Erratum to: On the viscosity solutions to Trudinger's equation

Tilak Bhattacharya and Leonardo Marazzi

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This note corrects the construction of a super-solution in Part II of Section 6 for the case $2 \leq p \leq n$. Please refer to the original article for this part of the discussion and the equation numbers. Let h be as defined in (1.3), $M = \sup_{P_T} h$ and $m = \inf_{P_T} h$.

We made use of scaling to render the second term on the left side of (6.9) small. However, the condition in (6.13), i.e.,

$$\lambda \log \left(\frac{M^{1/\lambda} + 2\hat{\varepsilon}}{\hat{h}(y, \hat{s}) + 2\hat{\varepsilon}} \right) < 1,$$

for small $\hat{\varepsilon}$, holds if $\log[(M + 2\varepsilon)/(h(y, s) + 2\varepsilon)] < 1$. Since $\varepsilon > 0$ is small, $M/m \leq e$. In this note, we provide an alternative that removes this restriction.

1. Super-solutions in Section 6 and Theorem 1.2

Part II: Super-solutions for $2 \leq p \leq n$. We construct a function $\eta > 0$ such that

$$\Delta_p \eta + (p-1)|D\eta|^p - (p-1)\eta_t \leq 0, \quad \text{in } \Omega_T \quad \text{and} \quad \eta \geq \log h \quad \text{on } P_T. \quad (1.1)$$

Recall from the paper that $m > 0$ and $\varepsilon > 0$, small, so that $m - 2\varepsilon > 0$.

Let $(y, s) \in P_T$ with $s > 0$. Then there are $\delta_0 > 0$ and $\tau_0 > 0$ such that $\forall (x, t) \in P_T$ with $|x - y| \leq \delta_0$ and $|s - t| \leq \tau_0$, we have

$$h(y, s) - \varepsilon \leq h(x, t) \leq h(y, s) + \varepsilon. \quad (1.2)$$

Lemma 1.1. *Let $A > 0$, $B > 0$, $\beta > 0$ and*

$$j(r) = \log \left(A - \frac{B}{r^\beta} \right), \quad \text{in } r > (B/A)^{1/\beta}.$$

Set $\Gamma = (n - p) - (p - 1)\beta$. Then,

$$\Delta_p j + (p - 1)|Dj|^p = \left(\frac{\Gamma}{r^p} \right) \left(\frac{\beta B}{Ar^\beta - B} \right)^{p-1}, \quad \text{in } r > (B/A)^{1/\beta}.$$

Proof. We make use of the following expression, i.e., in $r > 0$,

$$\Delta_p j + (p - 1)|Dj|^p = |j(r)'|^{p-2} \left((p - 1)j(r)'' + \frac{n - 1}{r}j(r)' \right) + (p - 1)|j(r)'|^p. \quad (1.3)$$

Differentiating,

$$\begin{aligned} j(r)' &= \frac{\beta B}{Ar^{\beta+1} - Br}, \quad rj(r)' = \frac{\beta B}{Ar^\beta - B} \quad \text{and} \\ j(r)'' &= -\beta B \left[\frac{(\beta + 1)Ar^\beta - B}{(Ar^{\beta+1} - Br)^2} \right]. \end{aligned} \quad (1.4)$$

Using (1.4),

$$j(r)'' = -\frac{j(r)'}{r} \left(\frac{A(\beta + 1)r^\beta - B}{Ar^\beta - B} \right). \quad (1.5)$$

Note that $j(r)' > 0$ and recall that $\Gamma = (n - p) - (p - 1)\beta$. Using (1.3), (1.4) and (1.5)

$$\begin{aligned} \Delta_p j &= (j(r)')^{p-2} \left[(n - 1)\frac{j(r)'}{r} - (p - 1)\frac{j(r)'}{r} \left(\frac{(\beta + 1)Ar^\beta - B}{Ar^\beta - B} \right) \right] \\ &= \frac{(j(r)')^{p-1}}{r} \left[(n - 1) - (p - 1) \left(\frac{\beta Ar^\beta + (Ar^\beta - B)}{Ar^\beta - B} \right) \right] \\ &= \frac{(j(r)')^{p-1}}{r} \left[\frac{(n - p)(Ar^\beta - B) - (p - 1)\beta Ar^\beta}{Ar^\beta - B} \right] \\ &= \frac{(j(r)')^{p-1}}{r} \left(\frac{\Gamma Ar^\beta - (n - p)B}{Ar^\beta - B} \right). \end{aligned} \quad (1.6)$$

Using (1.3), (1.4) and (1.6) we get

$$\begin{aligned} \Delta_p j + (p - 1)|Dj|^p &= \frac{(j(r)')^{p-1}}{r} \left[\frac{\Gamma Ar^\beta - (n - p)B}{Ar^\beta - B} + (p - 1)rj(r)' \right] \\ &= \frac{(j(r)')^{p-1}}{r} \left[\frac{\Gamma Ar^\beta - (n - p)B + (p - 1)\beta B}{Ar^\beta - B} \right] \\ &= \frac{\Gamma(j(r)')^{p-1}}{r} = \frac{\Gamma}{r^p} \left(\frac{\beta B}{Ar^\beta - B} \right)^{p-1}. \end{aligned}$$

The lemma follows. $\square$

We construct now a super-solution η as follows. Let β , $0 < \tau \leq \tau_0$, small, $\ell > 0$, $\theta > 0$, $\rho > 0$ such that

$$\begin{aligned} \beta &= \frac{n-1}{p-1}, \quad \ell = \frac{1}{\tau} \log \left(\frac{M+2\varepsilon}{h(y,s)+2\varepsilon} \right), \quad \log \left(\frac{1+\theta-2^{-\beta}}{\theta} \right) = \ell\tau, \quad \text{and} \\ 0 < \rho &\leq \min \left[\frac{\delta_0}{4}, \frac{1}{2\ell^{1/p}} \left(\frac{\beta}{[2^\beta(\theta+1)-1]} \right)^{(p-1)/p} \right]. \end{aligned} \quad (1.7)$$

With θ as in (1.7), set

$$F(\theta, \rho, r) = \log \left(\frac{1+\theta-(\rho/r)^\beta}{\theta} \right), \quad \rho \leq r \leq 2\rho. \quad (1.8)$$

Then F is increasing in r ,

$$\begin{aligned} \text{(i)} \quad & F(\theta, \rho, \rho) = 0, \quad \text{(ii)} \quad F(\theta, \rho, 2\rho) = \ell\tau, \quad \text{(iii)} \quad F \leq 0, \text{ for } r \leq \rho, \quad \text{and} \\ \text{(iv)} \quad & F \geq 0, \text{ for } r \geq \rho. \end{aligned} \quad (1.9)$$

Let (y, s) be as in (1.2). By the outer ball condition, there is a $\gamma_0 > 0$ such that for each $0 < \gamma \leq \gamma_0$, there is a $z \in \mathbb{R}^n \setminus \bar{\Omega}$ (depending on γ) with $B_\gamma(z) \subset \mathbb{R}^n \setminus \bar{\Omega}$ and $y \in \partial B_\gamma(z)$.

From hereon, τ and ρ satisfy (1.7) with the additional imposition that

$$0 < \rho < \gamma_0.$$

Set $r = |x - z|$ and $S_{\rho,\tau}(z) = (B_{2\rho}(z) \setminus B_\rho(z)) \times [s - \tau, s + \tau]$, the cylindrical shell centered at (z, s) . Clearly, $\bar{S}_{\rho,\tau}(z) \subset \bar{B}_{\delta_0}(y) \times [s - \tau, s + \tau]$. Let S^+ denote the part of $S_{\rho,\tau}(z)$ in $s \leq t \leq s + \tau$ and let S^- the part in $s - \tau \leq t \leq s$.

Define

$$\psi(r, t) = \begin{cases} \ell(s - t) + \ell\tau - F(\theta, \rho, r), & \forall (r, t) \in S^+, \\ \ell(t - s) + \ell\tau - F(\theta, \rho, r), & \forall (r, t) \in S^-. \end{cases} \quad (1.10)$$

Using (1.8) and (1.9), ψ is C^∞ in x , in $r > \rho$, and Lipschitz continuous in t . Moreover,

$$\begin{aligned} \psi(\rho, t) &= \ell(s + \tau - t) \quad \text{in } s \leq t \leq s + \tau, \quad \psi(\rho, t) = \ell(t - (s - \tau)) \\ &\quad \text{in } s - \tau \leq t \leq s, \quad \text{and} \quad \psi(r, s) = \ell\tau - F(\theta, \rho, r), \quad \text{in } \rho \leq r \leq 2\rho. \end{aligned}$$

Also, $\psi(r, s \pm \tau) \leq 0$, $r \geq \rho$, implying that $\psi(r, s \pm \tau) = 0$ if and only if $r = \rho$.

Next, recalling (1.8), (1.9) and (1.10) set

$$R = \{(r, t) \in S : \psi(r, t) \geq 0\}. \quad (1.11)$$

Call $R^+ = S^+ \cap R$ and $R^- = S^- \cap R$. It is clear that $R \subset \Omega_T \cap (B_{\delta_0}(y) \times [s - \tau, s + \tau])$. Here, $R \cap \{t = s\} = B_{2\rho}(y) \setminus B_\rho(y)$, is the widest section, and R narrows as $|t - s|$ increases to τ . See discussion following (1.10).

In $\bar{\Omega}_T$, define the *indent function*

$$\eta(x, t) = \begin{cases} \log(M + 2\varepsilon) - \psi(r, t), & \forall (x, t) \in R, \\ \log(M + 2\varepsilon), & \forall (x, t) \in \bar{\Omega}_T \setminus R. \end{cases} \quad (1.12)$$

Using (1.2), (1.7), (1.8) and (1.10) we see that

- (i) $\eta(y, s) = \log(h(y, s) + 2\varepsilon)$, (ii) $\eta(x, t) \leq \log(M + 2\varepsilon)$, $\forall (x, t) \in \overline{\Omega}_T$,
- (iii) $\log h(x, t) \leq \log(h(y, s) + 2\varepsilon) \leq \eta(x, t) \leq \log(M + 2\varepsilon)$,
 $\forall (x, t) \in R \cap P_T$, and
- (iv) $\eta(x, t) \geq \log h(x, t)$, $\forall (x, t) \in P_T$.

If we show that η is a super-solution in Ω_T , the observations listed above would imply that η is a super-solution of (1.1). First we show that η is a super-solution in $R \cap \Omega_T$. Lemma 5.1 of the original article will then show that η is a super-solution in Ω_T .

Case $t \neq s$: Using (1.10) and (1.12), we observe that $\eta_t = \ell$ in R^+ and $\eta_t = -\ell$ in R^- . The choice in (1.7) implies that $\Gamma = -(p-1)$. In Lemma 1.1, take $A = (1 + 1/\theta)$ and $B = \rho^\beta/\theta$ then (1.7), (1.8) and (1.12) imply in $\rho \leq r \leq 2\rho$ and $s - \tau \leq t \leq s + \tau$ that

$$\begin{aligned} \Delta_p \eta + (p-1)|D\eta|^p - (p-1)\eta_t &= \Delta_p F + (p-1)|DF|^p \pm (p-1)\ell \\ &\leq (p-1) \left(\ell - \frac{1}{r^p} \left(\frac{\beta B}{Ar^\beta - B} \right)^{p-1} \right) \\ &= (p-1) \left(\ell - \frac{1}{r^p} \left(\frac{\beta \rho^\beta}{\theta \left[\left(1 + \frac{1}{\theta}\right) r^\beta - \frac{\rho^\beta}{\theta} \right]} \right)^{p-1} \right) \leq 0, \end{aligned}$$

by taking $r = 2\rho$ in the above and using (1.7). The case $t = s$ is similar to the proof in Part II of Section 5 in the original article.

Tilak Bhattacharya
 Department of Mathematics
 Western Kentucky University
 Bowling Green
 KY 42101
 USA
 e-mail: tilak.bhattacharya@wku.edu

Leonardo Marazzi
 Department of Liberal Arts
 Savannah College of Arts and design
 Savannah
 GA 31401
 USA
 e-mail: lmarazzi@scad.edu