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Algebraic traveling waves for some family of reaction-diffusion equations including the Nagumo equations

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Abstract. We classify all possible algebraic traveling solutions for the family of second order reaction-diffusion equations

$$\frac{\partial u}{\partial t} = -d\, f(u)(f'(u) + r) + d\frac{\partial^2 u}{\partial x^2}$$

where f is a polynomial function and $d > 0$ and r are real constants. In particular, we provide all the algebraic traveling wave solutions of the celebrated Nagumo equation.

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1. Introduction and statement of the main results

In past years the study of reaction-diffusion equations received a lot of attention due to its widespread areas of application and the richness of their sets of solutions. Such equations arise in models of diverse natural phenomena, with biology and chemistry as main examples. Because of the strong connection between reaction-diffusion equations and applied sciences, the main research of this type of equations comes from models from natural phenomena. Reaction-diffusion models can explain important aspects of the spatial phenomena like the existence of wave solutions in the system, the formation of patterns in a homogeneous space, etc. There is a vast amount of literature on wave propagation in biological systems. For example the book by Britton [3] is a good complete introduction to the topic (see also all the references therein).

The simplest reaction-diffusion equation is in one spatial dimension in plane geometry:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + R(u),$$

where u is the unknown, D is the diffusion coefficient and R accounts for the local reaction. Well known models of reaction-diffusion equations in one spatial dimension are the Fisher's equation [4] (with the choice $R(u) = u(1-u)$), the Newell–Whitehead–Segel equation [9] (with the choice $R(u) = u(1-u^2)$) and the general Zeldovich equation [11] (with $R(u) = u(1-u)(u-\alpha)$ and $0 < \alpha < 1$), among many others.

Most of these nonlinear differential equations do not have explicit exact solutions. These explicit exact solutions of such equations are important to understand the dynamics of these equations. Among the possible solutions, the so-called traveling wave solutions have been widely studied. A traveling wave can be defined as a solution of a system of differential equations that travels as a constant speed with a fixed shape. It is important to point out that there are many different kinds of traveling waves in different systems of equations. Typical systems that have a traveling wave as a solution are the so-called excitable systems. One example of this type of systems in biology is the propagation of an action potential along the axon of a nerve.

An important distinction that we should make is between the two most important types of traveling waves in excitable systems. The first type are the ones that we denote as *traveling fronts* and in the phase portrait of a dynamical system: traveling fronts corresponds to a heteroclinic orbits. The second type are the ones called *traveling pulses*, and the phase space orbit for this solution corresponds to homoclinic orbits.

The theory of traveling wave solutions for these systems is a rapidly growing area of applied mathematics. From the point of view of applications, traveling wave solutions usually describe a transition process between steady states. The most typical type of transition occurs from one equilibrium state to another, but a more complicated behavior can occur.

There are various approaches for constructing traveling wave solutions. Some of these approaches are the Jacobi elliptic function method [7], the inverse scattering method [1] and the homogeneous balance method [10], to cite just a few. Most of the methods may sometimes fail or can only lead to a kind of special solution and these methods become very complex and difficult to solve as the degree of the nonlinearity increases. In [5] the authors gave a necessary and sufficient condition for a partial differential equation to have algebraic traveling waves (see below for a precise definition). More precisely, they showed that for a general n-th order partial differential equation a traveling wave solutions exist if and only if the associated n-dimensional first order ordinary differential equation has some invariant algebraic curve. More precisely, for $n = 2$, consider the 2-nd order partial differential equations of the form

$$\frac{\partial^2 u}{\partial x^2} = F\Big(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\Big), \tag{1}$$

where x and t are real variables and F is a smooth map. The traveling wave solutions of system (1) are particular solutions of the form $u = u(x,t) = U(x - ct)$, where $U(s)$ satisfies the boundary conditions

$$\lim_{s\to-\infty} U(s) = A \quad \text{and} \quad \lim_{s\to\infty} U(s) = B, \tag{2}$$

being A and B solutions, not necessarily different, of $F(u,0,0) = 0$ (if A and B are different the traveling wave is a traveling front, otherwise is a traveling pulse). Plugging $u(x,t) = U(x-ct)$ into (1) we get that $U(s)$ has to be a solution, defined for all $s \in \mathbb{R}$, of the 2-nd order ordinary differential equation

$$U^{(2)} = F(U, U', -cU') = \tilde{F}(U, U'), \tag{3}$$

where $U(s)$ and the derivatives are taken with respect to s. The parameter c is called the *speed* of the traveling wave solution.

Definition 1. We say that $u(x,t) = U(x-ct)$ is an algebraic traveling wave solution if $U(s)$ is a nonconstant function that satisfies (2) and (3) and there exists a real polynomial p such that $p(U(s), U'(s)) = 0$.

The main result that we will use is the following theorem, see [5] for its proof.

Theorem 2. *The partial differential equation* (1) *has an algebraic traveling wave solution with respect to c if and only if the first order differential system*

$$\begin{cases} y_1' = y_2, \\ y_2' = G_c(y_1, y_2), \end{cases} \tag{4}$$

where

$$G_c(y_1, y_2) = F(y_1, y_2, -cy_1) = \tilde{F}(y_1, y_2)$$

has an invariant algebraic curve containing the critical points $(A, 0)$ *and* $(B, 0)$ *and no other critical points between them.*

In this paper, using Theorem 2, we want to study the existence of algebraic traveling wave solutions for the second order reaction-diffusion equations

$$\frac{\partial u}{\partial t} = -df(u)(f'(u) + r) + d\frac{\partial^2 u}{\partial x^2}, \tag{5}$$

where f is a polynomial function of degree at most one, and $d > 0$ and r are real constants. Writing

$$f(x) = \sum_{j=0}^{n} a_j x^j, \quad n \geq 1 \tag{6}$$

we will restrict to the case in which $a_{n-1} \neq 0$ (note that $a_n \neq 0$). Therefore in this paper we will make the following assumption: writing $f(x)$ in system (5) as in (6), we have $a_{n-1} \neq 0$.

We will see that by studying the algebraic traveling wave solutions of (5) we will recover the results presented in [8, Ch 11]. In particular we will find all the algebraic traveling waves for the Zeldovich equation that arises in combustion theory which is a particular case of the Nagumo equation related with the FitzHugh–Nagumo model for the nerve action potentials. The Nagumo equation is a simplification for a model of signal transmission on nerve axis in a cardiac tissue. This system of equations is given by

$$\frac{\partial u}{\partial t} = a(u-u_1)(u-u_2)(u-u_3) + d\frac{\partial^2 u}{\partial x^2},$$

where $a, d > 0$ and $u_1 < u_2 < u_3$ with $u_1 + u_3 \neq 0$, are given real constants. Note that this equation can be written as in (5) with

$$f(u) = \sqrt{\frac{a}{2d}}(u-u_1)(u-u_3)$$

that satisfies the assumption because $a_1 = u_3 + u_1 \neq 0$. Our main theorem is the following.

Theorem 3. *The following statements hold for Eq.* (5) *under the assumption* $a_{n-1} \neq 0$.

a *If* $deg(f) = 1$, *Eq.* (5) *has no algebraic traveling wave solutions.*
b *If* $deg(f) \geq 2$, *Eq.* (5) *has algebraic traveling wave solutions if and only if* $c = dr$ *and the polynomial ordinary differential equation* $U'(s) - f(U(s)) = 0$ *has global solutions satisfying* (2) *with* A *and* B *being solutions of the equation* $f(x)(f'(x) + r) = 0$. *In this case the traveling wave solutions are of the form* $u(x,t) = U(x - ct)$.

Theorem 3 is proved in Sect. 2. In [5] the authors show that one algebraic traveling wave solution for system (5) with $\deg(f) \geq 2$ is indeed the one given in Theorem 3(b) but they are not able to show that it is the unique one. This is the main contribution of this paper and this allows us to classify all possible algebraic traveling wave solutions for system (5). In order to prove Theorem 3 we will show that each (weighted) homogeneous component of the algebraic invariant curve (see below for a definition) that gives rise to the algebraic traveling wave solution must fulfill an algebraic homological equation. The key point is to classify all the solutions of such homological equation with are (weighted) polynomials. It is important to note that Theorem 3 gives necessary and sufficient conditions for the partial differential equation (5) to have explicit algebraic traveling wave solutions, but Theorem 3 says nothing about the existence of non-algebraic traveling wave solutions.

2. Proof of Theorem 3

We separate the proof of Theorem 3 into two different results, one for $\deg(f) = 1$ and the other for $\deg(f) \geq 2$.

When $\deg(f) = 1$ we will use a work of Hayashi in [6] that characterizes the invariant algebraic curves of all systems of the form

$$x' = y, \quad y' = -g_1(x) - f_1(x)y, \tag{7}$$

where $g_1(x)$ and $f_1(x)$ are polynomials. More precisely, he proved the following result.

Theorem 4. *System* (7) *with* $deg(g_1) = deg(f_1) + 1$ *has an invariant algebraic curve* $g(x, y) = 0$ *if and only if* $g(x, y) = y - P(x)$, *where* $P(x)$ *satisfies*

$$g_1(x) = -(f_1(x) + P'(x))P(x) \tag{8}$$

where

1. *either* $P(x)$ *has degree one;*
2. *or* $P(x)$ *is such that* $P(x) + \int f_1(x)\, dx$ *is a polynomial of degree one.*

Theorem 5. *Equation* (5) *with* $deg(f) = 1$ *has no travelling wave solutions.*

Proof. The planar system (4) associated to system (5) with $f(x) = \gamma x + \beta$ with $\gamma, \beta \in \mathbb{R}$, $\gamma \neq 0$ is

$$\begin{aligned} x' &= y, \\ y' &= -\frac{c}{d} y + (\gamma x + \beta)(\gamma + r), \end{aligned} \tag{9}$$

which is of the form (7) with $g_1(x) = -(\gamma x + \beta)(\gamma + r)$ and $f_1(x) = c/d$. Note that $\deg(g_1) = \deg(f_1) + 1$. It follows from Theorem 4 that if system (9) has an invariant algebraic curve then it is of the form $g(x, y) = y - P(x)$ where $P(x)$ satisfies condition (8) and either $P(x)$ is of degree one or $P(x) = -\int f_1(x)\, dx + Q(x)$ being $Q(x)$ a polynomial of degree one. Since in this case $f_1(x)$ is constant, we get that both conditions on P can be written as $P(x)$ being a polynomial of degree one. We write it as $P(x) = Ax + B$ with $A, B \in \mathbb{R}$, $A \neq 0$. It follows from the condition (8) that

$$-\gamma(\gamma + r) = -\Big(\frac{c}{d} + A\Big)A \quad \text{and} \quad -\beta(\gamma + r) = -\Big(\frac{c}{d} + A\Big)B.$$

Solving it we get $B = A\beta/\gamma$ and

$$A = A^* = -\frac{1}{2}\left(\frac{c}{d} \pm \sqrt{\frac{c^2}{d^2} + 4(\gamma + r)\gamma}\right),$$

whenever the discriminant is non-negative. An invariant algebraic curve of system (9) must be of the form

$$g(x, y) = y - \frac{A^*}{\gamma}(\gamma x + \beta) = 0.$$

Note that from $x' = y$ we obtain

$$x'(s) = \frac{A^*}{\gamma}(\gamma x(s) + \beta)$$

and so for the function $U(s) = y_1(x) = x(s)$ we get the differential equation

$$U'(s) = \frac{A^*}{\gamma}(\gamma U(s) + \beta).$$

The non-constant solutions that are defined for all $s \in \mathbb{R}$ are

$$U(s) = -\frac{\beta}{\gamma} + \kappa \mathrm{e}^{A^* s},$$

for some constant $\kappa \neq 0$ (otherwise U is constant). Note that the function $U(s)$ does not satisfy condition (2), and so, in this case, no algebraic traveling wave solutions can exist. □

Now we consider the case in which $\deg(f) \geq 2$. We will prove the following result.

Theorem 6. *System* (5) *with* $\deg(f) \geq 2$ *and under the assumption (H1) has algebraic traveling wave solutions if and only if* $c = dr$*, and in this case* $p(U, U') = U'(s) - f(U(s))$.

In order to prove Theorem 6 we will first state and prove some results. Note that the planar system (4) associated to system (5) is

$$\begin{aligned} x' &= y, \\ y' &= -\frac{c}{d}y + f(x)(f'(x) + r), \end{aligned} \tag{10}$$

where $\deg(f) \geq 2$.

Theorem 7. *System* (10) *with* $\deg(f) \geq 2$ *has an invariant algebraic curve if and only if* $r = c/d$*. In this case the invariant algebraic curve is*

$$g(x, y) = y - f(x) = 0.$$

We note that to prove Theorem 6 is equivalent to prove Theorem 7 and so we will only prove Theorem 7.

Proof of Theorem 7. We will need some preliminary definitions and results.

For irreducible polynomials we have the following algebraic characterization of invariant algebraic curves: Given an irreducible polynomial $g(x, y)$ of degree $\deg(g)$ (not being a constant), we have that $g(x, y) = 0$ is an invariant algebraic curve for the system $x' = P(x, y)$, $y' = Q(x, y)$ for $P, Q \in \mathbb{C}[x, y]$, if there exists a polynomial $K(x, y)$ of degree at most $\max\{\text{degree}(P), \text{degree}(Q)\} - 1$, called the cofactor of g, such that

$$P(x, y)\frac{\partial g(x, y)}{\partial x} + Q(x, y)\frac{\partial g(x, y)}{\partial y} = K(x, y)g(x, y). \tag{11}$$

Note that $g(x, y) = 0$ is invariant by the system. □

Lemma 8. *If system* (10) *with* $n \geq 2$ *has an algebraic invariant curve, then* $K(x, y) = k_0(x)$*, where* $k_0(x)$ *is a polynomial of degree at most* $2n - 2$.

Proof. Since system (10) has degree $2n - 1$, the cofactor of an invariant algebraic curve $g(x, y) = 0$ must have degree at most $2n - 2$.

We write both g and K in their power series in the variable y as

$$K(x, y) = \sum_{j=0}^{2n-2} K_{2n-2-j}(x)y^j, \quad g = \sum_{j=0}^{\ell} g_j(x)y^j,$$

for some integer ℓ and where $K_{2n-2-j}(x)$ is a polynomial of degree at most $2n - 2 - j$ and g and K satisfy (11). Without loss of generality, since $g \neq 0$ we can assume that $g_\ell = g_\ell(x) \neq 0$. Moreover:

$$y\frac{\partial g}{\partial x} + \Big(-\frac{c}{d}y + f(x)(f'(x) + r)\Big)\frac{\partial g}{\partial y} = Kg. \tag{12}$$

We compute the coefficient of $y^{\ell+2n-2}$ in (12) and we get $g_\ell K_{2n-2} = 0$, that is $K_{2n-2} = 0$ because $g_\ell \neq 0$. Now we proceed by backwards induction on the degree in y of (12). Computing the coefficient of $y^{\ell+j}$ for $j = 2n-3, \ldots, 2$ in (12) and using that $g_\ell \neq 0$ we get $g_\ell k_j = 0$, that is, $K_j = 0$ for $j = 2n-3, \ldots, 2$. So, $K(x, y) = K_0(x) + K_1(x)y$. Computing the coefficient of $y^{\ell+1}$ in (12) we get

$$g_\ell'(x) = K_1 g_\ell$$

which yields $g_\ell = \kappa e^{\int K_1(x)\,dx}$, for $\kappa \in \mathbb{C}\backslash\{0\}$. Since g_ℓ must be a polynomial we get $K_1 = 0$. This implies that $K(x) = K_0(x)$ and completes the proof of the lemma. □

We introduce the change of variables

$$Y = y - f(x), \quad X = x$$

Then system (10) in these variables becomes

$$\begin{aligned} X' &= Y + f(X),\\ Y' &= -\frac{c}{d}(Y + f(X)) + f(X)f'(X) + f(X)r - f'(X)(Y + f(X))\\ &= -\Big(\frac{c}{d} + f'(X)\Big)Y + Af(X), \end{aligned} \tag{13}$$

where $A = r - c/d$.

Note that any irreducible Darboux polynomial $g(x, y) = \tilde{g}(X, Y)$ of system (10) is an irreducible Darboux polynomial of system (13) and vice-versa. Moreover, any irreducible Darboux polynomial $\tilde{g}$ of (13) satisfies

$$(Y + f(X))\frac{\partial \tilde{g}}{\partial \tilde{X}} + \bigg(-\Big(\frac{c}{d} + f'(X)\Big)Y + Af(X)\bigg)\frac{\partial \tilde{g}}{\partial \tilde{Y}} = K_0(X)\tilde{g}.$$

We write

$$f(X) = \sum_{j=0}^{n} a_j X^j, \quad K_0(X) = \sum_{j=0}^{2n-2} k_j X^j.$$

Now we introduce the weight-change of variables of the form

$$X = \mu^{-1}x_1, \quad Y = \mu^{-n}y_1, \quad t = \mu^{n-1}\tau$$

with $\mu \in \mathbb{R}\backslash\{0\}$. Then system (13) becomes

$$\begin{aligned} x_1' &= y_1 + \mu^n f(\mu^{-1}x_1) = y_1 + a_n x_1^n + a_{n-1}\mu x_1^{n-1} + \cdots + a_0\mu^n,\\ y_1' &= \mu^{n-1}\big(-\frac{c}{d} - f'(\mu^{-1}x_1)\big)y_1 + A\mu^{2n-1}f(\mu^{-1}x_1)\\ &= -\frac{c}{d}\mu^{n-1}y_1 - (na_n x_1^{n-1} + \cdots + a_1\mu^{n-1})y_1\\ &\quad + Aa_n\mu^{n-1}x_1^n + \cdots + A\mu^{2n-1}a_0\\ &= -na_n x_1^{n-1}y_1 - (n-1)a_{n-1}x_1^{n-2}y_1 - \cdots - a_1\mu^{n-1}y_1 - \frac{c}{d}\mu^{n-1}y_1\\ &\quad + Aa_n\mu^{n-1}x_1^n + \cdots + A\mu^{2n-1}a_0, \end{aligned} \tag{14}$$

where the prime denotes now derivative in τ.

A polynomial $G(x_1, y_1)$ is said to be *weight-homogeneous of degree* $N \in \mathbb{N}$ with respect to the weight exponent $s = (s_1, s_2)$ if for all $\mu \in \mathbb{R}\setminus\{0\}$ we have

$$G(\mu^{s_1} x_1, \mu^{s_2} y_1) = \mu^N G(x_1, y_1).$$

In our case, we set

$$G(x_1, y_1) = \mu^N g(\mu^{-1} x_1, \mu^{-n} y_1) = \sum_{i=0}^{N} \mu^i G_i(x_1, y_1)$$

where G_i is the weight homogeneous part with weight-degree $N - i$ of G, and N is the weight degree of G with the weight exponent $s = (1, n)$. Obviously $g = G|_{\mu=1}$. Take now $K(x) = \mu^{n-1} K_0(\mu^{-1} x_1)$, i.e.

$$\begin{aligned} K &= \mu^{n-1}(k_0 + k_1 \mu^{-1} x_1 + \cdots + k_{n-1} \mu^{-n+1} x_1^{n-1} \\ &\quad + k_n \mu^{-n} x_1^n + \cdots + k_{2n-2} \mu^{-2n+2} x_1^{2n-2}) \\ &= k_{2n-2} \mu^{-n+1} x_1^{2n-2} + \cdots + k_n \mu^{-1} x_1^n \\ &\quad + k_{n-1} x_1^{n-1} + k_{n-2} \mu x_1^{n-2} + \cdots + k_0 \mu^{n-1}. \end{aligned}$$

We note that $G = 0$ is an invariant algebraic curve of system (14) with cofactor K:

$$\begin{aligned} \frac{dG(x_1, y_1)}{d\tau} &= \mu^N \mu^{n-1} \frac{dg(\mu^{-1} x_1, \mu^{-n} y_1)}{dt} \\ &= \mu^N K g(\mu^{-1} x_1, \mu^{-n} y_1) = K G(x_1, y_1). \end{aligned}$$

From (11) we have

$$\begin{aligned} &\left(y_1 + \sum_{j=0}^{n} a_{n-j} \mu^j x_1^{n-1}\right) \sum_{i=0}^{N} \mu^i \frac{\partial G_i}{\partial x_1} - \left(y_1 \sum_{j=0}^{n-1} (n-j) a_{n-j} \mu^j x_1^{n-j-1}\right. \\ &\qquad \left. + \frac{c}{d} \mu^{n-1} y_1 - A \sum_{j=0}^{n} a_{n-j} \mu^{n-1+j} x_1^{n-j}\right) \sum_{i=0}^{N} \mu^i \frac{\partial G_i}{\partial y_1} \\ &= \sum_{j=0}^{2n-2} k_{2n-2-j} \mu^{-n+1+j} x_1^{2n-2-j} \sum_{i=0}^{N} \mu^i G_i. \end{aligned} \tag{15}$$

Equating the terms with μ^{-j} for $j = n+1, \ldots, 1$ we get $k_{2n-2} = \cdots = k_n = 0$ (note that if $G_0 = 0$, then g is a constant, which is not possible because $g = 0$ is an invariant algebraic curve). Moreover, equating the terms with μ^0 we get

$$L[G_0] = k_{n-1} x_1^{n-1} G_0 \tag{16}$$

where

$$L = (y_1 - a_n x_1^n) \frac{\partial}{\partial x_1} + n a_n x_1^{n-1} y_1 \frac{\partial}{\partial y_1}.$$

Now we use the method of characteristic curves for solving linear partial differential equations (see, for instance, Bleecker and Csordas [2]). The characteristic equations associated with the linear partial differential equation (16) are

$$\frac{dx_1}{dy_1} = \frac{y_1 + a_n x_1^n}{-na_n x_1^{n-1} y_1}.$$

This system has the general solution $y_1^2/2 + a_n x_1^n y_1 = \kappa$, where κ is a constant. According to the method of characteristics we make the change of variables

$$u = y_1^2/2 + a_n x_1^n y_1, \quad v = y_1. \tag{17}$$

Its inverse transformation is

$$x_1 = \Big(\frac{2u - v^2}{2a_n v}\Big)^{1/n}, \quad y_1 = v. \tag{18}$$

Under changes (17) and (18), the partial differential equation (16) becomes the following ordinary differential equation (for fixed u)

$$-na_n v \frac{d\overline{G}_0}{dv} = k_{n-1}\overline{G}_0,$$

where $\overline{G}_0$ is G_0 written in the variables u, v. In what follows we always write $\overline{\theta}$ to denote a function $\theta = \theta(x_1, y_1)$ written in the (u, v) variables, that is, $\overline{\theta} = \overline{\theta}(u, v)$. The above equation has the general solution

$$\overline{G}_0 = \overline{F}_0(u) v^{-k_{n-1}/(na_n)},$$

where $\overline{F}_0$ is an arbitrary smooth function in the variable u. In order that G_0 be a weight homogenous polynomial with weight degree N, since x_1 and y_1 have weight degrees 1 and n respectively we get that G_0 should be of weight degree $N = j + 2\ell n$ for some convenient $j, \ell \in \mathbb{N}$. So,

$$k_{n-1} = -jna_n, \quad G_0 = b_\ell\big(y_1^2/2 + a_n x_1^n y_1\big)^\ell y_1^j.$$

Without loss of generality we can assume that $b_\ell = 1$. We will prove by induction that

$$G_i = 0, \quad \ell = 0, \quad k_{n-1-i} = -j(n-i)a_{n-i} \quad \text{for } i = 1, \ldots, n-2. \tag{19}$$

Indeed, computing the term in μ in (15) we get

$$L[G_1] - k_{n-1}x_1^{n-1}G_1 + a_{n-1}x_1^{n-1}\frac{\partial G_0}{\partial x_1} \\ -(n-1)a_{n-1}x_1^{n-2}y_1\frac{\partial G_0}{\partial y_1} = k_{n-2}x_1^{n-2}G_0.$$

Using G_0 and doing some computations we obtain that

$$\begin{aligned} &L[G_1] + jna_n x_1^{n-2}G_1 \\ &\quad = R^{\ell-1}x_1^{n-2}y_1^j\Big[a_{n-1}\Big(-\ell n a_n x_1^n y_1 + (n-1)\ell(a_n x_1 y_1 + y_1^2) \\ &\qquad +(n-1)j\Big(\frac{y_1^2}{2} + a_n x_1^n y_1\Big)\Big) + k_{n-2}\Big(\frac{y_1^2}{2} + a_n x_1^n y_1\Big)\Big] \\ &\quad = R^{\ell-1}x_1^{n-2}y_1^j\Big[a_{n-1}\Big((n-1)\ell y_1^2 - \ell a_n x_1^n y_1 + (n-1)jR\Big) + k_{n-2}R\Big] \\ &\quad = R^{\ell-1}x_1^{n-2}y_1^j\Big[a_{n-1}\Big(-\ell R + (2n-1)\ell\frac{y_1^2}{2} + (n-1)jR\Big) + k_{n-2}R\Big] \end{aligned}$$

$$= R^{\ell-1}x_1^{n-2}y_1^j\Big[R\Big(k_{n-2}+\big((n-1)j-\ell\big)a_{n-1}\Big)+\frac{2n-1}{2}\ell a_{n-1}y_1^2\Big]$$
$$= B_1x_1^{n-2}R^\ell y_1^j + B_2x_1^{n-2}R^{\ell-1}y_1^{j+2},$$

where $R = y_1^2/2 + a_nx_1^ny_1$. Under changes (17) and (18), we can rewrite

$$L[G_1] + jna_nx_1^{n-1}G_1 = B_1x_1^{n-2}\big(y_1^2/2 + a_nx_1^ny_1\big)^\ell y_1^j + B_2x_1^{n-2}\big(y_1^2/2 + a_nx_1^ny_1\big)^{\ell-1}y_1^{j+2}$$

as the following ordinary differential equation (for fixed u)

$$\frac{d\overline{G}_1}{dv} = -\frac{j}{v}\overline{G}_1 + \frac{2^{1/n}B_1}{na_n^{(n-1)/n}}v^ju^\ell\frac{1}{v^{(n-1)/n}(2u-v^2)^{1/n}} + \frac{2^{1/n}B_2}{na_n^{(n-1)/n}}v^ju^{\ell-1}\frac{v^{(n+1)/n}}{(2u-v^2)^{1/n}}. \tag{20}$$

Note that it can be written as

$$\frac{d\overline{G}_1}{dv} = -\frac{j}{v}\overline{G}_1 + \tilde{G}_1(v),$$

whose general solution is

$$\overline{G}_1 = \overline{F}_1(u)v^j + v^j\int v^{-j}\tilde{G}_1(v)\,dv,$$

being $\overline{F}_1(u)$ a smooth function in the variable u. Hence, the solution of the linear differential equation (20) is

$$\begin{aligned}\overline{G}_1 &= \overline{F}_1(u)v^j + \frac{2^{1/n}B_1}{na_n^{(n-1)/n}}v^ju^\ell\int\frac{1}{v^{(n-1)/n}(2u-v^2)^{1/n}}\,dv\\ &\quad+\frac{2^{1/n}B_2}{na_n^{(n-1)/n}}v^ju^{\ell-1}\int\frac{v^{(n+1)/n}}{(2u-v^2)^{1/n}}\,dv\\ &= \overline{F}_1(u)v^j + \frac{2^{(1-n)/n}B_1}{a_n^{(n-1)/n}}v^{j+1/n}u^{\ell-1}(2u-v^2)^{(n-1)/n}\\ &\quad\times {}_2F_1\Big(1,\frac{2n-1}{2n},\frac{2n+1}{2n},\frac{v^2}{2u}\Big)\\ &\quad+\frac{2^{(1-n)/n}B_2}{(1+2n)a_n^{(n-1)/n}}u^{\ell-2}v^{j+2+1/n}(2u-v^2)^{(n-1)/n}\\ &\quad\times {}_2F_1\Big(1,\frac{4n-1}{2n},\frac{4n+1}{2n},\frac{v^2}{2u}\Big),\end{aligned} \tag{21}$$

where

$${}_2F_1(a,b,c,x) = \sum_{k=0}^\infty\frac{a(a+1)\cdots(a+k-1)}{b(b+1)\cdots(b+k-1)c(c+1)\cdots(c+k-1)}\frac{x^k}{k!} \tag{22}$$

is the hypergeometric function that is is well defined if b, c are not negative integers. In particular, it is a polynomial if and only if a is a negative integer. Note that in our case $a = 1$, $b = (2n-1)/2n$ and $c = (2n+1)/(2n)$ in the first hypergeometric function and $a = 1$, $b = (4n-1)/2n$ and $c = (4n+1)/(2n)$

in the second one. Hence, both hypergeometric functions are well defined and are never polynomials. Moreover, (21) can be rewritten as

$$\overline{G}_1 = \overline{F}_1(u)v^j + \frac{1}{a_n^{(n-1)/n}} u^{\ell-2} v^{j+1/n}(2u - v^2)^{(n-1)/n)}\Big(2^{1/n}B_1 u + ((2n-1)B_1 + B_2)\sum_{j\geq 1} \frac{2^{1/n-j}\prod_{r=2}^{j}(2rn-1)}{\prod_{r=1}^{j}(1+2rn)}\frac{v^{2j}}{u^{j-1}}\Big), \tag{23}$$

where $\prod_{r=2}^{j}(2rn-1) = 1$ whenever $j \leq 2$. Since $G_1(x_1, y_1) = \overline{G}_1(u, v)$ must be a weight-homogeneous polynomial with weight degree $N-1 = j-1$ (in the variables x_1, y_1) and n is a positive integer we must have $(2n-1)B_1 + B_2 = 0$. Taking into account the definition of B_1 and B_2 this implies

$$k_{n-2} = -\frac{a_{n-1}}{2}(2j(n-1) - \ell). \tag{24}$$

Then $B_1 = -\ell a_{n-1}/2$ and $B_2 = (2n-1)\ell a_{n-1}/2$. Imposing it in (23) we obtain

$$\overline{G}_1 = \overline{F}_1(u)v^j - \frac{2^{-1+1/n}}{a_n^{(n-1)/n}} a_{n-1}\ell u^{\ell-1} v^{j+1/n}(2u - v^2)^{(n-1)/n}. \tag{25}$$

Note that $\overline{G}_1$ must have weight-degree $N-1 = 2\ell n + j - 1$. Note that by (25) we have that the degree of $\overline{G}_1$ is

$$(\ell-1)2n + n(j+1/n) + 2(n-1) = 2n\ell + nj - 1.$$

Taking into account that it must be equal to $2\ell n + j - 1$ we must have $n = 1$, which is not possible because $n \geq 2$. So, $\overline{G}_{n-1} - \overline{F}_1(u)v^j$ must be zero. This implies that $\ell = 0$. But then also $\overline{F}_1 = 0$ and $G_1 = 0$. Since $\ell = 0$ it follows from (24) that $k_{n-2} = -j(n-1)a_{n-1}$. In short $N = j$ and

$$G_0 = y^j, \quad G_1 = 0, \quad k_{n-2} = -j(n-1)a_{n-1}.$$

Now proceeding inductively as we did for G_1 using that $\ell = 0$ we get that $G_i = 0$ and $k_{n-1-i} = -j(n-i)a_{n-i}$ for $i = 1, \ldots, n-2$. This yields (19).

Computing the term in μ^{n-1} in (15) and using (19) we get

$$L[G_{n-1}] + nja_n x_1^{n-1} G_{n-1} = -a_1 x_1 \frac{\partial G_0}{\partial x_1} + \frac{c}{d} y_1 \frac{\partial G_0}{\partial y_1} - A a_n x_1^n \frac{\partial G_0}{\partial y_1} + k_0 G_0.$$

Hence,

$$\begin{aligned} L[G_{n-1}] &= -nja_n x_1^{n-1} G_{n-1} + \frac{c}{d} j y_1^j - A a_n x_1^n j y_1^{j-1} + k_0 y_1^j \\ &= -nja_n x_1^{n-1} G_{n-1}\Big(k_0 + j\frac{c}{d}\Big) y_1^j - A a_n j x_1^n y_1^{j-1}. \end{aligned}$$

Under changes (17) and (18), we can rewrite the above equation as an ordinary differential equation (for fixed u) whose solution gives

$$\begin{aligned}\overline{G}_{n-1} &= \overline{F}_{n-1}(u)v^j - \frac{k_0 + jc/d}{na_n^{1/n}} 2^{(n-1)/n} v^j \int \frac{1}{v^{1/n}(2u - v^2)^{(n-1)/n}}\, dv \\ &\quad + \frac{2^{(n-1)/n} jAa_n^{(n-1)/n}}{n} v^j \int \frac{1}{v^{(n+1)/n}(2u - v^2)^{(n-1)/n}}\, dv \\ &= \overline{F}_{n-1}(u)v^j - \frac{k_0 + jc/d}{(n-1)a_n^{1/n}} 2^{-1/n} u^{-1} v^{j+1-1/n}(2u - v^2)^{1/n} \\ &\quad {}_2F_1\Big(1, \frac{n+1}{2n}, \frac{3n-1}{2n}, \frac{v^2}{2u}\Big) \\ &\quad - 2^{-1/n} jAa_n^{(n-1)/n} v^{j-1/n} u^{-1}(2u - v^2)^{1/n}\, {}_2F_1\Big(1, \frac{1}{2n}, \frac{2n-1}{2n}, \frac{v^2}{2u}\Big),\end{aligned}$$

where $\overline{F}_{n-1}$ is a smooth function in variable u and ${}_2F_1$ is the hypergeometric function introduced in (22). Again both hypergeometric functions are well defined and none of them are polynomials. Since G_{n-1} must be a polynomial, $j = N > 0$ and the above hypergeometric functions cannot combine to give a polynomial (in contrast to what happens in (21) because of the terms $v^{j+1-1/n}$ and $v^{j-1/n}$ whose exponents have different parity while the variable in the hypergeometric functions is the same: $v^2/(2u)$), we must have

$$k_0 + jc/d = 0 \quad \text{and} \quad A = r - \frac{c}{d} = 0.$$

Hence, $r = c/d$ and $k_0 = -jc/d$. Again, since G_{n-1} must be a weight-homogeneous polynomial of weight-degree $n(j-1)$ and y_1^j has weight degree nj we must have $\overline{F}_{n-1}(u) = 0$ and so $G_{n-1} = 0$.

Since G_0 does not depend on x_1, we have

$$L[G_n] + nja_n x_1^{n-1} G_n = 0$$

which yields $G_n = 0$. Proceeding inductively we get that $G = G_0$. Therefore, $g(X, Y) = G|_{\mu=1} = Y^j$. Since g must be irreducible we must have $j = 1$. Then the unique irreducible invariant algebraic curve is $Y = y - f(x) = 0$ and it occurs when $r = c/d$. It has cofactor $K = -(c/d + f'(x))$. This concludes the proof of the theorem. □

Proof of Theorem 3. Statement (a) in Theorem 3 follows directly from Theorem 5.

For statement (b), note that by Theorem 6, system (5) with $\deg(f) \geq 2$ and under the assumption $a_{n-1} \neq 0$, has a traveling wave solution if and only if $c = dr$ and in this case it must satisfy $U'(s) - f(U(s)) = 0$. The solution of this equation under adequate boundary conditions satisfying condition (2) with A and B being solutions of $f(x)(f'(x) + r) = 0$ defined for all $s \in \mathbb{R}$ give the algebraic traveling wave solutions $u(x,t) = U(x - ct)$ of Eq. (5) with speed $c = dr$. This concludes the proof. □

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