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Concentration of semi-classical solutions to the Chern–Simons–Schrödinger systems

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Abstract. In this paper we demonstrate the existence and concentration behavior of semi-classical solutions for the nonlinear Chern–Simons–Schrödinger systems with external potential. Combining the variational methods with concentration compactness principle, we prove the existence of a family of semi-classical solutions concentrating at the minimum points of the external potential.

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1. Introduction and main result

We study the concentration phenomenon of ground states to the following Chern–Simons–Schrödinger system (**CSS** system) in $H^1(\mathbb{R}^2)$

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u + A_0(u(x))u + \sum_{j=1}^{2} A_j^2(u(x))u = f(u), \\ \varepsilon\partial_1 A_0(u(x)) = A_2(u(x))|u|^2, \quad \varepsilon\partial_2 A_0(u(x)) = -A_1(u(x))|u|^2, \\ \varepsilon\big(\partial_1 A_2(u(x)) - \partial_2 A_1(u(x))\big) = -\frac{1}{2}u^2, \quad \partial_1 A_1(u(x)) + \partial_2 A_2(u(x)) = 0, \end{cases} \tag{1.1}$$

where the parameter $\varepsilon > 0$, $f(u) = |u|^{p-2}u$, $p > 6$ and the external potential $V(x)$ satisfies

(V) $V(x) \in C(\mathbb{R}^2, \mathbb{R})$ and $V_0 := \inf_{x\in\mathbb{R}^2} V(x) < V_\infty := \liminf_{|x|\to\infty} V(x)$.

This system arises in the investigation of the standing wave of Chern–Simons–Schrödinger system, proposed in [9,10] and [5] consists of the Schrödinger equation augmented by the gauge field, which describes the dynamics of large number of particles in a electromagnetic field. This feature of the model is important for the study of the high-temperature superconductor, fractional quantum Hall effect and Aharovnov-Bohm scattering. The Lagrangian density of the abelian Chern–Simons model provide **CSS** system

$$\begin{cases} \mathrm{i}D_0\phi + (D_1D_1 + D_2D_2)\phi = f(\phi), \\ \partial_0 A_1 - \partial_1 A_0 = -\mathrm{Im}(\bar{\phi} D_2 \phi), \\ \partial_0 A_2 - \partial_2 A_0 = \mathrm{Im}(\bar{\phi} D_1 \phi), \\ \partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2}|\phi|^2. \end{cases} \tag{1.2}$$

The **CSS** system (1.2) is invariant under the following gauge transformation $\phi \to \phi e^{\mathrm{i}\chi}, \quad A_\mu \to A_\mu - \partial_\mu \chi$ where $\chi : \mathbb{R}^{1+2} \to \mathbb{R}$ is an arbitrary C^∞ function. Blowing up time-dependent solutions were investigated by Berge et al. [1] and local wellposedness was studied by Liu et al. [13].

We suppose that the gauge field satisfies the Coulomb gauge condition $\partial_0 A_0 + \partial_1 A_1 + \partial_2 A_2 = 0$, and $A_\mu(x,t) = A_\mu(x)$, $\mu = 0, 1, 2$. Then the standing wave $\psi(x,t) = e^{\mathrm{i}\omega t}\, u(x)$ satisfies

$$\begin{cases} -\Delta u + \omega u + A_0 u + A_1^2 u + A_2^2 u = f(u), \\ \partial_1 A_0 = A_2 u^2, \quad \partial_2 A_0 = -A_1 u^2, \\ \partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2}|u|^2, \quad \partial_1 A_1 + \partial_2 A_2 = 0. \end{cases} \tag{1.3}$$

The existence of radial solutions to (1.3) has been investigated by Byeon et al. [2], under the assumptions of power type nonlinearities, see also [6] and [7]. A series of existence results of solitary waves has been established in [3,11,14,15,17,22]. We studied the existence, non-existence, and multiplicity of standing waves to the nonlinear **CSS** systems with an external potential $V(x)$ without the Ambrosetti–Rabinowitz condition in [18]. Multiplicity and concentration of radial solutions have established by using variational methods [17] in the general nonlinearities and Yuan [22] studied radial normalized solutions. Moreover, we show the existence of nontrivial solutions to Chern–Simons–Schrödinger systems (1.1) by using the concentration compactness principle with $V(x)$ is a constant and the argument of global compactness with $p > 4$, $V \in C(\mathbb{R}^2)$ and $0 < V_0 < V(x) < V_\infty$ in [19]. For the more physical background of **CSS** system, we refer to the references we mentioned above and [4,8].

Inspired by [2,18,19], and [20], the purpose of the present paper is to study the existence and concentration of ground state for system (1.1) where $p > 6$ and the external potential $V(x)$ satisfies condition (V). We can obtain the following result.

Theorem 1.1. *Let $p > 6$ and $V(x)$ satisfies condition* (V)*. Then for all $\varepsilon > 0$ small,*

(i) *System* (1.1) *has at least one least energy solution $u_\varepsilon \in H^1(\mathbb{R}^2)$.*
(ii) *There is a maximum point ξ_ε of u_ε such that as $\varepsilon \to 0$, $u_\varepsilon(\varepsilon x + \varepsilon \xi_\varepsilon)$ converges to a least energy solution of the limit problem in the form of* (1.3) *with*

$$\omega = V(\xi_0) = \inf_{\xi \in \mathbb{R}^2} V(\xi).$$

For this, we employ the variational method joined with Nehari manifolds and concentration compactness principle [12] to the corresponding energy functional. The difficulty arises in the non-local term A_α, $\alpha = 0, 1, 2$ depend on u and a lack of compactness in $\mathbb{R}^2$. For the concentration of semiclassical limits, we establish the regularity of weak solutions and the exponential decay of solutions at infinity.

The paper is organized as follows. In Sect. 2 we introduce the workframe and prove some technical lemmas. Especially, we show some important propositions of A_α, $\alpha = 0, 1, 2$. In Sect. 3 we prove the existence of ground states in Theorem 1.1 and the concentration of solutions in Theorem 1.1.

2. Preliminary

In this section, we discuss the variational framework for the future study. At end of section, we show the regularity results and exponential decay of weak solutions.

Let E^a denote the usual Sobolev space $H^1(\mathbb{R}^2)$ with

$$\|u\|_{E^a} = \left(\int_{\mathbb{R}^2} |\nabla u|^2 + a|u|^2\, dx\right)^{1/2},$$

where $a > 0$. By using $\partial_1 A_1 + \partial_2 A_2 = 0$, we observe that

$$\begin{aligned} 0 &= \partial_2\partial_1 A_0 - \partial_1\partial_2 A_0 = \partial_2(A_2 u^2) + \partial_1(A_1 u^2) \\ &= 2u(A_1\partial_1 u + A_2\partial_2 u) + u^2(\partial_1 A_1 + \partial_2 A_2). \end{aligned}$$

This implies that $\sum_{j=1}^2 A_j\partial_j u = 0$. Let us denote $A_\alpha(u(x)) = A_\alpha$ for $\alpha = 0, 1, 2$. Define the functional

$$J_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^2} \left(\varepsilon^2|\nabla u|^2 + V(x)|u|^2 + A_1^2|u|^2 + A_2^2|u|^2\right) dx - \frac{1}{p}\int_{\mathbb{R}^2} |u|^p\, dx. \tag{2.1}$$

Solutions of (1.1) can be obtained as critical points of J_ε. Also, if u is a solution of the following system

$$\begin{cases} -\Delta u + V(\varepsilon x)u + A_0 u + \sum_{j=1}^2 A_j^2 u = |u|^{p-2}u, \\ \partial_1 A_0 = A_2|u|^2, \quad \partial_2 A_0 = -A_1|u|^2, \\ \partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2}u^2, \quad \partial_1 A_1 + \partial_2 A_2 = 0, \end{cases} \tag{2.2}$$

by scaling $x \mapsto \varepsilon^{-1}x$ in $\mathbb{R}^2$, we have that $u(\varepsilon^{-1}x)$ is a solution for the system (1.1). Let E_ε to be the Hilbert subspace of $H^1(\mathbb{R}^2)$ under the norm

$$\|u\|_{E_\varepsilon} = \left(\int_{\mathbb{R}^2} |\nabla u|^2 + V(\varepsilon x)|u|^2\, dx\right)^{1/2} < +\infty.$$

We define the energy functional associated with (2.2),

$$\hat{J}_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^2} \left(|\nabla u|^2 + V(\varepsilon x)|u|^2 + A_1^2|u|^2 + A_2^2|u|^2\right) dx - \frac{1}{p}\int_{\mathbb{R}^2} |u|^p\, dx. \tag{2.3}$$

We have the derivative of $\hat{J}_\varepsilon$ in E_ε as follow:

$$\begin{aligned} &\langle \hat{J}_\varepsilon'(u), \eta\rangle \\ &\quad = \int_{\mathbb{R}^2} \left(\nabla u\nabla\eta + V(\varepsilon x)u\eta + (A_1^2 + A_2^2)u\eta + A_0 u\eta - |u|^{p-2}u\eta\right) dx, \end{aligned} \tag{2.4}$$

for all $\eta \in C_0^\infty(\mathbb{R}^2)$. Since

$$\begin{aligned}\int_{\mathbb{R}^2} A_0 u^2\,dx &= -2\int_{\mathbb{R}^2} A_0(\partial_1 A_2 - \partial_2 A_1)\,dx\\ &= 2\int_{\mathbb{R}^2}(A_2\partial_1 A_0 - A_1\partial_2 A_0)\,dx\\ &= 2\int_{\mathbb{R}^2}(A_1^2 + A_2^2)u^2\,dx,\end{aligned}$$

we obtain

$$\langle \hat{J}'_\varepsilon(u),\, u\rangle = \int_{\mathbb{R}^2}\Big(|\nabla u|^2 + V(\varepsilon x)|u|^2 + 3\big(A_1^2 + A_2^2\big)|u|^2 - |u|^p\Big)\,dx. \tag{2.5}$$

Let us consider the system

$$\begin{cases} -\Delta u + au + A_0 u + \sum_{j=1}^2 A_j^2 u = |u|^{p-2}u,\\ \partial_1 A_0 = A_2|u|^2, \quad \partial_2 A_0 = -A_1|u|^2,\\ \partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2}u^2, \quad \partial_1 A_1 + \partial_2 A_2 = 0 \end{cases} \tag{2.6}$$

to compare its energy with the one of (1.1). Define the functional

$$J_a(u) = \frac{1}{2}\int_{\mathbb{R}^2}\Big(|\nabla u|^2 + a|u|^2 + A_1^2|u|^2 + A_2^2|u|^2\Big)\,dx - \frac{1}{p}\int_{\mathbb{R}^2}|u|^p\,dx. \tag{2.7}$$

Let $V_\infty = \liminf_{|x|\to\infty} V(x)$. We will see that the system in the case $a = V_\infty$ play the role of the limit problem to (1.1).

The components A_j of the gauge field can be represented by solving the elliptic equations

$$\Delta A_1 = \partial_2\left(\frac{|u|^2}{2}\right), \quad \Delta A_2 = -\partial_1\left(\frac{|u|^2}{2}\right),$$

which provide

$$A_1 = A_1(u) = K_2 * \left(\frac{|u|^2}{2}\right) = -\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{x_2 - y_2}{|x-y|^2}\frac{|u|^2(y)}{2}\,dy, \tag{2.8}$$

$$A_2 = A_2(u) = -K_1 * \left(\frac{|u|^2}{2}\right) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{x_1 - y_1}{|x-y|^2}\frac{|u|^2(y)}{2}\,dy, \tag{2.9}$$

where $K_j = \frac{-x_j}{2\pi|x|^2}$, for $j = 1, 2$ and $*$ denotes the convolution. The identity $\Delta A_0 = \partial_1(A_2|u|^2) - \partial_2(A_1|u|^2)$, gives the following representation of the component A_0:

$$A_0 = A_0(u) = K_1 * (A_1|u|^2) - K_2 * (A_2|u|^2). \tag{2.10}$$

We know that $\hat{J}_\varepsilon$ is well defined in E_ε, $\hat{J}_\varepsilon \in C^1(E_\varepsilon)$, and the weak solution of (2.2) is the critical point of the functional $\hat{J}_\varepsilon$ from the following properties, which one can find the proofs in [19]. For the reader's convenience, we sketch the formal estimates.

Proposition 2.1. *Let* $1 < s < 2$ *and* $\frac{1}{s} - \frac{1}{q} = \frac{1}{2}$.

(i) *Then there is a constant C depending only on s and q such that*

$$\left(\int_{\mathbb{R}^2} \left|Tu(x)\right|^q dx\right)^{\frac{1}{q}} \leq C\left(\int_{\mathbb{R}^2} |u(x)|^s dx\right)^{\frac{1}{s}},$$

where the integral operator T is given by

$$Tu(x) := \int_{\mathbb{R}^2} \frac{u(y)}{|x-y|}\, dy.$$

(ii) *If $u \in H^1(\mathbb{R}^2)$, then we have that for $j = 1, 2$,*

$$\|A_j^2(u)\|_{L^q(\mathbb{R}^2)} \leq C\|u\|^2_{L^{2s}(\mathbb{R}^2)}$$

and

$$\|A_0(u)\|_{L^q(\mathbb{R}^2)} \leq C\|u\|^2_{L^{2s}(\mathbb{R}^2)}\|u\|^2_{L^4(\mathbb{R}^2)}.$$

(iii) *For $q' = \frac{q}{q-1}$, $j = 1, 2$*

$$\|A_j(u)u\|_{L^2(\mathbb{R}^2)} \leq \|A_j(u)\|_{L^{2q}(\mathbb{R}^2)}\|u\|_{L^{2q'}(\mathbb{R}^2)}.$$

Proof. (i) This is the Hardy-Lilltewood-Sobolev inequality.

(ii) Applying (i) to the gauge potential A_μ, $\mu = 0, 1, 2$, we have the results, see also [6].

(iii) The statement comes from the Hölder inequaity. That is,

$$\int_{\mathbb{R}^2} |A_j(u)|^2|u|^2\, dx \leq \left(\int_{\mathbb{R}^2} |A_j(u)|^{2q}\, dx\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^2} |u|^{\frac{2q}{q-1}}\, dx\right)^{\frac{q-1}{q}}.$$

□

We will need the following properties of the convergence for A_j.

Proposition 2.2. *Suppose that u_n converges to u a.e. in $\mathbb{R}^2$ and u_n converges weakly to u in $H^1(\mathbb{R}^2)$. Let $A_{\alpha,n} := A_\alpha(u_n(x))$, $\alpha = 0, 1, 2$. Then*

(i) *$A_{j,n}$ converges to $A_j(u(x))$ a.e. in $\mathbb{R}^2$.*

(ii) *$\int_{\mathbb{R}^2} A_{i,n}^2 u_n u\, dx$, $\int_{\mathbb{R}^2} A_{i,n}^2 |u|^2\, dx$, and $\int_{\mathbb{R}^2} A_{i,n}^2 |u_n|^2\, dx$ converge to $\int_{\mathbb{R}^2} A_i^2|u|^2$ dx, for $i = 1, 2$; $\int_{\mathbb{R}^2} A_{0,n} u_n u\, dx$ and $\int_{\mathbb{R}^2} A_{0,n}|u_n|^2\, dx$ converge to $\int_{\mathbb{R}^2} A_0|u|^2$ dx.*

(iii) *$\int_{\mathbb{R}^2} |A_i(u_n - u)|^2|u_n - u|^2\, dx = \int_{\mathbb{R}^2} |A_i(u_n)|^2|u_n|^2\, dx - \int_{\mathbb{R}^2} |A_i(u)|^2|u|^2\, dx + o_n(1)$, for $i = 1, 2$.*

Proof. The proof can be found in [19], which follows from the idea of Brezis-Lieb lemma, we sketch it here.

(i) We see that for $i = 1, 2$

$$|A_{i,n} - A_1| \leq |T\left(u_n^2 - u^2\right)| \leq \left\|u_n^2 - u^2\right\|_{L^4(B_R(x))}\left\|\frac{1}{x-y}\right\|_{L^{4/3}(B_R(x))} + \left\|u_n^2 - u^2\right\|_{L^{\frac{4}{3}}(B_R^c(x))}\left\|\frac{1}{x-y}\right\|_{L^4(B_R^c(x))},$$

where $T(u_n^2 - u^2) = \int_{\mathbb{R}^2} \frac{u_n^2(y) - u^2(y)}{|x-y|}\, dy$. Taking $n \to \infty$ and $R \to \infty$, we obtain that $A_{i,n}(x) \xrightarrow{n} A_i(x)$ and that $A_i^2(u_n(x))u_n(x) \xrightarrow{n} A_i^2(u(x))u(x)$, a.e. in $\mathbb{R}^2$.

(ii) By using the Hölder inequality we have that for $i = 1, 2$ and $q' = \frac{q}{q-1}$,

$$\left| \int_{\mathbb{R}^2} A_{i,n}^2 u_n(x) u(x)\, dx \right| \le \|A_i^2(u_n)\|_{L^q(\mathbb{R}^2)} \|u_n\|_{L^{2q'}(\mathbb{R}^2)} \|u\|_{L^{2q'}(\mathbb{R}^2)},$$

$$\left| \int_{\mathbb{R}^2} A_{i,n}^2 u^2(x)\, dx \right| \le \|A_i^2(u_n)\|_{L^q(\mathbb{R}^2)} \|u\|_{L^{2q'}(\mathbb{R}^2)}^2.$$

Thus, $\{A_{i,n}^2 u_n\}$, $\{A_{i,n}^2\}$ are bounded. The weak convergence implies that

$$\int_{\mathbb{R}^2} A_{i,n}^2 u^2\, dx, \int_{\mathbb{R}^2} A_{i,n}^2 u_n u\, dx \to \int_{\mathbb{R}^2} A_i^2 u^2\, dx.$$

Hence,

$$\begin{aligned} &\left| \int_{\mathbb{R}^2} A_{i,n}^2 |u_n|^2\, dx - \int_{\mathbb{R}^2} A_i^2 |u|^2\, dx \right| \\ &\quad \le \int_{\mathbb{R}^2} \left| \left(A_{i,n}^2 - A_i^2\right) |u_n|^2 \right| dx + \int_{\mathbb{R}^2} \left| A_i^2 \left(|u_n|^2 - |u|^2\right) \right| dx \\ &\quad \le \left(\int_{\mathbb{R}^2} \left(A_{i,n}^2 - A_i^2\right)^3 dx \right)^{\frac{1}{3}} \left(\int_{\mathbb{R}^2} |u_n|^3\, dx \right)^{\frac{2}{3}} \\ &\qquad + \left(\int_{\mathbb{R}^2} \left(|u_n|^2 - |u|^2\right)^{\frac{3}{2}} dx \right)^{\frac{2}{3}} \left(\int_{\mathbb{R}^2} A_i^6\, dx \right)^{\frac{1}{3}}. \end{aligned}$$

Since u_n converges to u a.e. in $\mathbb{R}^2$, (i), and Proposition 2.1, we have

$$\int_{\mathbb{R}^2} A_{i,n}^2 u_n^2\, dx \to \int_{\mathbb{R}^2} A_i^2 u^2\, dx.$$

Similarly, we can obtain $\int_{\mathbb{R}^2} A_{0,n} u_n u\, dx$ and $\int_{\mathbb{R}^2} A_{0,n} |u_n|^2\, dx$ converge to$\int_{\mathbb{R}^2} A_0 |u|^2\, dx$.

(iii) By using the Fatou lemma, we obtain that

$$\int_{\mathbb{R}^2} A_i^2 u^2\, dx \le \int_{\mathbb{R}^2} A_{i,n}^2 u_n^2\, dx.$$

Moreover, there exist small $\delta > 0$ and $C_1 > 0$ such that

$$\begin{aligned} h_\delta &:= \left[\left| A_{i,n}^2 u_n^2 - |A_{i,n} u_n - A_i u|^2 - A_i^2 u^2 \right| - \delta |A_{i,n} u_n - A_i u|^2 \right]_+ \\ &\le C_1 A_i^2 u^2. \end{aligned}$$

By using the Lebesgue Dominated Convergence Theorem, $\int_{\mathbb{R}^2} h_\delta \overset{n}{\to} 0$, we know that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^2} \left| A_{i,n}^2 u_n^2 - |A_{i,n} u_n - A_i u|^2 - A_i^2 u^2 \right| dx \le \delta C_2,$$

where $C_2 := \sup \int_{\mathbb{R}^2} |A_{i,n} u_n - A_i u|^2\, dx < \infty$. The desired result follows from $\delta \to 0$. $\square$

Let us define the Nehari manifold related to the functionals above and discuss the property of the least energy of the critical points. Let

$$\hat{\Sigma}_\varepsilon = \{w \in E_\varepsilon \backslash \{0\} : \langle \hat{J}'_\varepsilon(w), w\rangle = 0\},$$
$$\Sigma_a = \{w \in H^1(\mathbb{R}^2)\backslash\{0\} : \langle J'_a(w), w\rangle = 0\}.$$

Lemma 2.3. *Assume $p \geq 6$, then $\hat{\Sigma}_\varepsilon$ and Σ_a are smooth manifolds, where $a > 0$.*

Proof. Here we just give the proof of $\hat{\Sigma}_\varepsilon$, others are similar. Let

$$g(u) = \langle \hat{J}'_\varepsilon(u), u\rangle, \qquad u \in \hat{\Sigma}_\varepsilon.$$

Then

$$\langle g'(u), u\rangle = 2\int_{\mathbb{R}^2} \left(|\nabla u|^2 + V(\varepsilon x)u^2 + 9A_1^2u^2 + 9A_2^2u^2\right) dx - p\int_{\mathbb{R}^2} |u|^p\, dx.$$

Since $u \in \hat{\Sigma}_\varepsilon$, we have

$$\int_{\mathbb{R}^2} \left(|\nabla u|^2 + V(\varepsilon x)u^2 + 3A_1^2u^2 + 3A_2^2u^2\right) dx = \int_{\mathbb{R}^2} |u|^p\, dx.$$

Hence, if $p \geq 6$ we obtain

$$\langle g'(u), u\rangle = 2\int_{\mathbb{R}^2} \left(|\nabla u|^2 + V(\varepsilon x)u^2 + 9A_1^2u^2 + 9A_2^2u^2\right) dx - p\int_{\mathbb{R}^2} |u|^p\, dx < 0.$$

By the Implicit Function Theorem, $\hat{\Sigma}_\varepsilon$ is a smooth manifolds. $\square$

Now we can define critical values for the functionals on the corresponding manifolds. Define

$$c_a = \inf_{w\in\Sigma_a} J_a(w),\ c_a^* = \inf_{\gamma\in\Gamma_a}\max_{t\in[0,1]} J_a(\gamma(t)),\ c_a^{**} = \inf_{w\in H^1(\mathbb{R}^2)\backslash\{0\}}\max_{t\,\geq\,0} J_a(tw),$$

where $\Gamma_a := \{\gamma \in C([0,1], H^1(\mathbb{R}^2)) : \gamma(0) = 0, J_a(\gamma(1)) < 0\}$ and $a \in \{\varepsilon, \xi, \infty\}$. Similarly, we can define $\hat{c}_\varepsilon$, $\hat{c}^*_\varepsilon$, $\hat{c}^{**}_\varepsilon$ on $\hat{J}_\varepsilon$.

Lemma 2.4.

$$c_a = c_a^* = c_a^{**}, \quad \hat{c}_\varepsilon = \hat{c}^*_\varepsilon = \hat{c}^{**}_\varepsilon.$$

Proof. For convenience we drop the notation ε. Here, we only show the proof $\hat{c} = \hat{c}^* = \hat{c}^{**}$. The others are similar. First, we prove $\hat{c} = \hat{c}^{**}$. In fact, this will follow if we can prove that for any $u \in E_\varepsilon\backslash\{0\}$, the ray $R_t = \{tu : \ t \geq 0\}$ intersects the solution manifold $\hat{\Sigma}_\varepsilon$ once and only once at θu ($\theta > 0$) where $\hat{J}_\varepsilon(\theta u),\ \theta \geq 0$, achieves its maximum.

$$\langle \hat{J}'_\varepsilon(tu),\, tu\rangle = t^2\left(\int_{\mathbb{R}^2} \left(|\nabla u|^2 + V(\varepsilon x)u^2\right) dx \right.$$
$$\left. +3t^4\int_{\mathbb{R}^2} \left(A_1^2u^2 + A_2^2u^2\right) dx - t^{p-2}\int_{\mathbb{R}^2} |u|^p\, dx\right).$$

Let

$$h(t) = b_1 + t^4 b_2 - t^{p-2} b_3, \quad t \in [0,\, +\infty),$$

where

$$b_1 = \int_{\mathbb{R}^2} \left(|\nabla u|^2 + V(\varepsilon x)u^2\right) dx,\ b_2 = 3\int_{\mathbb{R}^2} \left(A_1^2 u^2 + A_2^2 u^2\right) dx,\ b_3 = \int_{\mathbb{R}^2} |u|^p\, dx.$$

We claim that there exists $t_0 \in (0, +\infty)$ such that $h(t_0) = 0$. Indeed, by simple computation, we have that

$$\begin{cases} h'' > 0, & t < t_1 := \left(\frac{12b_2}{(p-2)(p-3)b_3}\right)^{\frac{1}{p-6}}, \\ h'' < 0, & t > t_1 := \left(\frac{12b_2}{(p-2)(p-3)b_3}\right)^{\frac{1}{p-6}}. \end{cases}$$

Also, there exist $t_2 = 0$, $t_3 = \left(\frac{4b_2}{(p-2)b_3}\right)^{\frac{1}{p-6}}$ satisfying $t_2 < t_1 < t_3$, such that $h'(t) = 0$ and $h(t)$ is strictly decreasing for $t \geq t_3$ as well as strictly increasing for $t \leq t_3$. Since $h(t_2) = b_1 > 0$ and $h(t) \to -\infty$ as $t \to +\infty$, there exists an unique $t_0 > t_3$ such that $h(t_0) = 0$. Hence, the ray R_t intersects $\hat{\Sigma}_\varepsilon$ only once. We have shown that $\hat{c} = \hat{c}^{**}$.

Next, we prove $\hat{c}^* = \hat{c}^{**}$. It is clear that $\hat{c}^{**} \geq c^*$. Let us show $\hat{c}^{**} \leq \hat{c}^*$. Then, we can write

$$\hat{c}^{**} = \inf_{u \in K} \hat{J}_\varepsilon(u)$$

with

$$K = \{\bar{u} = \bar{t}u :\ u \in E_\varepsilon, u \neq 0, \bar{t} < \infty\}.$$

Let $\gamma \in \Gamma$ be a path. If for all $\gamma \in \Gamma$, $\gamma \cap K \neq \emptyset$, then the inequality is proved. If there exists $\gamma \in \Gamma$ such that $\gamma(t) \notin K$ for all $t \in [0, 1]$, then we have

$$\int_{\mathbb{R}^2} \left(|\nabla\gamma|^2 + V(\varepsilon x)\gamma^2 + 3A_1^2(\gamma)\gamma^2 + 3A_2^2(\gamma)\gamma^2\right) dx > \int_{\mathbb{R}^2} |\gamma|^p\, dx.$$

and if $p > 6$

$$\begin{aligned} \hat{J}_\varepsilon(\gamma) &= \frac{1}{2}\int_{\mathbb{R}^2} \left(|\nabla\gamma|^2 + V(\varepsilon x)\gamma^2 + A_1^2(\gamma_1)\gamma^2 + A_2^2(\gamma_2)\gamma^2\right) dx - \frac{1}{p}\int_{\mathbb{R}^2} |\gamma|^p\, dx \\ &> \frac{1}{2}\int_{\mathbb{R}^2} \left(|\nabla\gamma|^2 + V(\varepsilon x)\gamma^2 + A_1^2(\gamma_1)\gamma^2 + A_2^2(\gamma_2)\gamma^2\right) dx \\ &\quad - \frac{1}{p}\int_{\mathbb{R}^2} \left(|\nabla\gamma|^2 + V(\varepsilon x)\gamma^2 + 3A_1^2(\gamma)\gamma^2 + 3A_2^2(\gamma)\gamma^2\right) dx \\ &> 0, \end{aligned}$$

which contradicts the Mountain Pass characterization of $\hat{c}^*$. Consequently,

$$\hat{c}^* = \hat{c}^{**}.$$

□

Next, we will discuss the properties of the energy functionals depend on different parameters.

Lemma 2.5. *Suppose that $V_a(x)$ and $V_b(x)$ satisfy condition* (V). *If*

$$V_a(x) \le V_b(x), \tag{2.11}$$

then $c_{V_a} \le c_{V_b}$. Moreover, if the inequality in (2.11) *is strict and V_a and V_b are constants, then $c_{V_a} < c_{V_b}$.*

Proof. Let c_{V_a} be the corresponding critical value of the energy functional J_a. Define other related notation in the obvious way. Notice that $E^b \subset E^a$ and for any $u \in E^b$, $J_a(u) \le J_b(u)$. By Lemma 2.4,

$$c_{V_b} = \inf_{u \in E^b \setminus \{0\}} \max_{t \ge 0} J_b(tu) \ge \inf_{u \in E^a \setminus \{0\}} \max_{t \ge 0} J_a(tu) = c_{V_a}.$$

Next we prove the second assertion. Since V_a and V_b are constants, we get that $E^b = E^a = H^1(\mathbb{R}^2)$. Moreover, by [19], by there exists a ground state $u_b \in H^1(\mathbb{R}^2)$ such that $c(V_b) = J_b(u_b)$. Then, by Lemma 2.4, we have

$$c_{V_b} = J_b(u_b) = \max_{t \ge 0} J_b(tu_b) > \max_{t \ge 0} J_a(tu_b) \ge \inf_{u \in H^1(\mathbb{R}^2) \setminus \{0\}} \max_{t \ge 0} J_a(tu) = c_{V_a}.$$

□

Lemma 2.6. $\hat{c}_\varepsilon \ge c_{V_0}$. *Moreover,* $\limsup_{\varepsilon \to 0^+} \hat{c}_\varepsilon \le c_{V_0}$.

Proof. By Lemma 2.5, we have $\hat{c}_\varepsilon \ge c_{V_0}$. On the other hand, suppose $\bar{u}$ is a solution of the least energy of the following problem

$$\begin{cases} -\Delta u + V(\xi_0)u + A_0 u + \sum_{j=1}^{2} A_j^2 u = |u|^{p-2}u, \\ \partial_1 A_0 = A_2|u|^2, \quad \partial_2 A_0 = -A_1|u|^2, \\ \partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2}u^2, \quad \partial_1 A_1 + \partial_2 A_2 = 0. \end{cases}$$

That is, $J_{V(\xi_0)}(\bar{u}) = c_{V(\xi_0)}$ and $J'_{V(\xi_0)}(\bar{u}) = 0$. For any $R > 0$, take a cut-off function $\psi_R \in C_0^\infty(\mathbb{R}^2)$ such that $\psi_R \equiv 1$ in $B_R(0)$, $\psi_R \equiv 0$ in $B_{2R}^c(0)$, and $0 \le \psi_R \le 1$, $|\nabla \psi_R| \le c/R$. Let $u_R = \psi_R \bar{u}$, $u_\varepsilon(x) = u_R(x - \frac{\xi_0}{\varepsilon})$, and $t_\varepsilon > 0$ such that $\hat{c}_\varepsilon \le \hat{J}_\varepsilon(t_\varepsilon u_\varepsilon) = \max_{t \ge 0} \hat{J}_\varepsilon(tu_\varepsilon)$. We claim that $t_\varepsilon \to 1$ as $\varepsilon \to 0$. In fact, by the definition of t_ε, we have

$$t_\varepsilon^{2-p} \int_{\mathbb{R}^2} \left(|\nabla u_\varepsilon|^2 + V(\varepsilon x)u_\varepsilon^2\right) dx + 3t_\varepsilon^{6-p} \int_{\mathbb{R}^2} \left(A_1^2(u_\varepsilon)u_\varepsilon^2 + A_2^2(u_\varepsilon)u_\varepsilon^2\right) dx$$
$$= \int_{\mathbb{R}^2} |u_\varepsilon|^p \, dx.$$

Changing variable to $x - \frac{\xi_0}{\varepsilon}$, we have

$$t_\varepsilon^{2-p} \int_{\mathbb{R}^2} \left(|\nabla u_R|^2 + V(\varepsilon x + \xi_0)u_R^2\right) dx + 3t_\varepsilon^{6-p} \int_{\mathbb{R}^2} \left(A_1^2(u_R)u_R^2 + A_2^2(u_R)u_R^2\right) dx$$
$$= \int_{\mathbb{R}^2} |u_R|^p \, dx. \tag{2.12}$$

Since $J'_{V(\xi_0)}(u_R) = 0$, for R large enough, we have

$$\int_{\mathbb{R}^2} \left(|\nabla u_R|^2 + V(\xi_0)u_R^2\right) dx + 3\int_{\mathbb{R}^2} \left(A_1^2(u_R)u_R^2 + A_2^2(u_R)u_R^2\right) dx$$
$$= \int_{\mathbb{R}^2} |u_R|^p \, dx + o_R(1). \tag{2.13}$$

Then,

$$\left|\int_{\mathbb{R}^2} \Big(V(\varepsilon x + \xi_0) - V(\xi_0)u_R^2\Big) dx\right| \le \int_{B_{2R}} |V(\varepsilon x + \xi_0) - V(\xi_0)|u_R^2 \, dx$$
$$+ \int_{B_{2R}^c} |V(\varepsilon x + \xi_0) - V(\xi_0)|u_R^2 \, dx$$
$$< c\delta.$$

Hence,

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^2} V(\varepsilon x + \xi_0)u_R^2 \, dx = \int_{\mathbb{R}^2} V(\xi_0)u_R^2 \, dx. \tag{2.14}$$

By (2.12), (2.13), (2.14), and Proposition 2.2, we obtain

$$(1 - t_\varepsilon^{2-p}) \int_{\mathbb{R}^2} \left(|\nabla u_R|^2 + V(\xi_0)u_R^2\right) dx$$
$$+ t_\varepsilon^{2-p} o_\varepsilon(1) + 3(1 - t_\varepsilon^{6-p}) \int_{\mathbb{R}^2} \left(A_1^2(u_R)u_R^2 + A_2^2(u_R)u_R^2\right) dx = o_R(1).$$

Letting $R \to +\infty$, we have

$$(1 - t_\varepsilon^{2-p}) \int_{\mathbb{R}^2} \left(|\nabla \bar{u}|^2 + V(\xi_0)\bar{u}^2\right) dx$$
$$+ t_\varepsilon^{2-p} o_\varepsilon(1) + 3(1 - t_\varepsilon^{6-p}) \int_{\mathbb{R}^2} \left(A_1^2(\bar{u})\bar{u}^2 + A_2^2(\bar{u})\bar{u}^2\right) dx = 0.$$

If $t_\varepsilon \to \infty$, then $\bar{u} = 0$. It is absurd. Consequently, $t_\varepsilon \to 1$ as $\varepsilon \to 0^+$. Hence, letting $R \to +\infty$ and then $\varepsilon \to 0+$, we have $\hat{J}_\varepsilon(t_\varepsilon u_\varepsilon) \to J_{V(\xi_0)}(\bar{u})$ as $\varepsilon \to 0^+$. It follows for all $\xi_0 \in \mathbb{R}^2$

$$\limsup_{\varepsilon\to 0^+} \hat{c}_\varepsilon \le c_{V(\xi_0)}, \tag{2.15}$$

Since ξ_0 is arbitrary, (2.15) implies $\limsup_{\varepsilon\to 0} \hat{c}_\varepsilon \le c_{V_0}$. □

Proposition 2.7. *Let u be weak solution of* (1.1). *Then*

(i) $\lim_{|x|\to+\infty} u(x) = 0$ *and* $\lim_{|x|\to+\infty} \nabla u(x) = 0$;

(ii) *u satisfies the following exponential decay at infinity, i.e., there exist positive constant R, C, and δ such that $|u(x)| \le Ce^{-\delta|x|}$.*

Proof. (i) We might as well consider the solution of (2.2). Define

$$u_\gamma = \begin{cases} u, & |u(x)| \le \gamma, \\ \gamma, & u(x) \ge \gamma, \\ -\gamma, & u(x) \le -\gamma. \end{cases} \tag{2.16}$$

Then, we have $|u_\gamma| \le |u|$, $|\nabla u_\gamma| \le |\nabla u|$, and $\nabla u_\gamma \cdot \nabla u \ge 0$. We know that for $\beta > 0$,

$$\int_{\mathbb{R}^2} A_0(u)|u_\gamma|^{2(\beta+1)}\,dx \le \|A_0(u)\|_{L^q(\mathbb{R}^2)} \|u_\gamma\|^{2(\beta+1)}_{L^{2q'(\beta+1)}(\mathbb{R}^2)} \le C\|u\|^2_{L^{2s}(\mathbb{R}^2)}\|u\|^2_{L^4(\mathbb{R}^2)}\|u_\gamma\|^{2(\beta+1)}_{L^{2q'(\beta+1)}(\mathbb{R}^2)},$$

where $\frac{1}{s} - \frac{1}{2} = \frac{1}{q}$, $s \in (1,2)$, $q' = \frac{q}{q-1}$. Multiplying (2.2) by $|u_\gamma|^{2\beta}u_\gamma$ then integrating by parts and together with the above inequality, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left(|\nabla u|^2|u_\gamma|^{2\beta} + V(\varepsilon x)u^2|u_\gamma|^{2\beta}\right) dx \\
&\quad \le -\int_{\mathbb{R}^2} A_0 u^2 |u_\gamma|^{2\beta}\,dx + \int_{\mathbb{R}^2} |u|^{p-2}u^2|u_\gamma|^{2\beta}\,dx \\
&\quad \le \int_{\mathbb{R}^2} |A_0 u^2 |u_\gamma|^{2\beta}|\,dx + \int_{\mathbb{R}^2} |u|^{p-2}u^2|u_\gamma|^{2\beta}\,dx. \qquad (2.17)
\end{aligned}$$

We choose $q = \frac{t'}{p-2}$, where $t' > 2(p-2)$. Then, $q' = \frac{q}{q-1} = \frac{t'}{t'-p+2}$. By (2.17), Sobolev inequalities, Proposition 2.1 and $1+\beta^2 \le (1+\beta)^2$ for $\beta \ge 0$, we have

$$\begin{aligned}
&\left(\int_{\mathbb{R}^2} \left|u|u_\gamma|^\beta\right|^{t'} dx\right)^{\frac{2}{t'}} \\
&\quad \le C\int_{\mathbb{R}^2} \left(|\nabla(u|u_\gamma|^\beta)|^2 + V(\varepsilon x)u^2|u_\gamma|^{2\beta}\right) dx \\
&\quad \le C\int_{\mathbb{R}^2} \left(|\nabla u|^2|u_\gamma|^{2\beta} + \beta^2u^2|\nabla u_\gamma|^2|u_\gamma|^{2(\beta-1)}\right) dx + \int_{\mathbb{R}^2} V(\varepsilon x)u^2|u_\gamma|^{2\beta}\,dx \\
&\quad \le C(1+\beta)^2 \left(\int_{\mathbb{R}^2} |\nabla u|^2|u_\gamma|^{2\beta}\,dx + \int_{\mathbb{R}^2} V(\varepsilon x)u^2|u_\gamma|^{2\beta}\,dx\right) \\
&\quad \le C(1+\beta)^2\left(\|u\|^2_{L^{2s}(\mathbb{R}^2)}\|u\|^2_{L^4(\mathbb{R}^2)} + \|u\|^{p-2}\right)\|u\|^{2(\beta+1)}_{L^{2q'(\beta+1)}(\mathbb{R}^2)}.
\end{aligned}$$

By the Fatou's Lemma in γ, we have

$$\|u\|_{L^{(\beta+1)t'}(\mathbb{R}^2)} \le \left(C(1+\beta)^2\left(\|u\|^2_{L^{2s}(\mathbb{R}^2)}\|u\|^2_{L^4(\mathbb{R}^2)} + \|u\|^{p-2}\right)\right)^{\frac{1}{2(\beta+1)}} \cdot \|u\|_{L^{2q'(\beta+1)}(\mathbb{R}^2)}$$

Using the Moser iteration, letting $\beta_0 = \beta+1$, $2q'\beta_{m+1} = t'\beta_m$ for $m = 0,1,2,\ldots$, and $m \to \infty$, we obtain that $u \in L^t(\mathbb{R}^2)$, for all $t \ge 2$. By the Calderon-Zygmund inequality, we conclude that $u \in W^{2,t}(B_2(x_0)), \forall x_0 \in \mathbb{R}^2$. Next, by the interior L^t-estimates we have

$$\|u\|_{W^{2,t}(B_1(x_0))} \le C\left(\|u\|_{L^t(B_2(x_0))} + \|u\|^{p-1}_{L^{t(p-1)}(B_2(x_0))}\right).$$

Then, by Sobolev inequalities, for some $\tau \in (0,1)$,

$$\|u\|_{C^{1,\tau}(\overline{B_1(x_0)})} \le C\left(\|u\|_{L^t(B_2(x_0))} + \|u\|^{p-1}_{L^{t(p-1)}(B_2(x_0))}\right).$$

Letting $|x_0| \to \infty$, we have $\|u\|_{C^{1,\tau}(B_1(x_0))} \to 0$, which gives (i).

(ii) Define $\tilde{u} = Me^{-\theta(|x|-L)}$, where $M = \max\{|u(x)| : |x| = L\}$ for fix $\theta > 0$ satisfying $V_0 > \theta^2$. Then $\Delta\tilde{u} = (\theta^2 - \frac{\theta}{|x|})\tilde{u}$. Let us consider the difference

$$\phi_R = \begin{cases} 0, & x \in B_R^o, \\ b_1 u - \tilde{u}, & x \in \mathbb{R}^2 \backslash B_R^o. \end{cases}$$

with $b_1 > 0$. By (2.4), choosing $\eta = \phi_R$, we have

$$\begin{aligned} &\int_{\mathbb{R}^2} \left(|\nabla\phi_R|^2 + V(\varepsilon x)|\phi_R|^2\right) dx \\ &\quad \le \int_{\mathbb{R}^2} \left((\theta^2 - \frac{\theta}{|x|}) - V_0\right)\tilde{u}\phi_R \, dx + \int_{\mathbb{R}^2} b_1|u|^{p-2}u\phi_R \, dx + o_R(1). \end{aligned}$$

We choose $R > 0$ such that $|u|^{p-2} \le V_0 - \theta^2$ for $|x| > R$. Then,

$$\begin{aligned} \int_{|x|>R} V_0\phi_R^2 \, dx &\le \int_{|x|>R} \left(|\nabla\phi_R|^2 + V(\varepsilon x)|\phi_R|^2\right) dx \\ &\le \int_{|x|>R} (b_1 u - \tilde{u})(V_0 - \theta^2)\phi_R \, dx + o_R(1) \\ &= (V_0 - \theta^2)\int_{|x|>R} \phi_R^2 \, dx + o_R(1). \end{aligned}$$

This implies $\phi_R \equiv 0$ and gives the desired exponential decay.

3. Proof of Theorem 1.1

We demonstrate Theorem 1.1 in the section.

Part (i) We show the existence of ground states. By Lemma 2.4, there exists a sequence $\{\bar{u}_n\}$ be a minimizing sequence of $\hat{c}_\varepsilon$. Then, we can find a sequence $\{u_n\}$ such that $\{u_n\} \subset\subset \hat{\Sigma}_\varepsilon$, $\hat{J}_\varepsilon(u_n) \to \hat{c}_\varepsilon$, $\hat{J}'_\varepsilon(u_n) \to 0$, and $\|u_n - \bar{u}_n\|_{E_\varepsilon} \to 0$, as $n \to \infty$, which is a direct consequence of the Ekeland's Variational Principle. See [21].

Step 1. We show that $\{u_n\}$ is bounded in E_ε.

For n large enough, we have

$$\begin{aligned} \hat{c}_\varepsilon + 1 + \|u_n\| &\ge \hat{J}_\varepsilon(u_n) - \frac{1}{p}\langle \hat{J}'_\varepsilon(u_n), u_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\int_{\mathbb{R}^2} \left(|\nabla u_n|^2 + V(\varepsilon x)u_n^2\right) dx \\ &\quad + \left(\frac{1}{2} - \frac{3}{p}\right)\int_{\mathbb{R}^2} \left(A_{1,n}^2 u_n^2 + A_{2,n}^2 u_n^2\right) dx \\ &\ge \left(\frac{1}{2} - \frac{1}{p}\right)\int_{\mathbb{R}^2} \left(|\nabla u_n|^2 + V(\varepsilon x)u_n^2\right) dx \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|_{E_\varepsilon}^2. \end{aligned}$$

It follows that $\|u_n\|$ is bounded.

Then, there exist $u_0 \in E_\varepsilon$ and a subsequence of $\{u_n\}$, which still denoted by $\{u_n\}$, such that $u_n \rightharpoonup u_0$ weakly in E_ε as $n \to \infty$. Consequence, $u_n \to u_0$ strongly in $L^s_{loc}(\mathbb{R}^2)$, for $2 \le s < +\infty$ and almost everywhere in $\mathbb{R}^2$.

Step 2. We prove there exists $\eta > 0$ such that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} |u_n|^p \, dx > \eta. \tag{3.1}$$

Suppose by contradiction that (3.1) does not hold. Then,

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} |u_n|^p \, dx = 0. \tag{3.2}$$

Since $u_n \in \hat{\Sigma}_\varepsilon$, we have

$$\int_{\mathbb{R}^2} \left(|\nabla u_n|^2 + V(\varepsilon x)u_n^2\right) dx + 3\int_{\mathbb{R}^2} \left(A_{1,n}^2 u_n^2 + A_{2,n}^2 u_n^2\right) dx = \int_{\mathbb{R}^2} |u_n|^p \, dx,$$

where $A_{j,n} = A_j(u_n)$ for $j = 1, 2$. By (3.2) and the above equality, we have $\|u_n\|_{E_\varepsilon} \to 0$, as $n \to \infty$. Since $\{u_n\}$ is bounded, we have

$$\begin{aligned}
\hat{c}_\varepsilon &= \lim_{n\to\infty} \left(\hat{J}_\varepsilon(u_n) - \frac{1}{p}\langle \hat{J}'_\varepsilon(u_n), u_n\rangle\right) \\
&= \lim_{n\to\infty} \Bigg[\left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathbb{R}^2} \left(|\nabla u_n|^2 + V(\varepsilon x)u_n^2\right) dx \\
&\quad + \left(\frac{1}{2} - \frac{3}{p}\right) \int_{\mathbb{R}^2} \left(A_{1,n}^2 u_n^2 + A_{2,n}^2 u_n^2\right) dx\Bigg] \\
&= 0,
\end{aligned}$$

which contradicts Lemma 2.6.

Step 3. We show $u_0 \not\equiv 0$.

Otherwise,

$$u_n \to 0 \text{ strongly in } L^s_{loc}(\mathbb{R}^2), \text{ for } 2 \le s < +\infty. \tag{3.3}$$

By condition (V), we can choose $h > 0$ small enough such that

$$V_\infty - h > V_0. \tag{3.4}$$

By Lemma 2.5, we get

$$c_{V_\infty - h} > c_{V_0}. \tag{3.5}$$

Choose a constant $\rho > 0$ sufficiently large such that for $|x| > \rho$

$$V(x) > V_\infty - h. \tag{3.6}$$

From the proof of Lemma 2.4, there exists $\alpha_n > 0$ such that $\alpha_n u_n \in \Sigma_{V_\infty - h}$. We obtain that for some $b_1 > 0$, $b_2 > 0$ independent of n such that

$$\begin{aligned}
\alpha_n^p \int_{\mathbb{R}^2} |u_n|^p \, dx &= \alpha_n^2 \int_{\mathbb{R}^2} |\nabla u_n|^2 + (V_\infty - h)u_n^2 \, dx \\
&\quad + 3\alpha_n^6 \int_{\mathbb{R}^2} \left(A_{1,n}^2 u_n^2 + A_{2,n}^2 u_n^2\right) dx \\
&\le b_1 \alpha_n^2 + b_2 \alpha_n^6.
\end{aligned} \tag{3.7}$$

By (3.1) and (3.7), we obtain $\{\alpha_n\}$ is bounded. From (3.6), we have

$$\begin{aligned}
\hat{c}_\varepsilon &= \lim_{n\to\infty} \hat{J}_\varepsilon(u_n) = \lim_{n\to\infty} \max_{t\,\geq\,0} \hat{J}_\varepsilon(tu_n) \geq \limsup_{n\to\infty} \hat{J}_\varepsilon(\alpha_n u_n) \\
&= \limsup_{n\to\infty} \Big[\frac{\alpha_n^2}{2}\int_{\mathbb{R}^2} \Big(|\nabla u_n|^2 + V(\varepsilon x)|u_n|^2\Big)\, dx \\
&\quad + \frac{\alpha_n^6}{2}\int_{\mathbb{R}^2} \Big(A_{1,n}^2|u_n|^2 + A_{2,n}^2|u_n|^2\Big)\, dx - \frac{\alpha_n^p}{p}\int_{\mathbb{R}^2} |u_n|^p\, dx\Big] \\
&\geq \limsup_{n\to\infty} \Big[\frac{\alpha_n^2}{2}\int_{\mathbb{R}^2} \Big(|\nabla u_n|^2 + (V_\infty - h)|u_n|^2\Big)\, dx \\
&\quad + \frac{\alpha_n^6}{2}\int_{\mathbb{R}^2} \Big(A_{1,n}^2|u_n|^2 + A_{2,n}^2|u_n|^2\Big)\, dx - \frac{\alpha_n^p}{p}\int_{\mathbb{R}^2} |u_n|^p\, dx \\
&\quad + \frac{\alpha_n^2}{2}\int_{B_{\frac{\varrho}{\varepsilon}}} \Big(\big(V(\varepsilon x) - (V_\infty - h)\big)|u_n|^2\Big)\, dx\Big] \qquad (3.8)
\end{aligned}$$

By (3.3) and $\{\alpha_n\}$ is bounded, we obtain

$$\lim_{n\to\infty} \frac{\alpha_n^2}{2}\int_{B_{\frac{\varrho}{\varepsilon}}} \big(V(\varepsilon x) - (V_\infty - h)\big)|u_n|^2\, dx = 0. \qquad (3.9)$$

By (3.8), (3.9), and the boundedness of $\{\alpha_n\}$, we have $\hat{c}_\varepsilon \geq c_{V_\infty - h}$, which is impossible for small h according to (3.5) and Lemma 2.6.

Step 4. We prove $u_0 \in \hat{\Sigma}_\varepsilon$ and u_0 is a positive ground state of (2.2).

We observe that $u_n \rightharpoonup u_0$ in E_ε, $u_n \to u_0$ a.e. in $\mathbb{R}^2$ as $n \to \infty$. Proposition 2.2 gives $u_0 \in \hat{\Sigma}_\varepsilon$. By Fatou's Lemma, we obtain

$$\begin{aligned}
\hat{c}_\varepsilon &= \lim_{n\to\infty} \Big(\hat{J}_\varepsilon(u_n) - \frac{1}{p}\langle \hat{J}'_\varepsilon(u_n), u_n\rangle\Big) \\
&= \lim_{n\to\infty} \Big[\Big(\frac{1}{2} - \frac{1}{p}\Big)\int_{\mathbb{R}^2} \big(|\nabla u_n|^2 + V(\varepsilon x)u_n^2\big)\, dx \\
&\quad + \Big(\frac{1}{2} - \frac{3}{p}\Big)\int_{\mathbb{R}^2} \big(A_{1,n}^2 u_n^2 + A_{2,n}^2 u_n^2\big)\, dx\Big] \\
&\geq \Big(\frac{1}{2} - \frac{1}{p}\Big)\int_{\mathbb{R}^2} \big(|\nabla u_0|^2 + V(\varepsilon x)u_0^2\big)\, dx \\
&\quad + \Big(\frac{1}{2} - \frac{3}{p}\Big)\int_{\mathbb{R}^2} \big(A_1^2 u_0^2 + A_2^2 u_0^2\big)\, dx \\
&= \hat{J}_\varepsilon(u_0) \geq \hat{c}_\varepsilon.
\end{aligned}$$

This implies that $\hat{J}_\varepsilon(u_0) = \hat{c}_\varepsilon$ and hence $|u_0|$ is a positive ground state of (2.2). □

Part (ii) Suppose that $\varepsilon_k \to 0^+$ as $k \to \infty$. We shall show that there exists a sequence of points $\{\xi_k\}$ in $\mathbb{R}^2$ such that most of the mass of $v_k = v_{\varepsilon_k}$ is contained in a ball centered at ξ_k and $\{\varepsilon_k \xi_k\}$ is bounded. Then the limit ξ of $\{\varepsilon_k \xi_k\}$ verifies $c_{V(\xi)}$ is the least energy of the functional $J_{V(\xi)}$.

Let v_ε be a nonnengative ground state of (2.2), and $u_\varepsilon(x) = v_\varepsilon(\frac{x}{\varepsilon})$ be a ground state of (1.1).

Notice that for any v on the manifold $\hat{\Sigma}_\varepsilon$, we have

$$\hat{J}_\varepsilon(v) = \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathbb{R}^2} \left(|\nabla v|^2 + V(\varepsilon x)v^2\right) dx + \left(\frac{1}{2} - \frac{3}{p}\right) \int_{\mathbb{R}^2} \left(A_1^2 v^2 + A_2^2 v^2\right) dx.$$

Define a measure μ_ε by

$$\mu_\varepsilon(\Omega) = \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\Omega} \left(|\nabla v_\varepsilon|^2 + V(\varepsilon x)v_\varepsilon^2\right) dx + \left(\frac{1}{2} - \frac{3}{p}\right) \int_{\Omega} \left(A_1^2(v_\varepsilon) v_\varepsilon^2 + A_2^2(v_\varepsilon) v_\varepsilon^2\right) dx.$$

By using Lemma 2.6, up to a subsequence, we assume that as $\varepsilon_k \to 0^+$, $(k \to \infty)$,

$$\mu_k(\mathbb{R}^2) = \mu_{\varepsilon_k}(\mathbb{R}^2) = \hat{c}_{\varepsilon_k} \to c_{V_0}.$$

It follows that $\{v_\varepsilon\}$ is bounded in E_ε when ε small enough. By the Concentration Compactness Lemma in [12] and [16], there exists a subsequence of $\{\mu_k\}$, which we will always denote by $\{\mu_k\}$, satisfying one of the three following possibilities:

(1) *Compactness* There is a sequence $\{\xi_k\} \subset \mathbb{R}^2$ such that for any $\delta > 0$ there exists a radius $\rho > 0$ such that

$$\int_{B_\rho(\xi_k)} d\mu_k \geq c_{V_0} - \delta, \quad \text{for all } k. \tag{3.10}$$

(2) *Vanishing* There exists a sequence of $\{\varepsilon_k\}$ that tends to zero such that for all $\rho > 0$

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_\rho(y)} d\mu_k = 0.$$

(3) *Dichotomy* There exist a constant $\bar{c}$ with $0 < \bar{c} < c_{V_0}$, sequences $\{\rho_k\} \to \infty$, $\{\xi_k\} \subset \mathbb{R}^2$, and two nonnegative measures μ_k^1 and μ_k^2 satisfying the following:

$$\begin{gathered} 0 \leq \mu_k^1 + \mu_k^2 \leq \mu_k, \\ \sup(\mu_k^1) \subset B_{\rho_k}(\xi_k), \qquad \sup(\mu_k^2) \subset B^c_{2\rho_k}(\xi_k), \\ \mu_k^1(\mathbb{R}^2) \to \bar{c}, \quad \mu_k^2(\mathbb{R}^2) \to c_{V_0} - \bar{c}, \quad \text{as } k \to \infty. \end{gathered}$$

Proposition 3.1. *Neither vanishing* (2) *nor dichotomy* (3) *occurs.*

Proof. Claim 1. Vanishing (2) does not occur.

Otherwise, $\{v_k\}$ i.e. $\{v_{\varepsilon_k}\}$, is also vanishing. That is, there exists a subsequence of $\{v_k\}$, such that for all $\rho > 0$,

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_\rho(y)} \left(|\nabla v_k|^2 + V(\varepsilon_k x)v_k^2\right) dx = 0.$$

By the Lions' Lemma [12], $v_k \to 0$, in $L^s(\mathbb{R}^2)$, $s \geq 2$. By using

$$0 = \langle \hat{J}'_{\varepsilon_k}(v_k), v_k\rangle = \int_{\mathbb{R}^2} \left(|\nabla v_k|^2 + V(\varepsilon_k x)v_k^2 + 3A_{1,k}^2 v_k^2 + 3A_{2,k}^2 v_k^2 - |v_k|^p\right) dx$$

and $\int_{\mathbb{R}^2} |v_k|^p \, dx \to 0$ as $k \to \infty$, where $A_{1,k} := A_1(v_k) = A_1(v_{\varepsilon_k})$ and $A_{2,k} := A_2(v_k) = A_2(v_{\varepsilon_k})$, we obtain

$$\lim_{k\to\infty} \int_{\mathbb{R}^2} \left(|\nabla v_k|^2 + V(\varepsilon_k x)v_k^2 + 3A_{1,k}^2 v_k^2 + 3A_{2,k}^2 v_k^2\right) dx = 0.$$

Thus,

$$\begin{aligned} 0 &= \lim_{k\to\infty} \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathbb{R}^2} \left(|\nabla v_k|^2 + V(\varepsilon_k x)v_k^2\right) dx \\ &\quad + \left(\frac{1}{2} - \frac{3}{p}\right) \int_{\mathbb{R}^2} \left(A_{1,k}^2 v_k^2 + A_{2,k}^2 v_k^2\right) dx \\ &= \lim_{k\to\infty} \hat{c}_{\varepsilon_k} = c_{V_0} > 0. \end{aligned}$$

It is absurd. Thus, Claim 1 holds.

Claim 2. Dichotomy (3) does not occur.

Note that $\varepsilon_k \to 0$ as $k \to \infty$. let us define a cut-off function $\eta_k \in C_0^1(\mathbb{R}^2)$ such that $\eta_k \equiv 1$ in $B_{\rho_k}(\xi_k)$, $\eta_k \equiv 0$ in $B^c_{2\rho_k}(\xi_k)$, and $0 \leq \eta_k \leq 1$, $|\nabla \eta_k| \leq 2/\rho_k$, where $\xi_k \in \mathbb{R}^2$. Let $v_k = v_{\varepsilon_k} := v_{1,k} + v_{2,k}$, where

$$v_{1,k} := v_{1,\varepsilon_k} = \eta_k v_{\varepsilon_k}, \quad v_{2,k} := v_{2,\varepsilon_k} = (1-\eta_k)v_{\varepsilon_k}.$$

If the Dichotomy case happens, then, as $k \to \infty$,

$$\hat{J}_{\varepsilon_k}(v_{1,k}) \geq \mu_k(B_{\rho_k}(\xi_k)) \geq \mu_k^1(B_{\rho_k}(\xi_k)) = \mu_k^1(\mathbb{R}^2) \to \bar{c} \tag{3.11}$$

and

$$\hat{J}_{\varepsilon_k}(v_{2,k}) \geq \mu_k(B^c_{2\rho_k}(\xi_k)) \geq \mu_k^2(B^c_{2\rho_k}(\xi_k)) = \mu_k^2(\mathbb{R}^2) \to c_{V_0} - \bar{c}. \tag{3.12}$$

Set $\Omega_k := B_{2\rho_k}(\xi_k) \backslash B_{\rho_k}(\xi_k)$. Then, as $k \to \infty$

$$\begin{aligned} &\left(\frac{1}{2} - \frac{1}{p}\right) \int_{\Omega_k} \left(|\nabla v_k|^2 + V(\varepsilon_k x)v_k^2\right) dx + \left(\frac{1}{2} - \frac{3}{p}\right) \int_{\Omega_k} \left(A_{1,k}^2 v_k^2 + A_{2,k}^2 v_k^2\right) dx \\ &\quad = \mu_k(\Omega_k) = \mu_k(\mathbb{R}^2) - \mu_k(B_{\rho_k}(\xi_k)) - \mu_k(B^c_{2\rho_k}(\xi_k)) \\ &\quad \leq \mu_k(\mathbb{R}^2) - \mu_k^1(\mathbb{R}^2) - \mu_k^2(\mathbb{R}^2) \\ &\quad \to 0. \end{aligned} \tag{3.13}$$

Thus, by the Sobolev inequalities, we have $\int_{\Omega_k} |v_k|^p \, dx \to 0$ as $k \to \infty$. Consequently,

$$\int_{\mathbb{R}^2} |v_k|^p \, dx = \int_{\mathbb{R}^2} |v_{1,k}|^p \, dx + \int_{\mathbb{R}^2} |v_{2,k}|^p \, dx + o(1). \tag{3.14}$$

By (3.13), we obtain

$$\int_{\mathbb{R}^2} \left(|\nabla v_k|^2 + V(\varepsilon_k x)v_k^2\right) dx = \int_{\mathbb{R}^2} \left(|\nabla v_{1,k}|^2 + V(\varepsilon_k x)v_{1,k}^2\right) dx + \int_{\mathbb{R}^2} \left(|\nabla v_{2,k}|^2 + V(kx)v_{2,k}^2\right) dx + o(1). \tag{3.15}$$

We notice that $v_{2,k}$ converges to 0 a.e. in $\mathbb{R}^2$, and $A_j(v_{2,k}) \to 0$ a.e. in $\mathbb{R}^2$ for $j = 1, 2$, as $k \to \infty$. Since $\|(1-\eta_k)v_k\|$ is bounded and $\mathrm{supp}((1-\eta_k)v_k) \subset B^c_{\rho_k}$, then Proposition 2.1 gives for $j = 1$, 2

$$\begin{aligned} |A_j((1-\eta_k)v_k)| &\le C\|v_k^2\|_{L^{\frac{4}{3}}(B^c_{\rho_k}(x))} \left(\int_{B^c_{\rho_k}(x)} \frac{dy}{|x-y|^4}\, dy\right)^{\frac{1}{4}} \\ &\le C\frac{1}{\rho_k^{1/2}} \xrightarrow{k} 0. \end{aligned}$$

and

$$\begin{aligned} &\left|\int_{\mathbb{R}^2} K_j(x-y)(1-\eta_k)\eta_k|v_k(y)|^2\, dy\right| \\ &\quad \le \|v_k^2\|_{L^{\frac{4}{3}}(\Omega_k)} \left(\int_{\Omega_k} \frac{dy}{|x-y|^4}\, dy\right)^{\frac{1}{4}} \le C\frac{1}{\rho_k^{1/2}} \xrightarrow{k} 0. \end{aligned} \tag{3.16}$$

Since $\|v_k\| \le C$, for $j = 1$, 2

$$\lim_{k\to\infty} A_j(v_{2,k}) = 0, \tag{3.17}$$

$$\lim_{k\to\infty} \int_{\mathbb{R}^2} A_j(v_{1,k})A_j(v_{2,k})|v_{1,k}|^2\, dx = 0, \tag{3.18}$$

$$\lim_{k\to\infty} \int_{\mathbb{R}^2} |A_j(v_{2,k})|^2|v_{1,k}|^2\, dx = 0. \tag{3.19}$$

By (3.16)

$$\begin{aligned} A_{1,k} &= -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{x_2-y_2}{|x-y|^2}\frac{1}{2}|v_{1,k}+v_{2,k}|^2\, dy \\ &= A_1(v_{1,k}) + A_1(v_{2,k}) - \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{x_2-y_2}{|x-y|^2} v_{1,k}v_{2,k}\, dy \\ &= A_1(v_{1,k}) + A_1(v_{2,k}) + o(1), \end{aligned}$$

we have

$$\begin{aligned} \int_{\mathbb{R}^2} A_1^2(v_k)|v_k|^2\, dx &= \int_{\mathbb{R}^2} \left(A_1(v_{1,k}) + A_1(v_{2,k}) + o(1)\right)^2 |v_{1,k}+v_{2,k}|^2\, dx \\ &= \int_{\mathbb{R}^2} \big[A_1^2(v_{1,k})|v_{1,k}|^2 + A_1^2(v_{2,k})|v_{2,k}|^2 \\ &\quad + 2A_1(v_{1,k})A_1(v_{2,k})\left(|v_{1,k}|^2 + |v_{2,k}|^2\right) + A_1^2(v_{1,k})|v_{2,k}|^2 \\ &\quad + A_1^2(v_{2,k})|v_{1,k}|^2 + 2\left(A_1^2(v_{1,k}) + A_1^2(v_{2,k})\right)v_{1,k}v_{2,k} \\ &\quad + 4A_1(v_{1,k})A_1(v_{2,k})v_{1,k}v_{2,k}\big]\, dx + o(1). \end{aligned}$$

Hence, by using (3.17), (3.18), (3.19), and $v_{2,k}$ converges to zero a.e. in $\mathbb{R}^2$, we get

$$\int_{\mathbb{R}^2} A_1^2(v_k)|v_k|^2\,dx = \int_{\mathbb{R}^2} A_1^2(v_{1,k})|v_{1,k}|^2\,dx + \int_{\mathbb{R}^2} A_1^2(v_{2,k})|v_{2,k}|^2\,dx + o(1). \tag{3.20}$$

Similarly, we have

$$\int_{\mathbb{R}^2} A_2^2(v_k)|v_k|^2\,dx = \int_{\mathbb{R}^2} A_2^2(v_{1,k})|v_{1,k}|^2\,dx + \int_{\mathbb{R}^2} A_2^2(v_{2,k})|v_{2,k}|^2\,dx + o(1). \tag{3.21}$$

Then, by (3.14), (3.15), (3.20), and (3.21), we get

$$\begin{aligned} c_{V_0} &= \lim_{k\to 0^+} \hat{J}_{\varepsilon_k}(v_k) = \lim_{k\to 0^+} \big(\hat{J}_{\varepsilon_k}(v_{1,k}) + \hat{J}_{\varepsilon_k}(v_{2,k}) + o(1)\big) \\ &\geq \liminf_{k\to 0^+} \hat{J}_{\varepsilon_k}(v_{1,k}) + \liminf_{k\to 0^+} \hat{J}_{\varepsilon_k}(v_{2,k}) \\ &\geq \bar{c} + (c_{V_0} - \bar{c}) = c_{V_0}. \end{aligned}$$

Consequently,

$$\lim_{k\to 0^+} \hat{J}_{\varepsilon_k}(v_{1,k}) = \bar{c}, \ \lim_{k\to 0^+} \hat{J}_{\varepsilon_k}(v_{2,k}) = c_{V_0} - \bar{c}. \tag{3.22}$$

Define

$$\begin{aligned} I_k^1 &= \int_{\mathbb{R}^2} \big(|\nabla v_{1,k}|^2 + V(\varepsilon_k x) v_{1,k}^2\big)\,dx \\ &\quad + 3\int_{\mathbb{R}^2} \big(A_1^2(v_{1,k})v_{1,k}^2 + A_2^2(v_{1,k})v_{1,k}^2\big)\,dx - \int_{\mathbb{R}^2} |v_{1,k}|^p\,dx \end{aligned}$$

and

$$\begin{aligned} I_k^2 &= \int_{\mathbb{R}^2} \big(|\nabla v_{2,k}|^2 + V(\varepsilon_k x) v_{2,k}^2\big)\,dx \\ &\quad + 3\int_{\mathbb{R}^2} \big(A_1^2(v_{2,k})v_{2,k}^2 + A_2^2(v_{2,k})v_{2,k}^2\big)\,dx - \int_{\mathbb{R}^2} |v_{2,k}|^p\,dx. \end{aligned}$$

Since $v_{\varepsilon_k} \in \hat{\Sigma}_{\varepsilon_k}$, (3.14), (3.15), (3.20), and (3.21), we obtain

$$I_k^1 = -I_k^2 + o(1). \tag{3.23}$$

Next we show (3.23) is not true. By Lemma 2.4, $\exists \theta_1 > 0$, such that $\theta_1 v_{1,\varepsilon} \in \hat{\Sigma}_\varepsilon$, and then

$$\begin{aligned} &\theta_1^2 \int_{\mathbb{R}^2} \big(|\nabla v_{1,\varepsilon}|^2 + V(\varepsilon x) v_{1,\varepsilon}^2\big)\,dx + 3\theta_1^6 \int_{\mathbb{R}^2} \big[A_1^2(v_{1,\varepsilon})v_{1,\varepsilon}^2 + A_2^2(v_{1,\varepsilon})v_{1,\varepsilon}^2\big]\,dx \\ &\quad = \theta_1^p \int_{\mathbb{R}^2} |v_{1,\varepsilon}|^p\,dx. \end{aligned} \tag{3.24}$$

Case 1 Up to a subsequence, $I_k^1 \le 0$.

By (3.24), we have

$$\begin{aligned}
&\theta_1^{2-p}\int_{\mathbb{R}^2}\left(|\nabla v_{1,k}|^2+V(\varepsilon_k x)v_{1,k}^2\right)dx+3\theta_1^{6-p}\int_{\mathbb{R}^2}\left[A_1^2(v_{1,k})v_{1,k}^2+A_2^2(v_{1,k})v_{1,k}^2\right]dx\\
&\quad=\int_{\mathbb{R}^2}|v_{1,k}|^p\,dx\\
&\quad\ge\int_{\mathbb{R}^2}\left(|\nabla v_{1,k}|^2+V(\varepsilon_k x)v_{1,k}^2\right)dx+3\int_{\mathbb{R}^2}\left[A_1^2(v_{1,k})v_{1,k}^2+A_2^2(v_{1,k})v_{1,k}^2\right]dx.
\end{aligned}$$

Let $b_1=\int_{\mathbb{R}^2}\left(|\nabla v_{1,k}|^2+V(\varepsilon_k x)v_{1,k}^2\right)dx$ and $b_2=\int_{\mathbb{R}^2}\left[A_1^2(v_{1,k})v_{1,k}^2+A_2^2(v_{1,k})\right.$ $\left.v_{1,k}^2\right]dx$. Since $\lambda(t)=t^{2-p}b_1+t^{6-p}b_2$ is strictly decreasing on any interval where $\lambda(t)>0$. It yields that $\theta_1\le 1$. Hence, by (3.22), as $k\to 0^+$

$$\hat{c}_{\varepsilon_k}\le \hat{J}_{\varepsilon_k}(\theta_1 v_{1,k})\le \hat{J}_{\varepsilon_k}(v_{1,k})\to \bar{c}<c_{V_0},$$

which contradicts $\lim_{k\to\infty}\hat{c}_{\varepsilon_k}=c_{V_0}>\bar{c}$.

Case 2 Up to a subsequence, $I_k^2\le 0$.

We can repeat the arguments of previous case.

Case 3 Up to a subsequence, $I_k^1>0$ and $I_k^2>0$.

By (3.23), we obtain $I_k^1=o_n(1)$ and $I_k^2=o(1)$. If $\theta_1\le 1+o(1)$, we can can argue as in the Case 1. Assume that $\lim_{k\to 0^+}\theta_1=\theta_0>1$. We claim, up to a subsequence, $\lim_{k\to 0^+}(b_1+b_2)>0$. Otherwise, $\lim_{k\to 0^+}\int_{\mathbb{R}^2}\left(|\nabla v_{1,k}|^2+\right.$ $\left.V(\varepsilon_k x)v_{1,k}^2\right)dx=0$. By Sobolev embedding theorem, we have $\lim_{k\to 0^+}\int_{\mathbb{R}^2}|v_{1,k}|^s$ $dx=0$, for $2\le s<+\infty$. Hence, $\bar{c}=\lim_{k\to 0^+}\hat{J}_{\varepsilon_k}(v_{1,k})=0$. This is impossible. Then

$$\begin{aligned}
0&=\lim_{k\to\infty}I_k^1=\lim_{k\to 0^+}(b_1+b_2-\theta_1^{2-p}b_1-\theta_1^{6-p}b_2)\\
&\ge\lim_{k\to\infty}(1-\theta_1^{6-p})(b_1+b_2)=(1-\theta_0^{6-p})\lim_{k\to 0^+}(b_1+b_2)\\
&>0.
\end{aligned}$$

Then, we have a contradiction. We prove Claim 2 and Proposition 3.1. □

Define

$$w_k(x):=v_k(x+\xi_k)=u_k(\varepsilon_k x+\varepsilon_k\xi_k),$$

where the sequence $\{\xi_k\}$ is the one we obtained in (3.10). Then, $w_k(x)$ is a positive ground state of

$$\begin{cases}-\Delta w_k+V(\varepsilon_k x+\varepsilon_k\xi_k)w_k+A_0(w_k)w_k+\sum_{j=1}^2 A_j^2(w_k)w_k=|w_k|^{p-2}w_k,\\ \partial_1 A_0(w_k)=A_2(w_k)|w_k|^2,\quad \partial_2 A_0(w_k)=-A_1(w_k)|w_k|^2,\\ \partial_1 A_2(w_k)-\partial_2 A_1(w_k)=-\frac{1}{2}w_k^2,\quad \partial_1 A_1(w_k)+\partial_2 A_2(w_k)=0.\end{cases}\tag{3.25}$$

Lemma 3.2. *If* (V) *holds, then the sequence* $\{\varepsilon_k\xi_k\}$ *is bounded as* $k\to\infty$.

Proof. Assume that after there is a subsequence $\{\varepsilon_k\xi_k\}$ such that $\varepsilon_k\xi_k\to\infty$ as $\varepsilon_k\to 0^+$. Because $\hat{c}_\varepsilon$ is bounded, $\{w_k\}$ is also bounded in E_ε. Hence, up to a subsequence, there exists $w_0\in E_\varepsilon$ such that $w_k\rightharpoonup w_0$ weakly in E_ε as $k\to\infty$. Consequently, $w_k\to w_0$ strongly in $L^s_{loc}(\mathbb{R}^2)$, for $2\le s<+\infty$ and almost everywhere in $\mathbb{R}^2$. By (3.10), for any $\delta > 0$, there exists $\rho > 0$ such that

$$\left(\frac{1}{2}-\frac{1}{p}\right)\int_{B_\rho^c(\xi_k)}\left(|\nabla w_k|^2+V(\varepsilon_k x+\varepsilon_k\xi_k)w_k^2\right)dx \le \mu_k(B^c_{\rho_k}(\xi_k)) < \delta.$$

Then, by the Sobolev embedding theorem, we get

$$w_k \to w_0 \ \text{ in } L^s(\mathbb{R}^2) \text{ for any } s\in[2,+\infty). \tag{3.26}$$

We notice that

$$\begin{aligned}
&\left[\left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathbb{R}^2}\left(|\nabla w_0|^2+V_\infty w_0^2\right)dx\right.\\
&\qquad\left.+\left(\frac{1}{2}-\frac{3}{p}\right)\int_{\mathbb{R}^2}\left(A_1^2(w_0)w_0^2+A_2^2(w_0)w_0^2\right)dx\right]\\
&\ge \limsup_{k\to\infty}\left[\left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathbb{R}^2}\left(|\nabla w_k|^2+V(\varepsilon_k x+\varepsilon_k\xi_k)w_k^2\right)dx\right.\\
&\qquad\left.+\left(\frac{1}{2}-\frac{3}{p}\right)\int_{\mathbb{R}^2}\left(A_1^2(w_k)w_k^2+A_2^2(w_k)w_k^2\right)dx\right]\\
&= \limsup_{k\to\infty}\hat{c}_{\varepsilon_k} \ge c_{V_0} > 0.
\end{aligned}$$

Hence, $w_0(x)\not\equiv 0$. Take $h>0$ such that (3.5) holds. From (3.26), we obtain

$$\begin{aligned}
&-\Delta w_0+(V_\infty-h)w_0+A_0(w_0)w_0+\sum_{j=1}^{2}A_j^2(w_0)w_0-|w_0|^{p-2}w_0\\
&\quad\le 0 \ \ in\ H^{-1}(\mathbb{R}^2).
\end{aligned}$$

Especially,

$$\begin{aligned}
&\int_{\mathbb{R}^2}\left(|\nabla w_0|^2+(V_\infty-h)|w_0|^2\right)dx+3\int_{\mathbb{R}^2}\left(A_1^2(w_0)|w_0|^2+A_2^2(w_0)|w_0|^2\right)dx\\
&\quad<\frac{1}{p}\int_{\mathbb{R}^2}|w_0|^p\,dx,
\end{aligned} \tag{3.27}$$

since $w_0\not\equiv 0$. Choose $\theta>0$ such that $\theta w_0\in\Sigma_{V_\infty-h}$. Then, by (3.27), we have $\theta<1$. From $\varepsilon_k\xi_k\to\infty$ as $k\to\infty$, we have

$$\begin{aligned}
c_{V_\infty-h} &\le \frac{\theta^2}{2}\int_{\mathbb{R}^2}\left(|\nabla w_0|^2+(V_\infty-h)|w_0|^2\right)dx\\
&\qquad+\frac{\theta^6}{2}\int_{\mathbb{R}^2}\left(A_1^2(w_0)|w_0|^2+A_2^2(w_0)|w_0|^2\right)dx-\frac{\theta^p}{p}\int_{\mathbb{R}^2}|w_0|^p\,dx\\
&\le \liminf_{k\to\infty}\left[\frac{\theta^2}{2}\int_{\mathbb{R}^2}\left(|\nabla w_k|^2+V(\varepsilon_k x+\varepsilon_k\xi_k)|w_k|^2\right)dx\right.\\
&\qquad\left.+\frac{\theta^6}{2}\int_{\mathbb{R}^2}\left(A_1^2(w_k)|w_k|^2+A_2^2(w_k)|w_k|^2\right)dx-\frac{\theta^p}{p}\int_{\mathbb{R}^2}|w_k|^p\,dx\right]\\
&= \liminf_{k\to\infty}\lambda(\theta),
\end{aligned}$$

where $\lambda(\theta) := \frac{\theta^2}{2}b_1 + \frac{\theta^6}{2}b_2 - \frac{\theta^p}{p}(b_1+3b_2)$. We know that $b_1 + b_2 > 0$, we can prove that $\frac{d\lambda(\theta)}{d\theta} = b_1\theta + 3b_2\theta^5 - (b_1+3b_2)\theta^{p-1} > 0$, for $\theta \in (0,1)$. Hence, $\lambda(\theta) < \lambda(1)$ for $\theta \in (0,1)$. This and Lemma 2.6 imply

$$c_{V_\infty - h} \le \liminf_{k\to\infty} \lambda(1) = \lim_{\varepsilon_k \to 0^+} \hat{c}_{\varepsilon_k} \le c_{V_0},$$

which contradicts (3.5). □

From the above Lemma, we notice that for any sequence $\{\varepsilon'_k\} \to 0$, there exists a subsequence $\{\varepsilon_k\}$ such that $\bar{x}_k := \varepsilon_k\xi_k \to \xi_0$, $w_k \rightharpoonup w_0$ ($w_0 \ge 0$ and $w_0 \not\equiv 0$) weakly in E_ε as $\varepsilon_k \to 0^+$. Furthermore, (3.26) is true.

Lemma 3.3. *$c_{V(\xi_0)} = \inf_{x\in\mathbb{R}^2} c_{V(x)}$. Moreover, $w_k \to w_0$ strongly in E_ε, as $k \to \infty$.*

Proof. From elliptic regularity theory and (3.26), $w_k \to w_0$ in C^2_{loc} and

$$-\Delta w_0 + V(\xi_0)w_0 + A_0(w_0)w_0 + \sum_{j=1}^{2} A_j^2(w_0)w_0 = |w_0|^{p-2}w_0, \quad x \in \mathbb{R}^2.$$

Consequently, by (3.10) and (3.26), we have

$$\begin{aligned} c_{V(\xi_0)} \le &\left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathbb{R}^2}\left(|\nabla w_0|^2 + V(\xi_0)w_0^2\right)dx && (3.28)\\ &+\left(\frac{1}{2}-\frac{3}{p}\right)\int_{\mathbb{R}^2}\left(A_1^2(w_0)w_0^2 + A_2^2(w_0)w_0^2\right)dx && (3.29)\\ &\le \liminf_{k\to\infty}\left[\left(\frac{1}{2}-\frac{1}{p}\right)\int_{\mathbb{R}^2}\left(|\nabla w_k|^2 + V(\varepsilon_k x + \bar{x}_k)w_k^2\right)dx\right. && (3.30)\\ &\left.+\left(\frac{1}{2}-\frac{3}{p}\right)\int_{\mathbb{R}^2}\left(A_1^2(w_k)w_k^2 + A_2^2(w_k)w_k^2\right)dx\right] && (3.31)\\ &= \liminf_{k\to\infty}\hat{c}_{\varepsilon_k} \le \inf_{\xi\in\mathbb{R}^2} c_{V(\xi)}, && (3.32)\end{aligned}$$

which yields that $c_{V(\xi_0)} = \inf_{x\in\mathbb{R}^2} c_{V(x)}$. By (3.28), Proposition 2.2, and (3.26), we have

$$\lim_{k\to\infty}\int_{\mathbb{R}^2}\left(|\nabla w_k|^2 + V(\varepsilon_k x + \bar{x}_k)w_k^2\right)dx = \int_{\mathbb{R}^2}\left(|\nabla w_0|^2 + V(\xi_0)w_0^2\right)dx$$

From this and $w_k \rightharpoonup w_0$ weakly in E_ε as $k \to \infty$, we obtain $w_k \to w_0$ strongly in $H^1(\mathbb{R}^2)$, as $k \to \infty$. □

Theorem 3.4. *There exists a maximum point ξ_ε of $|u_\varepsilon|$ such that $u_\varepsilon(x+\xi_\varepsilon)$ converges to a least energy solution of (1.3) in $H^1(\mathbb{R}^2)$.*

Proof. We note that w_0 obtain in the proof of Lemma 3.3 satisfies the following system

$$\begin{cases} -\Delta w_0 + V(\xi_0)w_0 + A_0(w_0)w_0 + \sum_{j=1}^2 A_j^2(w_0)w_0 = |w_0|^{p-2}w_0, \\ \partial_1 A_0 = A_2|w_0|^2, \quad \partial_2 A_0 = -A_1|w_0|^2, \\ \partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2}w_0^2, \quad \partial_1 A_1 + \partial_2 A_2 = 0. \end{cases}$$

Since w_0 has exponential decay at infinity and C^2-convergence, w_k decays to zero at infinity. By the similar proof of Proposition 2.7, w_0 has maximum point. Let $\hat{p} \in \mathbb{R}^2$ and R, $\delta > 0$ such that

$$w_0(\hat{p}) = \max_{x \in \mathbb{R}^2} w_0 \geq \delta \tag{3.33}$$

and $0 < w_0(x) \leq \frac{\delta}{4}$ for $|x| \geq R$. Since

$$w_k \to w_0 \text{ in the sense } C^2_{loc}(\mathbb{R}^2), \tag{3.34}$$

w_k converges to zero at infinity. Take $\hat{p}_k$ satisfying $w_k(\hat{p}_k) = \max_{x \in \mathbb{R}^2} w_k(x)$. From (3.33), $\hat{p}_k \in \bar{B}_R(0)$. We claim that the maximum points of w_k converge to the same point. Indeed, recall that $\bar{w}_k(x) = w_k(\frac{x}{\varepsilon_k})$ is a solution of (1.1) where ε_k take the place of ε and their maximum points $\bar{p}_k$ are given by $\bar{p}_k = \varepsilon_k \hat{p}_k + \varepsilon_k \xi_k$. Hence, as $\varepsilon_k \xi_k \to \xi_0$, we obtain $\bar{p}_k \to \xi_0$ with $c_{V(\xi_0)} = \inf_{x \in \mathbb{R}^2} c_{V(x)}$. Therefore, w_k concentrates near ξ_0. □

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