A}(1+\varepsilon_1)\left(\frac{1-\varepsilon_0}{\omega_{N-1,A}}\right)^{1/(D-1)} \ln\frac{R_0}{r} + \alpha A(\varepsilon_1)\right) x^A dx \\
&\leq C R_0^{\alpha_{D,A}(1+\varepsilon_1)((1-\varepsilon_0)/\omega_{N-1,A})^{1/(D-1)}} (\omega_{N-1,A}) \\
&\quad \times \int_0^{R_0} r^{D-1-\alpha_{D,A}(1+\varepsilon_1)((1-\varepsilon_0)/\omega_{N-1,A})^{1/(D-1)}} dr \\
&\leq C R_0^D \omega_{N-1,A} \leq C\left(\int_{B^*_{R_0}} 1 dx\right) \leq C\int_{\mathbb{R}^N_*} |u|^D x^A dx. \qquad (4.4)
\end{aligned}$$

where $\varepsilon_1 = (1+\varepsilon_0)^{1/(D-1)} - 1$.

Case 2: $0 < R_1 < R_0$.

We have

$$\begin{aligned}
I &= \int_{B^*_{R_0}} \frac{\Phi_D\left(\alpha_{D,A}|u|^{\frac{D}{D-1}}\right)}{1+u^{\frac{D}{D-1}}} x^A dx \\
&= \int_{B^*_{R_1}} \frac{\Phi_D\left(\alpha_{D,A}|u|^{\frac{D}{D-1}}\right)}{1+u^{\frac{D}{D-1}}} x^A dx + \int_{B^*_{R_0}\setminus B^*_{R_1}} \frac{\Phi_D\left(\alpha_{D,A}|u|^{\frac{D}{D-1}}\right)}{1+u^{\frac{D}{D-1}}} x^A dx \\
&= I_1 + I_2.
\end{aligned}$$

Using (4.2), we get

$$u(r) - u(R_0) \leq \left(\frac{\varepsilon_0}{\omega_{N-1,A}}\right)^{1/D} \left(\ln\frac{R_0}{r}\right)^{\frac{D-1}{D}} \quad \text{for } r \geq R_1.$$

Hence

$$u(r) \leq 1 + \left(\frac{\varepsilon_0}{\omega_{N-1,A}}\right)^{1/D} \left(\ln\frac{R_0}{r}\right)^{\frac{D-1}{D}}.$$

Then, we have

$$u^{\frac{D}{D-1}}(r) \leq (1+\varepsilon_2)\left(\frac{\varepsilon_0}{\omega_{N-1,A}}\right)^{\frac{1}{D-1}} \ln\frac{R_0}{r} + A(\varepsilon_2).$$

So

$$\begin{aligned}
I_2 &= \int_{B^*_{R_0}\setminus B^*_{R_1}} \frac{\Phi_D\left(\alpha_{D,A}|u|^{\frac{D}{D-1}}\right)}{1+u^{\frac{D}{D-1}}} x^A dx \\
&\leq C\omega_{N-1,A}\int_{R_1}^{R_0} \exp\left(\alpha_{D,A}(1+\varepsilon_2)\left(\frac{\varepsilon_0}{\omega_{N-1,A}}\right)^{\frac{1}{D-1}} \ln\frac{R_0}{r} + \alpha A(\varepsilon_2)\right) r^{D-1}dr \\
&\leq C\omega_{N-1,A} R_0^{\alpha_{D,A}(1+\varepsilon_2)\left(\varepsilon_0/\omega_{N-1,A}\right)^{\frac{1}{D-1}}} \\
&\quad\times \frac{R_0^{D-\alpha_{D,A}(1+\varepsilon_2)\left(\varepsilon_0/\omega_{N-1,A}\right)^{\frac{1}{D-1}}} - R_1^{D-\alpha_{D,A}(1+\varepsilon_2)\left(\varepsilon_0/\omega_{N-1,A}\right)^{\frac{1}{D-1}}}}{D-\alpha_{D,A}(1+\varepsilon)\left(\varepsilon_0/\omega_{N-1,A}\right)^{\frac{1}{D-1}}} \\
&\leq \frac{C\omega_{N-1,A}}{D-\alpha_{D,A}(1+\varepsilon_2)\left(\varepsilon_0/\omega_{N-1,A}\right)^{\frac{1}{D-1}}}\left(R_0^D - R_1^D\right) \\
&\leq C\omega_{N-1,A}\left(\int_{R_1}^{R_0} 1dr\right) \leq C\int_{B^*_{R_0}\setminus B^*_{R_1}} x^A dx \leq C\int_{\mathbb{R}^N_*} |u|^D x^A dx.,
\end{aligned}$$

where $\varepsilon_2 > 0$ is such that $D-\alpha_{D,A}(1+\varepsilon_2)\left(\varepsilon_0/\omega_{N-1,A}\right)^{\frac{1}{D-1}} > 0$.

So, we now just need to estimate $I_1 = \int_{B^*_{R_1}} \frac{\Phi_D\left(\alpha_{D,A}|u|^{\frac{D}{D-1}}\right)}{1+u^{\frac{D}{D-1}}} x^A dx$ with $u(R_1) > 1$.

First, we define

$$v(r) = u(r) - u(R_1) \text{ on } 0 \leq r \leq R_1.$$

It's clear that $\int_{B^*_{R_1}} |\nabla v|^N x^A dx = \int_{B^*_{R_1}} |\nabla u|^N x^A dx \leq 1-\varepsilon_0$.

Moreover, for $0 \leq r \leq R_1$:

$$u^{\frac{D}{D-1}}(r) \leq (1+\varepsilon)v^{\frac{D}{D-1}}(r) + A(\varepsilon)u^{\frac{D}{D-1}}(R_1)$$

with

$$0 < \varepsilon = \left(\frac{1}{1-\varepsilon_0}\right)^{\frac{1}{D-1}} - 1.$$

Hence

$$\begin{aligned}
I_1 &= \int_{B^*_{R_1}} \frac{\Phi_D\left(\alpha_{D,A}|u|^{\frac{D}{D-1}}\right)}{1+u^{\frac{D}{D-1}}} x^A dx \\
&\leq \frac{e^{\alpha_{D,A}A(\varepsilon)u^{\frac{D}{D-1}}(R_1)}}{u^{\frac{D}{D-1}}(R_1)} \int_{B^*_{R_1}} e^{\alpha_{D,A}(1+\varepsilon)v^{\frac{D}{D-1}}(r)} x^A dx
\end{aligned}$$

$$= \frac{e^{\alpha_{D,A} A(\varepsilon) u^{\frac{D}{D-1}}(R_1)}}{u^{\frac{D}{D-1}}(R_1)} \int_{B^*_{R_1}} e^{\alpha_{D,A} w^{\frac{D}{D-1}}(r)} x^A dx \tag{4.5}$$

where $w = (1+\varepsilon)^{\frac{D-1}{D}} v$.

It's clear that $\text{supp}(w) \subset B^*_{R_1}$ and $\int_{B^*_{R_1}} |\nabla w|^D x^A dx = (1+\varepsilon)^{D-1} \int_{B^*_{R_1}} |\nabla v|^D x^A dx \leq (1+\varepsilon)^{D-1}(1-\varepsilon_0) = 1$. Noting that in this case

$$A(\varepsilon) = \left(1 - \frac{1}{(1+\varepsilon)^{D-1}}\right)^{\frac{1}{1-D}} = \varepsilon_0^{\frac{1}{1-D}}$$

Hence, using Theorem 1.1, we have

$$\int_{B^*_{R_1}} e^{\alpha_{D,A} w^{\frac{D}{D-1}}(r)} x^A dx \leq C \int_{B^*_{R_1}} x^A dx \leq C R_1^D \int_{B^*_1} x^A dx. \tag{4.6}$$

Also, using Theorem 2.1, we have

$$\begin{aligned}
&\frac{\exp\left(\alpha_{D,A} A(\varepsilon) u^{\frac{D}{D-1}}(R_1)\right)}{u^{\frac{D}{D-1}}(R_1)} R_1^D \int_{B^*_1} x^A dx \\
&= \frac{\exp\left(\frac{\alpha_{D,A}}{\varepsilon_0^{\frac{1}{D-1}}} u^{\frac{D}{D-1}}(R_1)\right)}{u^{\frac{D}{D-1}}(R_1)} R_1^D \int_{B^*_1} x^A dx \\
&\leq C \varepsilon_0^{\frac{D}{D-1}} \int_{\mathbb{R}^N_* \setminus B^*_{R_1}} |u|^D x^A dx \\
&\leq C \int_{\mathbb{R}^N_*} |u|^D x^A dx. \qquad (4.7)
\end{aligned}$$

By (4.5), (4.6) and (4.7), the proof is now completed.

The fact that $\alpha_{D,A}$ is sharp can be showed as in the proof of Theorem 1.1. Now, to show that the power $\frac{D}{D-1}$ in the denominator of (1.4) is also optimal, again we will consider the Moser sequence. We have for sufficiently large n:

$$\begin{aligned}
&\int_{\mathbb{R}^N_*} |M_n|^D x^A dx \\
&= (\omega_{N-1,A}) \int_0^1 |M_n(r)|^D r^{D-1} dr \\
&= (\omega_{N-1,A}) \int_0^{e^{-\frac{n}{D}}} \left(\frac{1}{\omega_{N-1,A}}\right) \left(\frac{n}{D}\right)^{D-1} r^{D-1} dr \\
&\quad + (\omega_{N-1,A}) \int_{e^{-\frac{n}{D}}}^1 \left(\frac{D}{n\omega_{N-1,A}}\right) \left|\log\left(\frac{1}{r}\right)\right|^D r^{D-1} dr \\
&\lesssim e^{-n} n^{D-1} + \frac{1}{n} \lesssim \frac{1}{n}.
\end{aligned}$$

Also,

$$\begin{aligned}
&\int_{\mathbb{R}_*^N} \frac{\Phi_D\left(\alpha_{D,A}\left|M_n\right|^{\frac{D}{D-1}}\right)}{(1+|M_n|^p)} x^A dx \\
&\quad \gtrsim \int_0^{e^{-\frac{n}{D}}} \frac{\Phi_D\left(\alpha_{D,A}\left(\frac{1}{\omega_{N-1,A}}\right)^{\frac{1}{D-1}}\frac{n}{D}\right)}{\left(1+\left|\left(\frac{1}{\omega_{N-1,A}}\right)^{\frac{1}{D}}\left(\frac{n}{D}\right)^{\frac{D-1}{D}}\right|^p\right)} r^{D-1}dr \\
&\quad \gtrsim \int_0^{e^{-\frac{n}{N-\beta}}} \frac{\Phi_D(n)}{n^{\frac{p(D-1)}{D}}} r^{D-1}dr \gtrsim \frac{\Phi_D(n)e^{-n}}{n^{\frac{p(D-1)}{D}}} \gtrsim \frac{1}{n^{\frac{p(D-1)}{D}}}
\end{aligned}$$

Hence, we need

$$\frac{1}{n^{\frac{p(D-1)}{D}}} \lesssim \frac{1}{n} \Rightarrow p \geq \frac{D}{D-1}.$$

□

We next prove Theorem 1.3:

Proof of Theorem 1.3. It is clear that Statement 1 without the exact asymptotic behavior $\frac{1}{\alpha_{D,A}-\alpha}$ is an easy consequence of Theorem 1.2 while Statement 3 can be deduced from Statement 2 easily.

First, let $0 < \alpha \lessapprox \alpha_{D,A}$ and $u \in C_c^\infty\left(\overline{\mathbb{R}_*^N}\right) : \int_{\mathbb{R}_*^N} |\nabla u|^D x^A dx \leq 1$. Then

$$\int_{\mathbb{R}_*^N} \Phi_D\left(\alpha\left|u\right|^{\frac{D}{D-1}}\right) x^A dx = \int_{|u|\leq 1} + \int_{|u|>1} \Phi_D\left(\alpha\left|u\right|^{\frac{D}{D-1}}\right) x^A dx.$$

The first integral is easy to estimate:

$$\begin{aligned}
&\int_{|u|\leq 1} \Phi_D\left(\alpha\left|u\right|^{\frac{D}{D-1}}\right) x^A dx \\
&\quad = \int_{|u|\leq 1} \sum_{k=D-1}^{\infty} \frac{\alpha^k}{k!} \left|u\right|^{k\frac{D}{D-1}} x^A dx \\
&\quad \leq \sum_{k=D-1}^{\infty} \frac{\alpha^k}{k!} \int_{\mathbb{R}_*^N} |u|^D x^A dx \\
&\quad \leq \frac{C_0(D,A)}{\alpha_{D,A}-\alpha} \int_{\mathbb{R}_*^N} |u|^D x^A dx.
\end{aligned}$$

Also, when $|u| > 1$:

$$\begin{aligned}
&\int_{|u|>1} \Phi_D\left(\alpha\left|u\right|^{\frac{D}{D-1}}\right) x^A dx \\
&\quad \leq \int_{|u|>1} \exp\left(\alpha\left|u\right|^{\frac{D}{D-1}}\right) x^A dx \\
&\quad \leq \int_{|u|>1} \exp\left(\alpha_{D,A}\left|u\right|^{\frac{D}{D-1}}\right) \exp\left(\left[\alpha-\alpha_{D,A}\right]\left|u\right|^{\frac{D}{D-1}}\right) x^A dx
\end{aligned}$$

$$\leq C_0(D,A)\int_{|u|>1}\frac{\exp\left(\alpha_{D,A}|u|^{\frac{D}{D-1}}\right)}{(\alpha_{D,A}-\alpha)|u|^{\frac{D}{D-1}}}x^A dx$$
$$\leq \frac{C_1(D,A)}{\alpha_{D,A}-\alpha}\int_{|u|>1}\frac{\Phi_D\left(\alpha_{D,A}|u|^{\frac{D}{D-1}}\right)}{1+|u|^{\frac{D}{D-1}}}x^A dx$$
$$\leq \frac{C_2(D,A)}{\alpha_{D,A}-\alpha}.$$

Next, we will prove Statement 2. Let $u\in C_c^\infty\left(\overline{\mathbb{R}_*^N}\right):\int_{\mathbb{R}_*^N}|\nabla u|^D x^A dx<1$. If $\int_{\mathbb{R}_*^N}|\nabla u|^D x^A dx\leq\left(\frac{M-1}{M}\right)^D$, then with $v=\frac{M}{M-1}u$, we get $\int_{\mathbb{R}_*^N}|\nabla v|^D x^A dx$ ≤ 1 and

$$\int_{\mathbb{R}_*^N}\Phi_D\left(M_{D,A}(u)\,\alpha_{D,A}|u|^{\frac{D}{D-1}}\right)x^A dx$$
$$=\int_{\mathbb{R}_*^N}\Phi_D\left(M_{D,A}(u)\,\alpha_{D,A}\left(\frac{M-1}{M}\right)^{\frac{D}{D-1}}|v|^{\frac{D}{D-1}}\right)x^A dx$$
$$\leq\int_{\mathbb{R}_*^N}\Phi_D\left(\left(\frac{M-1}{M}\right)\alpha_{D,A}|v|^{\frac{D}{D-1}}\right)x^A dx\leq C_0(M,D,A)\int_{\mathbb{R}_*^N}|v|^D x^A dx$$
$$\leq C_1(M,D,A)\int_{\mathbb{R}_*^N}|u|^D x^A dx\leq C_1(M,D,A)\frac{\int_{\mathbb{R}_*^N}|u|^D x^A dx}{1-\int_{\mathbb{R}_*^N}|\nabla u|^D x^A dx}.$$

where

$$M_{D,A}(u)=\frac{M^{\frac{1}{D-1}}}{\left(M-1+\int_{\mathbb{R}_*^N}|\nabla u|^D x^A dx\right)^{\frac{1}{D-1}}}.$$

If $1>\int_{\mathbb{R}_*^N}|\nabla u|^D x^A dx>\left(\frac{M-1}{M}\right)^D$, then we set $w=\frac{u}{\left[\int_{\mathbb{R}_*^N}|\nabla u|^D x^A dx\right]^{\frac{1}{D}}}$,

$$\alpha=M_{D,A}(u)\left[\int_{\mathbb{R}_*^N}|\nabla u|^D x^A dx\right]^{\frac{1}{D-1}}\alpha_{D,A},$$

and note that

$$\alpha_{D,A}>\alpha\geq\frac{M^{\frac{1}{D-1}}}{\left(M-1+\left(\frac{M-1}{M}\right)^D\right)^{\frac{1}{D-1}}}\left[\left(\frac{M-1}{M}\right)^D\right]^{\frac{1}{D-1}}\alpha_{D,A}.$$

$$\frac{1}{\alpha_{D,A}-\alpha} \leq C\left(M,D,A\right)\frac{1}{1-\int_{\mathbb{R}_*^N}\left|\nabla u\right|^D x^A dx}.$$

Hence, by Statement 1, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}_*^N}\Phi_D\left(M_{D,A}\left(u\right)\alpha_{D,A}\left|u\right|^{\frac{D}{D-1}}\right)x^A dx \\
&\quad\leq C_0\left(D,A\right)\frac{1}{\alpha_{D,A}-\alpha}\frac{\int_{\mathbb{R}_*^N}\left|u\right|^D x^A dx}{\int_{\mathbb{R}_*^N}\left|\nabla u\right|^D x^A dx} \\
&\quad\leq C_1\left(M,D,A\right)\frac{\int_{\mathbb{R}_*^N}\left|u\right|^D x^A dx}{1-\int_{\mathbb{R}_*^N}\left|\nabla u\right|^D x^A dx}.
\end{aligned}$$

□

We now prove the equivalence of the three versions of the Trudinger-Moser inequalities:

Proof of Theorem 1.4. We will first show that

$$ITM_M = \frac{M}{M-1}\sup_{\alpha\in(0,\alpha_{D,A})}\left[\frac{1-\left(\frac{\alpha}{\alpha_{D,A}}\right)^{D-1}}{\left(\frac{\alpha}{\alpha_{D,A}}\right)^{D-1}}\right]STM\left(\alpha\right).$$

Indeed, first, for any $u \in C_c^\infty\left(\overline{\mathbb{R}_*^N}\right)\setminus\{0\} : \int_{\mathbb{R}_*^N}\left|\nabla u\right|^D x^A dx = 1 = \int_{\mathbb{R}_*^N}\left|u\right|^D x^A dx$, we set

$$v\left(x\right) = \eta u\left(\lambda x\right).$$

Then it is easy to verify that

$$\int_{\mathbb{R}_*^N}\left|\nabla v\right|^D x^A dx = \eta^D$$

and

$$\int_{\mathbb{R}_*^N}\left|v\right|^D x^A dx = \frac{\eta^D}{\lambda^D}.$$

Also,

$$\int_{\mathbb{R}_*^N}\Phi_D\left(\alpha\left|u\right|^{\frac{D}{D-1}}\right)x^A dx = \lambda^D\int_{\mathbb{R}_*^N}\Phi_D\left(\frac{\alpha}{\eta^{\frac{D}{D-1}}}\left|v\right|^{\frac{D}{D-1}}\right)x^A dx.$$

Hence, if we choose η and λ such that

$$\frac{\alpha}{\eta^{\frac{D}{D-1}}} = M_{D,A}\left(v\right)\alpha_{D,A},$$

that is

$$\eta^D = \frac{M-1}{M\left(\frac{\alpha_{D,A}}{\alpha}\right)^{D-1}-1} = \left(\frac{\alpha}{\alpha_{D,A}}\right)^{D-1}\frac{M-1+\eta^D}{M},$$

and

$$\frac{\eta^D}{\lambda^D} = 1-\eta^D,$$

we get

$$\begin{aligned}
&\int_{\mathbb{R}^N_*}\Phi_D\left(\alpha\left|u\right|^{\frac{D}{D-1}}\right)x^A dx \\
&\quad = \frac{\eta^D}{1-\eta^D}\int_{\mathbb{R}^N_*}\Phi_D\left(M_{D,A}\left(v\right)\alpha_{D,A}\left|v\right|^{\frac{D}{D-1}}\right)x^A dx \\
&\quad \leq \frac{M-1}{M}.\frac{\left(\frac{\alpha}{\alpha_{D,A}}\right)^{D-1}}{1-\left(\frac{\alpha}{\alpha_{D,A}}\right)^{D-1}}ITM_M.
\end{aligned}$$

Hence, by Lemma 2.4

$$STM\left(\alpha\right) \leq \frac{M-1}{M}.\frac{\left(\frac{\alpha}{\alpha_{D,A}}\right)^{D-1}}{1-\left(\frac{\alpha}{\alpha_{D,A}}\right)^{D-1}}ITM_M.$$

Now, we note here that we can reverse the above process. Hence, we can deduce that

$$ITM_M = \frac{M}{M-1}\sup_{\alpha\in(0,\alpha_{D,A})}\left[\frac{1-\left(\frac{\alpha}{\alpha_{D,A}}\right)^{D-1}}{\left(\frac{\alpha}{\alpha_{D,A}}\right)^{D-1}}\right]STM\left(\alpha\right).$$

Also, we can argue as in [13] and as above to get that

$$TM = \sup_{\alpha\in(0,\alpha_{D,A})}\left[\frac{1-\left(\frac{\alpha}{\alpha_{D,A}}\right)^{D-1}}{\left(\frac{\alpha}{\alpha_{D,A}}\right)^{D-1}}\right]STM\left(\alpha\right).$$

□

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