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Global bifurcation for problem with mean curvature operator on general domain

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Abstract. We establish the existence of nontrivial nonnegative solution for the following 0-Dirichlet problem with mean curvature operator in the Minkowski space

$$\begin{cases} -\text{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right)=\lambda f(x,u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a general bounded domain of $\mathbb{R}^N$. By bifurcation and topological methods, we determine the interval of parameter λ in which the above problem has nontrivial nonnegative solution according to sublinear or linear nonlinearity at zero.

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1. Introduction

The aim of this paper is to study the existence of nontrivial nonnegative solution of the following problem by bifurcation and topological methods

$$\begin{cases} -\text{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right)=\lambda f(x,u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where λ is a real parameter, Ω is a general C^2 bounded domain of $\mathbb{R}^N$ with $N \geq 1$ and $f : \overline{\Omega} \times [0, d] \to \mathbb{R}_+$ is a continuous function with d denoting the diameter of Ω and $\mathbb{R}_+ = [0, +\infty)$.

The study of spacelike submanifolds of codimension one in the flat Minkowski space $\mathbb{L}^{N+1}$ with prescribed mean extrinsic curvature can lead to the type of problems (1.1), where

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$$\mathbb{L}^{N+1} = \left\{(x,t) : x \in \mathbb{R}^N, t \in \mathbb{R}\right\}$$

endowed with the Lorentzian metric

$$\sum_{i=1}^{N} (dx_i)^2 - (dt)^2,$$

where (x,t) is the canonical coordinate in $\mathbb{R}^{N+1}$ (see [2]). This kind of problems are originated from differential geometry or classical relativity.

There are a large amount of papers in the literature on the existence and on qualitative properties of solutions for this type of problems; see [1,10,18] for zero or constant curvature, [3–6,9,15] for variable curvature. In particular, Bartnik and Simon [2] proved the existence of one strictly spacelike solution when $\lambda = 1$ and f is bounded. Recently, using Leray-Schauder degree argument and critical point theory, the authors of [7] obtained some existence results for positive radial solutions of problem (1.1) with $\lambda = 1$ and $\Omega = B_R(0) := \left\{x \in \mathbb{R}^N : |x| < R\right\}$ for some constant $R > 0$. In [8], they also established some nonexistence, existence and multiplicity results for positive radial solutions of problem (1.1) with $\lambda f(x,s) = \lambda\mu(|x|)s^q$, where $q > 1$, $\mu : [0,+\infty) \to \mathbb{R}$ is continuous and strictly positive on $(0,+\infty)$. Recently, the author of this paper [11] studied the nonexistence, existence and multiplicity positive radial solutions of problem (1.1) on the unit ball via bifurcation analysis method (see [16]).

By a solution u of problem (1.1) we understand that it is a function which belongs to $C^2(\Omega) \cap C^1\left(\overline{\Omega}\right)$ with $|\nabla u| < 1$ in Ω such that problem (1.1) is satisfied. Of course, it is also a strictly spacelike solution (see [2]). For any $u \in C^1\left(\overline{\Omega}\right)$ with $u = 0$ on $\partial\Omega$ and fixed $y \in \partial\Omega$, we can see that

$$u(x) = \int_0^1 \frac{\partial u}{\partial t}\left(tx + (1-t)y\right)\, dt = \int_0^1 \nabla u\left(tx + (1-t)y\right) \cdot (x-y)\, dt.$$

It follows that $\|u\|_\infty \le \|\nabla u\|_\infty d$, where $\|\cdot\|_\infty$ denotes the usual sup-norm on $\overline{\Omega}$. Moreover, if u is a solution of problem (1.1), one has that $|u| < d$ in Ω. So we have that $\|u\|_\infty < d$ for any solution u of problem (1.1). Let $\Lambda = \lambda \max_{\overline{\Omega}\times[0,d]} f(x,s)$ for any fixed $\lambda > 0$. From Theorem 3.5 of [2], we know that $\|\nabla u\|_\infty < 1$.

Now, we state the following hypothesis on the nonlinearity f:

(H_f) there exists $f_0 \in (0,+\infty]$ such that

$$\lim_{s\to 0^+} \frac{f(x,s)}{s} = f_0$$

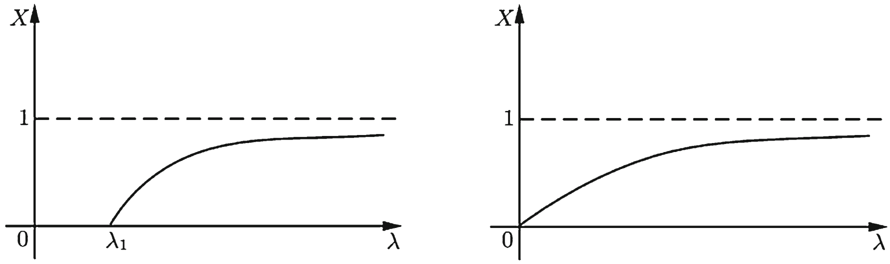
uniformly for $x \in \Omega$.

Let λ_1 denote the first eigenvalue of

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

It is well known that λ_1 is simple, isolated and the associated eigenfunction has one sign in Ω. Let

$$X = \left\{u \in C^1(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\right\}$$



(a) $f_0 = 1$ **(b)** $f_0 = +\infty$

Figure 1. Bifurcation diagrams of Theorems 1.1 and 1.2. **a** $f_0 = 1$, **b** $f_0 = +\infty$

with the norm $\|u\| = \|\nabla u\|_\infty$. From the fact of $\|u\|_\infty \le \|\nabla u\|_\infty d$, it is easy to verify that the norm $\|u\|$ is equivalent to the the usual norm $\max_{\overline{\Omega}} |u| + \max_{\overline{\Omega}} |\nabla u|$.

The following theorem is the first main result of this paper.

Theorem 1.1. *Let* (H_f) *hold with* $f_0 = 1$. *Then there is an unbounded component* $\mathscr{C}$ *of the set of nontrivial nonnegative solutions of problem (1.1) bifurcating from* $(\lambda_1, 0)$ *such that* $\mathscr{C} \subseteq ((\mathbb{R}_+ \times X) \cup \{(\lambda_1, 0)\})$.

It follows from Theorem 1.1 that problem (1.1) possesses at least one nontrivial nonnegative solution for any $\lambda \in (\lambda_1, +\infty)$, see (a) of Fig. 1.

The condition of $f_0 = 1$ shows that f is linear at 0 and $f(x, 0) = 0$ for any $x \in \Omega$. As for sublinear case at 0, i.e., $f_0 = +\infty$, we have the following theorem.

Theorem 1.2. *Assume that* (H_f) *holds with* $f_0 = +\infty$. *Then there is an unbounded component* $\mathscr{C}$ *of the set of nontrivial nonnegative solutions of problem* (1.1) *emanating from* $(0, 0)$ *such that* $\mathscr{C} \subseteq ((\mathbb{R}_+ \times X) \cup \{(0, 0)\})$.

Theorem 1.2 gives that problem (1.1) has at least one nontrivial nonnegative solution for any $\lambda \in (0, +\infty)$, see (b) of Fig. 1. Clearly, Theorem 1.2 improves the corresponding results of [7] even in the case of $\lambda = 1$ or $\Omega = B_R(0)$. Moreover, one has $f_0 = +\infty$ if f is bounded. So if f is bounded and $\lambda = 1$, problem (1.1) has at least one nontrivial nonnegative solution, which is essentially the conclusion of Theorem 3.6 of [2]. So Theorem 1.2 also improves the corresponding results of [2].

The rest of this paper is arranged as follows. In Sect. 2, we prove Theorem 1.1. Section 3 is devoted to the proof of Theorem 1.2.

2. Proof of Theorem 1.1

For any $t \in (0, 1]$, we first consider the following auxiliary problem

$$\begin{cases} -\mathrm{div}\left(\frac{\nabla u}{\sqrt{1 - t|\nabla u|^2}}\right) = g(x) \ \text{in} \ \ \Omega, \\ u = 0 \qquad\qquad\qquad\qquad\quad \text{on} \ \partial\Omega \end{cases} \tag{2.1}$$

for a given $g \in C\left(\overline{\Omega}\right)$. Letting $v = \sqrt{t}u$, problem (2.1) is equivalent to

$$\begin{cases} -\mathrm{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) = \sqrt{t}g(x) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

By Theorem 3.6 of [2], we know that there exists a unique strictly spacelike solution $v \in C^2\left(\overline{\Omega}\right)$ to problem (2.2) which is denoted by $\Psi(\sqrt{t}g)$. So $u = v/\sqrt{t}$ is the unique solution of problem (2.1).

We also consider the following auxiliary problem

$$\begin{cases} -\Delta u = h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.3}$$

for a given $h \in L^{p/(p-1)}(\Omega)$, where $p \in (1, 2^*]$ with

$$2^* = \begin{cases} \frac{2N}{N-2} & \text{if } N \geq 3, \\ +\infty & \text{if } N = 1, 2. \end{cases}$$

By an argument similar to that of Theorem 8.3 of [14], we can easily show that there is a unique weak solution u to problem (2.3) for every given $h \in L^{p/(p-1)}(\Omega)$. Let $\Phi(h)$ denote the unique solution to problem (2.3) for a given $h \in L^{p/(p-1)}(\Omega)$. Clearly, $\Phi : L^{p/(p-1)}(\Omega) \to H_0^1(\Omega)$ is continuous and linear. Moreover, from Theorem 8.34 of [14] we know that if $h \in L^\infty(\Omega)$ problem (2.3) has a unique solution in $C^{1,\alpha}\left(\overline{\Omega}\right)$ with some constant $\alpha \in (0, 1)$. So $\Phi : L^\infty(\Omega) \to C^1\left(\overline{\Omega}\right)$ is completely continuous and linear.

For any $g \in C\left(\overline{\Omega}\right)$, define

$$G(t, g) = \begin{cases} \frac{\Psi(\sqrt{t}g)}{\sqrt{t}} & \text{if } t \in (0, 1], \\ \Phi(g) & \text{if } t = 0. \end{cases}$$

Then we can show that:

Lemma 2.1. *$G : [0, 1] \times C\left(\overline{\Omega}\right) \longrightarrow X$ is completely continuous.*

Proof. We first show the continuity of G. For any $g_n, g \in C\left(\overline{\Omega}\right)$ and $t_n, t \in [0, 1]$ with $g_n \to g$ in $C\left(\overline{\Omega}\right)$ and $t_n \to t$ in $[0, 1]$ as $n \to +\infty$, it suffices to show that $u_n := G(t_n, g_n) \to u := G(t, g)$ in X.

If $t > 0$, without loss of generality, we can assume that $t_n > t/2$ for any $n \in \mathbb{N}$. By Theorem 3.6 of [2], $u_n\sqrt{t_n} := v_n, u\sqrt{t} := v \in C^2\left(\overline{\Omega}\right)$ and $\|v_n\| \leq 1 - \theta < 1$ for any $n \in \mathbb{N}$ and some positive constant θ which only depends on g and Ω. Theorem 13.7 of [14] gives an a priori estimate for $\|v_n\|_{C^{1,\alpha}(\overline{\Omega})}$ for some $\alpha \in (0, 1)$. So, up to a subsequence, there exists $w \in C^1\left(\overline{\Omega}\right)$ such that $v_n \to w$ in $C^1\left(\overline{\Omega}\right)$ as $n \to +\infty$. From Lemma 1.3 of [2] we have that w is the maximum point of

$$I(w) = \int_\Omega \left(\sqrt{1 - |\nabla w|^2} - \sqrt{t}g(x)w\right) dx$$

in $\mathscr{C}(\Omega) := \left\{w \in C^{0,1}(\Omega) : w = 0 \text{ on } \partial\Omega \text{ and } |\nabla w| \leq 1 \text{ a.e. in } \Omega\right\}$. Further, Proposition 1.1 of [2] implies that w is also the unique maximum point of

I in $\mathscr{C}(\Omega)$. So we have that $w = v$ and $v_n \to v$ in X as $n \to +\infty$. It follows that $u_n \to u$ in X as $n \to +\infty$.

If $t = 0$ and there exists a subsequence t_{n_i} of t_n such that $t_{n_i} = 0$, then $u_{n_i} = G(t_{n_i}, g_{n_i}) = \Phi(g_{n_i}) \to \Phi(g) = u$ in X as $i \to +\infty$. So next we can assume that $t = 0$ and $t_n > 0$ for any $n \in \mathbb{N}$. From Theorem 3.6 of [2] we know that problem (2.2) has only trivial solution $v = 0$ when $t = 0$. Then reasoning as the above, we can show that $v_n \to 0$ in X as $n \to +\infty$.

Note that u_n satisfies

$$\begin{cases} -\frac{1}{\sqrt{1-|\nabla v_n|^2}} \sum_{i,j=1}^{n} \left(\delta_{ij} + \frac{\nabla_i v_n \nabla_j v_n}{1-|\nabla v_n|^2}\right) \nabla_{ij} u_n = g_n(x) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

The fact of $\|v_n\| \le 1 - \theta < 1$ guarantees that the above problem is a priori uniformly elliptic. So Theorem 13.7 of [14] implies an a priori estimate for $\|v_n\|_{C^{1,\alpha}(\overline{\Omega})}$. Further, by the argument of [14, Theorem 11.4], we can see that $\|u_n\|_{C^{2,\alpha}(\overline{\Omega})} \le C$ for some positive constant C. So, up to a subsequence, there exists $w \in C^2\left(\overline{\Omega}\right)$ such that $u_n \to w$ in $C^2\left(\overline{\Omega}\right)$ as $n \to +\infty$. Letting $n \to +\infty$ in (2.4), we obtain that

$$\begin{cases} -\Delta w = g(x) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence, one has that $w = \Phi(g) = G(0, g) = u$. Furthermore, we obtain that $u_n \to u$ in X as $n \to +\infty$.

Now we show the compactness of G. It is enough to show that G satisfies

(a) $G(t, \cdot)$ is compact for any $t \in [0, 1]$;
(b) for any $\varepsilon > 0$ and $g \in C\left(\overline{\Omega}\right)$, there exists $\delta > 0$ such that $\|G(t_1, g) - G(t_2, g)\| < \varepsilon$ when $|t_1 - t_2| < \delta$ with any $t_1, t_2 \in [0, 1]$.

Clearly, $G(t, \cdot)$ is compact for any $t \in [0, 1]$. So we only need to show (b). Suppose, by contradiction, that there exist $\varepsilon_0 > 0$, $g_0 \in \left(\overline{\Omega}\right)$ such that for any $n \in \mathbb{N}$, existing t_n', $t_n'' \in [0, 1]$ with $|t_n' - t_n''| < 1/n$ such that

$$\|G(t_n', g_0) - G(t_n'', g_0)\| \ge \varepsilon_0. \tag{2.5}$$

Clearly, up to a subsequence, we have $t_n' \to t_0 \in [0, 1]$ as $n \to +\infty$. It implies that $t_n'' \to t_0 \in [0, 1]$ as $n \to +\infty$. Letting $n \to +\infty$ in (2.5), in view of the continuity of G, we have that

$$0 = \lim_{n \to +\infty} \|G(t_n', g_0) - G(t_n'', g_0)\| \ge \varepsilon_0,$$

which is a contradiction. □

For any fixed λ, consider the following problem

$$\begin{cases} -\operatorname{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

Clearly, problem (2.6) is equivalent to the operator equation $u = \Psi(\lambda u) := \Psi_\lambda(u)$. From Lemma 2.1 we see that $\Psi_\lambda : X \to X$ is complete continuous.

Moreover, by virtue of Lemma 2.1, we can obtain the following topological degree jumping result.

Lemma 2.2. *For any $r > 0$, we have that*

$$\deg\left(I - \Psi_\lambda, B_r(0), 0\right) = \begin{cases} 1 & \text{if } \lambda \in (0, \lambda_1)\,, \\ -1 & \text{if } \lambda \in (\lambda_1, \lambda_1 + \delta) \end{cases}$$

for some $\delta > 0$.

Proof. Since λ_1 is isolated, we can choose δ small enough such that there has not any eigenvalue of problem (1.2) in $(\lambda_1, \lambda_1 + \delta)$.

We first show the Leray-Schauder degree $\deg\left(I - G(t, \lambda\cdot), B_r(0), 0\right)$ is well defined for any $\lambda \in (0, \lambda_1 + \delta) \setminus \{\lambda_1\}$ and $t \in [0, 1]$. It is obvious for $t = 0$. So it is enough to show that $u = G(t, \lambda u)$ has no solution with $\|u\| = r$ for r sufficiently small and any $t \in (0, 1]$. Otherwise, there exists a sequence $\{u_n\}$ such that $u_n = \Psi_\lambda\left(\sqrt{t}u_n\right)/\sqrt{t}$ and $\|u_n\| \to 0$ as $n \to +\infty$. Let $w_n := u_n/\|u_n\|$, then by an argument similar to that of Lemma 2.1, we can show that for some convenient subsequence $w_n \to w$ as $n \to +\infty$ and w verifies problem (1.2) with $\|w\| = 1$. This implies that λ is an eigenvalue of problem (1.2), which is a contradiction.

Now from the invariance of the degree under homotopies and Lemma 2.1 we obtain that

$$\begin{aligned} \deg\left(I - \Psi_\lambda, B_r(0), 0\right) &= \deg\left(I - G(1, \lambda\cdot), B_r(0), 0\right) \\ &= \deg\left(I - G(0, \lambda\cdot), B_r(0), 0\right) = \deg\left(I - \lambda\Phi, B_r(0), 0\right). \end{aligned}$$

Since Φ is compact and linear, by Theorem 8.10 of [13], we have that

$$\deg\left(I - \lambda\Phi, B_r(0), 0\right) = \begin{cases} 1 & \text{if } \lambda \in (0, \lambda_1)\,, \\ -1 & \text{if } \lambda \in (\lambda_1, \lambda_1 + \delta)\,. \end{cases}$$

Therefore, we obtain that

$$\deg\left(I - \Psi_\lambda, B_r(0), 0\right) = \begin{cases} 1 & \text{if } \lambda \in (0, \lambda_1)\,, \\ -1 & \text{if } \lambda \in (\lambda_1, \lambda_1 + \delta)\,. \end{cases}$$

This completes the proof. □

Now we can give the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $\xi : \Omega \times [0, d] \to \mathbb{R}$ be such that

$$f(x, s) = s + \xi(x, s)$$

with

$$\lim_{s \to 0^+} \frac{\xi(x, s)}{s} = 0$$

uniformly for $x \in \Omega$. Let us consider

$$\begin{cases} -\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 - |\nabla u|^2}}\right) = \lambda u + \lambda \xi(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.7}$$

as a bifurcation problem from the trivial solution axis.

Define

$$F(\lambda, u) = \lambda u + \lambda\xi(x, u) + \operatorname{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right)$$

for any $(\lambda, u) \in \mathbb{R} \times X$. Then, by some simple calculations, we have that

$$F_u(\lambda, 0)v = \lim_{t\to 0} \frac{F(\lambda, tv)}{t} = \lambda v + \Delta v.$$

It follows that if $(\mu, 0)$ is a bifurcation point of problem (2.7), μ is an eigenvalue of problem (1.2).

For any $s \in [0, 1]$, we consider the following problem

$$\begin{cases} -\operatorname{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) = \lambda u + \lambda s\xi(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.8}$$

Then problem (2.8) is equivalent to

$$u = \Psi\left(\lambda u + \lambda s\xi(x, u)\right) := F_\lambda(s, u).$$

In view of Lemma 2.1, $F_\lambda : [0, 1] \times X \to X$ is completely continuous. In particular, $H_\lambda := F_\lambda(1, \cdot) : X \to X$ is completely continuous.

Let

$$\widetilde{\xi}(x, w) = \max_{0\le s\le w} |\xi(x, s)| \text{ for any } x \in \Omega.$$

Then $\widetilde{\xi}$ is nondecreasing with respect to w and

$$\lim_{w\to 0^+} \frac{\widetilde{\xi}(x, w)}{w} = 0. \tag{2.9}$$

Further it follows from (2.9) that

$$\left|\frac{\xi(x, u)}{\|u\|}\right| \le \frac{\widetilde{\xi}(x, u)}{\|u\|} \le \frac{\widetilde{\xi}\left(x, \|u\|_\infty\right)}{\|u\|} \le d\frac{\widetilde{\xi}(x, d\|u\|)}{d\|u\|} \to 0 \text{ as } \|u\| \to 0 \tag{2.10}$$

uniformly in $x \in \Omega$.

By (2.10) and an argument similar to that of Lemma 2.2, we can show that the Leray-Schauder degree $\deg\left(I - F_\lambda(s, \cdot), B_r(0), 0\right)$ is well defined for $\lambda \in (0, \lambda_1 + \delta) \setminus \{\lambda_1\}$. From the invariance of the degree under homotopies we obtain that

$$\begin{aligned}\deg\left(I - H_\lambda, B_r(0), 0\right) &= \deg\left(I - F_\lambda(1, \cdot), B_r(0), 0\right) = \deg\left(I - F_\lambda(0, \cdot), B_r(0), 0\right) \\ &= \deg\left(I - \Psi_\lambda, B_r(0), 0\right).\end{aligned}$$

So by Lemma 2.2, we have that

$$\deg\left(I - H_\lambda, B_r(0), 0\right) = \begin{cases} 1 & \text{if } \lambda \in (0, \lambda_1), \\ -1 & \text{if } \lambda \in (\lambda_1, \lambda_1 + \delta). \end{cases}$$

By the global bifurcation Theorem of [17], there exists a continuum $\mathscr{C}$ of nontrivial solution of problem (1.1) bifurcating from $(\lambda_1, 0)$ which is either unbounded or $\mathscr{C} \cap (\mathbb{R}\setminus\{\lambda_1\} \times \{0\}) \ne \emptyset$. Since (0,0) is the only solution of problem (1.1) for $\lambda = 0$ and 0 is not an eigenvalue of problem (1.2), so $\mathscr{C} \cap (\{0\} \times X) = \emptyset$. By Lemma 1.2 of [2], we have $u \ge 0$ for any $(\lambda, u) \in \mathscr{C}$.

We claim that $\mathscr{C}\cap(\mathbb{R}\setminus\{\lambda_1\}\times\{0\})=\emptyset$. Otherwise, there exists a nontrivial solution sequence $(\lambda_n,u_n)\in\mathscr{C}$ and $\mu\neq\lambda_1$ such that $\lambda_n\to\mu$ and $u_n\to 0$ as $n\to+\infty$. Let $w_n:=u_n/\|u_n\|$, by (2.10) and an argument similar to that of Lemma 2.1, we can show that $w_n\to w$ as $n\to+\infty$ and w verifies problem (1.2) with $\|w\|=1$. It follows that $\mu=\lambda_1$, a contradiction. Therefore, $\mathscr{C}$ is unbounded. The fact of $\|u\|<1$ for any $(\lambda,u)\in\mathscr{C}$ implies that the projection of $\mathscr{C}$ on $\mathbb{R}_+$ is unbounded. □

3. Proof of Theorem 1.2

In this section, on the basis of Theorem 1.1, we prove Theorem 1.2.
Proof of Theorem 1.2. For any $n\in\mathbb{N}$, define

$$f^n(x,s)=\begin{cases} ns, & s\in\left[0,\frac{1}{n}\right],\\ \left(f\left(x,\frac{2}{n}\right)-1\right)ns+2-f\left(x,\frac{2}{n}\right), & s\in\left(\frac{1}{n},\frac{2}{n}\right),\\ f(x,s), & s\in\left[\frac{2}{n},+\infty\right). \end{cases}$$

Clearly, we can see that $\lim_{n\to+\infty}f^n(x,s)=f(x,s)$ and $f_0^n=n$. Now, we can apply Theorem 1.1 to the following problem

$$\begin{cases} -\mathrm{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right)=\lambda f^n(x,u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Then there exists a sequence unbounded continua $\mathscr{C}_n$ of the set of nontrivial nonnegative solutions of problem (3.1) emanating from $(\lambda_1/n,0)$ such that

$$\mathscr{C}_n\subseteq\left((\mathbb{R}_+\times X)\cup\{(\lambda_1/n,0)\}\right).$$

Taking $z^*=(0,0)$, clearly $z^*\in\liminf_{n\to+\infty}\mathscr{C}_n$. The compactness of Ψ implies that $\left(\cup_{n=1}^{+\infty}\mathscr{C}_n\right)\cap B_R$ is pre-compact. Lemma 2.5 of [12] implies that $\mathscr{C}=\limsup_{n\to+\infty}\mathscr{C}_n$ is unbounded and connected such that $z^*\in\mathscr{C}$.

For any $(\lambda,u)\in\mathscr{C}$, the definition of superior limit (see [19]) shows that there exists a sequence $(\lambda_n,u_n)\in\mathscr{C}_n$ such that $(\lambda_n,u_n)\to(\lambda,u)$ as $n\to+\infty$. Clearly, one has that

$$u_n=\Psi\left(\lambda_n f^n\left(x,u_n\right)\right).$$

Letting $n\to+\infty$, we get that

$$u=\Psi\left(\lambda f\left(x,u\right)\right).$$

It follows that u is a solution of problem (1.1). Thus, u is a solution of problem (1.1) for any $(\lambda,u)\in\mathscr{C}$. Clearly, u is nonnegative for any $(\lambda,u)\in\mathscr{C}$ because $u_n\geq 0$ in Ω.

Next we show that u is nontrivial for any $(\lambda,u)\in\mathscr{C}\setminus\{(0,0)\}$. It is sufficient to show that $\mathscr{C}\cap((0,+\infty)\times\{0\})=\emptyset$. Suppose on the contrary that there exists $\mu>0$ such that $(\mu,0)\in\mathscr{C}$. There exists N_0 such that $\mu>\lambda_1/n$ for any $n>N_0$. It follows that $(\mu,0)\notin\mathscr{C}_n$ for any $n>N_0$. So $(\mu,0)\notin\mathscr{C}$, an absurd. □

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